On Dimensional Analysis : "The Bridgman-Buckingham Theorems"[1]

by

K. S. Sarkaria

ABSTRACT. The theorems of Dimensional Analysis become obvious if one thinks of the set of m-tuples of positive real numbers as a vector space over the reals.

1. INTRODUCTION. There is a fascinating thing called *dimensional reasoning* which pervades all of physics and engineeering. It goes back to Galileo, but came into its own only in 1915, when, in a short but striking note [8], Rayleigh listed in quick order a wealth of physical laws which can be obtained easily by using only dimensional reasoning. The ensuing intense interest led to an analysis of the basic notions, "physical quantities" and "physical laws", and a new subject was born with the appearance of Bridgman's classic [2]. Later on, these foundational questions were probed further in Birkhoff [1].

The subject is simple enough, there being essentially only two theorems. The first, "Bridgman's", stems from his postulate that the ratio of two measurements of a physical quantity is an invariant, and deduces from it the existence of dimensional symbols. The second, "Buckingham's", tells precisely how much information about a physical law can be squeezed out by using dimensional reasoning only. My object is to give quick proofs of these two theorems. However, since the beauty of Dimensional Analysis lies in its examples, which moreover are what motivate these otherwise dry general theorems, I will start with an example, indeed the very first one on Rayleigh's list.

Dispersion relation for water waves. Intuition suggests a physical law $v = f(\lambda, g, h)$ amongst the physical quantities v = wave speed, λ = wave length, g = gravitational acceleration, h = water depth. Adjoining their dimensional symbols (it is convenient usually to take mass, length and time as primary quantities) we get $LT^{-1}v = f(L\lambda, LT^{-2}g, Lh)$. This equation represents the fact that our physical law does not depend on the size of the units. In particular we can make $L\lambda = 1$ and

$LT^{-2}g = 1$ by choosing $L = \lambda^{-1}$ and $T = \lambda^{-1/2}g^{1/2}$. So our physical law is $\lambda^{-1/2}g^{-1/2}\ v = f(1,\ 1,\ h/\lambda)$, i.e. $v = C(h/\lambda)\sqrt{g\lambda}$, where $C(x)$ is an unknown function of just one dimensionless quantity h/λ. If the depth h is much bigger than λ, then its value should not matter, i.e. *for deep water waves* $v = C\sqrt{g\lambda}$, where now the unknown C is just a constant.

We seem to have got something by doing almost nothing ! Indeed, dimensional reasoning is valuable precisely because it is a quick means of obtaining partial, and occasionally the *only* known, information about a physical problem. For water waves, there however does exist a full blown theory — see e.g. Lighthill [7] — in which this dispersion relationship is obtained by solving a simple boundary value problem. This gives better results : $C(x) = (1/\sqrt{2\pi})\tanh^{1/2}(2\pi x)$ (Stokes 1847) and so $C = 1/\sqrt{2\pi}$ (Green 1839) for the deep water case. This is typical of what one can expect in general : dimensional reasoning reduces the task to determining a function involving a lesser number of quantities, and Buckingham's theorem tells us how small this number can be.

2. "POSITIVE" VECTOR SPACES. The main point of this note is that the theorems of Dimensional Analysis are in fact about vector spaces $\mathbb{P}^m$ of m-tuples of *positive* real numbers. In $\mathbb{P}^m$ "addition" is componentwise multiplication and "scalar multiplication by r" same as taking the rth power of all components. Taking log of all the components gives a linear isomorphism $\mathbb{P}^m \xrightarrow{\cong} \mathbb{R}^m$, using which we will also transfer the usual metric and measure of $\mathbb{R}^m$ to $\mathbb{P}^m$. It is natural to use these vector spaces $\mathbb{P}^m$ because, in Dimensional Analysis, one deals only with physical quantities whose values are positive real numbers[2]. We will assume also that all functions are *continuous*[3] unless mentioned otherwise.

