Plain Geometry & Relativity, part V

28. *Matter is but extension, and is differentiated only by its various motions.* `Let us summarize where we are` now in our efforts to answer those three questions—Note 25—of manifolds starting from this dictum of Descartes.

(28.1) We saw `closed manifolds emerge magically`—and it seems in all their diversity—*in euclidean n-spaces from their own periodic motions.*[1] Even simple physical systems have many more degrees of freedom n than visual space, but we can here work in just the one cartesian space of all sequences of real numbers having only finitely many nonzero entries.[2]

(28.2) That pragmatic restriction—any observer's space at his time $t = 1$ is *an n-ball* of radius $c < \infty$ around him—led us to an *absolute time τ* defined on, and preserved by the linear reflections of *the cone* over his n-ball. The *flow lines* of a motion of the n-ball project its moving points on the successive absolute spaces τ = constant. *Their chords are parallel to rays of the cone,* besides we require that this motion proceeds via homeomorphisms.

(28.21) *For $n > 4$ this requirement implies that there is a perturbation of the motion of the n-ball with lipschitz homeomorphisms.* We sketched in Note 23 why this seems true and gives one half of : `A closed manifold emerges in a periodic relativistic motion if and only if it admits a lipschitz structure.` For the other half we showed that a 2-ball emerges from a smooth motion for any $n \geq 2$, from which it is clear any closed smooth manifold occurs, hoping that similar constructions work for closed lipschitz manifolds.

(28.22) Further, using a `lipschitz yang-mills theory`, Donaldson and Sullivan have shown that some closed 4-manifolds do not admit any lipschitz structure. On the other hand, using `a bieberbach theorem for relativistic crystallography`, the latter had shown long before that outside dimension 4 all manifolds admit a unique lipschitz structure. However I have yet to understand these technologies to my full satisfaction.

(28.3) From that complicated hidden motion we only got `static` closed manifolds in our n-ball of radius $c < \infty$. `What sets them moving` is more pragmatism : *only a compact interval of absolute time for each snapshot.* Depending on the *scale* at which we are discerning the hidden motion, there is say *a number* $1 < s < \infty$ – very big in macrophysics, almost 1 for microphysics – and *the τth frame of this moving picture uses* $[\tau/s, \tau s]$*,* i.e., the actual hidden motion is replaced with the one having this restriction as a period to make this snapshot. Since there is in fact no periodicity the closed manifolds move and occasionally coalesce or bifurcate in this movie : cobordism or intrinsic `homology arises naturally from motion.` Not only that, as our discernment of the hidden motion becomes finer, what was a minimal manifold can get partitioned, say into a foliation, and its compact leaves will be now natural candidates for cartesian matter at this smaller scale s, et cetera.

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[1] As `persistent minimal sets` of these motions (more details of this and some other things still to be done) so our manifolds are *connected* but may not be orientable.

[2] This space $\mathbb{R}^\infty$ was used in the last two `parallel notes` उ and ब.

29. `A cartesian motion` is a partition of conical spacetime into infinite arcs, its *flow lines,* that cut each copy $\tau =$ constant of the absolute space once and only once, inducing homeomorphisms between these level surfaces, and have chords parallel to rays. If all flow lines tend to the origin as $\tau \to 0$ it is a `deformation`[3] of the basic example, *cartesian rest,* which has as flow lines all rays of the cone. We'll now discuss why this definition is reasonable.

(29.1) The `continuity` of flow lines does not for $n > 1$ guarantee that the bijections induced between the spaces $\tau =$ constant are bicontinuous, that was a separate condition. To obtain manifolds that are smooth, piecewise linear or lipschitz we'll also use cartesian motions with flow lines and homeomorphisms smooth, piecewise linear or lipschitz. The condition, `chords parallel to rays`, which kicks in for $c < \infty$ implies then that *the flow lines are lipschitz.* This being an open condition it persists under perturbations and so we'll be able to approximate – if $c < \infty$, $n \neq 4$ and the given motion is periodic – by arbitrarily close cartesian motions with homeomorphisms lipschitz.[4]

(29.2) *The* `initial condition` *at* $\tau \to 0$ *ensures the flow lines cut each* $t =$ *constant of any observer once and only once* :- With the continuity of flow lines it gives one cut, and there can't be two because the chords of these n-balls are not parallel to even the rays of the *closed* cone over them. □

In other words, *the time of any observer is strictly increasing and takes all possible values on flow lines.* Conversely for $c < \infty$ this implies the initial condition, and that chords of flow lines are parallel to the rays of the closed cone[5] :- The time t of S takes all values on it, so the flow line must start from O. It can't have a chord $\overrightarrow{P_1 P_2}$ which extended on either side exits the cone, for then we can find a nearby chord of the cone which separates O and P_2 from P_1 in the plane of these points, so the time t' of the ray S' through this chord's mid-point would have a lesser value at P_2 than at P_1. □

(29.3) We call cartesian motions preserved by all homotheties *steady*, and those preserved by some homothety other than the identity `periodic in time`. *If* $c < \infty$ *the restriction of a deformation to times* $[t/s, st]$ *of an observer extends to a periodic deformation preserved by multiplication by* s^2 :- For $s > 1$ we must use homothetic patterns of flow lines over the intervals $\ldots, [t/s^3, t/s]$ and $[st, s^3 t], \ldots$ which is okay since concatenation preserves continuity, the sum of two vectors parallel to rays is parallel to a ray, and the new flow lines are defined for all $t > 0$, so the initial condition also holds. □ Likewise, we can replace any portion of a motion by a homothetically equivalent portion of another, and concatenation works just as well in the everything lipschitz or piecewise linear

[3]Talking of deformations, I still don't know – see ट – if a `cantorian PG&R` ties up with his IUTT, but, even as Mochizuki finished his 8 talks in Kyoto, I saw it was child's play to win a game on p. 787 of the Notices of the A.M.S. of August 2016 by using deformations of addition (for example, if the ith kid guesses the number which makes the total i mod 10)!

