“213, 16A” and Mathematics

by

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This paper is based on ‘lectures’ in simple everyday language that I’ve been giving to mostly non-mathematical visitors on some motifs – see photographs below – in our house. Hopefully this written version shall appeal to a wider audience.[1]

1. “Perfectly proportioned.” The defining property of a *golden rectangle* is that, if we delete from it a square on a smaller side, then we are left with a similar rectangle, i.e., one having the same length-to-breadth ratio, and this ratio, that is, the common length-to-breadth ratio of all golden rectangles, is called *the golden ratio.*

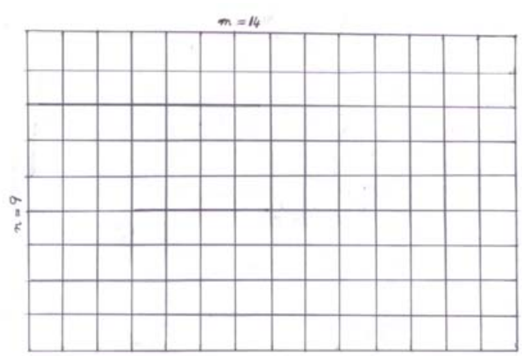
The two murals near the front door depict golden rectangles. Thus, deleting the blues gives a smaller golden rectangle, then, by throwing away the bluish greens, another which is still smaller, and after three more deletions, one arrives finally in these murals at the tiny white rectangles. Indeed, the inherent limitations of masonry, or for that matter, any practical form of visual representation, clearly dictate that one would need to stop, sooner or later, at some point or the other. However, what the spiralling in these murals is intended to evoke is that, in our God-given unlimited imagination, there is, just as clearly, no need to stop, now or ever. *We can continue deleting a square on a smaller side for as*

[1] Also, I’ll now be able to refer my inquisitors to this paper, which policy should, if nothing else, save those of our visitors who pose their queries more out of politeness than curiosity, from being bored to death by my possibly long-winded answers!

long as we wish! And therein, in this evoked idea, as I'll presently show, lies the real beauty of these depictions, a beauty which transcends by far their mere visuals, and in my humble, but admittedly prejudiced, mathematician's opinion, is far more fetching than anything the best painter has ever painted, or the best poet ever written.

In fact these 'mere visuals' already have – as you can confirm from the web – a huge fan following. From Phidias, through Leonardo da Vinci, to Le Corbusier, the architect who planned Chandigarh, the proportions of the golden rectangle have been raved about by all sorts of aesthetes, with Corbusier even opining that the navel divides a perfectly proportioned human body in the golden ratio, and then the two parts in turn are divided in the same ratio at the throat and the knees respectively! I've nothing to add to all this, but I do have something to say about an assertion that one finds on quite a few websites, and also on a panel in the City Museum in Sector 10, namely that, 'the golden ratio is equal to 1.618.' This is simply not true!

*The golden ratio is **not** equal to 1.618* (recall that 1.618 is decimal notation for 1618 by 1000). To see this, we note that, if the length-to-breadth ratio of our rectangle were 1618 by 1000, we could have subdivided each side of our rectangle into either 1618 or 1000 appropriately chosen equal units. Then, by drawing parallels through these subdivision points, we could have partitioned the rectangle into 1618×1000 equal squares. The deletion of the blues would have entailed throwing away 1000×1000 of these small squares, and then, that of the bluish greens, 618×618 more of these small squares, etcetra. So, since we have finitely many small squares in all, and are throwing away some at each step, we would have been done (even in our imagination) in finitely many steps. This contradicts the fact that the deletion process can (in our imagination) continue for as long as we wish. So, the length-to-breadth ratio of our rectangle is not 1618 by 1000, i.e., the golden ratio is not equal to 1.618.

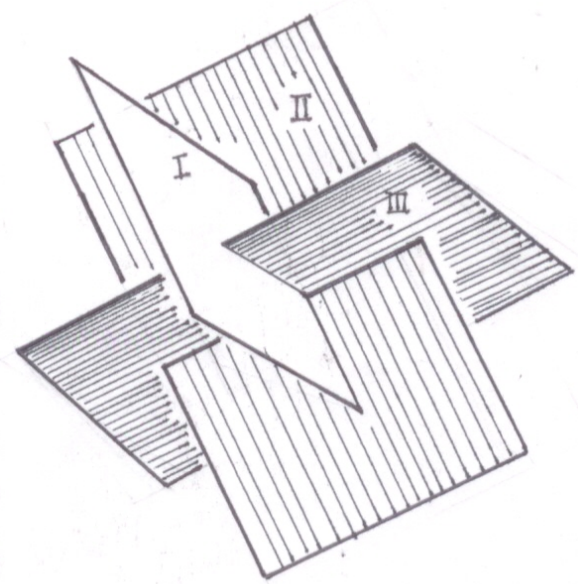


Remarkably 1618 and 1000 played no special role in this reasoning, it works just as well with any two whole numbers m and n! Therefore: *the golden ratio is not equal to any fractional number* m/n (m by n), or equivalently, *a golden rectangle cannot be partitioned into finitely many equal squares* in above way! A truly mind-blowing result! For, these squares can be as tiny as we want, even the size of a meson! Again, between any two fractional numbers, howsoever close, there is an infinity of fractional numbers, yet, on this already more than overcrowded ruler, there is somehow still some room left for numbers like the golden ratio that are not fractional!

There is much more that can be said about this argument (see Notes) but I'll move on now to a water feature, located on the other side of the front door, past that garden path, roman aqueduct, overhead water tank, and rockery with a stony brook hurrying down through its bonsai, cacti and white-washed temple into that small water pond.



2. "Beesmukhi" (bees = twenty, mukhi = faced) is what its Bhojpuri speaking sculptors called the concrete statue in this small water pond. It depicts – once we have sublimated those concrete rods into zero thickness edges in our boundless imagination and put down a triangle of our own into each triangular boundary – an *icosahedron* (icosa = twenty, hedron = faced), that is, a regular solid having 20 equilateral (all 3 sides equal) triangular faces, with 5 faces at each of its $(20 \times 3)/5 = 12$ vertices. We also see, thanks to the omission of the faces (triangular glass panes were considered briefly), the 'innards' of this statue, which reveal the secret of an icosahedron's existence: *its* $12 = 3 \times 4$ *vertices are the vertices of 3 identical rectangles* (those concrete slabs) *each inserted perpendicularly half-way through a parallel central slit of the next in cyclic order* (see the line drawing accompanying this photo, the cyclic order is I, II, III, I)!



To verify this, we note that the 12 vertices are equidistant from the common center of the 3 rectangles; and each is incident to 5 edges (concrete rods), while each edge has 2 vertices, so there are in all $(12 \times 5)/2 = 30$ edges. Of these, the 24 which join a vertex to one of the two nearest vertices of the next rectangle have clearly, by symmetry, the same length *a*. Also, the remaining 6 edges, being the shorter sides of the identical rectangles, have the same length *b*. In general, *a* is *not* equal to *b,* and only 8 of the obtained triangles are equilateral with sides $\{a, a, a\}$, the remaining 12 are only isosceles (two sides equal) with sides $\{a, a, b\}$. However, *the above construction gives an icosahedron, provided the three identical rectangles have a special length-to-breadth ratio.* For, if we keep *b* fixed, then *a* increases or decreases with *l,* the common length of the rectangles, being patently much *bigger* than *b* when the rectangles are very long, but is *lesser* than *b* when the rectangles approach 'squarehood' (the angle between the equal sides of the isosceles triangles is then almost a right angle), so there is an optimal *l* at which $a = b$.

Once again, it is time to move on, so I won't elaborate further (but see Notes) on the above argument, and shall only mention without proof that *the above 'special' ratio turns out to be none other than the golden ratio!* Which reminds me, we should have (because syntactically correct oxymorons are dime a dozen) proved the existence of a golden rectangle also, not gullibly accepted its defining prescription at face value (and the missing beauties of the next 'lecture' shall drive home this existential point further). Luckily, this particular oversight is easily rectified. We note that, the length-to-breadth ratio, of the rectangle obtained by deleting a square on a smaller side from a rectangle with fixed breadth *b* and shrinking length *l,* is equal to 1 when this varying length is 2*b*, and becomes arbitrarily large as *l* approaches *b,* so there is an optimal length *l* at which the rectangle obtained after deletion has also the same ratio *l/b*.

Also, here explicitly is the omitted general prescription for a *regular solid* {p,q}: its finitely many vertices should be on a sphere, its faces should be equal *regular*—i.e., with consecutive vertices equally spaced on a circle—*polygons* with p sides and vertices, and there should be q of these faces at each vertex. Clearly, p and q must be whole numbers 3 or bigger, and there exists a regular polygon with p sides for any such p, so, from the delicate way in which we settled the case {3,5} (icosahedron), it is natural to fear that the general existence problem for regular solids must be hard. Surprisingly, this is *not* the case, but, since it would be best to first meet the Egyptian beauty (no, not Cleopatra!) which inspired this unexpectedly easy solution, let's move on a bit, past this glorious sextet of golden barrels – which is almost as old as this fifty-year-old house – towards that small gate, and look back at the new addition on the roof.

3. "Pyramid." The four topmost glass panels suggest a small pyramid with a square base and equilateral triangular sloping faces, but only two of these extend to those much bigger but similar sloping triangular walls of glass, so, more imaginatively, you might like to think in terms of a bigger pyramid of the same kind that has only managed to free itself partially from the confines of the edifice! [The remaining walls of this stair-head room, dubbed "*Kuttiya*" (cottage) by the carpenters, are mostly opaque and plumb, and its interior is 'ethnic' and spartan: brick and salvaged circa 1958 marble-chips flooring, raised sitting area also in brickwork with in-built chessboard, etc.] A quick continuity argument confirms that, if the apex is at a suitable height – in fact at height equal to the radius of the base – above the center of the square base, then the sloping and obviously isosceles triangular faces of the pyramid shall be equilateral.

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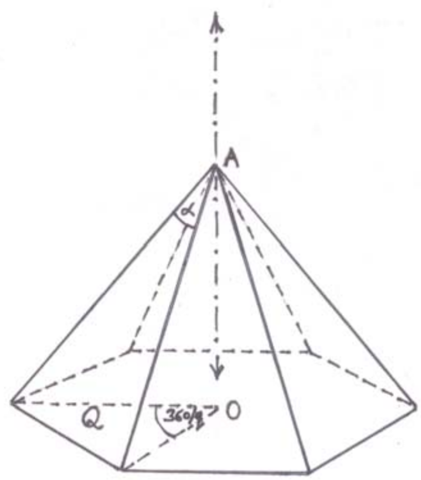
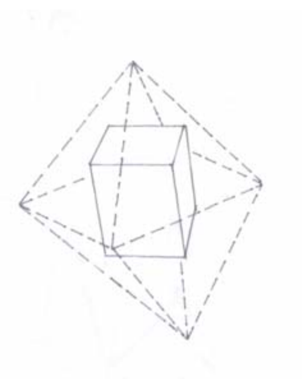
More generally, *for a variable pyramid, with base a fixed regular polygon* Q *with* q *sides, and apex* A *moving up and down on the perpendicular line through the center* O *of the base, the angle* α *subtended at the apex by a polygonal edge takes all values less than* 360/q *degrees, and only these values.* For, the equal edges of this isosceles triangle (see figure) become very long, so their included angle α becomes very small, as A recedes towards infinity, while α increases towards 360/q when A approaches O. (In particular, since 60 is less than 360/4 = 90, there exists a square pyramid for which α is 60 degrees, i.e., one with sloping isosceles faces equilateral).

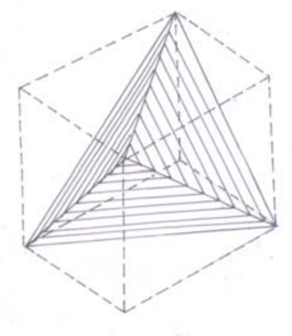


This generalization was motivated by the fact that, *a regular solid* {p,q} *can exist only if there exists a pyramid of the above kind with* α *equal to an angle of a regular polygon with p sides.* Indeed, any pyramid with A a vertex of {p,q} and vertices of Q equidistant points on the q incident edges shall do the job.

We invoke next the 'theorem' that, *the sum of the angles of a triangle is a straight angle* (180 degrees), which you probably still remember from school? If not, you can safely take it on faith, for it is logically equivalent to *Euclid's fifth postulate,* the main unproved 'axiom' of his geometry. Now that Euclid's name has come up, let me remark that all these beautiful things that I'm telling you about, about golden rectangles and regular solids and pyramids and such, they are all in that great treatise on geometry that was written by this Egyptian, virtually under the shadow of the Great Pyramid, about 2 to 3 centuries *before* the advent of the fair and beauteous Cleopatra. Coming back to our problem, we note that a regular polygon with p sides can be cut up into p-2 triangles, so the sum of its angles is (p-2) $\times$ 180 degrees, so each is ((p-2) $\times$ 180)/p degrees. Therefore, *a regular solid* {p,q} *can exist only if* ((p-2) $\times$ 180)/p < 360/q, a very strong constraint indeed, for this inequality can be rewritten q(p-2) < 2p, i.e. qp – 2q – 2p < 0, i.e. qp – 2q – 2p + 4 < 4, i.e. (q-2)(p-2) < 4, *i.e., the product of the whole numbers* q-2 *and* p-2 *must be less than four, i.e.* {p,q} = {3,3}, {3,4}, {4,3}, {3,5} *or* {5,3}!

