THE RIEMANN HYPOTHESIS

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1. Introduction

In my Abel lecture [1] at the ICM in Rio de Janeiro 2018, I explained how to solve a long-standing mathematical problem that had emerged from physics. The problem was to understand the fine structure constant α.

The full details are contained in [2] which has been submitted to proceedings A of the Royal Society. The techniques developed in [2] are a novel fusion of ideas of von Neumann and Hirzebruch. They are sophisticated and powerful, based on an infinite iteration of exponentials, while having an inherent simplicity.

Attacking the mystery of α was the motivation, but the power and universality of the methods indicated that they should solve other hard problems, or at least shed new light on them if they are insoluble. In expanding my Abel Lecture for the ICM Proceedings I speculated that the techniques of [2] might lead to the new subject of Arithmetic Physics.

The Riemann Hypothesis RH is the assertion that $\zeta(s)$ has no zeros in the critical strip $0 < Re(s) < 1$, off the critical line $Re(s) = 1/2$. It is one of the most famous unsolved problems in mathematics and a formidable challenge for the programme envisaged in [1]. I believe it will live up to this challenge, and this paper will provide the proof.

The proof depends on a new function $T(s)$, the Todd function, named by Hirzebruch after my teacher J.A.Todd. Its definition and properties are all in [2] but, in section 2, I will review and clarify them. In section 3 I will use the function $T(s)$ to prove RH. In section 4, entitled *Deus ex Machina*, I will try to explain the mystery of this simple proof of RH. Finally, in section 5, I will place this paper in the broader context of Arithmetic Physics as envisaged in [1].

2. The Todd function

In this section I summarize the properties of the Todd function $T(s)$, constructed in [2].

T is what I will call a **weakly analytic function** meaning that it is a weak limit of a family of analytic functions. So, on any compact set K in $\mathbb{C}$, T is analytic. If K is convex, T is actually a polynomial of some degree $k(K)$. For example a step function is weakly analytic and, for any closed interval K on the line, the degree is 0. This shows that a weakly analytic function can have compact support, in contrast to an analytic function. Weakly analytic functions are weakly dense in L^2 and in their weak duals. They are well adapted for Fourier transforms on all L^p spaces. They are also composable: a weakly analytic function of a weakly analytic function is weekly analytic.

Define $K[a]$ to be the closed rectangle

$$|Re(s-1/2)| \leq \frac{1}{4}, \quad |Im(s)| \leq a. \tag{2.1}$$

Then, on $K[a]$, T is a polynomial of degree $k\{a\} = k(K[a])$.

This terminology is formally equivalent to that of Hirzebruch [3], with his Todd polynomials. But Hirzebruch worked with formal power series and did not require convergence. That was adequate for his applications which were essentially algebraic and arithmetic, as the appearance of the Bernoulli numbers later showed.

However, to relate to von Neumann's analytical theory it is necessary to take weak limits as has just been done. This provides the crucial link between algebra/arithmetic and analysis which is at the heart of the ζ function.

This makes it reasonable to expect that RH might emerge naturally from the fusion of the different techniques in [2].

I return now to other properties of $T(s)$ explained in [2]:

2.2 T is real i.e. $T(\bar{s}) = T\bar{(}s)$.

2.3 $T(1) = 1$

2.4 T maps the critical strip into the critical strip and the critical line into the critical line.

(This is not explicitly stated in [2] but it is included in the mimicry principle 7.6, which asserts that T is compatible with any analytic formula, so in particular $Im(T(s-1/2)= T(Im(s-1/2)).)$

The main result of [2], identifying α with $1/$ж, was

2.5 on $Re(s) = 1/2$, $Im(s) > 0$, T is a monotone increasing function of $Im(s)$ whose limit, as $Im(s)$ tends to infinity, is ж.

As was noted above, on a given compact convex set, the Todd polynomials stabilize as the degree increases. In [3] Hirzebruch expressed this stability in the form of an equation:

2.6 if f and g are power series with no constant term, then

$$T\{[1+f(s)]\cdot[1+g(s)]\} = T\{1+f(s)+g(s)\}.$$

Remark. Weakly analytic functions have a formal expansion as a power series near the origin. Formula 2.6 is just the linear approximation of this expansion (more precisely this is on the branched double cover of the complex s-plane given by $\sqrt{s}$). This implies

2.6 $T(\sqrt{s}) = \sqrt{T(s)}$ or

2.7 $\sqrt{T(1+s)} = T(1+s/2)$

which gives us the uniform constant $1/2$ needed in 3.3 of section 3.

3. The proof of RH

In this section I will use the Todd function $T(s)$ to prove RH. The proof will be by contradiction : assume there is a zero b inside the critical strip but off the critical line. To prove RH, it is then sufficient to show that the existence of b leads to a contradiction.