3. PHYSICAL QUANTITIES are of various *kinds* (masses, viscosities, etc.) subject to the following hypotheses.

(i) *Ratios constant.* Following Bridgman [2] we assume that the ratio of measured values of two physical quantities of the same kind is independent of the units used.

(ii) *Finiteness.* We assume that there is an n, and choose it to be the least such, so that the measured value of any quantity of a given kind Q is a fixed (continuous) function $Q(p_1, \cdots, p_n)$ of those of some quantities of n chosen primary kinds (in [2] this function is assumed differentiable). Quantities of the primary kind are measured directly by comparison with chosen primary units; quantities of the remaining secondary kinds[4] are usually measured by indirect methods.

4. "BRIDGMAN'S THEOREM". *The function* $Q : \mathbb{P}^n \longrightarrow \mathbb{P}$ *postulated above must in fact be an affine* $\mathbb{R}$*-linear function.*

That is, if $T(p_1, \dots, p_n) = Q(p_1, \dots, p_n) \div Q(1, \dots, 1)$, then $T : \mathbb{P}^n \longrightarrow \mathbb{P}$ is a linear map, i.e. we must have

$$T(p_1, \dots, p_n) = p_1^{t_1} p_2^{t_2} \cdots p_n^{t_n}$$

for fixed $t_i \in \mathbb{R}$. This monomial or *dimensional symbol* characterizes the kind Q. Note also that the function $Q(p_1, \dots, p_n)$ postulated in (ii) was onto, which happens iff the indices t_i are not all zero. However the above statement remains true if we regard, as we shall from here on, the ordinary positive numbers as physical quantities of an extra *dimensionless kind* having $t_i = 0$ for all i.

Proof. Recall that a group homomorphism between real vector spaces is continuous iff it is a linear map.[5] By (i)

$$\frac{T(p_1, \dots, p_n)}{T(p_1', \dots, p_n')} = \frac{T(\alpha_1 p_1, \dots, \alpha_n p_n)}{T(\alpha_1 p_1', \dots, \alpha_n p_n')} .$$

Put $p_1' = \dots = p_n' = 1$ to see that T is a group homomorphism. *q.e.d.*

5. RESCALING GROUP REPRESENTATIONS. As in Birkhoff [1] we fix a $\mathbb{P}^n = G$ and denote its elements by $\alpha = (\alpha_1, \dots, \alpha_n)$. Choosing n primary kinds is interpreted as identifying G with the set $\mathbb{P}^n$ of n-tuples $(p_1, \dots, p_n)$ of measured values of such quantities. These are measured by direct comparison with primary units. So, if the ith primary unit becomes α_i times smaller, the value p_i becomes α_i times larger. Hence, the group operation $(\alpha_1, \dots, \alpha_n)(p_1, \dots, p_n) \longmapsto (\alpha_1 p_1, \dots, \alpha_n p_n)$ interprets as the rescaling action of G on $\mathbb{P}^n$. More generally, given any m kinds, the set of all m-tuples $(Q_1, \dots, Q_m)$ of measured values of such quantities constitutes a $\mathbb{P}^m$, and if primary units become $\alpha = (\alpha_1, \dots, \alpha_n)$ times smaller, the measured value Q_i

becomes $T_i(\alpha)$ times bigger, $T_i : G \longrightarrow \mathbb{P}$ being the $\mathbb{R}$-linear map of § 4. So, if $\mathbf{T} = (T_1, \ldots, T_m): G \longrightarrow \mathbb{P}^m$, the translations $\mathbf{T}(\alpha)(.)$ of $\mathbb{P}^m$ interpret as the *rescaling action of* G *on* $\mathbb{P}^m$.