To mop up note that, *given any* {p,q} *we can make a* {q,p} *as follows*: for each vertex of {p,q} take the regular polygon whose vertices are the q centers of the incident faces, the solid enclosed by all these polygons is a {q,p}. We already know that {3,5} or *icosahedron* exists, and pasting (see figure) together the bases of two equal square pyramids with equilateral sloping faces gives us a {3,4} or *octahedron* (eight equilateral triangular faces, four at each vertex). So by this method we can also make a {5,3} or *dodecahedron* (twelve regular pentagonal faces, three at each vertex) and {4,3} or *hexahedron* (six square faces, three at each vertex), respectively. The otherwise so rich English language has, very curiously, a popular synonym, *cube*, only for the regular solid {4,3}. Even the simplest regular solid {3,3}, which can be realized by choosing any four mutually non-adjacent vertices of a cube (see figure) goes only by its rather pedantic name, *tetrahedron* (four equilateral triangular faces, three at each vertex).

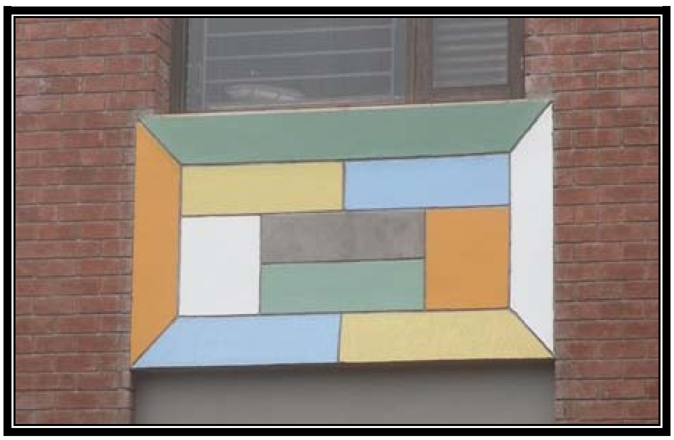




These five beauties are all there in this house! That "white-washed temple" on the rockery was a {3,3} (did you notice?), and we've discussed "Beesmukhi" = {3,5} at length, and this glass "Pyramid" (believe me!) is really the tip of a big {3,4} trying to free itself from the confines of this edifice. On the first-floor terrace, there are the **"Eyes"** (photo above) or the skylights of that new and skew ground-floor living room. These depict the two pieces of a {4,3} (cube) that one obtains when it is cut perpendularly to its main diagonal, and there is much of interest (see Notes) that can be said about these hexagonal sections, as well as the grills showing beneath them. The {5,3} is also in the house, but as befits its special metaphysical significance, in a different avatar.

To explain this rather cryptic remark, I recall that Aflatoon (i.e. Plato, in Punjabi) had associated {3,3} with *Fire*, {4,3} with *Earth*, {3,4} with *Air*, {3,5} with *Water* – the 'Four Elements' – and {5,3} with the *Universe* which contains everything, as well as the *Quintessence* of everything (One is All, All is One). Even as metaphysics, this association looks somewhat arbitrary, for, we saw that {p,q} and {q,p} are practically the two sides of the same coin, then why are their associates so different? However, mankind always has, and always shall, try – with of course varying success – to find in mathematical certainty the basis for all sorts of beliefs and facts. Indeed, a large number of today's theoretical physicists seem to be (if not openly declared, then closet) Platonists. For instance, with just a wee bit of poetic license (see Notes for elaboration) one can say that 'the special Lie group E_8', in the symmetries of which String Theory is currently seeking the truth about quarks and practically everything else, is nothing but a higher-dimensional regular solid. Aflatoon would have approved!

4. “Miss Universe.” That wall-painting with bevelled frame under my office window depicts a ‘*topologist’s dodecahedron’*. Exactly one of the four sides of each cemented panel is subdivided into two by a fifth vertex, so these panels are non-regular pentagons. Further, 5 different colours have been assigned to 5 pairs of panels, but the eleventh panel is left uncoloured, which should alert you to the fact that there is a twelfth face, which is – of course! – the obviously uncoloured and missing ‘glass pane’ in front of this painting. With this understood, you can check that there are, at each of the 20 vertices, exactly 3 of these pentagonal faces, so what we have here is a rather severely squashed {5,3}. Yet, this distortion is mild enough, and of no import, to *a topologist, i.e.*, ‘*a person who can’t distinguish between his/her coffee-cup and doughnut*’ (see Notes for more on this and other examples of topologically equivalent shapes).

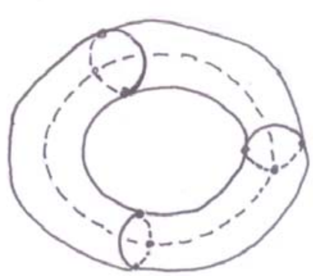


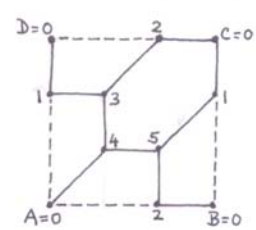
More seriously, topology is an extreme generalization/simplification of Euclid’s geometry in which distance and angle become bit players, only continuity matters. Accordingly, two shapes are *topologically equivalent* if there exists a one-one onto correspondence between them which is continuous in both directions. So a ‘*topologist’s polygon’* with p sides may not (unlike the pentagons of this mural) even be planar, and its specified p boundary edges, joining one after another its specified p vertices, can be very crooked. This implies that a *closed surface* bounding any reasonably smooth body admits (infinitely many) *polygonal subdivisions* into finitely many topologist’s polygons. By choosing one point (these shall be the vertices of the new subdivision) in each face of a polygonal subdivision, and for each pair of faces having a common edge a curve (these shall be the edges of the new subdivision) across this edge between the two points, we get a *dual polygonal subdivision* of the same surface with each face containing just one vertex of the original subdivision. For example, this construction applied to the mural above would give us a ‘*topologist’s icosahedron* {3,5}’. More generally a ‘*topologist’s* {p,q}’ shall be any reasonably smooth body with a polygonal subdivision of its bounding surface into finitely many topologist’s p-sided polygons, q at each vertex.

Topology frees us from the shackles of Euclid’s fifth postulate, which had a decisive say in the last argument, so it seems there can’t be *just five possibilities, with* p *and* q *at least three, for which one has a topologists’s* {p,q} *on a ball?* However the answer remains *“yes”*, but there are brave new worlds (to wit, pretzels!) on which the remaining lost beauties {p,q} dwell. We denote by V, E and F the number of vertices, edges and faces of a polygonal subdivision of a surface. If it is a topologist's {p,q}, each face has p edges and each vertex belongs to q edges, while each edge belongs to 2 faces and has 2 vertices, so pF = 2E = qV = say t, so V – E + F = t/q – t/2 + t/p = t(2p – pq + 2q)/2pq. Therefore, *if* V – E + F *is positive, then* pq – 2p – 2q is negative, which implies (assuming that both p and q are at least three) as before the same conclusion, viz., {p,q} = {3,3}, {3,4}, {4,3}, {3,5} *or* {5,3}. At first sight our new hypothesis appears useless, but looks can be deceptive! *The number* V – E + F *is the same for* ***all*** *polygonal subdivisions of a closed surface,* i.e., it is a *topological invariant* of the surface, and can be easily computed by using any convenient polygonal subdivision of the same.

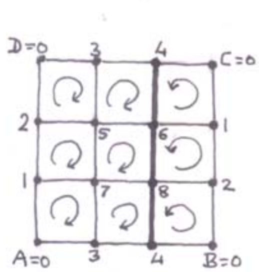
To get a feel of the proof subdivide any polygonal subdivision, with V vertices, E edges and F faces, further by putting a new vertex (a) inside a face with u edges and joining it to the vertices of this face, or (b) on an edge and joining it to the vertices of the two, say u- and v-sided, faces to which the edge belongs. In case (a) the new subdivision has V + 1 vertices, E + u edges and F + u – 1 faces, while in case (b) it has V + 1 vertices, E + u + v – 3 edges and F + u + v – 4 faces; so V – E + F stays put under both operations. The required invariance is then a corollary of the fact (see Notes for more details) that any two polygonal subdivisions P and Q of a surface are related to each other by a finite sequence of polygonal subdivisions $P = P_1$, P_2 , ... , $P_n = Q$, such that at each step either P_{i+1} is such an elementary subdivision of P_i or vice versa.

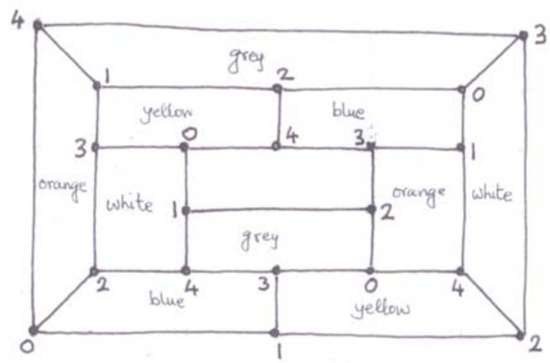
In particular, V – E + F = 2 for the five regular solids, which are topologically equivalent to a ball, so this formula holds for any polygonal subdivision of its surface. The lost beauties can only be on surfaces whose V – E + F is non-positive, and one such we've already mentioned, viz., the surface of a doughnut or a coffee cup or, if you please, a pretzel with 1 hole. It has V – E + F = 0 and the two figures below depict a {4,4} and a {6,3} on it. More generally, the surface of a pretzel, or a block of wood, with t holes has V – E + F = 2 – 2t, which is a very negative number if t is very big, which prompts the query: *can all the missing beauties* {p,q} *be found on surfaces of pretzels?* As we'll see later, the answer is *"yes"*, and just like Plato's five, these {p,q}'s are also geometrically regular, if we are willing to use distances not satisfying Euclid's fifth postulate!





In the last figure we used the fact that, *the surface of a doughnut can be obtained by glueing opposite sides of a square*: AB ≡ DC converts the square into a cylinder, and AD ≡ BC corresponds to bringing the circular ends of this cylinder together in space. If we alter our glueing slightly to AB ≡ DC and AD ≡ CB something very remarkable happens: we obtain (in our imagination) a shape, called *Klein's Bottle,* which is locally just like any closed surface, but which cannot possibly exist in 3-dimensional space! To see this note that if we put a circular arrow, on each face of a polygonal subdivision of a closed surface, which is clockwise when seen from the direction of the enclosed body, then each edge occurs with opposite directions with respect to the arrows on its two faces, but Klein's Bottle does not obey this condition, for example, the choice of arrows depicted in the next figure is bad on the heavy edges. We can hope to find Klein's Bottle only in a higher-dimensional euclidean space!





Just so there is no confusion here, let me emphasize once again that geometry is really played out in the mind, it is a logical development of some axioms, and its admittedly helpful, but necessarily imperfect figures are quite dispensable. One goes from 2- to 3-dimensional space simply by postulating a new direction perpendicular to the old ones, and using this device again and again, *we can do Euclid's geometry in* 4, 5, *or as many finite dimensions,* n, *as we like.* We'll denote this *euclidean n-dimensional space* by E^n, and connected and closed shapes which topologically are just like E^n locally shall be called *n-dimensional manifolds*. Examples: an *n-dimensional sphere* S^n, i.e., a subset of E^{n+1} consisting of all points at a fixed distance from a fixed point; again, any closed surface of euclidean 3-dimensional space is a 2-dimensional manifold; but so too is Klein's Bottle; likewise, by glueing pairs of faces of a subdivision of the surface of a ball one can obtain many 3-dimensional manifolds, above all, *Poincaré's manifold* P^3, which can be obtained, from the body bounded by this bevelled wall-painting and its 'glass pane' by identifying the 6 like coloured pairs of faces as above.

Whatever one might think of Plato's {5,3} = Universe/Quintessence – or its 'improvement' {5,3}/{3,5} = Universe/Quintessence? – *this offspring* P^3 *of* {5,3} *has been at the centre of the topological universe for a 100 years and more!* It was known that S^2 was the only 2-dimensional manifold in which every loop bounded. For some time Poincaré thought that a like statement was true in dimension 3, till he found P^3, which has this property but, unlike S^3, is not *simply connected*: it has loops which cannot be shrunk to a point. Thus was born *Poincaré's Conjecture* – 'a simply connected closed 3-dimensional manifold is topologically equivalent to S^3' – the (erstwhile) holy grail of topology which spawned numerous beautiful results and finally a solution by Perelman so natural that it promises much more. Or, for that matter, take the embeddability of P^3 in E^4 which (unlike the embeddability of the Klein Bottle in E^3 that we disposed off in short order) turned out to be extremely delicate: the answer is "no" if the embedding is required to be the slightest bit reasonable, nevertheless Freedman showed that there is a very crumpled copy of P^3 in E^4, and this unexpected "yes" unlocked the door to the complete classification of simply connected 4-manifolds! I'm sure the last few sentences have gone completely above your head; to be quite frank, I don't really understand all this myself, all I know is that there is scintillating beauty here, but despite trying off and on, I've only caught glimmers of it so far, but enough to keep on trying.