[4]The `deforming homeomorphisms`, that change each ray to the flow line through the same point on $\tau = 1$, are made lipschitz step by step, using as scaffolding a crystallographic tiling of spacetime, $n = 4$ excluded because of those – see ড, ਚ – `switching difficulties`.

[5]So this more general definition permits photon-like instantaneous motion on some intervals of absolute time, but we'll use our open and particle-like definition, on each flow line the `elapsed time` defined by $\int_O^P \tau(d\mathbf{r})$ – see Note 15 – is also strictly increasing.

context, but some sandpapering is needed for smoothness.

Similarly, *the restriction of any cartesian motion to absolute times* $[\tau/s, s\tau]$ *extends periodically, but we may lose the initial condition,* for example, consider cartesian rest for $(0, \tau_0]$ followed by all lines parallel to a ray S. An infinite repetition will play a key rôle again in the torus tricks needed to get also some measure of spatial periodicity into a given cartesian motion.

30. Periodicity of a cartesian motion in fact makes sense with respect to any transformation of spacetime which preserves this notion, say a *linear reflection* of the cone, or else the `time reversal` of all its rays in a curved mirror $\tau = a$, or any composition of these, see Note 23. Clearly, *a homothety is a composition of two time reversals,* but why do these *nonlinear* factors also preserve the notion of cartesian motion and the cayley distance of the cone? We'll see why, also we'll see that cayley distance is born from the age-old definitions of `adding and multiplying segments` that are given in elementary classes.

(30.1) *The above involutions mirror cartesian motions to cartesian motions.* This will follow easily once we have checked the following.

(30.11) *A line parallel to the boundary and cutting the mirror in one point is switched with the other such line on the same plane through the origin* :-

This plane—of the given line L and the origin—is shown below, P being the one point of the mirror on the line. So, *if the mirror is flat,* it cannot contain this plane – in this trivial case the line stays put – it cuts it in the ray S through P. Any point A of the plane reflects to the point A' such that the mid-point of AA' is on S. It follows that the mirror image L' of our line is the other line M of this plane through P which is parallel to the boundary of the cone.

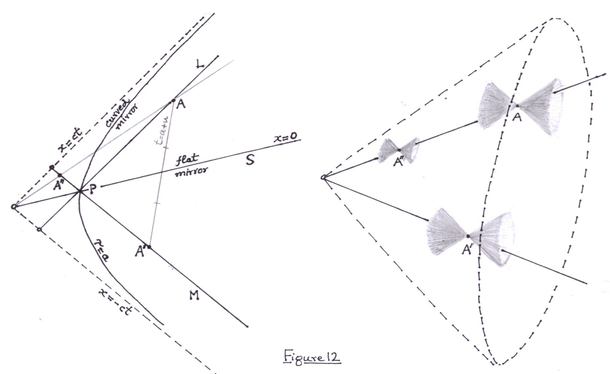

Figure 12

If the mirror is curved, $\tau = a$, then any point A reflects to the point A'' on the same ray such that $\tau(A)\tau(A'') = a^2$. We recall that $\tau^2 = t^2 - \frac{x^2}{c^2}$ in the coordinates (t, x) of S. So if $A = (a + u, cu)$ is on the line L with slope c through $P = (a, 0)$ we have $\tau^2(A) = a^2 + 2au$. Therefore $\tau^2(A'') = \frac{a^4}{a^2+2au}$ and $\frac{\tau(A'')}{\tau(A)} = \frac{a}{a+2u}$. Hence $A'' = (\frac{a(a+u)}{a+2u}, \frac{acu}{a+2u}) = (a - \frac{au}{a+2u}, \frac{cau}{a+2u})$, the point on the ray through A and the line M of slope $-c$ through P. The mirror image L'' is again M, but now as A runs over L linearly in the direction τ increasing, its image A'' describes M non-linearly in the direction τ decreasing. $\square$

So these involutions of spacetime not only preserve its product structure, they map a line parallel to its boundary to another such line. The hypersurface of all such lines through A is mapped to the hypersurface at A' or A''. A segment joining A to B is parallel to a ray iff B is in a component of the complement of this hypersurface through which the ray of A passes. Using this we see that the mirror images of all flow lines obey the chord condition. □

(30.12) In fact, *the above switching property fixes the curved mirrors and so* τ *up to a constant multiple.* Also, $\tau(B'') < \tau(A'')$ iff $\tau(B) > \tau(A)$ was fine above because our flow lines are *unoriented.* And, as is shown below, the segment $B''A''$ is *not* the reversal of AB unless B is on the ray of A. Hence, *there is no piecewise linear cartesian motion other than rest which is preserved by a time reversal!* Which suggests that, in this unfolding tale about the cartesian genesis of closed manifolds, this non-linear doubling of the symmetries of spacetime will tie up with the `cohomological obstruction` to piecewise linearity.