Given b, take $a = b$ in 2.1 then, on the rectangle $K[a]$, T is a polynomial of degree $k\{a\}$. Consider the composite function of s, given by

$$F(s) = T\{1+\zeta(s+b)\} - 1 \tag{3.1}$$

From its construction, and the hypothesis that $\zeta(b) = 0$, it follows that

3.2 F is analytic at $s = 0$ and $F(0) = 0$.

Now take $f = g = F$ in 2.6 and we deduce the identity

3.3 $F(s) = 2F(s)$.

Since $\mathbb{C}$ is not of characteristic 2, it follows that $F(s)$ is identically zero. 2.3 ensures that T is not the zero polynomial and so it is invertible in the field of meromorphic functions of s. The identity $F(s) = 0$ then implies the identity $\zeta(s) = 0$. This is clearly not the case and gives the required contradiction.

This completes the proof of RH.

The proof of RH that has just been given is sometimes referred to as the search for the *first Siegel zero*. The idea is to assume there is a counterexample to RH, study the first such zero b, and hope to derive a contradiction.

This is exactly what we did. Using the composite function $F(s)$ of 3.1 with a zero at b, off the critical line, we found another zero b' which halves the distance $|s-\frac{1}{2}|$ to the critical line. Continuing this process gives an infinite sequence of distinct zeros, converging to a point (on the critical line).But an analytic function which vanishes on such an infinite sequence must be identically zero. Applying this to $F(s)$ (using 2.8 now instead of 2.6) shows that $F(s)$ is identically zero and this then leads to a contradiction as argued in the last few lines after 3.3.

Remark. This *Siegel* version of the proof can be viewed as a *renormalized* version of Fermat's proof of infinite descent. As is well known, the Fermat descent may not improve on the hypothetical solution. But our use of the Hirzebruch/von Neumann process of infinite ascent cancels the Fermat descent and enables us to derive a contradiction. What is crucial to make this work is establishing a uniform inequality. In our case the uniform factor is the 1/2 that appears in 2.8.

4. Deux ex machina

The proof of RH in section 3 looks deceptively easy, even magical, so in this section I will look behind the scenes and explain the magic. Clearly the function T is the secret key that unlocks the doors, so I must explain its secret.

In [1] I fused together the algebraic work of Hirzebruch, as summarized above, and the analytical work of von Neumann, enabling me to get the best of both worlds. In brief the merits of the two worlds are:

4.1 Hirzebruch worked with explicit polynomials T

4.2 von Neumann worked with the unique hyperfinite factor A.

Von Neumann's work is clearly deep since A is constructed by an infinite limit of exponential operations. Hirzebruch's work is deceptively simple, like that of all good magicians. But look carefully behind the scenes and it becomes clear that here too there is an infinite limit of exponentials. This time the limit is given by a sequence of discrete steps and the process is formal and algebraic. There is more detail in section 4 of [1].

The fusion between the work of Hirzebruch and that of von Neumann involves a passage from the discrete to the continuous, the transition from algebra to analysis. Although explained in [2], the new presentation in section 2 of this paper makes it clearer. The notion of a weakly analytic function captures the essence of the fusion.

I hope this brief explanation shows why the new technique is both powerful and natural. It should also have removed the mystery behind the short proof of RH.

In the final section 5 I will put this paper into the general context of Arithmetic Physics envisaged in [1].

5. Final Comments

In this final section I will comment on possible future developments in Arithmetic Physics. These comments are on two levels.

At the first level there are firm expectations. At the second level there are speculations.

Starting with the first level, some comments on RH. Using our new machinery, RH and the mystery of α, were solved. But RH was a problem over the rational field $\mathbb{Q}$, and there are many generalizations to other fields or algebras. I firmly anticipate much work in this direction.

There are also logical issues that will emerge. To be explicit, the proof of RH in this paper is by contradiction and this is not accepted as valid in ZF, it does require choice. I fully expect that the most general version of the Riemann Hypothesis will be an undecidable problem in the Gödel sense.

RH should be the bench mark for other famous problems in mathematics, such as the Birch-Swinnerton Dyer conjectures. I expect most cases will be undecidable.

I now pass to the second level. Following the example of α, and the more difficult case of the Gravitational constant G (see 2.6 in [2]), I expect that mathematical physics will face issues where logical undecidability will get entangled with the notion of randomness.

In 4-dimensional smooth geometry I expect the famous $11/8$ conjecture of Donaldson theory will prove to be undecidable, as will the smooth Poincare conjecture.

References

[1] M.F.Atiyah *Arithmetic Physics*. Proceedings of ICM Rio de Janeiro 2018

[2] M.F.Atiyah *The Fine Structure Constant*. submitted to Proc.Roy. Soc A 2018

[3] F.Hirzebruch *Topological methods in algebraic geometry (with appendices by R.L.E.Schwarzenberger, and A.Borel)*. Springer 1966