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Aequationes Mathematicae

Roots of translations

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Summary. Given q and k we find all functions (for $\mathbb{R}$, also all continuous functions) $f : X \to X$, where $X = \mathbb{N}$, $\mathbb{Z}$ or $\mathbb{R}$, such that $f^q(n) = n + k \ \forall n$.

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1. A recent e-mail from Dhruv, an engineering senior and a nephew of mine, drew my attention to the 'year problem' from the International Mathematical Olympiad held in 1987 at Havana, Cuba: *prove that there is no function f from $\mathbb{N}$ to $\mathbb{N}$ such that $f(f(n)) = n + 1987$ for all n.* In other words, the translation of the nonnegative integers $\mathbb{N}$ by 1987 does not have a 'square root' $f : \mathbb{N} \to \mathbb{N}$. More generally, one has the following, where q and k denote any positive integers.

2. Proposition. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f^q(n) = n + k$ for all n in $\mathbb{N}$ if and only if q divides k.*

To see this we partition $\mathbb{N}$ into k *cosets,* two numbers being in the same coset C iff their difference is divisible by k. We note that f is obviously one-to-one, and must map cosets into cosets. This last follows from

$$f(n + k) = f(n) + k,$$

which holds since both sides are equal to $f^{q+1}(n)$. This formula also shows that $f(n)-n$ is constant as n runs over a coset C. We shall call this positive or negative constant c the *increment* of f on the coset C. The cardinality k set of cosets partitions into disjoint *orbits* $\{C_0, C_1, C_2, \dots\}$, where C_{i+1} denotes the coset into which the coset C_i is mapped by f. Since after q iterations f maps a coset back into itself, the cardinality of any orbit is either exactly q, or else a proper divisor d of q. In the first case, the sum of the increments of f on the cosets of an orbit is exactly k. In the second case, q/d times this sum of increments is equal to k. There must be an orbit of the second kind in case q does not divide k. Consider any coset of such an orbit. The map f^d maps each member n of this coset into n plus a constant $f^d(n) - n$, the aforementioned sum of increments, which is now a proper divisor of k. This contradicts the fact that k must divide the difference $f^d(n) - n$ of two numbers in the same coset. So this case is ruled out. In case q divides k, the translation by k/q is obviously a qth root of the translation by k. We can in fact list all the qth roots.

3. Proposition. *If q divides k, then a function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f^q(n) = n+k$ for all n in $\mathbb{N}$ is necessarily one of the functions f_π defined below.*

Here π denotes a partition of the cardinality $k = qr$ set of cosets into r totally ordered cardinality q subsets. Let $(C_0, C_1, \ldots, C_{q-1})$ be any of these totally ordered subsets, with $C_i = \{m_i + tk : t \geq 0\}, 0 \leq i < q$. Then $f_\pi : \mathbb{N} \to \mathbb{N}$ is the function – cf. Fig. 1, where $k = 6$ and $q = 3$ – such that $f_\pi(m_i + tk) = m_{i+1} + tk$ for $0 \leq i < q-1$ while $f_\pi(m_{q-1} + tk) = m_0 + (t+1)k$.