(30.13) *Any other line* R *parallel to a ray* S *reverses to an* R'' *which is coplanar, strictly convex, tangent to* S *at the origin, and asymptotic to the line* L'' *parallel to the boundary whose reversal* L *has the same end as* R :-

In coordinates (t, x) of the observer S such that R consists of all points $A = (u, b)$, $\frac{b}{c} < u < \infty$, its reversal R'' in the curved mirror $\tau = a$ consists of all points $A'' = \frac{a^2c^2}{c^2u^2-b^2}(u, b)$. As u increases both coordinates of A'' decrease to 0, the second at a faster rate, so the graph of R'' is strictly convex downwards and approaches the origin tangent to S. As $u \to \frac{b}{c}$, R and L approach their common end E, so R'' approaches the line L''. □

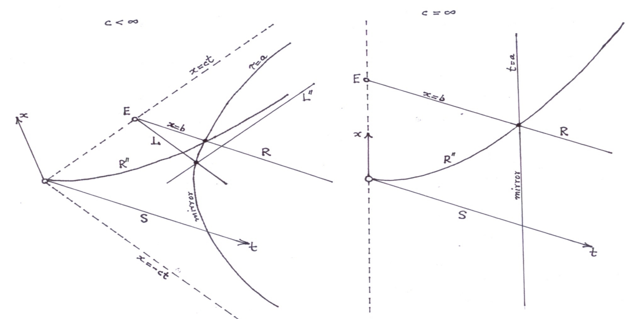

Figure 13

(30.14) Eliminating $u = \frac{bt}{x}$ we see that $A'' = (t, x)$ satisfies $bx^2 + c^2a^2x - c^2bt^2 = 0$, so R'' lies on a hyperbola. *This non-linearity persists in the classical limit* $c \to \infty$: now $A'' = \frac{a^2}{u^2}(u, b), 0 < u < \infty$, so R'' is the $t > 0$ portion of the parabola $a^2x - bt^2 = 0$ in the coordinates of the observer S.[6] So, *the cartesian motion with flow lines straight and parallel to* S *always reverses to one whose flow lines, other than this ray, are conics tangent to it at the origin.*

[6]From note 23 : *even for* $c = \infty$ *the hidden product structure is different from that of any observer.* Things are not always easier now, classical notions often have simpler relativistic deformations. From page 2 of PG&R text : *ball geometry 'explains' euclidean geometry,* the classical limit of the naturally defined linear reflections of the cone restrict to the euclidean reflections on the flat $t = 1$ of half-space.

On the other hand, a linear reflection in a mirror containing S preserves this motion, and, *in a flat mirror not containing S, it reflects to a motion with flow lines straight but, unless $c = \infty$, not parallel to each other* : they diverge towards infinity because signals are reaching S' at a finite speed.

(30.15) Indeed, the lines parallel to S form one half of this observer's product structure, which therefore reverses to the arcs above together with the reversals of his balls. *The reversal B'' of any ball $B = \{(\mathbf{x}, t) : t = d\}$ in $\tau = a$ is given by translating $\tau = \frac{a^2}{2d}$ parallel to S by $\frac{a^2}{2d}$* :- A obeys $f(t, \mathbf{x}) = 0$ iff A'' obeys $f(\frac{a^2c^2}{c^2t^2-\mathbf{x}^2}t, \frac{a^2c^2}{c^2t^2-\mathbf{x}^2}\mathbf{x}) = 0$, so B'' is the subset of all $(\mathbf{x}, t)$ such that $\frac{a^2c^2}{c^2t^2-\mathbf{x}^2}t = d$, i.e., $dc^2t^2 - a^2c^2t - d\mathbf{x}^2 = 0$, i.e., $(t - \frac{a^2}{2d})^2 - \frac{\mathbf{x}^2}{c^2} = (\frac{a^2}{2d})^2$. $\square$

(30.16) *So let's extend the principle of mirror relativity to reversals!* The hidden product structure consists of the rays and the level surfaces of absolute time. The point on ray S and surface $\tau = a$ will be denoted S_a. We had called S an observer, we'll now think of each S_a as an `alien` associate who can also reverse in time. So we have, *an n-ball's worth of observers, each a line's worth of aliens.* Mimicking the enunciation we used in PG&R text : *any other alien S_b observes the curved mirror image of what S_a observes, under the time reversal switching these points.* This too is dictated by the aesthetics of Note 2. Also, the composition of two time reversals is a homothety, and our instruments can simulate a species for which time is apparently speeded up or slowed down. The postulate implies that our physical laws, scaled by a suitable factor, coincide with their laws. However, this extension shall really come into play when we come to the cartesian genesis of elementary particles. If we can hear orientation dependent `characteristic numbers` of a closed manifold, it says aliens can hear the same manifold born with the opposite orientation. Further, I've heard said that *we too* can hear these anti-particles! Therefore, but more importantly just for the fun of it, let's reflect some more on reflections.

(30.17) We first recall why, *any observer deems the rulers of another shrunk up to, and his clock slower by, the same factor* :- Any observer S puts himself at the center of a euclidean ball whose radius is increasing in proportion c to the time on his clock : the disjoint union of all these balls, a right cone, is spacetime as he sees it. *Mirror relativity identifies this multitude of right cones, one for each observer, with just one cone* : the unique[7] linear reflection of the right cone of S which switches its axis S with another ray S' transfers the *product structure*[8] *of S* to another representing how S' sees spacetime. For example PQ below represents the distance between S and S' as measured by S at his time OP. It reflects to $P'Q'$ which has the same length as measured by S' at the same time OP' on his clock. However, $P'Q'$ is not in a ball of S, and even its spatial component[9] $P'R$ is more than PQ. Likewise, OP' is not parallel to S, and even its temporal component OR is more than OP. The said factors are the same, i.e., $PQ/P'R = OP/OR$, by similar triangles. $\square$

[7] Since its *flat mirror contains all midpoints of chords of the cone parallel to PP'.*

[8] Each point has its *ball* and *parallel* to S (but a ball and parallel may be disjoint); this structure is preserved by the *many* linear reflections of the cone which preserve S.