Uncannily enough, this offspring P^3 of "Miss Universe" has also been weighed by physicists as the global shape of the spatial universe – one can almost hear Aflatoon's: "I told you so!" – as revealed to us by those mysterious radio waves constantly coming to Earth because of that 'Big Bang' aeons ago. However, I'm sure Plato would have been more impressed by what mathematicians have gotten out of his beloved {5,3}, to my mind there is nothing in any other field which is remotely as deep or attractive.

INTERMISSION

The following is based on tea-time conversations – for example, in this 'new and skew' room I mentioned before – with parents of school-going children and others.

A common reaction to these 'lectures' is that they are unlike any my listeners heard in school or college. At which I blurt out that, at least in all of India north of the Narbada, this is the common reaction to anyone saying anything mathematical in any manner. The unfortunate fact quite simply is that the "mathematics" which is being taught, learnt and done in our schools, colleges and even universities, is as close to mathematics as Johnny Walker's "chan-chen chin-chon-chun" is to Chinese![2] The tail constantly wags the dog in these parts, *and* the baby gets thrown out with the bathwater every day: "mathematics" means monotonously memorizing methods for mindlessly doing the stereotypical questions in that looming examination at the tail-end of the year, and reasoning – such a waste of precious time! – finds no place in this inane and often panic-stricken drill.

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[2] Bollywood comedians were and are prone to mimicking alien languages to raise laughs. A further note for mathematicians in developed countries: beware of drawing parallels with shortcomings in your own system, believe me, they are all insignificant in comparison! 'The law of large numbers' implies a few of us do manage to learn mathematics despite this system, and this tiny percentage then tends to drift to your shores; but in India itself, it is not this handful, but the mainstream of "mathematicians" – a totally different species really! – which is very much in control of the whole show from top to bottom.

Mathematics is all about formulating and trying to reason out solutions of clearly stated logical problems. Crosswords, chess or bridge problems, and sudokus qualify – and some fine mathematics is indeed tied closely to games – but more attention is rightly paid to logical issues and problems that have arisen from our attempts to understand more basic things like space, time, and motion. To my students confused about this word, I always used to say: *'proof' simply means 'the reasons why'* and you all know what that means! For example, when someone asks "why are you carrying an umbrella?" you go "because it was raining earlier" if that happens to be your reason, it's that simple! You have to do exactly the same, no less no more, that is, just clearly give your reasons in simple everyday language, if you want to fully answer any question in this subject; put another way: *there are only 'proof questions' in mathematics, none other!* Their classification into sums, examples, exercises, problems, propositions, lemmas, theorems and so on, is done in your book only for organizational purposes – a highlighted theorem is likely to be used again, an exercise probably not – and does not indicate any basic difference in the way in which their answers have to be penned: the reasons why, that is, a proof, is always a must! And neither is this classification a reliable indicator of difficulty: a prominently displayed theorem may be easy to prove without looking at the argument given in the book – always a wise policy to adopt till a problem has utterly defeated you – while a nondescript exercise might have you running around in circles for many days or even weeks. Also, I used to repeatedly remind my students: *mathematical symbols are convenient abbreviations meant to shorten sentences, not a license to murder them!* Your argument, whether right or wrong, complete or incomplete, should at least be in grammatically correct sentences. Just like a string, say, e.g. et al. $ etc. @ % op. cit. i.e., of ordinary abbreviations all by themselves, is gibberish (unless you are writing in some code to a fellow-spy) so is almost certainly that extended string of mathematical symbols without any ordinary words between them which you handed in as your answer (unless you and your teacher are two computers who are communicating with each other using a purely symbolic language). *If you can't read out the full form of what you just wrote as an ordinary grammatically sound sentence, then you need to rewrite it.* Indeed, in my humble opinion, it is almost always a good policy (especially so for a beginner) to try to use the minimum amount of notation, our mathematics tends to become that much better; anyway, there is certainly no direct correlation between the density per square inch of mathematical symbols and the quality of mathematics!

On the other hand, I am all for a liberal use of another kind of abbreviations – *a picture speaks a thousand words!* – in teaching, learning and doing mathematics. Indeed, via just four motifs and some quickly-drawn related sketches, I've conveyed to you, in next to no time and without any jargon, essentially complete proofs of some very famous results of mathematics.[3] That I have gone easy on name-dropping was partly because of

[3] For example, the existence of non-fractional or 'irrational numbers' was proved by the Pythagoreans, and is still amongst the 3-4 deepest things in school mathematics; yet the proof which I've given in "Perfectly proportioned" is perfectly easy! Moreover this pictorial method – which, much to my bafflement, I've not seen used in any text-book – is quite general, and repeatedly deleting a square on the smaller side of a rectangle or 'Euclid's algorithm' also gives us, when it terminates, the 'highest common factor' of the sides, or equivalently, the subdivision of the rectangle into equal squares of largest size. Again, the theorem that there are five and only five regular solids, was the probable aim of Euclid's long treatise, and comes at its very end, and is not even taught in schools and most colleges. The informal continuity argument which I

lack of time, but more out of a fear that you may not confuse the rose with a name, or for that matter a picture, of the rose. Such admonishments are no longer necessary if you have grasped the beauty of the patterns of thought, that is, the mathematics evoked by these motifs. Why, you are smelling the rose now! Your richly deserved reward for the effort that you have put in: *mathematics is never a spectator sport, one has to do some mental work of one's own even to 'see' already-done mathematics.* Were my motifs merely making statements they would have demanded much less, but this was not my intention, and non-trivial effort is sometimes needed to understand even some points of an argument which are labelled 'obvious', 'clear' or 'trivial'. These 'trivial' gaps are inevitable in almost any argument of a readable length, and until and unless these missing steps also become clear to you, you obviously have not grasped the argument. *Yes, reasoning correctly is the same as redefining things* – if 'things' means finite sets of statements and 'redefining' addition or deletion at each step of a statement which follows trivially from the others – *but mathematics is more: it is the* ***art*** *of redefining things*, for example, euclidean geometry is a never-ending logical ***thumri*** [4] on just one non self-evident statement, the fifth postulate! Here too a part of the kick, a great part I think, lies in the suddenness with which yet another lovely and unexpected reformulation arises ever so often out of nowhere, as is exemplified by some gems of this art-form that I have chosen to submit for your consideration via these motifs.

To really appreciate mathematics, one has to 'do' mathematics. If you are game, you can start trying today, and no prerequisites are needed, for, there are all sorts of logical problems, more natural than crosswords and sudokos, so more typical of what mathematicians prefer to think about, which can be found aplenty, and at all levels, from the internet or elsewhere.[5] If you are the sort who doesn't give up easily, then pretty soon will come the day when you'll be able to tell the world of one which was, oh! so innocent-looking, yet led you such a long and merry dance, before it finally revealed its secret, but not before you had a rather inspired idea of your very own! *Then, and only then, shall ye know why, to those who are intensely involved in mathematics, its beauty is palpable!* There is no silver bullet, but personally, I try to understand most problems via pictures – *a suitable figure being already the battle half-won!* – and find the same helpful in explaining my solutions. However, I must warn you that a figure not suitable for the task in hand can just as easily lead us astray, and there are times when those underlying patterns of thought can be 'seen' better and more easily without any visual aid.

Obviously hard work is a must, but *don't do maths 'seriously' : its only a game!* A mathematician at his most creative is like a child 'in the zone' with his video-game : as

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used in "Beesmukhi" for the existence of an icosahedron (or golden rectangle) was used by Eudoxus in a more formal guise to define multiplication of lengths, so similarity of triangles, to which topic the hardest of Euclid's books, Book V, was devoted; and much later, in the hands of Dedekind in the nineteenth century, this 'completeness property of the line' led to a complete arithmetization of geometry, and people started calling the points of the number line 'real numbers'. Et cetera (see Notes).

[4] Light classical Indian vocal music in which a female singer typically repeats, virtually fondles, just one sentence, say a declaration of her love for Krishna, in wildly different ways.

[5] For example, the problems from the last IMO, a competition for gifted high school students, are posted with my solutions to the same at my website on www.kssarkaria.org/docs/imo2009.pdf.

J. L. Synge has suggested, *"the human mind is at its best when playing"!* So this its-only-a-game approach may, just may, increase your chances of solving that problem. Besides, it has the extra bonus of softening the disappointment of failure, which – let's face it – is always a distinct possibility whenever we do anything worth doing. Pertinent too perhaps is the fact that, *mathematics* ***is*** *a game* – just like say "dots and squares" (which shall occur in a motif below) but with more symbols and moves – in the precise sense that mathematical statements can be written (though often very cumbersomely) as sentences of a purely symbolic language, and mathematical proofs can be written as finite (but usually extremely long : it would take reams of paper to actually write out some very simple arguments this way) sequences of these symbolic sentences, each produced from the sentence preceding it by one of a handful of prescribed moves.

As we walk now after our cuppa towards the staircase, I note that walking in the opposite direction towards the skew room – see the last two pics, right to left – one has the pleasant feeling that one is in a moving ship turning left! Also I recall our structural engineer's "easy method" for calculating the area of this room or any quadrilateral: multiply the average lengths of the two pairs of opposite sides! Easy, but easily seen to be wrong, after which I'd asked – *this is typical of mathematics, one question leads to another* – as to exactly which were the quadrilaterals for which it was correct, which had led in turn (see Notes for the full story) to the roof-top motifs that we'll discuss next.

Emerging out now from the "Pyramid" stair-head on to the roof-top, and looking down over its parapet, we can espy the first-floor sundial **"IX to V"** (see Notes for more on the geometry of the shadow of the circular hole in its gnomon, and the circa 1958 in-laid Star of David faintly discernible around it) and way back, atop that roman aqueduct you saw before, a brand-new little white house which bing has built for himself … but I'm getting ahead of myself here … let me return to the motifs on the roof-top.

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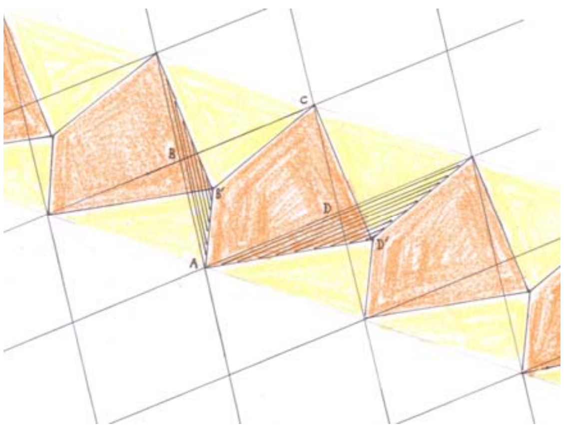
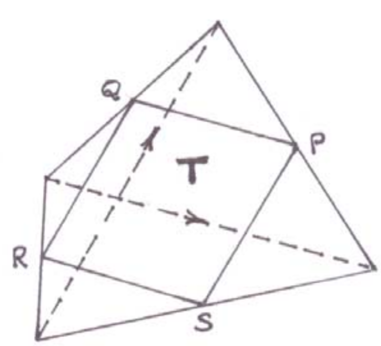
5. "Four half-turns." The area of a rectangle is length times breadth, using which one sees that the area of a triangle is half the perpendicular from a vertex to the opposite side times the length of that side. Which in turn shows that, if we deform a quadrilateral by bodily translating a diagonal parallel to itself, *resp.* parallel to the other diagonal, then the areas of the triangles on either side of this diagonal, *resp*. the other diagonal, stay put. Therefore, *the area of a quadrilateral stays put under a deformation effected by any translation of a diagonal*. In particular, a translation B′D′ to BD after which AC and BD bisect each other, converts any given quadrilateral AB′CD′ to a parallelogram ABCD (opposite sides parallel and equal) of the same area. By drawing two families of equally spaced parallel lines we now partition the entire plane—see figure below—into congruent parallelograms, of which ABCD is only one, and use B $\to$ B′ and D $\to$ D′ to translate back the entire line containing BD to the parallel straight line containing B′D′, so as to simultaneously deform a whole row of these congruent parallelograms back to the row of congruent orange quadrilaterals, of which AB′CD′ is only one. Next we observe that the two shaded triangles have sides of lengths {AB+DC, AB′, D′C} and {AD+BC, AD′, B′C}. Since any side of a triangle is less than the sum of the other two, it follows that, *this deformation increases the average length of a pair of opposite sides of a parallelogram.* For a non-rectangular parallelogram the product of these averages was already bigger than the area, so we conclude that, *the "easy method" of calculating the area of a quadrilateral is correct only for rectangles,* and in all other cases it gives a value which is strictly bigger than the actual area.

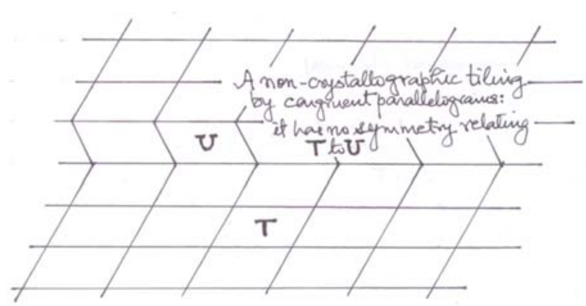


Besides, we have stumbled on a striking fact: *the plane can be tiled by congruent copies of any given quadrilateral!* To see this, we deform *all* parallelograms having 'the same chessboard-colour' as ABCD in the same way: the spaces left between the ensuing orange quadrilaterals will be congruent yellow quadrilaterals, namely, the fusions of the yellow triangles of each strip with those of an adjacent strip. Such a plane tiling is depicted ('building up', as it were) in an island in the red brick roof-top. I note that, though the quadrilateral used in this motif is *convex*, i.e., with all angles less than 180 degrees, this was not needed in our proof, a non-convex quadrilateral tiles too.