The same letter $\mathbf{T}$ (and later likewise $\mathbf{U}$, $\mathbf{V}$, etc.) will also denote this *representation of* G *by translations.*[6] The linear map $\mathbf{T} : G \longrightarrow \mathbb{P}^m$, and thus this representation $\mathbf{T}$, are characterized by the $m \times n$ *matrix* $[t_{ij}]$ over $\mathbb{R}$ formed by the indices of the m symbols $p_1^{t_{i1}} p_2^{t_{i2}} \cdots p_n^{t_{in}}$. The *orbits* are the subspace $\mathbf{T}(G)$ and its translates. By the *dimension* of this representation, we mean that of the underlying vector space, so $\dim(\mathbf{T}) = m$. And, by its *rank* r, we mean the dimension of its orbits, so $r = \operatorname{rank}(\mathbf{T}) = \operatorname{rank}\,[t_{ij}]$. Note that two such representations $\mathbf{T}$ and $\mathbf{U}$ are G-homeomorphic iff they have the same dimension and the same rank.

It is useful to allow this matrix to be perfectly general, e.g., in the following, it is the G-set $\mathbb{P}^m$ of *all possible* m*-tuples of values of* m *given physical quantities*, not necessarily of distinct kinds, which is interpreted as $\mathbf{T}$, so some rows of $[t_{ij}]$ may be equal to each other.

6. PHYSICAL LAWS are *relations* $L \subset \mathbf{T}$ (zero-sets $\psi(Q_1, \ldots, Q_m) = 0$ of functions $\mathbb{P}^m \xrightarrow{\psi} \mathbb{R}$, etc.) subject to the following hypotheses.

(a) *Closed.* We assume that if the relation is not true for some values $(Q_1, \ldots, Q_m) \in \mathbb{P}^m$ of the physical quantities, then it is not true also for nearby values, i.e. that the complement of L is open.

(b) *Invariant* (cf. Birkhoff [1], pp. 89-90). We assume that the relation is valid in all units, i.e. that if $(Q_1, \ldots, Q_m) \in L$, then $\mathbf{T}(\alpha)(Q_1, \ldots, Q_m) \in L$ for all $\alpha \in G$.

7. "BUCKINGHAM'S THEOREM". *Above* $L \subset \mathbf{T}$, $\dim(\mathbf{T}) = m$, $\operatorname{rank}(\mathbf{T}) = r$, *is a linear pull-back of a law between* m-r *dimensionless quantities.*

Proof. Take any linear surjection $\Pi : \mathbb{P}^m \longrightarrow \mathbb{P}^{m-r}$ whose kernel is the r-dimensional subspace $\mathbf{T}(G)$.[7] By (b), L is a union of orbits of $\mathbf{T}$, i.e. translates of $\mathbf{T}(G)$, so $L = \Pi^{-1}(\Pi(L))$, the pull-back under Π of the closed subset $\Pi(L)$ of the trivial representation $\mathbf{U} = \mathbb{P}^{m-r}$. *q.e.d.*

Equivalently, a law given as the zero-set of ψ surjects linearly on a law given as the zero-set of the function $\psi\circ\Pi^*$ of m-r dimensionless quantities.[8] Geometrically, Buckingham's theorem tells us that, upto homeomorphism, *physical laws are cylinders* $L \cong \mathbb{P}^r \times \Pi(L)$.[p] Recall next that, being a closed Euclidean set, $\Pi(L)$ is the zero-set of a smooth real-valued function $\phi(\Pi_1, \ldots, \Pi_{m-r})$, and so L of the smooth function $(\phi\circ\Pi)(Q_1, \ldots, Q_m)$ constant on orbits of $\mathbf{T}$; so our notion of physical law is essentially the same as of Bridgman [2]. Note finally that *changing primary kinds* interprets as identifying G with another $\mathbf{V}$ having $\dim(\mathbf{V}) = \mathrm{rank}(\mathbf{V}) = n$; so each $[t_{ij}]$ gets multiplied from the right by the $n \times n$ matrix $[v_{ij}]^{-1}$, but the orbits of $\mathbf{T}$ are unchanged.

NOTES

1. Based on talk delivered 17.12.99 in the "Instructional School on Linear Algebra", Panjab University, Chandigarh.

2. For example, norms of vectorial/tensorial physical quantities; also, for non-relativistic macroscopic measurements, we can fix a frame, and focus on just one orthant of values of their components.