[9] In ball of P'; the parallel to S through P' may not cut his ball at time OP.

Calculation of this factor in terms of v, the speed PQ/OP of S' as measured by S, equivalently $P'Q'/OP'$ of S as measured by S' :- We'll use coordinates of S. Let EF be the diameter of the ball of S extending PQ (perpendicular diameters reflect to equal and parallel chords). The parallelogram $\{O, E', 2P', F'\}$ has sides of slope $\pm c$ and one diagonal has slope v, so the other diagonal extending $Q'P'$ *has slope* c^2/v, equivalently $1/\gamma = \sqrt{1 - v^2/c^2}$ in

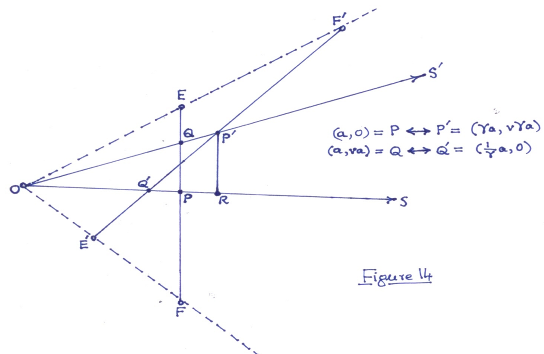

Figure 14

. $\square$

In the same vein, S may deem a point `invisible` to S' if no parallel to S through its ball goes through its mirror image. *This invisible-to-S' subset of a ball of S consists of all points which are not in the ellipsoid with centre on ray S', all perpendicular diameters equal to that of the ball, but the one in this plane is shrunk by* $1/\gamma$:- Linearity and $P \mapsto P', P' \mapsto P$ imply $(t, x, \mathbf{y}) \mapsto (\gamma t - \frac{\gamma v}{c^2}x, v\gamma t - \gamma x, \mathbf{y})$ which preserves $c^2t^2 - x^2 - \mathbf{y}^2$. So $(t, x, \mathbf{y})$ is in the hyperplane of the ball of S' iff $\gamma t - \frac{\gamma v}{c^2}x = a$, i.e. $t = \frac{a}{\gamma} + \frac{v}{c^2}x$, and is the mirror image of the point $(a, \overline{x} := v\gamma[\frac{a}{\gamma} + \frac{v}{c^2}x] - \gamma x = va - \frac{x}{\gamma}, \mathbf{y})$ in the hyperplane of the ball of S, which satisfies $\gamma^2(\overline{x} - va)^2 + \mathbf{y}^2 < a^2c^2$ iff $x^2 + \mathbf{y}^2 < a^2c^2$. $\square$ We note that P is not in this ellipsoid iff $\gamma v \geq c$, i.e., *iff* $\sqrt{2}v \geq c$, *then each observer may think that he is invisible to the other!* $\square$ Finally, here is a construction of P' via the time reversal that keeps P fixed :-

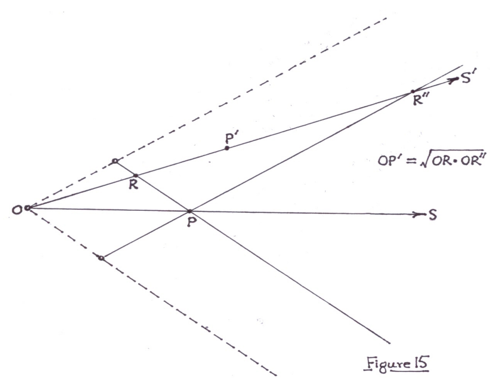

Figure 15

(30.18) *Of lines through P parallel to the boundary two are cut by any other ray, the reversal that keeps P fixed switches these cuts,* see (30.11). So any point

T is mapped to T'' on the same ray such that $OT \cdot OT'' = OR \cdot OR''$ where R and R'' denote these cuts, and its fixed point on this ray, i.e., the point P' on the curved mirror through P, is given by $OP' = \sqrt{OR \cdot OR''}$. Thus, *a single reversal, defined directly in this way for $c < \infty$, gives all the linear reflections of the cone,* and lays bare its hidden product structure. $\square$

Any alien S_a perceives spacetime as a euclidean ball around him whose radius is growing in proportion c to a `two-valued time` on his two-way clock, with both orientations equally natural. (30.16) *identifies all these right cones—one for each observer, two for each alien—with just one cone* : Let the right cone of S be one of the right cones of S_1; the unique linear reflections which switch S with any other ray S' then identify the right cones of the observers S' with one of the right cones of the aliens S'_1; the time reversal in the curved mirror $\tau = 1$ converts these to the other product structure of these aliens; and finally, the unique time reversals which switch $\tau = 1$ and any other $\tau = a$ give us the pairs of right cones of the remaining aliens. *So any oriented alien $\mathcal{A}$ perceives the right conical structure of another, e.g., his alter ego $\mathcal{A}^*$ with the other orientation, as distorted,* and can jump to misconceptions, *even for $c = \infty$* :-

For example, two aliens are related by a time reversal, necessarily unique, iff they are associated to the same observer S. Any point T at time t and a distance r from S in the right cone of $\mathcal{A}$, reverses to a T'' which is at the same time and distance from S in the right conical structure of $\mathcal{A}''$, but $\mathcal{A}$ deems these measurements of $\mathcal{A}''$ to be his own spatial and temporal components for T'', and for $c < \infty$ he may also deem an annulus of his ball through T invisible to $\mathcal{A}''$ in the same sense of this word as used before.[10]