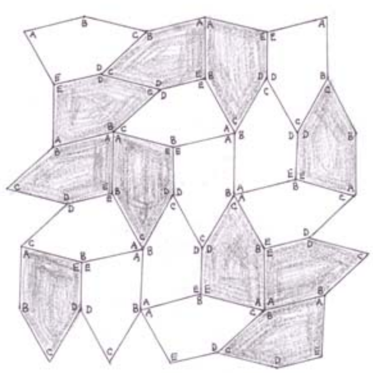
As does any triangle, because two copies make a parallelogram; but all pentagons do not tile. For example, *a regular pentagon does not tile the plane*: all its angles are 108 degrees, and 108 does not divide 360, so no number of these can fit at a vertex. It seems it was Hilbert who suggested, more than a 100 years ago, that it shouldn't be so hard, *to classify all convex pentagons which tile the plane*, but the progress since then on this problem has been painfully slow and somewhat funny! A longish list of types of convex pentagons that tile the plane has been dressed up, and every 10-15 years or so, someone has claimed that the list was now complete, till someone else came along and cooked the said claim, simply by adding yet one more type, that had previously been overlooked, to the allegedly complete list! The analogous classification of convex hexagons which tile was much easier, and a short and indeed complete list is known since long; and, when the number of sides is more, it is easier still with list really short: *the plane cannot be tiled by congruent copies of any convex polygon with more than six sides* (see Notes)!

If a tiling has a symmetry relating any pair of tiles, then its symmetries are said to form a *crystallographic group.* Our quadrilateral tiling is of this type because, *when a tile executes a half-turn around the mid-point of one of its edges*, *one obtains the tile sharing that edge with it*, which (is an easy method for laying the tiling, and) shows that the euclidean motion responsible for the (unique, because our quadrilateral has sides of distinct lengths) congruence **T** ≡ **U** between any two tiles must, automatically, map tiles to tiles! This symmetry is the composition of any sequence of planar half-turns taking our 'dancing quadrilateral' from **T** to **U**, so our group is generated by half-turns around the mid-points of all the edges of the tiling, but we'll show more: *our group is generated by the four half-turns* {p, q, r, s} *of a single tile!* I.e., we'll show that words in these four letters, interpreted right to left as compositions of the corresponding half-turns, give us all the symmetries of our tiling, it being understood that there is also an *empty word* 1 with no letters which represents the do-nothing (or identity) motion. However the same symmetry is given by many words: *we have the relations* pp = 1, qq = 1, rr = 1, ss = 1 *and* pqrs = 1, and all their obvious consequences. The first four relations merely say that doing a half-turn twice is the same as doing nothing. More generally, it is easy to check that a half-turn p of the plane around P, followed by a half-turn q around Q, is the same as translating each point of the plane by an amount equal to, and in the direction parallel to, the directed segment 2PQ. We'll also need this pretty proposition—see figure—from school geometry: *the mid-points* {P, Q, R, S} *of the sides of any quadrilateral* **T** *are the vertices of a parallelogram* PQRS, *whose sides are parallel to, and half as long as its diagonals.* So pq = sr = t_1 , the translation by the directed diagonal 2QP = 2RS; so pqrs = srrs = ss = 1, the fifth relation; and qp = rs is the inverse translation by 2PQ = 2SR. Likewise, qr = ps = t_2 , the translation by 2RQ = 2SP, and rq = sp is the inverse translation by 2QR = 2PS. The required result follows because, starting from this **T**, we can obviously reach any **U** by making, if need be, any one of these four half-turns, followed by a suitable number of translations t_1 and t_2 or their inverses.





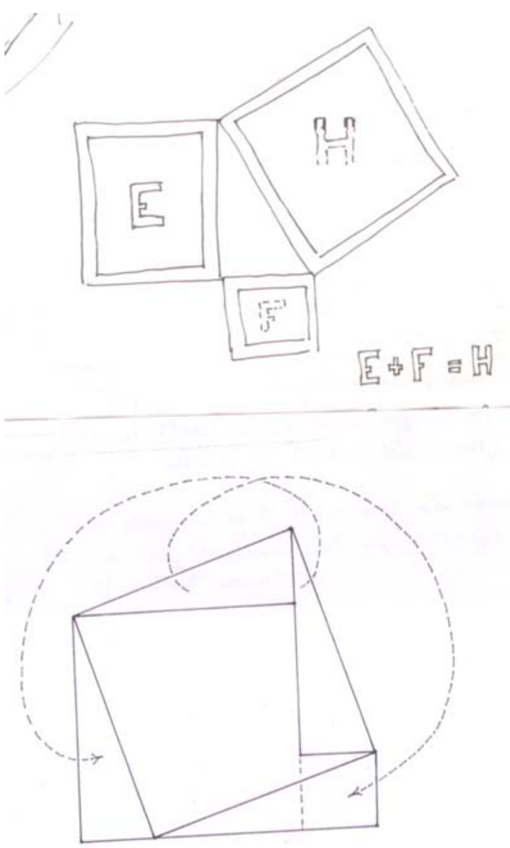
It has been known for 200 years that, upto a composition-preserving bijection, *there are exactly 16 more of these crystallographic groups.* By 1900, the classification of the analogously defined spatial groups, which are the ones that really matter in the study of crystalline substances, was also essentially complete, there being now, *exactly 230 spatial crystallographic groups.* This had entailed arguments much more delicate than in the planar case, which led Hilbert to include, in his famous list of problems from that year, *the determination of the crystallographic groups of euclidean n-space*, as the first part of his Eighteenth Problem. By 1930 or so (see Notes for more) it was known that (like our 2-dimensional example above) *crystallographic groups of n-space are finitely generated and have n independent translations*, and that, *there are only finitely many of these groups*, but this number is still unknown for n > 4. The second part of the Eighteenth Problem was about not necessarily crystallographic tilings, and had asked, amongst other things, for a particular 3-dimensional example. From the fact that Hilbert did *not* ask for a similar planar example, it is likely that he had expected the answer 'yes' to the following question: *if congruent copies of a convex polygon tile the plane, then can they also tile it crystallographically?* The second motif on this roof-top, that is, **"Marjorie's example,"** shows that *the answer is 'no'!*



Since one turns through 360 degrees as one goes around a convex polygon, the five angles of such a pentagon ABCDE obey $A + B + C + D + E = (5 \times 180) - 360 = 540$ degrees, and, in general, they obey no other condition. To make a tiling it is necessary – otherwise its copies won't fit at a vertex – that some of these angles, possibly with repetitions, should add up to 360 degrees. Again, it is necessary that some of the edges {AB, BC, CD, DE, EA} be equal: otherwise, the two tiles across any edge have to be reflections of each other, so we can tile only if the five angles are divisors of 360 adding up to 540, that is, never. A longish list of *fitting conditions*, sufficient for tiling, is now available. The pentagon used here obeys the conditions $B + 2E = 360$, $C + 2D = 360$ and $EA = AB = BC = CD$ that were shown by Marjorie Rice to be sufficient (see Notes) in 1975, and the fifth edge DE is shorter. So, in *any* tiling by congruent copies of this pentagon, one has triads of tiles like the one in the upper left corner. The reflection in DE is the sole euclidean motion that throws the white tile of this triad on the one below it. For the tiling to be crystallographic, this reflection must map the third tile of the triad on itself, which is impossible because our pentagon is congruent to itself only via the identity map. *So this is a counterexample to a conjecture of Hilbert's; also, it is an* *example of what can be achieved by sheer enthusiasm!* For Marjorie Rice, you see, was a housewife who had studied maths only till her high school 35 long years before, when she chanced one day upon an article on pentagonal tilings in the *"Scientific American,"* that led her to start making more and more of these tilings on her own.

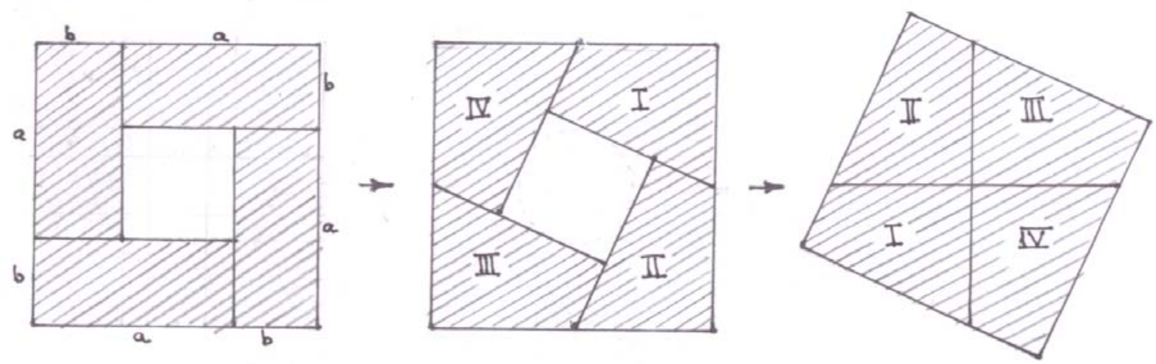
We'll now descend the stairs back to the ground floor, and exit towards the annexe, to view the line-mural on the northern wall of this house (but, for a really up-close view, we should have gone instead to the adjoining roof-top of the annexe).

6. "Grecian origami." I'll resist repeating in full a favourite story of mine about how I had tracked down a famous, but forgotten, mural of Jeanneret's in this city about five years ago, but shall recall that his raised red-brick mural (see Notes for the original photo) was in a state of advanced neglect (with almost two full letters missing) when I'd finally re-found it and hurriedly made that day (for I didn't have a camera on me) the sketch that is shown below. You'll immediately recognize in its **E + F = H** the statement of, *the one theorem that we all remember from school!*



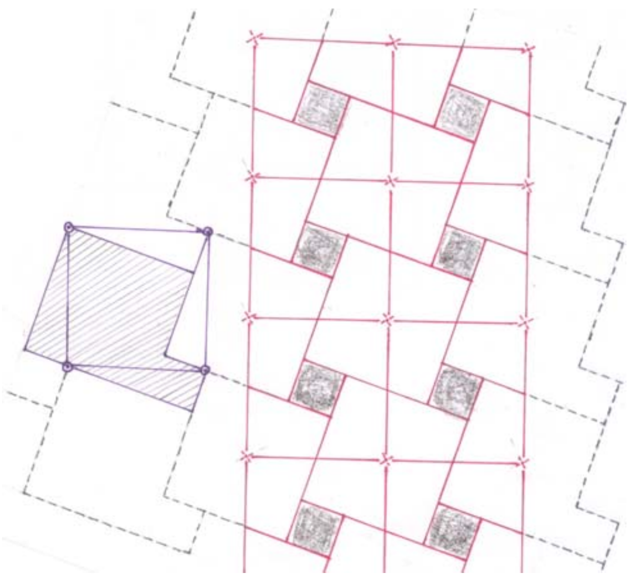
Above that balcony with a 'jeanneret jaali' – a knotted (see Notes) brick lattice de rigueur in Chandigarh houses built till the late sixties – and a vertical version of a more traditional jaali once popular on the parapets of Punjab's villages, there is a mural which is even simpler – grooves in cement, coloured purple – than Jeanneret's, and scores over it inasmuch as, *it depicts the same theorem with a complete proof!* Simply cut out – with a scissors if you will! – the two congruent right triangles from the big tilted square on their hypotenuses, and translate the triangle at the top to the bottom, and the one on the right to the extreme left, to obtain that shoe-shaped union of the squares on the smaller sides, which can be cut further, if need be, into these two squares.

More generally, the paper-less Greek ancients knew that, *any finite set of polygons can be converted into a single square by paper-cutting and pasting* (= 'uncutting') *moves!* Indeed, the mural has already taught us that *a sum of two squares is equal to a square* in the sense of this 'grecian origami' (as against japanese origami, which uses paper-folding and unfolding moves), and a polygon can obviously be cut into triangles, and a triangle converted easily into a rectangle, which can be cut into equal squares and a rectangle which is not too long (length/breadth ≤ 5 will do). We cut such a rectangle parallel to its length and breadth to make four similar and equal rectangles, and lay these four as shown below, so it'll suffice to show that this (shaded) *difference of two squares is also equal to a square.* To see this we check that, because our rectangles are not too long, the square hole can be rotated around its centre by any amount we like *within* the big square, and that this rotation can be realized by cutting and re-pasting four equal triangles. We rotate the hole thus to the new position in which the clockwise extensions of its arms meet the edges of the bigger square at their mid-points, and cut along these extensions, then the resulting four congruent pieces shall refit to form a square.