We know already that an $f : \mathbb{N} \to \mathbb{N}$ satisfying $f^q(n) = n + k$ for all n in $\mathbb{N}$ determines a partition of the cardinality k set of cosets into cyclically ordered cardinality q subsets – orbits – with f injecting each coset of an orbit into the cyclically next orbit with a constant increment. If a number belongs to the image of f so do all bigger numbers in the same coset, but all numbers of all the cosets of an orbit cannot be in the image: otherwise, by applying f^{-q} repeatedly we can make any number negative, whereas $\mathbb{N}$ has only nonnegative numbers. So in each orbit there is a coset C_0 whose least element m_0 is not in the image of f, let $C_1, C_2, \ldots, C_{q-1}$ be its remaining cosets in cyclic order after C_0. We assert that $f^{q-1}(m_0)$ must be the least element m_{q-1} of C_{q-1}. This follows because the constant increment of f on C_{q-1} equals $f^q(m_0) - f^{q-1}(m_0) = m_0 + k - f^{q-1}(m_0)$, so m_{q-1} is mapped by f to $m_{q-1} + m_0 + k - f^{q-1}(m_0)$ which would be smaller than $m_0 + k$ if $f^{q-1}(m_0)$ were bigger than m_{q-1}, which is not possible because the only such element m_0 of this coset is not in the image of f. Having thus proved the assertion $f^{q-1}(m_0) = m_{q-1}$ we can now use it, and the constancy of increment on the previous coset, to show in a similar manner that $f^{q-2}(m_0)$ must be the smallest element m_{q-2} of C_{q-2}, and so on. Thus C_0 is the unique coset of orbit whose minimal element is not in the image of f, our way of totally ordering the orbit is unambiguous, and f coincides with f_π where π is the partition into totally ordered cardinality q sets thus determined by f.

4. Proposition. *The translation of $\mathbb{N}$ by $k = qr$ has exactly $k!/r!$ qth roots mapping $\mathbb{N}$ to $\mathbb{N}$.*

This follows from the above because r disjoint totally ordered cardinality q parts can be concatenated in $r!$ distinct ways to form a total ordering of the cardinality k set, and each of the $k!$ total orderings of this set occurs once and only once as such a concatenation. For example, $+6 : \mathbb{N} \to \mathbb{N}$ has $6!/2! = 360$ cube

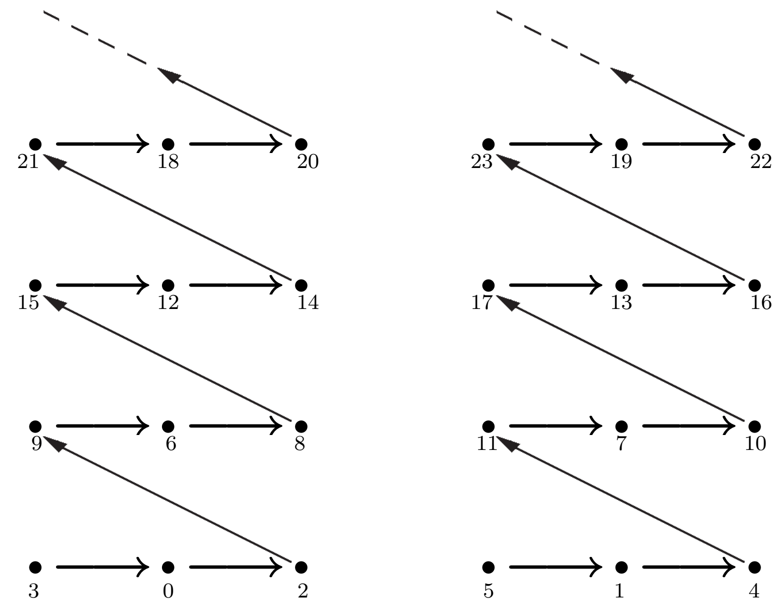

Fig. 1

roots $\mathbb{N} \to \mathbb{N}$, of which one is displayed in Fig. 1 above.

This finiteness of the number of qth roots hinges on the fact that we are constrained to remain in $\mathbb{N}$ which is bounded below (and all of the above generalizes from $\mathbb{N} = \mathbb{Z}_0$ to $\mathbb{Z}_t = \{n \in \mathbb{Z} : n \geq t\}$). Without this constraint the number of qth roots is zero or infinite.