To compute this distortion we think of Figure 14 now as *the right cone of $\mathcal{A}$.*[11] So the point A of S to which $\{\mathcal{A}, \mathcal{A}^*\}$ correspond has time 1, and $\mathcal{A}$ is related to $\mathcal{A}''$ at the point A'' with time a^2 by the reversal in the curved mirror through P. More generally, if P' is the point of this mirror on the ray OT, then T reverses to T'' on this ray such that $\frac{OT}{OT''} = \frac{OT^2}{OT \cdot OT''} = \frac{OT^2}{OP'^2} = \frac{OT^2}{OQ^2}\frac{OQ^2}{OP'^2} = \frac{t^2}{a^2}(1 - \frac{r^2/t^2}{c^2})$[12], *the ratio by which the time and distance-from-S measurements of $\mathcal{A}''$ are deemed off by $\mathcal{A}$.* For $c < \infty$, T is deemed invisible to $\mathcal{A}''$ by $\mathcal{A}$ iff his distance from S to T'' is $\geq ct$, the radius of the ball of T, i.e., iff $r \geq ct\frac{t^2}{a^2}(1 - \frac{r^2/t^2}{c^2})$. The case of equality re-arranged with $g = \frac{ct}{r}$ shows that, *the ratio g of the outer to inner radii of any invisible-to-$\mathcal{A}''$ annulus satisfies $g^2 - \frac{a^2}{t^2}g - 1 = 0$,* in particular at time $t = a$ of $\mathcal{A}$, it is precisely the `golden ratio!` $\square$

(30.19) *As a group all compositions of reversals is the nonabelian double cover of the positive numbers under multiplcation*:- A composition κ of two reversals

[10] *"When I use a word,"* Humpty-Dumpty said, *"it means just what I chose it to mean,"* and Synge has said, in his turn, that relativity has much the same appeal as Alice in Wonderland: aliens and invisibility enhance this fairy tale charm!

[11] Previously, it was *the right cone of the observer S,* in which the time u at A is such that $S_u = \{\mathcal{A}, \mathcal{A}^*\}$. Even if $u = 1$, this may be the right cone of $\mathcal{A}^*$, in which the right cone of $\mathcal{A}$ is awfully distorted; and its about $\tau = 1$ *reversed rays have only a fictitious origin at infinity.* The hidden product structure is preserved by reversals, but the so-called *absolute time τ is invariant only if we limit ourseles to linear reflections of the cone.*

[12] i.e., once again, $T = (t, \mathbf{x}) \mapsto \frac{a^2c^2}{c^2t^2 - \mathbf{x}^2}(t, \mathbf{x}) = T''$, where $\mathbf{x}^2 := r^2$.

$\alpha \circ \beta$ multiplies the right cones of any alien $\{\mathcal{A}, \mathcal{A}^*\}$ with the same number and its inverse, viz., *the squared ratio k of his time at the curved mirrors of α and β.* For $\beta \circ \alpha$, these two numbers interchange, our group is nonabelian. Moreover, $(\alpha \circ \beta) \circ (\gamma \circ \delta)$, etc., multiply the two cones by the product of these numbers for $\alpha \circ \beta$, $\gamma \circ \delta$, etc., thereby giving us two isomorphisms of the *index two subgroup of all even compositions* with the group of positive reals under multiplication, which are related to each other by inversion. $\Box$

Though the *ratio of times*[13] at an ordered pair of points of a ray is the same up to inversion in all the right conical structures, even a *homothetical oriented alien* $\kappa(\mathcal{A})$ can be grossly misunderstood by $\mathcal{A}$:- The time and distance of any point $\kappa(T)$ in the right cone of the oriented alien $\kappa(\mathcal{A})$ are exactly the same[14] as those of the point T in the right cone of $\mathcal{A}$, but, in the same sense as before, $\mathcal{A}$ *deems the time and distance measurements of $\kappa(\mathcal{A})$ to be off by the above factor k, and if $k > 1$ he may also deem points at distance ct/k or more in his ball of radius ct to be invisible to $\kappa(\mathcal{A})$!* $\Box$

(30.191) *Distortion of an alien $\{\mathcal{B}, \mathcal{B}^*\}$ on another ray S'* :- To see this $\mathcal{A}$ can use, after[15] a reversal α about an apt time a, or the homothety κ multiplying his right cone by a^2, the linear reflection f switching S and S'. Therefore, $T = (t, x, \mathbf{y}) \mapsto \frac{a^2c^2}{c^2t^2 - x^2 - \mathbf{y}^2}(t, x, \mathbf{y}) \mapsto \frac{a^2c^2}{c^2t^2 - x^2 - \mathbf{y}^2}(\gamma t - \frac{\gamma v}{c^2}x, v\gamma t - \gamma x, \mathbf{y}) = (f \circ \alpha)(T)$, or $T = (t, x, \mathbf{y}) \mapsto a^2(t, x, \mathbf{y}) \mapsto a^2(\gamma t - \frac{\gamma v}{c^2}x, v\gamma t - \gamma x, \mathbf{y}) = (f \circ \kappa)(T)$.[16] So $\mathcal{A}$ deems the time of $\mathcal{B} = (f \circ \alpha)(\mathcal{A})$ to be off by the factor $\frac{t^2}{a^2}(1 - \frac{r^2/t^2}{c^2})\gamma$, and his distance-to-ray measurement off by the same factor in the plane of S and S', but only by $\frac{t^2}{a^2}(1 - \frac{r^2/t^2}{c^2})$ in directions perpendicular to this plane. Further, he may also deem the point T invisible to $\mathcal{B}$ if the distance of $(f \circ \alpha)(T)$ from S is ct or more, and these invisible-to-$\mathcal{B}$ subsets of his balls can be calculated from the formula above. Likewise, $\mathcal{A}$ deems the time and distance-to-ray measurements of the alter ego $\mathcal{B}^*$ to be off by, and up to the factor $a^2\gamma$, and he may also deem the points of his balls, not in $1/a^2$ times the ellipsoids of (30.17) with centre on S', to be invisible to this oriented alien. $\Box$