A line segment and a polygon are also called, respectively, a 1-dimensional and a 2-dimensional polytope, and quite generally, an *n-dimensional polytope* is a region of n-dimensional space bounded by finitely many (n-1)-dimensional polytopes, and one can play n-dimensional grecian origami with these! The *grothendieck group* K(n) has as elements all finite sets of n-dimensional polytopes or their negatives, with two such sets deemed same if they are related to each other by pasting and cutting, and these operations are used to define addition and subtraction in K(n). *This is a natural generalization of the geometric definition of numbers*: K(1) is clearly the real numbers as defined in school with the usual addition, and it is easy to see that addition is associative, commutative, and obeys the cancellation rules in any K(n). The result we proved above shows that on K(2) too, there is a notion of 'greater than' which is well-behaved with respect to addition, and which is 'complete' (given any partition into two parts, with anything in the first less than anything in the second, exactly one of the parts has a greatest or smallest element), so, K(2) *is isomorphic to the additive group of real numbers* K(1), more precisely, there is a unique one-one, onto, and order- and addition-preserving *signed-area* map K(2) → K(1) which assigns to the unit square the number 1. As against this, for $n \geq 3$, the groups K(n) are not totally ordered (but their structure is still not completely worked out for $n \geq 5$), so, *there are 3-dimensional polytopes of the same volume which are not related to each other by any finite sequence of cutting and pasting moves.* Indeed, responding to Hilbert's Third Problem, Dehn had shown in the year 1900 itself that, *the regular tetrahedron cannot be converted into a cube by any such sequence of moves* (see Notes)!

From our vantage point under this old mango tree, that we had chosen to view the pythagorean mural, if you'll now kindly turn, and peer out over the boundary wall, you'll notice many copies of the last two drawings in the humble concrete of the **"Pythagoras Drive"** which a car needs to take from the main road to enter this house!



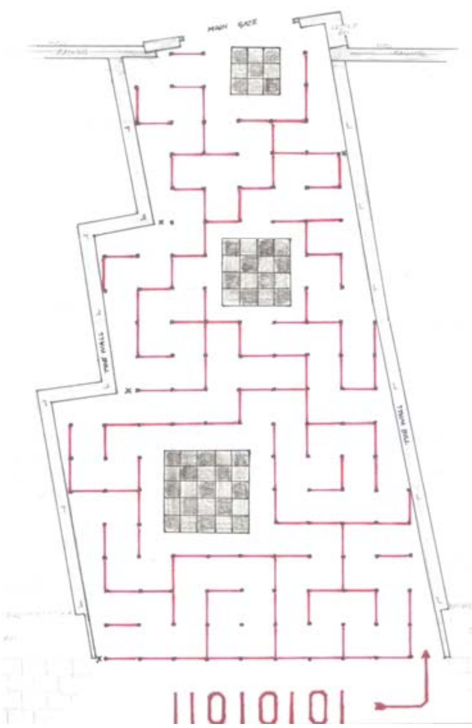
As we walk out on this driveway, let me point out how intimately its red-brick pattern is tied with the mural, which is re-drawn above, with the big tilted square now shown straight-up, and I've shaded its shoe-shape, the better to display how, *congruent copies of this shoe, when translated vertically and horizontally by amounts equal to the hypotenuse, fit snugly with each other to give us a tiling of the entire plane!* The crosses mark the centres of the bigger square constituents of some of these shoe-tiles, joining them one gets the big square panels of the driveway, whose central black granite tiles depict the smaller square constituents, or the toes, of some of these shoes. While making this driveway, the tricky part was to lay these small granite tiles accurately, so that the clockwise extensions of their arms would hit the sides of the big panels in their mid-points; once this was done, the brick pattern was easy enough to complete, and then finally, concrete was poured into the remaining congruent quadrilateral spaces. Indeed, you must have noted some 'warm-up panels' of this driveway inside the gate, in which extensions are not required to hit the mid-points, e.g., there was one depicting, *the first of the trio of diagrams on the last page, which proves* $(a-b)^2 + 4ab = (a+b)^2$.

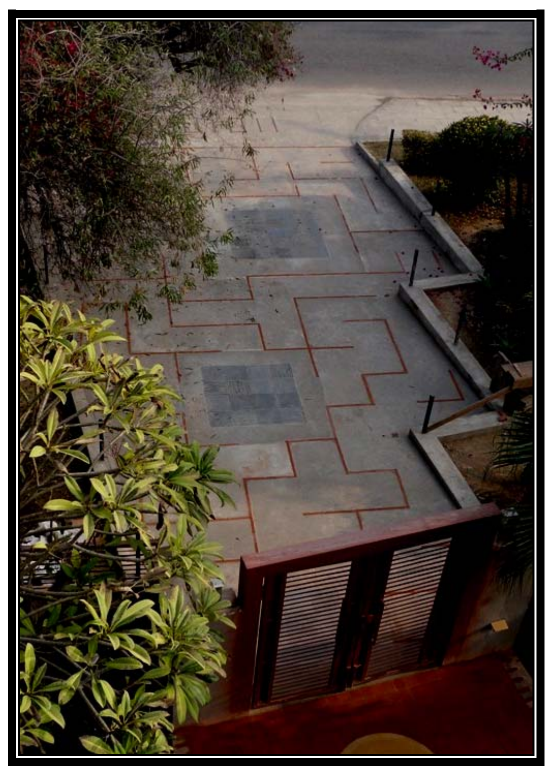
It also solves the discrete pythagorean problem, in which one seeks all pairs of squares made from square tiles of unit size, whose sum is also such a square. The diagram shows that the difference of two such squares, both even or both odd, is made up of a whole number of unit tiles, and is a perfect square if and only if ab is a perfect square; for example, a = 4 and b = 1 give the solution $3^2 + 4^2 = 5^2$; a = 9 and b = 4 give $5^2 + 12^2 = 13^2$; and so on. We can just as easily solve some higher-dimensional problems, e.g., *is the sum of two cubes a cube in the sense of grecian origami? Yes!* Using two-dimensional origami (with the remaining dimension basically a passenger) we can convert the cubes, and the cube having the same volume as their sum, to boxes of height one; then, using two-dimensional origami again, convert the sum of the bases of the first two boxes to that of the third box. Repeating this, we see in fact that, *the same is true for any two 'n-dimensional cubes'*, i.e., n-fold 'powers' of segments, in the same sense as a plane is often considered a 'product' of two chosen lines or axes. More generally, this cartesian product defines a multiplication $K(n) \times K(m) \to K(n+m)$ which is still not fully understood, however $K(1) \times K(1) \to K(2)$ followed by the area isomorphism $K(2) \to K(1)$ is obviously the multiplication of numbers. However, the higher-dimensional analogues of the discrete pythagorean problem are more challenging, and the general assertion that, *for any* $n \geq 3$, *the equation* $x^n + y^n = z^n$ *has no solution in positive whole numbers* x, y *and* z – which Fermat (in)famously could not leave us a proof of because there was not enough space in that margin! – has been proved only recently after all these centuries by Wiles, but we definitely don't have enough space to explain his proof here.

Returning to the tiling that we're standing on, I note that it is crystallographic with no symmetry other than its defining translations, and that there are 8 vertices on each of its congruent shoe-tiles – the angles of this non-convex octagon are 270, 90, 90, 180, 90, 180, 90 and 90 degrees in clockwise cyclic order – but at some vertices there are 3 tiles, at others only 2. Unlike, say, the usual brick pattern all over these walls, which depicts a tiling by non-convex hexagons, 3 at each vertex. So, in analogy with a question about polygonal subdivisions of surfaces that we posed in "Miss Universe" we ask: *is there a tiling of the plane by topological p-gons, q at each vertex?* Note that we have thrown away all geometric shackles, our tiles are now allowed to be not only non-convex but practically arbitrary, and we aren't demanding any congruence between them, let alone crystallographicity of tiling, moreover, not being confined now to a closed surface of a bounded body – we have the plane's infinitude at our disposal – intuition suggests no obstruction, it seems very reasonable to expect the answer 'yes'? However some care is needed; you can soon find out for yourself, by just trying to build these tilings, that, *the answer is 'no' for* {p,q} = {3,3}, {3,4}, {4,3}, {3,5} *and* {5,3}: you'll be *forced* to go into the third dimension (and there you'll get the same old examples)! *For all other values of* $p \geq 3$ *and* $q \geq 3$ *the answer is 'yes'*, as we'll see in the 'lecture' after the next, and then we'll use our proof to solve the previous question about closed surfaces.

It is time to move on, but we won't re-enter the house just yet, we'll make a right turn on this sidewalk to take a look at that other driveway which goes till the main gate of "213, 16A", for there awaits us, again in its humble concrete, our next motif (there is also some 'concrete mathematics', viz., a repeat of "four half-turns", which can be seen outside the small southern gate of this house, which opens on the side street).

7. “Amazing curves!” As you can see – a line-drawing and a roof-top view are also given – the same materials are used again: three square ‘islands’ are made from granite tiles of unit size, and around them swirls an extended red-brick pattern, which turns this driveway into *a maze* of concrete paths, complete with an arrow indicating its point of entry. Its two-dimensionality (no view-obstructing high hedges here) renders this maze easy to thread, and I’ve seen many ‘children’ (of all ages!) go zig-zagging gleefully and triumphantly up this maze to the main gate, though it must be said that quite a few of them don’t know what that initial 11010101 in dark red brick signifies.



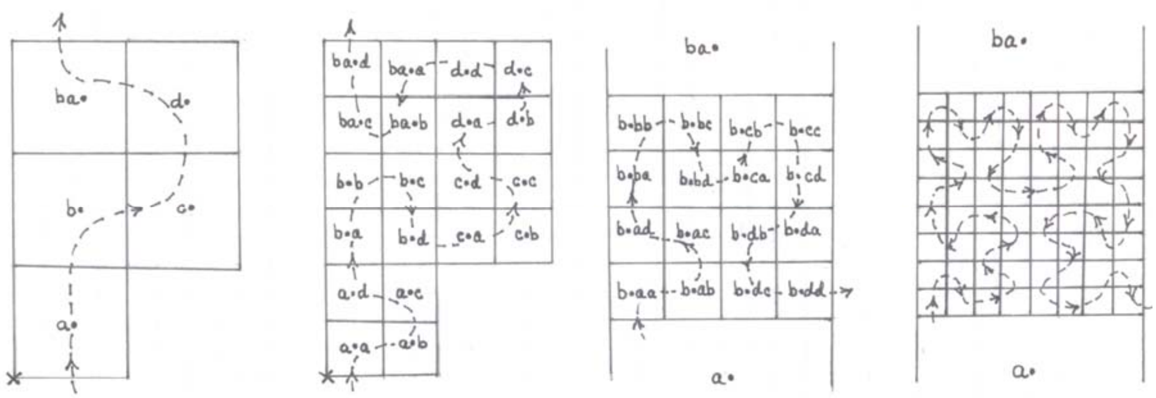


I’ll come to this in a little while, but first, I must urge you to look harder, for then you’ll realize (this is especially clear in the line-drawing) that something even more ‘childish’ has been going on here: *a dots-and-squares game!* Doubtless, you have played this game in school and/or college between (or maybe even during some particularly boring) classes, but here are its rules anyway. On a chosen playing area – a rectangular piece of paper usually, but here the region between these toe-walls converging to the main gate of the house, minus the three tiled islands – one marks all the vertices – here one-inch square dots of dark brick – of a square grid; then the players take turns putting in the edges – here one-inch thick in red brick – of the grid between them, with a player having the option of an extra turn every time he/she completes a square; and the winner, if any, at the end is the person who has made the most squares. However there is something peculiar about this particular game: *no player has bothered to 'take' a square!* Maybe the players are rank tyros, or maybe, though it is much less likely, accomplished grandmasters playing a deep cat-and-mouse strategy, for, after all, *dots-and-squares is still unsolved* (see Notes)? No, the real reason is something else again: *the players have become so fascinated by an incredible thought tied closely to the emerging and meandering snakes of squares* – in a typical game, each of these gets taken in one go towards the end, and these climactic moves usually decide who wins or loses – *that they have lost their competitive edge entirely, and have started thinking of something much more interesting than who shall win or lose this silly game!* Leaving you wondering for the moment as to what this 'incredible thought' could possibly be, I'll finish the description by mentioning that – like that arrow, and the initial mysterious string of 1's and 0's – the four interdicting crosses in darker brick, that you see near the toe-walls, are there only to turn this peculiar game position into an appealing maze.

Long familiarity often makes us forget that *whole numbers are abstractions* (and it is this that makes them useful): sure, five ducks, five pens, five poems, etc., are tangible enough, but, as Cantor reminded us in the nineteenth century, *five*, of and by itself, is the commonality between, or alternatively, *the equivalence class of* ***all*** *these sets, any two of which are related to each other by a one-one onto or bijective correspondence*, and, when we think about it, it is nothing short of a miracle that a human child can intuitively grasp this abstract notion 'five' in kindergarten: no other species even comes close! The usual notation for these notions is also interesting: for the first ten non-negative whole numbers we use the ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, and then, *we denote any number by counting off powers of ten, from the highest downwards, in that number*, for example, the number of this house has 2 second powers, 1 first power, and 3 zeroth powers of ten (a zeroth power is always unity by convention), and that's what 213 signifies. However, *the choice of ten as 'base', though customary, is in fact quite arbitrary: any number bigger than one would serve just as well* (the choice ten was probably made because we have as many digits on our hands, but there have been ancient civilizations that got along just fine with other bases)! For instance, if we want to decrease the number of symbols, then the least choice, two, is obviously the best. Once again, let us agree to use 0 and 1 for the first two whole numbers, but after that, any other number will now be denoted by counting off powers of two – that is, 1, 2, 4, 8, 16, 32, 64, 128, and so on – from the highest downwards, in that number, for example, 11010101 *is the house number in base two notation,* because, as you can check, $213 = (1 \times 2^7) + (1 \times 2^6) + (0 \times 2^5) + (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$.

