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A "RIEMANN HYPOTHESIS" FOR TRIANGULABLE MANIFOLDS

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ABSTRACT. Given a triangulable manifold we show how to find a triangulation whose characteristic polynomial has roots which are either real or on the line $\operatorname{Re} z = 1/2$.

If K is a (finite) simplicial complex, then $f_K(z)$ will denote the polynomial $\chi/2 - f_0(K) \cdot z + f_1(K) \cdot z^2 - \cdots$; here χ is the Euler characteristic of the underlying space $M = |K|$ and $f_i(K)$ is the number of i-simplices in K.

THEOREM. *If M is any closed triangulable manifold, then it admits a triangulation K for which all the nonreal zeros of $f_K(z)$ lie on the line* $\operatorname{Re} z = 1/2$.

PROOF. If L is any triangulation of M^m, then one has the *functional equation* $f_L(z) = (-1)^{m+1} f_L(1-z)$. (This fact is well known and is a concise way of writing the Dehn-Sommerville equations (see e.g. [**1**, p. 101]): it was observed by Klee [**2**] that these equations hold if the link of each i-simplex of L has the same Euler characteristic as an $(m-i-1)$-dimensional sphere, e.g. if L triangulates a closed m-manifold.) So the roots of $f_L(z)$ are symmetrically situated about the real axis and the line $\operatorname{Re} z = 1/2$.

For each integer $q \geqslant 0$ we construct a simplicial complex L_q as follows: $L_0 = L$ is any triangulation of M^m and L_{q+1} is obtained by deriving an m-simplex of L_q. We note that

$$\begin{aligned} f_{L_q}(z) &= f_L(z) - qz + q(m+1)z^2 - q\binom{m+1}{2}z^3 + \cdots \\ &\quad + (-1)^{m+1} q \binom{m+1}{m} z^{m+1} - (-1)^{m+1} q z^{m+1} \\ &= f_L(z) - qz(1-z)^{m+1} - (-1)^{m+1} q z^{m+1}(1-z). \end{aligned}$$

We assert that *for all q sufficiently big $K = L_q$ is a triangulation of M^m such that $f_K(z)$ has distinct roots of which all but* 2 *lie on the line* $\operatorname{Re} z = 1/2$. It is clear that the remaining 2 roots must then be equal to $1/2 \pm \kappa$ for some $\kappa > 0$; if $\chi = 0$ these exceptional roots are obviously 0 and 1.

Note that $f_K(1-z) = (-1)^{m+1} f_K(z)$ and $f_K(\bar{z}) = \overline{f_K(z)}$ imply that for m odd (resp. m even) $f_K(z)$ takes real (resp. purely imaginary) values on the line $\operatorname{Re} z = 1/2$; the same is also true for the degree $m+1$ polynomial

$$-z(1-z)^{m+1} - (-1)^{m+1} z^{m+1}(1-z) = q^{-1} f_K(z) - q^{-1} \cdot f_L(z).$$

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Next we observe that the $m - 1$ roots of $-z(1 - z)^{m+1} - (-1)^{m+1}z^{m+1}(1 - z)$ other than 0 and 1 satisfy $|z/(1 - z)| = 1$, i.e. lie on the line $\operatorname{Re} z = 1/2$. So for q big the neighbouring polynomial $q^{-1}f_K(z)$ must also have $m - 1$ roots on the line $\operatorname{Re} z = 1/2$. Q.E.D.

REMARK. Let L be a triangulation of M^m and let $C(q, m + 1)$, $q \geqslant m + 2$, be a cyclic triangulation (see e.g. [**1**, p. 82]) of the sphere S^m. By omitting an m-simplex each from L and $C(q, m + 1)$ and then identifying their boundaries, one gets a triangulation L^q of M^m. One can verify (using equation (13) on p. 172 of [**1**] to examine the roots of the polynomial of $C(q, m + 1)$) that if $m \geqslant 5$ and q is sufficiently big, then $f_{L^q}(z)$ has some roots which are neither real nor on the line $\operatorname{Re} z = 1/2$.

The "Riemann hypothesis" considered above is related to the lower and upper bound conjectures for manifolds and is amongst the problems posed in §6 of [**3**].

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