(30.192) Though $f \circ \alpha$ and $f \circ \kappa$ both map the point A on which the alien $\{\mathcal{A}, \mathcal{A}^*\}$ lives, to the point B of $\{\mathcal{B}, \mathcal{B}^*\}$, they are very different transformations.

[13] The cayley distance between the two aliens is the log of the $c/2$th power of the *bigger* ratio, while the squares of *both* ratios give $\{k, k^{-1}\}$. Modulo reversals, an alien can explain this rescaling by saying the other is using *different units* of absolute time, the primary *physical quantity* in the sense of *On dimensional analysis* (1999) for cartesian $c < \infty$ physics. With reversals thrown in, this is moot, but we have the discrete and dimensionless topological invariants of the created manifold-matter.

[14] And this, in fact, is so for *any* composition κ of reversals and linear reflections.

[15] Or *before* : $\alpha \circ f = f \circ \alpha$ is true for any function α of the cone to itself which on each ray restricts to the same but *arbitrary function* $\tau \mapsto \alpha(\tau)$ of numbers, because the linear reflection f maps rays to rays and preserves τ. This gives many interesting deformations, for example, the beautiful theorem of Sarkovskii joins the fray, but $\alpha(\tau) = a^2/\tau$ are the only decreasing homeomorphisms of positive numbers that preserve cayley distance.

[16] We are again in, and Figure 16 shows, the right cone of $\mathcal{A}$ with his τ, v is the slope of S', $\gamma(v)$ as in (30.17), and the $\mathbf{x}$ of (30.18) has been split into $(x, \mathbf{y})$ along and perpendicular to the plane of S and S', so $x^2 + \mathbf{y}^2 = r^2$. The perceived *distortion depends, but only up to an orthogonal transformation of his cone, on how $\mathcal{A}$ is seeing $\{\mathcal{B}, \mathcal{B}^*\}$,* for example, the size and shape of the invisible subsets are fixed, but not how they sit in his balls.

The orientation-preserving non-linear `half-turn` $f \circ \alpha$ keeps the `axis` in which the mirrors of f and α intersect[17] fixed, and is its own inverse. The orientation-reversing linear `glide reflection` $f \circ \kappa$ has for $a \neq 1$ no periodic points, but preserves the mirror of f and multiplies it by a^2. A `translation` $f \circ g$, i.e., a composition of two linear reflections of the cone, is identity on the intersection of their flat mirrors and preserves the complementary subspace spanned by the directions in which they reflect.[18] This subspace is 2-dimensional if $f \neq g$, and $f \circ g$ multiplies one of the boundary rays in it by a number bigger than one, and the other by its inverse, for example, if $g(t, x, \mathbf{y}) = (t, -x, \mathbf{y})$, then $f \circ g$ multiplies the boundary rays $(t, ct, \mathbf{0})$ and $(t, -ct, \mathbf{0})$ by $\sqrt{\frac{c+v}{c-v}}$ and $\sqrt{\frac{c-v}{c+v}}$, while, for the inverse translation $g \circ f$, these *proper values* are switched.

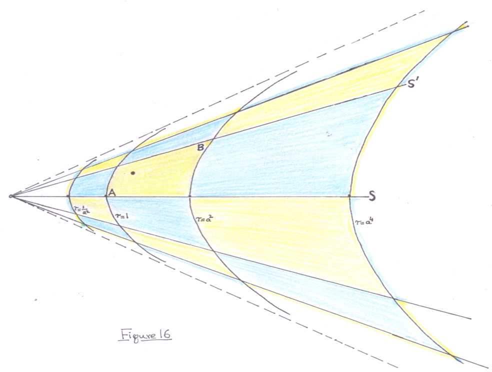

Figure 16

(30.193) *As a group all compositions of linear reflections parallel to a given plane is the nonabelian double cover of the positive numbers* :- $f \circ g$ multiplies the two boundary rays of the right cone of any $\mathcal{A}$ parallel to this plane with the same number and its inverse, for $(f \circ g) \circ (h \circ k)$ these proper values multiply, so giving two[19] isomorphisms of the index two subgroup of translations, with the multiplicative group of positive reals, related by inversion. □

(30.194) So $x = 0$ is mapped by the translations $(f \circ g)^i$ to $x = v_i t$, where $\frac{c+v_i}{c-v_i} = (\frac{c+v}{c-v})^i$, while the homotheties κ^j take $\tau = 1$ to $\tau = a^{2j}$. These flat and curved hypersurfaces give a subdivision of the cone of $\mathcal{A}$ which restricts to the deformed `rectangular tiling` of his 2-*cone* $\mathbf{y} = \mathbf{0}$ shown above. Dividing by these symmetries gives a 2-*torus*; and if we divide it by f or the glide reflection

[17] This codimension two axis cuts the plane of S and S' in that *black dot* with $\mathcal{A}$-coordinates $a\sqrt{\frac{2}{1+\gamma}}(\frac{\gamma+1}{2}, \frac{\gamma v}{2})$ inside the tile with vertices $A = (1, 0), A' = (\gamma, \gamma v), B = (a^2\gamma, a^2\gamma v), B' = (a^2, 0)$, it is the point on the ray through the mid-point of AA' with $\tau = a$.