There is a whiff here of "Pythagoras Drive" too: *the three tiled islands depict the smallest solution of the discrete pythagorean problem,* $3^2 + 4^2 = 5^2$. In this context, observe that an odd square has a central tile, an even square has none, and the number of tiles in each square ring is a multiple of four. So, the sum of two odd squares is not a square, a point tacitly used when we'd solved this problem; also note that, in *base four notation, using the letters a, b, c and d* (instead of the more hackneyed 0, 1, 2 and 3 respectively), this observation translates into saying that, the right-most letter of the word denoting an odd square is *b*, and that it is *a* for an even square.

Our dots-and-squares players, who were incidentally named Georg and David, had not neglected maths in school, so knew that not only whole numbers, but, *any non-negative real number can be written as an infinite decimal,* and likewise, in other bases too, e.g., *in base four*. We subdivide the non-negative number line into *segments* (end-points are deemed in them) of unit length, and denote each of them by the base four notation of its initial whole number followed by • ('decimal point'); then, we subdivide these segments into four equal segments, and denote them, respectively, by appending *a*, *b*, *c* or *d* to the notation of the subdivided segment; next, we subdivide these into four equal segments, and denote them, respectively, by appending *a, b, c* or *d* to the notation of the subdivided segment; ad infinitum. Having done this, *we denote a real number* x *by* $[x_1 \ldots x_t \bullet x_{t+1}\, x_{t+2} \ldots]$ – where each x_i is *a, b, c* or *d*; thus, except for that dot and the enclosing brackets, this is a right-infinite word formed from four letters – *if and only if it is contained in all the segments denoted by the finite truncations of this infinite word,* that is, x should be in $x_1 \ldots x_t \bullet$, as well as in $x_1 \ldots x_t \bullet x_{t+1}$, as well as in $x_1 \ldots x_t \bullet x_{t+1}\, x_{t+2}$, et cetera. I've added the brackets to emphasize that, *some pairs of words are to be identified,* for, to positive end points of all these segments, we have obviously assigned not one, but two words; this too is exactly as in decimal expansions, where, for example, we know that 13 • 45000 … is to be identified with 13 • 44999 …

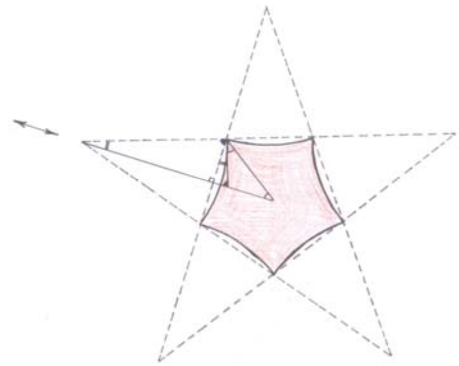


Now consider a *snake of squares* (boundaries are deemed parts of squares) of unit size that a dots-and-squares player can take thus: he puts in the final edge of the first square which also becomes the third edge of the next square, then he puts in the final edge of this square which also becomes the third edge of the next square, etc. The first diagram above shows five of the twenty squares en prise in the game on this driveway, which can be taken in this manner in the indicated order; but clearly, one has arbitrarily long finite snakes, as well as, infinite ones that cover the entire plane. The 'incredible thought' they inspired was this: *using a snake, one can label points of the plane, in a manner entirely analogous to the base four notation above for points of the line!* We denote the chosen snake's squares, in order of capture, by the base four notation of the corresponding whole number followed by • ; this order is indicated by the dotted curve in the first picture, which starts from an extant edge with a crossed vertex on the first square. We now subdivide these squares into four equal squares, and build a snake – see the second picture – from all these: its first square is the one having the cross on it, then the other three in the first big square in the order which lets us exit it on the same side as before; then come the four small squares of the second big square in the order which lets us exit it on the same side as before; etc. (it is easily checked that the order indicated by the directed dotted curve of the second picture is the unique such order). We denote the small squares within each big square, in this order, by appending a, b, c and d to the notation for the big square. Next, we cut up all these small squares into four equal and still smaller squares – because of congestion this step is shown in the third picture only for the square b • on a magnified scale – and make, exactly as before, a snake from all these; and these are denoted by appending a, b, c and d in this order as before; and so on (things became so crowded that only the dotted curve indicating this portion of the next snake is drawn next, the names of the squares are omitted). With all these squares now unambiguously named (exactly like those segments were before) we make the parallel definition: *we denote a point* X *of the plane by* $[[x_1 \ldots x_t \bullet x_{t+1}\, x_{t+2} \ldots]]$ *if and only if it is contained in all the squares denoted by the finite truncations of this infinite word,* that is, X should be in $x_1 \ldots x_t \bullet$, as well as in $x_1 \ldots x_t \bullet x_{t+1}$, as well as in $x_1 \ldots x_t \bullet x_{t+1}\, x_{t+2}$, et cetera. *The double brackets indicate that there is need to make more identifications now than in the one-dimensional case*: to points X on the interiors of the edges of our squares we have assigned two words, but to those on the vertices, as many as four. Thus $x = [x_1 \ldots x_t \bullet x_{t+1}\, x_{t+2} \ldots] \to [[x_1 \ldots x_t \bullet x_{t+1}\, x_{t+2} \ldots]] = X$ is well-defined but not one-one, and obviously X is varying continuously, i.e., is tracing a plane 'curve', as x varies, *a curve which, amazingly, visits each and every point of all the squares of the chosen snake* (these variations of X can however be shown to have no specified direction, see Notes, so one can justifiably argue that this is an inappropriate usage of the word curve)!

This neat definition, of the younger player David Hilbert, generalizes painlessly to all dimensions n – one uses base 2^n now because an n-dimensional cube subdivides into so many cubes of half the size – to yield, *a continuous mapping of the unit segment onto the* n*-dimensional unit cube!* For $n \geq 2$ this is magical: by identifying finite subsets of a segment we can increase dimension as much as we like! This was known already to the older Georg Cantor, who had given a (non-continuous) one-one onto map between a segment and a square, and had observed that the magic begins at n = 1: *the space* **C** *of all infinite sequences of* 0's *and* 1's *is zero-dimensional,* and becomes the one-dimensional segment only post those innocuous two-fold identifications of base 2 expansions! To see this zero-dimensionality, switch 1's to 2's and interpret these sequences of 0's and 2's as points of the segment in base 3, this exhibits **C** as the 'dust' remaining after this: *delete the middle third to get two disjoint segments, then delete their middle thirds, etc.* The segment, the square, the cube, etc., so all the *continuums* like manifolds that one encounters in most of mathematics, can be constructed from this primordial cantorian dust **C** by making finite-to-one identifications! For, *base* 2^n *expansions can be read off from the base* 2 *expansion,* for example, the point 1101010011… of **C** goes to the point x of the segment having base 4 expansion [•(11)(01)(01)(00)(11)…] = [• dbbad ...], which goes to the point X = [[• dbbad ...]] of the unit square defined by using a snake with one square, with say the top edge final; likewise, grouping terms in threes gives the base 8 expansion, so, by the generalized definition we mentioned, a point in the cube, etc. In all likelihood, Cantor already knew of *a simpler, but less neat, construction of a continuous map from the segment onto a square*: the base 2 surjection **C** → [0,1] induces a surjection **C** × **C** → [0,1] × [0,1], but **C** × **C** can be identified with **C** (see Notes), so by linearly extending this function on those deleted thirds we obtain a continuous surjection [0,1] → [0,1] × [0,1]. Also, he most certainly knew that, **C** represents a bigger *infinite whole number* – in the sense of his definition of five given earlier – than the set of all finite whole numbers, because any one-one function from the latter to the former misses the sequence of **C** whose nth term is not the nth term of the nth sequence! Mathematics of today can all be written formally in *Set Theory*, the language which Cantor invented, and Hilbert and others developed, to talk expressly of these infinite whole numbers.

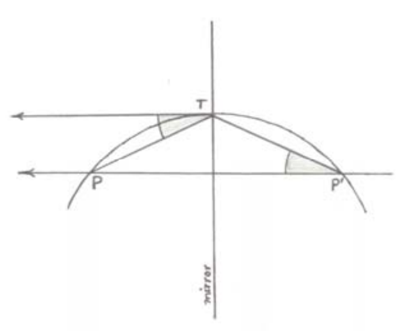
8. "Magic Carpet." Now that you've traversed the maze, and are past the main gate, you'll note that we're back where this tour began: there, to the right on that wall in front, is "Perfectly proportioned," which involved the golden ratio, after which we'd gone tripping down that stone path to the left to see "Beesmukhi," whose three rectangles were golden too, which is the same as saying – consider the pentagon whose five vertices are the five neighbours of any vertex of this icosahedron – that golden ratio = diagonal ÷ side for any regular pentagon. You'll note also, if you look down, that you're being given a red-carpet welcome, and that, in this welcoming mat that you're standing on, there is again, *a regular pentagon, but this time, one whose edges are circular arcs*; as Poincaré showed, this seemingly frivolous frill has amazingly far-reaching implications!

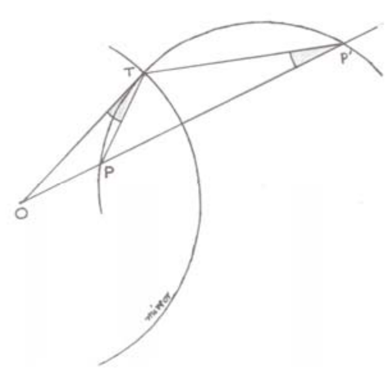
More precisely, our five circular edges have as their centres the five vertices of the pentagram obtained by extending the edges of an ordinary regular pentagon, from which fact it follows—see the next diagram—that, *the angle between the tangents to the two circular edges at each vertex is 72 degrees*, that is, one-fifths of 360 degrees; and likewise, starting from an ordinary regular polygon with p ≥ 6 edges, one gets a regular polygon with p circular edges making an angle of 360/p degrees at each vertex.

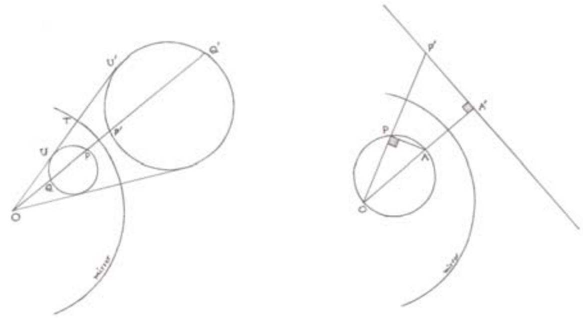


By moving their centres equally towards or away from the centre we can make these circular edges more or less curved, so we see further that, *for all* p ≥ 3 *and* q ≥ 3, *other than the five exceptions*{p,q} = {3,3}, {3,4}, {3,5}, {4,3} *and* {5,3}, *there is a regular polygon with* p *circular edges making an angle of* 360/q *degrees at each vertex.* The five exceptions come about because we don't want edges bending outwards, so 360/q cannot be bigger than the angle ((p-2)×180)/p of an ordinary regular p-sided polygon, and while discussing "Pyramid," we've seen that this happens if and only if (p-2)(q-2) < 4, i.e., in these five cases. However, we've *not* excluded the cases when the edges are not curved at all, which happens if and only if these two angles are equal, i.e. (p-2)(q-2) = 4, i.e. {p,q} = {3,6}, {4,4} or {6,3}. In other words, we've adopted the convention that 'a regular polygon with circular edges' may, in particular, have straight edges; in fact the key to the following is to, *think of straight lines as if they were "infinite circles" passing through an additional "point at infinity,"* for this intuition suggests rather naturally the appropriate generalization of some concepts to 'all' circles.

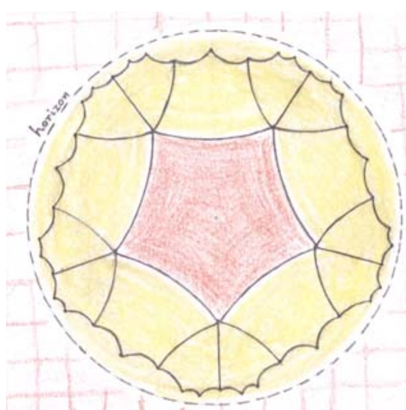
For example, the case {4,4} is that of a square, and the red half-brick square tiling that you see around the curved pentagon is 'seen' also by someone who has ensconced herself in a square changing-room with mirrored walls, which prompts this question: *do multiple reflections in the circular edges of this single curved pentagon determine likewise a tiling by infinitely many curved pentagons, five at each vertex?* The answer is *yes* (!) provided we define *reflection in a circle* suitably: the image of any point P shall be the point P′ lying on all circles through P which cut the mirror perpendicularly.