5. Proposition. *There is an $f : \mathbb{Z} \to \mathbb{Z}$ satisfying $f^q(n) = n + k$ for all n in $\mathbb{N}$ iff q divides k, and when q divides k there are infinitely many such f.*

We might as well suppose $f^q(n) = n + k\ \forall n \in \mathbb{Z}$, because $f(n + k) = f(n) + k$ holds for all nonnegative n, and if we redefine f on negative integers so as to make this formula valid for all integers, then the new f will do the job. An argument just like that used in Section 2 shows that a qth root f of any translation of $\mathbb{Z}$ by k – note that translations, and so their roots, are now bijections – must partition the cardinality k set of cosets of $\mathbb{Z}$ into cyclically ordered cardinality q subsets – so q must divide k – $(C_0, C_1, \ldots, C_{q-1})$ with f mapping each coset bijectively on the cyclically next coset with a constant increment. For any arbitrary choice $m_i \in C_i$ of numbers, one in each coset – note that the number of such choices is infinite – there is one and only one such $f : \mathbb{Z} \to \mathbb{Z}$ with $f(m_0) = m_1, \ldots, f(m_{q-2}) = m_{q-1}, f(m_{q-1}) = m_0 + k$.

A similar argument shows that qth roots of a translation of the reals $\mathbb{R}$ by k always exist, with each cyclically permuting cardinality q mutually disjoint sets whose union is the infinite set of all cosets (a coset being a subset of $\mathbb{R}$ of the type $\{a + tk : t \in \mathbb{Z}\}$). Most of these qth roots are discontinuous, those which are continuous are classified by the next result.

6. Proposition. *A continuous $f : \mathbb{R} \to \mathbb{R}$ satisfies $f^q(n) = n + k$ for all n in $\mathbb{R}$ if and only if f is one of the homeomorphisms $\phi = \phi(f_0, f_1, \ldots, f_{q-2})$ of $\mathbb{R}$ defined below.*

Here $f_0 : [0, a_1] \to [a_1, a_2], f_1 : [a_1, a_2] \to [a_2, a_3], \ldots, f_{q-2} : [a_{q-2}, a_{q-1}] \to [a_{q-1}, k]$ are any increasing homeomorphisms. If x is in the domain of any of these functions, $\phi(x)$ shall be the value of that function on x, if x is in $[a_{q-1}, k]$ then $\phi(x) = (f_{q-2} \circ \cdots f_1 \circ f_0)^{-1}(x) + k$, and for the remaining real numbers we define $\phi(x)$ in such a way that $\phi(x + k) = \phi(x) + k$ holds for all $x \in \mathbb{R}$.

Since the continuous f has no fixed point, its graph is either above or below the 45° line. The latter case is ruled out because then the graph of its qth iterate, the translation by the positive number k, would also be below this line. Since it is one-to-one, the continuous f is strictly increasing or strictly decreasing. However f is also onto, which rules out that it is strictly decreasing: if it were with say (x, y), $y > x$ on its graph then no number less than x could be in its image. Hence we have seen that f is strictly increasing with $f(x) > x$ for all x. Let $a_1 = f(0)$, $a_2 = f(a_1)$, …, $a_{q-1} = f(a_{q-2})$. Here, since $f(a_{q-1}) = f^q(0) = k$, we are assured of $0 < a_1 < a_2 \cdots < a_{q-1} < k$. So $f = \phi(f_0, f_1, \ldots, f_{q-2})$ where the f_i's are the restrictions of f on the subintervals of $[0, k]$ mentioned in the previous paragraph.

The infinitude of cosets ensures that translations of the nonnegative reals $\mathbb{R}_0$ always have qth roots mapping $\mathbb{R}_0$ to $\mathbb{R}_0$, however cosets of $\mathbb{R}_0$ have minimal elements, so the nature of the root on each orbit of q cosets is as discussed in Section 3, the continuous roots are the restrictions to $\mathbb{R}_0$ of those just described.

Added in proof. The geometrical method of this note in fact yields far-reaching generalizations of the above results. These, and some other applications of this method, are posted on my website.

References

I owe these to the referee and Professor L. Reich. The problem treated here seems somehow related to the similar "multiplicative" problem addressed (for $q = 2$) by Allouche et al., and Targonski's book is a general source for problems and results from iteration theory.

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