[18] Likewise, the subspace spanned by the directions of *any* number of linear reflections of the cone is the complement of the intersection of their mirrors with respect to the *non-degenerate symmetric form of* τ^2*,* with cone *all* null vectors, etc. This algebra is useful, but, for us, *only that open connected cone is spacetime,* in particular, we don't use the linear reflections, also called 'time reversals', which interchange it with its missing half.

[19] *The proper values of* $f \circ g$ *are also switched for* $\mathcal{A}^*$*,* but, as in (30.91), the log of the cth power of the bigger is a *cayley distance* $\frac{c}{2} \log \frac{c+v}{c-v}$*;* by which the rays parallel to the plane get translated; *for* $c \to \infty$*,* $(f \circ g)(1, x, \mathbf{y}) \to (1, x + v, \mathbf{y})$ *and* $\frac{c}{2} \log \frac{c+v}{c-v} \to v$.

$f \circ \sqrt{\kappa}$ a *möbius strip*[20] or *klein bottle*; while further division by the half-turn $f \circ \alpha$ wraps it twice with a branch point over another torus, etc.

(30.195) The two discrete subgroups of positive numbers coincide iff $\frac{c+v}{c-v} = a^{\pm 2}$, but even for this `square tiling` of the 2-cone, no composition of linear reflections and reversals can interchange adjacent sides [21] of its tiles, so, *its symmetry group is no bigger.* Again, if v is fixed and c is big, then a is almost 1 for squarehood, so, *in the classical limit there is no such tiling*; but, if we make *no* additional demands, we can always keep both v and a fixed, and straighten the limiting subdivision, of the half-space $t > 0$ by flats $x = ivt$ and $t = a^{2j}$, by a suitable $(t, x, \mathbf{y}) \mapsto (\log_C t, \frac{x}{t}, \frac{\mathbf{y}}{t})$ to obtain, on the entire euclidean plane $\mathbf{y} = 0$ of $\mathcal{A}$, *an ordinary tiling by squares of size* $v \times v$.

(30.196) Staring us in the face also from Figure 16 is a magical `stairway to heaven`[22] which is a concomitant of $c < \infty$! The curves suggest points of constant height on the boundary of a cone of one dimension more, which it is natural to put inside the cone over the ball B^{n+1} with the same centre and radius, for, the linear reflections of the cone over B^n not only extend to it, with their rotations they give all of them ... till finally we are in the ball B^∞ of radius $c < \infty$, where this stairway ends, because, we can shift each guest in an infinite hotel to the next room, and put a new arrival in the first.

[20]Ditto if we divide $S^1 \times S^1$ by the involution which switches its factors, so, *a möbius strip is the space of quadratic homogenous equations* $ax^2+bxy+cy^2 = 0$ *over* $\mathbb{R}$ *with* $b^2-4ac \geq 0$:- the `quadratic formula` tells us this condition is necessary and sufficient for factorization over $\mathbb{R}$, and $S^1 = \mathbb{R} \cup \infty$. $\square$ Attaching the remaining equations with complex conjugate roots, an open 2-disk, then completes *the manifold* $\mathbb{R}P^2$ *of all real quadratic equations,* but a like dissection of $\mathbb{R}P^n$ for $n \geq 3$ is more involved. Unlike the circle, `the symmetric powers of a 2-manifold are manifolds` :- a suspect link is the join of a sphere and a circle divided by its antipodal action, but this is a circle too. $\square$ This implies, *the multiplication of n linear equations in x and y over* $\mathbb{C}$*, an injective map from the nth symmetric power of* $S^2 = \mathbb{C} \cup \infty$ *to the manifold* $\mathbb{C}P^n$ *of all degree n equations, is surjective,* that is, `the fundamental theorem of algebra!` The relativistic analogues of this bijection, for 2-manifolds $M^2 = B^2/\Gamma$ – the inverse map is how *'Poincaré had solved any polynomial equation by using automorphic functions'* – are once again 'well-known', but not to me! These old memories had resurfaced after a conversation with Keerti about three months ago; the symmetric powers of 2-manifolds also figure in some attractive *'numerologies'* about particles, going back to Majorana, in which, for example, a beautiful recent paper of Atiyah and Manton also indulges.

[21]Despite the fact that, *there is no cayley isometry of the cone other than these compositions*, and, *all four sides of these tilings do have the same cayley length.* My plan–see page 3–was to go over cartesian motions, cayley distance and segments rather quickly in (30.1), (30.2) and (30.3), but things went truly for a toss after the advent of the aliens in (30.16)! Fearing that it might be some time before I return to these topics, let me remind you that the factor $\frac{c}{2}$ in cayley distance made its classical limit the euclidean distance on $t = 1$, but, because of it, *the cayley distance between distinct times blows up as* $c \to \infty$*.* Again, the points at a constant cayley distance can be funny, for example, *the inscribed cayley circle of our square tile touches its boundary in segments*; but, *distinct cayley circles of a disk intersect in at most two points,* for this distance is equivalent to its `conformal metric`, which has genuine but eccentric circles. Anyway, from the square tiling of the 2-cone we can make trivalent `bricklaying patterns`, and it is not hard to calculate the cayley diameter of a star of a vertex, so the fourth proof in ਉਂਗਲੀਆਂ ਅਤੇ ਟਾਇਲਾਂ (2015) still works. There is also a `cayley-invariant volume`, and one can probably find for the 2-cone also, *all quadrilaterals for which this area is equal to the product of the average cayley lengths of the opposite sides,* etc.