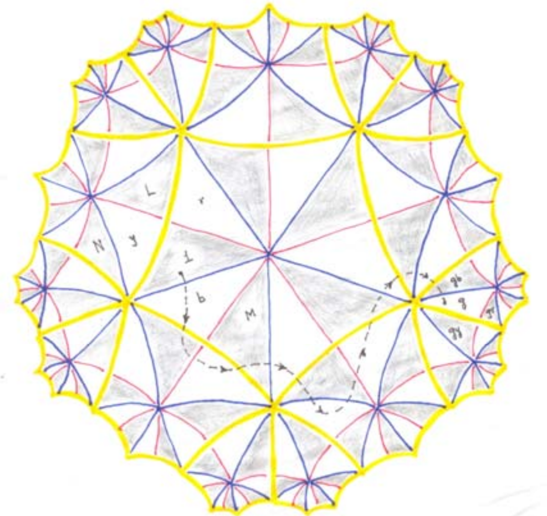
So, *reflection in a finite circle switches its centre* O *with the point at infinity, keeps its own points fixed, and preserves any perpendicular circle*—say, a line through O—*as a whole, but interchanges all pairs of points* {P,P′} *on it such that* OP′ *times* OP *is the square of its radius.* For, a circle cuts the mirror perpendicularly at T if and only if OT is its tangent at T, i.e., OP′T is similar to OTP, i.e., OP′/OT = OT/OP, i.e., $OP'\times OP = OT^2$. *The remaining circles are paired with distinct circles.* If OU ≠ OT is tangent to the circle at U—see diagram—then it expands/contracts to the circle obtained by multiplying all rays from O by OT^2/OU^2, because $OQ\times OP = OU^2$ and $OP'/OQ = OT^2/OU^2 = OQ'/OP$ imply $OP'\times OP = OT^2 = OQ'\times OQ$. Likewise, when O is inside the circle, then reversing and multiplying all rays from O by the constant $OT^2/(OQ\times OP)$ – here PQ is any chord of the circle through O – shall give the other circle. Finally, a finite circle through O gets paired with the line (infinite circle) perpendicular to the diameter OA at A′, because similarity of OAP and OP′A′ gives OP′×OP = OA′×OA. Though not distance-preserving, *reflection in a circle maps any angle at* P *to an equal but opposite angle at* P′: by continuity of P → P′ it suffices to check this off the mirror, when the circles through P and P′ tangent to the two directions enclosing the given angle at P, being perpendicular to the mirror, are preserved by reflection in it, so the angle at P is mapped to the equal but opposite angle between these circles at their second intersection P′.



The round yellow *'halo'*, in which the red curved pentagon is inlaid, is such that its *'horizon'* cuts the edge-circles perpendicularly, so its size is ordained by the pythagorean theorem: *(radius of horizon)*^{2} + *(radius of an edge)*^{2} = *(distance from the centre to the centre of an edge)*^{2}. Likewise, any regular genuinely curved polygon has a bounded – but limitless, the horizon is not assumed to be in it – halo, while that of an ordinary regular polygon shall be deemed to be the entire plane. The diagram above shows only the five primary reflections of the red pentagon in its edges, and two new reflections of each of these, but this suggests already that, *the halo of any regular curved polygon is the union of its multiple reflections in edges!* That these polygonal reflections are all in the halo, have circular edges perpendicular to the horizon, and the same angle at each vertex, is clear enough, because reflection in any edge keeps it fixed, preserves the perpendicular horizon as a whole, maps circles to circles, and preserves angles. However, as the diagram shows, these iterated reflections can contract in size very rapidly, so it is not obvious that any point P of the halo is in one of these reflections. To see that it is, we use the fact that the centres of the edges of all the reflections are *outside* the perpendicular horizon, so the radii of the edges which cut the segment joining the centre to P are all bigger than the positive distance from P to the horizon, so contraction in size is bounded for reflections in these particular edges, so only finitely many reflections are needed to cover this segment and find a polygonal reflection which contains P.

To precise this result, *we'll now indulge in some "grecian origami" involving these multiple reflections:* we make a separate paper copy of each of these (infinitely many) polygonal reflections, and glue two of these paper polygons along an edge if and only if reflection in this edge switches their originals! This gives us a space locally like any closed surface, except possibly near the vertices v: *we get a surface if and only if the angle of the regular curved polygon is a fractional number.* For, a polygonal reflection incident to v returns to itself after some, say q, reflections in edges incident to v, only if the qth, but no smaller, multiple of the angle is a multiple of 360 degrees, that is, the angle is 360r/q where the whole numbers r and q have no common factors. We dub this, the *riemann surface* of a regular curved polygon with p sides and angle 360r/q, and note that, *it carries a tiling by infinitely many p-sided paper polygons q at each vertex*; for example, all ordinary regular polygons have riemann surfaces; and the one tiled by ordinary regular paper pentagons has ten at each vertex, because the angle is now 108 degrees, so r = 3 and q = 10. We examine next, the *covering map* f from the riemann surface onto the halo, which identifies each paper polygon with its original. When r ≥ 2, this map is obviously not one-one: the pre-image v of f(v) 'branches off' into r distinct pre-images when f(v) is varied slightly. That there is no branching when r = 1 is however not reason enough to conclude that, *the halo of a regular curved polygon with p sides and angle 360/q is tiled by its multiple reflections in edges.* It is conceivable that f might fail to be one-one in a more subtle way: a sequence of edge-reflections covering a long loop, starting and ending at a point P of the halo, might bring us back to a different polygonal reflection containing P? The answer is 'no' because, *the halo is simply connected* – any loop at P can be shrunk continuously to the constant loop at P – from which fact it follows easily that f is a topological equivalence when r = 1.

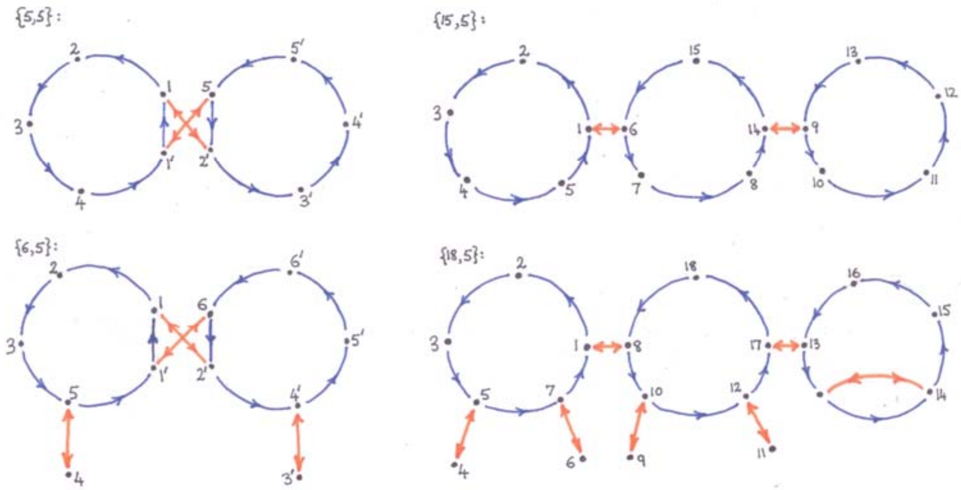
Moreover, *our tiling is 'regular' in its curved geometry*: any vertex v of the red tile **T** can be mapped to any other vertex w by a composition g of reflections in circles which preserves the tiling! Choose any sequence of tiles, starting with **T**, and ending with a tile **U** incident to w, such that each shares an edge with the preceding. From the way in which the tiling was defined, we know that the composition g of reflections in this sequence of shared edges maps **T** onto **U**, and by using preliminary reflections in the lines of symmetry of **T** we can also ensure g(v) = w. Since g maps circles perpendicular to a circle C to ones perpendicular to g(C), we also know that, if P reflects in C to P′, then g(P) shall reflect in g(C) to g(P′). So g maps the primary reflections of **T** onto the corresponding primary reflections of **U**, and likewise the secondary reflections of **T** are mapped to those of **U**, et cetera, so g preserves the tiling.



We'll now subdivide the central tile by its lines of symmetry, extend this subdivision to all tiles by reflections in edges, and *identify* (!) each curved triangle with the unique symmetry g which maps any chosen *basic triangle* 1 to it. This subdivision is shown above for the pentagonal tiles of the last picture: their boundaries are in *yellow*, the new edges within them incident to their vertices are in *blue*, and the remaining edges are in *red*. So, the edges of any triangle have different colours, and symmetries preserve colours of edges. Using this we'll now show that, *the group of symmetries is generated by the reflections* {r, y, b} *in the edges of the basic triangle*, i.e. (see "Four half-turns") any g is equal to a word in these three letters interpreted right to left as a composition of reflections. Indeed, r, y and b are the triangles sharing a red, yellow and blue edge with the triangle 1; so, the corresponding three triangles adjacent to any triangle h are hr, hy and hb; so, g is equal to the word whose letters give in succession the colours of the shared edges of *any* path of triangles from 1 to g*;* for example, the dotted path in the figure shows byrybybbrybybyb = g. Thus there are infinitely many ways of writing the same symmetry, for instance the identity 1, as such a word, but things are really not all that bad: *any relation* w = 1 *can be reduced to* 1 = 1 *by using the obvious relations* $r^2 = 1$, $y^2 = 1$, $b^2 = 1$, $L^2 = 1$, $M^p = 1$ *and* $N^q = 1$, where L = ry, M = br and N = yb. This again uses the *simple-connectivity* of the halo: any loop of triangles can be built up from small loops enclosing at most one vertex of the triangular tiling, and the listed relations are easily seen to be all those that can arise from such small loops. The *shaded triangles* form the subgroup of all orientation-preserving symmetries—compositions of an even number of reflections—which is generated by any two of the three 'rotations' above, say L and N, subject only to the relations $L^2 = 1$, $N^q = 1$ and $(LN)^p = 1$.

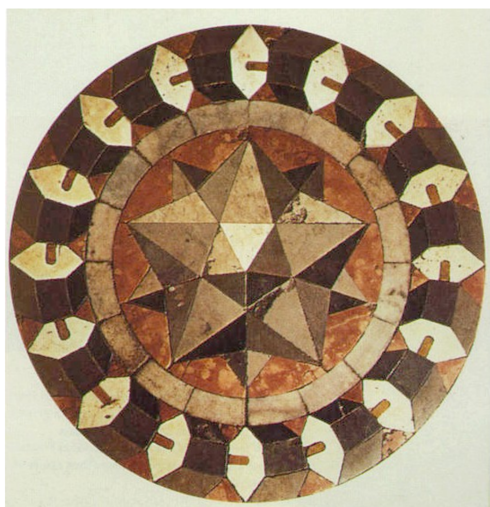
The above groups are *infinite*, but this does not rule out that, by imposing additional relations, one can't get, *finite groups generated by two elements* L *and* N *whose orders* (lowest powers equal to the identity) *are* 2 *and* q, *while that of* LN *is* p? Given such a group, we can double its size by a new generator y subject to $y^2 = 1$, $r^2 = 1$

and $b^2 = 1$, where r = Ly and b = yN, and then indulge again in some (purely topological) “grecian origami” imitating the last picture: each element g of this doubled group is deemed a triangle with red-blue-yellow edges, and we glue the red edges of all the pairs {g, gr}, the blue of all pairs {g, gb}, and the yellow of all {g, gy}. The resulting space is a closed orientable surface—the boundary of a pretzel with holes—because each edge belongs to two triangles, each vertex to 2p, 2q or 4 triangles forming a loop around it, depending on whether the vertex is opposite a yellow, red or blue edge, and the shaded triangles of the initial group can be unambiguously deemed clockwise. Displayed in glorious yellow on the surface just described we’ll plainly see, *finitely many* p*-sided topologist’s polygons*, q *at each vertex*, and at the same time, displayed in fiery red, the dual topologist’s {q,p}! Moreover, the orientation-preserving symmetries of either give us back the initial group, and the cardinality of the doubled group gives, via an obvious euler characteristic calculation (cf. p.8) the number of holes in the pretzel too!



Finding such finite groups might appear formidable, but it is really a none-too-hard problem about something from school, viz., *permutations*, i.e., one-one functions of a finite set on itself. The job is only, *to give, for each* $q \geq 3$ *and* $p \geq 3$, *permutations* N *of order q, and* L *of order* 2, *such that* N *followed by* L *is of order* p, for the finiteness of the underlying set ensures that the generated group is finite! The figure above indicates one way of making these examples for $q \geq 3$ and $p \geq q$, which suffices, because p and q can be interchanged. In each case, the finite set consists of dots and, *the functions* N *and* L *are defined by the blue and orange arrows,* respectively, dots without arrows being fixed points; and the numbering indicates the cycles of LN, i.e., N followed by L. In the upper left diagram N has the two cycles of length q = 5, connected by an ‘orange cross,’ the two transpositions of L, and it turns out that LN has also two cycles of length q, viz., (12345) and (1′2′3′4′5′). There are q-2 fixed points of L in each, and by ‘dangling’ a new orange transposition to a new dot, as in the lower diagram, from the same number of these in the two cycles, we can increase the lengths of these cycles equally by this number, so this gives us the needed examples for $q \geq 3$, $q \leq p \leq 2q-2$. In the upper right diagram, N has t = 3 orbits of length q = 5 connected by the t-1 transpositions of L, and it turns out that LN is a cycle of size tq. By an orange double-arrow ‘shorting’ two consecutive fixed points of L on this cycle, or by up to q-1 danglers from them – the diagram on the bottom right has both! – we can vary the size of this cycle from tq-1 (now LN also has a fixed point) through (t+1)q-1, which completes the job because any $t \geq 2$ can be used.