3. The context being measurement, one should perhaps only assume measurability; this leads however – see note 5 – to the same results.

4. Tying secondary units suitably to the primary units one can ensure $Q(1, \ldots, 1) = 1$; we omit this "normalization" hypothesis because it will not work for dimensionless quantities defined later.

5. This was proved well before Bridgman [2] by Cauchy [4], 1821, and is very easy : any solution $F : \mathbb{R} \to \mathbb{R}$ of $F(x + y) = F(x) + F(y)$ obeys $F(t) = t.F(1)$ for all $t \in \mathbb{Q}$, and so for all $t \in \mathbb{R}$, provided F is continuous. Hamel [6], 1905, pointed out that one obtains all solutions by assigning arbitrary values F(b) to the elements b of a $\mathbb{Q}$-basis of $\mathbb{R}$, and Fréchet [5], 1913, proved that a measurable solution F of this equation is automatically continuous. For example, the solution F which assigns the value 1 to one b and 0 to all others is non-measurable, which shows, as Im(F) is countable, that codimension one $\mathbb{Q}$-vector subspaces of $\mathbb{R}$ are non-measurable. For more on Cauchy's equation, Hamel bases, and non-measurable sets, we refer to the beautiful papers of Sierpinski and Banach in Volume 1 of the Fundamenta Mathematica.

6. A translation of $\mathbb{P}^m$ extends uniquely to a "positive linear transformation" of $\mathbb{R}^m$, i.e. one which maps the orthant $\mathbb{P}^m \subset \mathbb{R}^m$ onto itself. Conversely, a continuous representation $\mathbf{T}$ of G by such transformations of $\mathbb{R}^m$ restricts to one by translations of $\mathbb{P}^m$. For, the m coordinate rays, being the extreme rays of the convex cone $\mathbb{P}^m$, must map on each other under such a transformation; so, since G is connected, and $\mathbf{T}(id) = id$, each coordinate ray must in fact map onto itself.

7. Equivalently, make m-r rows of $[t_{ij}]$ zero by row operations, i.e. by left multiplication by a nonsingular m × m matrix $[\pi_{ki}]$, and define Π by the submatrix provided by the corresponding rows of $[\pi_{ki}]$.

8. Birkhoff [1] points out that this was proved much before Buckingham [3] by Vaschy [9], 1892. The dual linear injection $\Pi^* : \mathbb{P}^{m-r} \to \mathbb{P}^m$ is defined by the transpose of the matrix of $\Pi : \mathbb{P}^m \longrightarrow \mathbb{P}^{m-r}$. An analysis of the calculus proofs of [2] and [3] reveals that at heart only a Π is constructed, the rest is just unnecessary baggage.

9. To understand the topology of L further it is necessary to go beyond Dimensional Analysis into the specifics of the physical law.

REFERENCES

[1] G. Birkhoff, *Hydrodynamics : A Study in Logic, Fact and Similitude*, Princeton (1960).

[2] P. W. Bridgman, *Dimensional Analysis*, Yale (1920).

[3] E. Buckingham, *On physically similar systems*, Physical Review 4 (1914) 354-376.

[4] A-L. Cauchy, *Cours d'Analyse de l'École Polytechnique*, Paris (1821).

[5] M. Fréchet, *Pri la funkcia equacio* $f(x+y) = f(x) + f(y)$, L'Enseignement Math. 15 (1913) 390-393.

[6] G. Hamel, *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung* $f(x+y) = f(x)+f(y)$, Math. Ann. 60 (1905) 459-462.

[7] J. Lighthill, *Waves in Fluids*, Cambridge (1978).

[8] Lord Rayleigh, *The principle of similitude*, Nature 95 (1915) 66-68.

[9] A. Vaschy, Annales Télégraphiques 19 (1892) 25-28.

Department of Mathematics,
Panjab University,
Chandigarh 160014, INDIA.