[22]My name for what is usually *Poincaré extension,* for example, in Beardon's book.

(30.197) So these ball geometries arise plainly, one after the other, from a mere segment $[0, c)$. Then, in Note 23, had dawned on us the realization that, *associated to any cartesian motion there is a canonical* `partition` $\mathcal{F}$ *of the ball* B^n *into path connected subsets* M *that are* `topologically homogenous`.[23] Any M inherits its charts from the motion itself, and if `compact` we had deemed it to be `cartesian matter` provided it is `persistent`, i.e., appears also in any perturbation of the cartesian motion.

(30.198) *Calculus, more precisely lipschitz calculus, is also a child of* $c < \infty$. If n is big enough, or even not four, any cartesian motion can be perturbed over a compact time interval by an arbitrarily small amount to one whose homeomorphisms are lipschitz. The proof needs `a cayley lattice` *of the* $(n+1)$*-cone with quotient a closed parallelizable manifold.* Given this scaffolding, it is like simplicial approximation, except that it is nearby homeomorphisms, not just maps that are sought. *For* $c = \infty$ *any two translations commute,* so n of these with a homothety give one, with quotient an $(n+1)$-dimensional torus, but now that all-important chord condition on the flow lines of the cartesian motion is not available, so the results are different : *it is only for* $c < \infty$ *that cartesian matter is necessarily lipschitz-smooth.*

(30.199) So, just from a segment, a whole world, see Note 28, has sprung up, *but what about fractional dimensions?* It is topological homogeneity that is the basic feature of manifolds, the other, that they should be locally euclidean, is our natural desire to not stray too far from home. There seems no reason why all cartesian matter should be locally euclidean, but about path connected homogenous fractals, I know very little; however the Bing-Borsuk conjecture, see Note 25, suggests there may be surprises. Also: *what dimension is best?* In the lipschitz context we saw that `hausdorff dimension` is natural, but there are certainly other candidates.[24]

We have stayed at home, yet in these n-balls have popped up naturally it seems all manifolds and some other path connected and homogenous compacta. This cartesian matter can be examined, if the birthing motion was smooth enough, using only the elementary tools of the calculus of several variables. Besides we have the option $n = \infty$ to sidestep or delay `knotty questions`, still without giving up the creature comforts of home.[25]

Coming back to that cayley lattice, the natural idea, 'lets make c an integer and search over $\mathbb{Z}$', needs to be finessed, as Fricke pointed out long ago. Due to their quadratic nature the hyperboloids may not have enough integral points, so

[23]For, once again, these `minimal sets` M are the equivalence classes generated by the *binary relation* R on the ball B^n defined by xRy iff x and y are on the same `orbit`, i.e., iff they are the projections of two points on the same flow line, say, at absolute times τ_1 and τ_2, but then the homeomorphism $\phi_{\tau_1 \tau_2}$ of B^n given by the motion throws x on y, and besides, it maps each equivalence class to itself, so it preserves any M and its complement $B^n \setminus M$. $\square$

[24]More general and natural might be `von Neumann dimension` : from the defining relation R of $\mathcal{F}$, one should make nice $\mathbb{C}^*$-algebras with involution given maybe by reversal. Also, this reminds me of the `smoothing operators` *for closed foliated manifolds* $(M, \mathcal{F})$ that I had played around a lot with in the 1970s, but it was Connes and Skandalis, a bit later, who had got a full-blown index formula for foliations by using such a dimension.

[25]For an example of these comforts, see *From calculus to cyclic cohomology* (1995).

we search quadratic extensions, and this suffices to find a lattice with quotient a closed manifold; but to get parallelizability the search was extended into the étale or grainy nature of this homotopy type. Maybe this can be avoided, but it would be even nicer if we could understand this 'grainy nature' with the same cartesian clarity. Which brings me to the last note of this year.

(30.199...) The dots indicate I'm keeping my options open in case I want to add another tid-bit before (30.2), but you might object: *dots mean* 9 *recurring from school and that this is* (30.2) *already,* written in a long-winded way. *Cantor opened a wonderful world by simply giving up this school dictat* ! Even with 9 recurring, it was now not (30.2) for him, or even something lesser than it by an infinitesimal, it was just an infinite sequence (with one point) of ten things, and on all such sequences – a power of the cardinality ten set – we have only the product topology. *What can one do with this mere dust? Almost anything!* is the short answer : *not just the real line, all interesting spaces you can think of, for example, all manifolds, are but quotient spaces of this dust.* Besides, it is not hard to show this, and that is precisely the rub, the bewildering number of nice ways[26] in which we can lift reality to this combinatorial dust. Though various criteria have guided what seems 'best' it would be fair to say that what followed from *Hensel's discovery of those wonderful fields in such dust* is so far the clear winner. From (30.196) and (30.197) we know how, starting with a mere segment, we can bring into being naturally manifolds and all, so we ask, *is there an equally natural* `cantorian P G & R` *of which this is only a functorial quotient?* Many possibilities come to mind, but when it comes to reading the mind of God almost, it is best to proceed slowly and in humility ...

K S Sarkaria (*contd.*)

[26]See, for example *"Amazing curves!"* (2010); to all those motifs has recently been added the above mural *"P G & R"* on a verandah roof. December 31, 2016.