In fact, these non-platonic 'lost beauties' exist in great profusion! For instance, we used two blue pentagonal cycles linked by an orange cross, but an arbitrarily long train of this type works too, so, *there are infinitely many finite groups of type* {5,5} (and the same is true for any non-platonic pair)! Moreover, an orange cross *within* a single pentagonal blue cycle gives the *smallest* such group, which has 60 elements, namely, all the 'even permutations' of the five dots, so this particular {5,5} has t = 60 in the notation used in "Miss Universe" (p. 8), so, *twelve pentagons, five at each vertex, on a pretzel with four holes,* and the bigger finite groups of this type will entail using pretzels with more and more holes. Most importantly, *all these finite groups are obtainable by putting extra relations on the group of symmetries, of the tiling of its halo given by the multiple reflections of our red regular curved pentagon*, for, we have proved that this infinite group has only the stated relations. So, if we are willing to think of these surfaces intrinsically, rather than being 'of' pretzels 'in' ordinary three-dimensional space, *these infinitely many finite* {5,5}*'s are all regular in the curved geometry they inherit naturally from that of the halo via these extra relations!* Indeed, each of these relations is of the form g = 1, where g is a fixed point free composition of an even number of reflections in circles, and the surface in question is topologically equivalent to the space obtained by identifying all points of the halo which are related to each other by such g's.



No, the above picture is not – alas! – of another motif in "213, 16A," but of marble inlay work done long ago (around 1420) by Uccello in far away Venice. It shows a *dodecahedragram,* which is obtained by extending the faces of a regular dodecahedron {5,3}, just like a pentagram is obtained by extending the sides of a regular pentagon; but, whereas a pentagram is only a pentagon with self-intersections, *a dodecahedragram is the smallest* {5,5} *with self-intersections!* Its 12 vertices are the apices of the pyramids on the dodecahedron's faces, which have expanded into the pentagrams which are its 12 pentagonal (with self-intersections) faces, while its 30 edges are the extensions of those of the dodecahedron. So, *topology jumps from genus zero to that of a surface with genus four just by the act of extending the faces of a dodecahedron* (this may be tied to a result of Freedman mentioned at the end of "Miss Universe"), as this banished beauty {5,5} makes a brave, but inevitably blemished (by self-intersections) attempt to display all its charms with the five platonic beauties in ordinary 3-dimensional space!

The regularity of the non-platonic {p,q}'s with respect to the curved geometry of the halo tells us that the parallel postulate must be invalid in it, and that is so: now *lines* are circular paths perpendicular to the horizon, and through a point not on a given line there are many non-intersecting lines! However, we still have a *distance* which is invariant under reflections in all lines, and Poincaré pointed out that it can be derived from the assumption that local measurements at ordinary distance r from the centre are made by using rulers which have *shrunk* (!) by the factor $R^2 - r^2$, where R is the radius of the halo. Which is related to his rejection of the age-old notions of separate space and time in favour of a new principle of relativity to give Maxwell's theory of light a coordinate-free meaning, for the distance we mentioned is just the right one if space-time is viewed projectively (see Notes). This last 'lecture' started with a pentagon and a pentagram, and has ended with their spatial analogues, but the game has barely begun! The dodecahedron is now almost crying out for *its* faces to be curved inwards, and starting with Poincaré himself, down to Thurston and others, magical things have been discovered thus; likewise, Gromov and others have lately been busy playing 'dots-and-squares' in halos; but here is where I'll stop, except for the following tailpiece.

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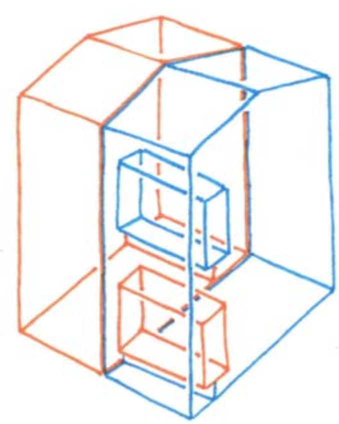
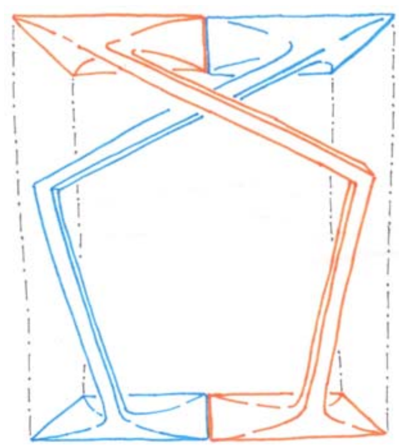
bing's ome

One afternoon I was telling someone with a math background how that little white **"house with two rooms"** and a sloping red roof – it was mentioned fleetingly on p.14, see also pp.3,5 for older pictures of this area – atop that roman aqueduct was, in fact, an example constructed by Bing of a space which is, *contractible but not collapsible!*



Its glass windows allow us to see that it is peculiar: *one enters its rear room via a corridor attached by a flange to the front room, and vice versa!* This is shown above in the line-drawing, more precisely, the position depicted is the one obtaining *after* the slightly overhanging roof has been 'pushed in' a wee bit to make it flush with the walls. There being no free edges left, we can't collapse it any further. On the other hand, filling the house with sand, and then excavating out the same shows, just as plainly, that the 3-dimensional ball, which of course is collapsible, collapses also to Bing's house!

At this point I realized that I had not one, but *three* listeners, and that I had been talking above the heads, in more ways than one, of the other two! Two small girls, a precocious child of about ten, and her equally bright-eyed younger sister, who was probably still not of school-going age, were also with us (they and their parents were amongst a number of guests we had over for tea that day). To make amends, I switched to my story about that *chotta aadmi* (small man) who had, as a matter of fact, made this fully-functional *paani ki taanki* (overhead water tank), as well as this roman aqueduct with its thirteen (true) arches, all by himself, and who was, to tell the truth, still pretty much running the whole show, more or less … There was an indulgent and bemused look on the older child's face as I spun out this yarn, but her little sister was impressed! She asked, "So he lives in the house with the two rooms, uncle?" "No, *beta*," I replied, "that's the pump-house. He lives, like all elves do, far far underground. That little cave in the wall, from where that tiny road comes out to that ramp leading to the spiral staircase winding around and up this *taanki*, and finally, that arch bridge to the aqueduct, I would suspect that that's his route to and from work, but like most humans I can't *see* elves, so I'm not sure." From the eagerness growing on the older sister's face, it was clear that she had much surer knowledge than me, and sure enough, she soon took over from me entirely the further education of her sister on these rather technical matters pertaining to the habits of hobbits and elves and even ents! Her sister listened dutifully enough, but I couldn't help noticing that she was, all the while, using her eyes, and so taking in the details on the rockery, more than her rather busy teacher. As the tiny stony brook hurtled down its rocks, and around that island with the tetrahedral *agni mandir* and the sacred *peepul* tree, into that water pond with "Beesmukhi" and its spout, and that double-curved bridge over it that takes one into the plant house.



With the children busy in their play, the conversation returned to mathematics, and my companion now pointed out something which had been before my eyes all along, but which – duh! – I had not *seen* before: the mother golden-barrel in the background has its five prickly 'pups' arranged in pentagonal symmetry around it, just like that around each vertex of that icosahedron! In return, I explained why the above statue is called, **"pi two is abelian!"** Minus the jargon, this basic theorem from topology says just this: *any continuous mapping, of the top and bottom of the box shown in the line-diagram, which is defined in the same way on the two blue, and on the two orange squares, and which takes their boundaries to just one point, can be extended to a continuous mapping of the box* *which takes all its vertical faces to this point.* To see this, extend in the obvious way to the two disjoint—but flared enormously near their ends—blue and orange square tubes, which join these pairs of squares, and map the rest of the box to the point!

By now we were sitting on the marble-quilted *chabutra* (platform) shown below, our legs dangling over the polished rounded edges of what had remained of the huge tile from which the halo of "Magic Carpet" had been cut, and were doing our maths on a yellow curved pentagonal table-top, made from the very piece that had been cut out from this halo to inlay its red counterpart! From this chabutra one can enjoy the passing show around and past *both* the entrances of the greenhouse, in particular, the cantilever roof above the main door (p.1) is also visible, and my companion did not fail to notice that it was just like the top of "pi two is abelian!" This is so because this motif in concrete was originally intended to be done there on a bigger scale; likewise, something like the statue (p.33) near the other entrance to this plant-shed, which I therefore call **"möbius's balcony,"** was once supposed to frame an intended balcony outside my office's rear window. Since my companion already knew a simple[6] method for making this *one-sided surface* – take a strip of paper, twist one end through 180 degrees, and glue to the other – there was nothing much I had to say about it. So, we just sat in silence, the better to enjoy another very pleasant and surprising feature of this place, which, frankly, was unintended too, it is pure serendipity: when the brook is flowing, there is almost as much sound of running water coming from the *rear* of this chabutra as from its front!

On our way out, we paused at the spot from where the sound of running water is transmitted via the metallic frame to the rear of the chabutra, to ponder the patterns made by fast-flowing water from the fall, over the pebbles of that gorge, as it about-turns into a subterranean tunnel and pool which feed the stony brook. It was about then that I told my companion, "That method of yours does not really give us a möbius strip of paper, indeed it can *never* be made, for, the material making it would prescribe an inner side, while we want a one-sided surface!" Now this is serious, not only am I faulting another method, it

[6] Making the möbius strip *in concrete* is however not as simple. I'll give specific attributions about the mentioned motifs in the Notes, but since these are far from ready, I'll like to express here and now my gratitude to Asa Ram, Miraj, Guni Laal, Ram Prakash, and all the other artisans who were involved in this work. Also, these motifs could not possibly have come into being without either the indulgence, or the organisational abilities of my wife, so here's a very big "Thank you!" to you too, Minni.

seems I didn't come clean on mine either, but my companion saw the point I was trying to get across. Neither the concrete statue, nor the most delicate of models in the thinnest of papers shall give a möbius strip, for the simple reason that, *it is an abstract idea: it exists in our imagination only*: there alone is to be found that ethereal paper with zero thickness, the gossamer necessary to 'make' *any* mathematical surface. So, the "house with two rooms" atop the aqueduct, that too is not mathematician R. H. Bing's example either, only an evocation of the same, and so on down the line, for *all* the motifs. They are all flawed depictions of flawless forms that can't be seen as such – "*like elves,*" my companion teased me – but the fact that these imperfect depictions are so efficient in evoking in our minds these precise ideas, suggests that these 'elves of mathematics' are somewhat more important and natural, and in fact, most of these logical constructs arose historically from a contemplation of the patterns of nature. So it is not surprising that, there is not only great beauty in these logically perfect thought-patterns[7] of mathematics, they have also given us an ever-deeper understanding of nature's patterns.

Conversation over tea that day had flowed smoothly on everything under the sun, but both were waning now, yet the older sister was chattering away with the grown-ups, but her sibling had left the skew room long ago. Fearing that she might be too near the water-pond in this fading light, I was relieved to find her by the aqueduct, but was taken aback by her posture: *half-kneeling and stockstill, she was sort of squinting, steadily and fixedly, into that little cave!* Setting myself down slowly on the grass besides her, I asked, "What is it, *beta*, what are you looking at?" Her voice was soft, but the two words were clear enough, and made my eyes dart to the "house with two rooms," but it was not that, it was this cave only that was holding all her attention, as she repeated,*"bing's ome,"* and moved a finger slowly towards it. The thought flashed that she was pulling my leg in return, but there was not a trace of naughtiness on that innocent face, and the thought was dismissed, as I looked at exactly where she was looking, but I saw only what you see in the photo above (which however was taken on a later day, in better light).

[7] In the preceding 'lectures' I have used the motifs merely as gateways towards the much deeper beauty in these patterns of thought, but, as I mentioned on p. 13, non-trivial effort is needed even to 'see' already-done mathematics. Therefore, to keep the focus on these very beautiful but demanding arguments, it was necessary to minimize all distraction; so only a bare amount of terminology was used, and just a few names were mentioned; but a more extended glossary, and a bibliography, will be given in the Notes.

While she remained in that fascinated state for about thirty seconds more, when, all of a sudden, her body relaxed and returned to normal, and it was clear that whatever she had seen, or thought she had seen, was no longer there. The happy smile the child smiled at me then presumed that I had also seen what she had seen, so I only smiled back. However later, on many evenings at about the same time, I have sat down on the same grass at the same place, and tried many contortions of face and squintings of eyes, hoping to espy, *for myself*, that elf which that little girl had apparently seen, but I would be lying if I were to tell you that I have seen it too, for I haven't, *not yet,* anyways!

I remember reading that, try as they might, a small but significant percentage of people can't see the hidden image in even the simplest of those *"Magic Eye"* pictures, so who knows, *bing with a small 'bee'* might be somewhere in the previous photograph, only you aren't squinting at it properly? More seriously, I hope you managed to catch at least a glimpse of the shimmering beauty of the elves of mathematics, which indeed I *have* seen, and which I've tried to share with you, via these motifs, in this paper.

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