# AN "ODD" SCHWARZ INQUALITY

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Soon after learning, more than a year ago, of the statement of the following problem from Vineet Kahlon, I had obtained a solution of the same which is somewhat neater and shorter than that[1] which he subsequently showed me. The object of this note is merely to put on record the new features of my proof.

**Problem.** *Show that the distance* AB *from an* $A = (a_1, \dots, a_n)$, *with all* $a_i$'s *odd integers, to line* $(t, 2t, \dots, nt)$, *is least when* $A = (1, \dots, 1)$.

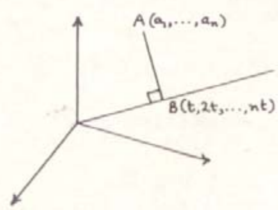


*Fig.* 1

I too had noted, since $(1^2 + 2^2 + \dots + n^2).AB^2 = \Delta_{a_1 \dots a_n}$, where $\Delta_{a_1 \dots a_n} = (1^2 + 2^2 + \dots + n^2)(a_1{}^2 + \dots + a_n{}^2) - (a_1 + 2a_2 + \dots + na_n)^2$, that the required result is equivalent to the "odd" Schwarz **inequality** $\Delta_{a_1 \dots a_n} \geq \Delta_{1 \dots 1}$, and that this in turn will follow inductively by using Lagrange's identity $\Delta_{a_1 \dots a_n} = \Delta_{a_1 \dots a_{n-1}} + \sum_{1 \le i \le n-1}$

[1]This has now appeared as [1] which also gives references to background literature.

$(ia_n - na_i)^2$, once we have checked, for all $a_i$ odd, that

$$\sum_{1\le i\le n-1} (ia_n - na_i)^2 \ \ge\ 1^2 + 2^2 + \ldots + (n-1)^2 . \qquad (1)$$

**Proof of (1).** We note that the contribution of each pair $(i,n-i)$ to the left side of (1) obeys

$$(ia_n - na_i)^2 + ((n-i)a_n - na_{n-i})^2 \ \ge\ \zeta_i^{\,2} + (n - \zeta_i)^2$$

where $0 \le \zeta_i \le n-1$ is defined by $ia_n = \zeta_i \bmod n$. This follows because the sum of the numbers $x = ia_n - na_i$ and $y = (n-i)a_n - na_{n-i}$ is a *nonzero* — because odd $a_n \ne a_i + a_{n-i}$ even — multiple of $n$, thus ruling out the possibilities $(x, y) = (-\zeta_i, +\zeta_i), (-(n-\zeta_i), +(n-\zeta_i))$.

So the left side of (1) is $\ge S := \frac{1}{2} \sum_{1\le i\le n-1} [\zeta_i^{\,2} + (n - \zeta_i)^2] = \frac{1}{2} \sum_{0\le\zeta\le n-1} [\zeta^2 + (n - \zeta)^2]\, i_\zeta$ where $i_\zeta$ denotes the number of $i$'s such that $\zeta_i = \zeta$. To see $S \ge$ right side of (1) we proceed as follows.

In case $i_\zeta = 1$ for all nonzero $\zeta$ (so $i_0 = 0$) then $S$ coincides with the right side of (1). Otherwise the possible values of $\zeta_i$ (these being in the image of the homomorphism of $\mathbb{Z}/n$ provided by multiplication by $a_n$) are the multiples of a divisor $h \ge 2$ of $n$, and we'll have $i_\zeta = h$ for each of these except $i_0 = h-1$ (and $i_\zeta = 0$ for all non-multiples $\zeta$ of $h$). So $S$ = shaded area of *Fig. 2*, where $\vec{h}(\zeta)$ is the broken line function having same values as $\frac{1}{2}[\zeta^2 + (n - \zeta)^2]$ at multiples of $h$. Adding/subtracting the rectangles marked +/− we see that this equals

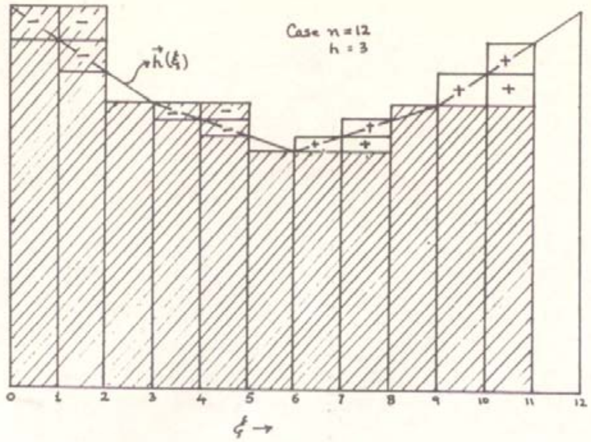


*Fig.* 2

$\sum_{1 \le \xi \le n-1} \vec{h}(\xi)$. (Alternatively, adding the n/h sums of A.P.'s, $\sum_{kh \le \xi < (k+1)h} \vec{h}(\xi) = \frac{1}{2}[(h+1)\vec{h}(kh) + (h-1)\vec{h}((k+1)h)]$, gives tne requisite $\sum_{0 \le \xi < n} \vec{h}(\xi) = h. \sum_{0 \le k < n/h} \vec{h}(kh)$.) So result follows because $\vec{h}(\xi) > \frac{1}{2}[\xi^2 + (n-\xi)^2]$ (a concave-up function) at non-multiples of h. **q.e.d.**

The oddness of the $a_i$'s played only a minor rôle in the above, e.g. the same argument shows, for any $q \ge 2$, that $\Delta_{a_1 \ldots a_n} \ge \Delta_{1 \ldots 1}$ if all $a_i = 1 \bmod q$. It also shows, under same hypotheses, that $\Delta_{f;a_1 \ldots a_n} \ge \Delta_{f;1 \ldots 1}$ where $\Delta_{f;a_1 \ldots a_n} := \Delta_{f;a_1 \ldots a_{n-1}} + \Sigma_{1 \le i \le n-1} f(|ia_n - na_i|)$, with $f(x)$ being now any increasing and concave-up function of $x \ge 0$.

## *Reference*

[1] V.Kahlon, *View obstruction problem*, Res. Bull. Panjab Univ. 45 (1995) 175-180.