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Sierksma's Dutch Cheese Problem

by

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§1. Introduction

The object of this note is to give a proof of the following conjecture of Sierksma [15], 1979 (= Reay [7], Problem 14 a), where σ^i_j denotes the simplicial complex consisting of all faces, of an i-simplex σ^i, of dimensions $\leq j$.

(1.1) Dutch Cheese Problem [(1)]. For any (general position) linear map $f: \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)} \to \mathbb{R}^\ell$, there exist at least $((q-1)!)^\ell$ pairwise disjoint q-tuples $\{\sigma_1, \ldots, \sigma_q\}$ of simplices of $\sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)}$ such that $f(\sigma_1) \cap \ldots \cap f(\sigma_q) \neq \phi$.

This improves on a theorem of Tverberg [17], 1966, which is equivalent to saying that, for $q \geq 2$, there always exists at least one such q-tuple. The case $q = 2$ is Radon's Theorem [6], 1921. It is easily seen -- see §2 -- that the bound $((q-1)!)^\ell$ is the best possible.

We will infact prove a more general result which applies also to continuous maps f. Let us recall some definitions -- cf. [9] -- before stating this generalization.

For any simplicial complex K, the q-fold cartesian product of K, i.e. the product ${}^1K \times \ldots \times {}^qK$ of q disjoint copies of K, will also be denoted by $K \times \ldots \times K$ (q times) or K^q. Its cells $\sigma_1 \times \ldots \times \sigma_q$, $\sigma_i \in K$, $\sigma_i \neq \phi$, will usually be identified with ordered q-tuples $(\sigma_1, \ldots, \sigma_q)$ of nonempty simplices of K. The qth product configuration $K^q_\#$ is the sub cell complex of K^q obtained by deleting all cells $(\sigma_1, \ldots, \sigma_q)$ for which $\sigma_i \cap \sigma_j \neq \phi$ for some $i \neq j$. We will also need the qth deleted product $X^q_\#$ of a topological space X, i.e. the subspace of its q-fold cartesian product $X^q = X \times \ldots \times X$ (q times) obtained by deleting all points of the type $(x, \ldots, x)$, $x \in X$.

Consider now the case when $K = \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)}$ and $X = \mathbb{R}^\ell$. We note that this K has a $((q-1)\ell)$-dimensional qth product configuration $K^q_\#$. Also, that $(q-1)\ell$ is the codimension of the ℓ-dimensional diagonal $\Delta = (\mathbb{R}^\ell)^q \setminus (\mathbb{R}^\ell)^q_\#$ in the q-fold cartesian

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(1) So named because of the prize offered by Sierksma for its solution.

product $(\mathbb{R}^\ell)^q$ of $\mathbb{R}^\ell$. Any $f: K \to \mathbb{R}^\ell$ induces an $f^q: K^q \to (\mathbb{R}^\ell)^q$, $f^q(x_1, \ldots, x_q) = (f(x_1), \ldots, f(x_q))$. A cardinality q subset $\{x_1, \ldots, x_q\}$ of points (of the space) of K is called (i) <u>separated</u> if $(x_1, \ldots, x_q) \in K^q_\#$, and (ii) <u>a q-uple point of f</u> if $f(x_1) = \ldots = f(x_q)$, i.e. if $f^q(x_1, \ldots, x_q) \in \Delta$. Now suppose that $f: K \to \mathbb{R}^\ell$ is a <u>general position continuous map</u>, i.e. is a general position linear map with respect to some simplicial subdivision of K. ~~Then,~~ Since $\dim K^q_\# = \operatorname{codim} \Delta$, it follows that the set of separated q-uple points of under an f is finite, and, for any such q-uple point $\{x_1, \ldots, x_q\}$, if $x_i \in \operatorname{int} \sigma_i$, then $(\sigma_1, \ldots, \sigma_q)$ is a top-dimensional cell of $K^q_\#$. Further if f is linear ~~In the linear case~~, the correspondence $\{x_1, \ldots, x_q\} \mapsto \{\sigma_1, \ldots, \sigma_q\}$ is one-one; so the following result contains (1.1).

(1.2) Generalized " Tverberg-Sierksma " Theorem. <u>A general position continuous map $f: \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)} \to \mathbb{R}^\ell$ must have at least $((q-1)!)^\ell$ separated q-uple points.</u>

Such general position continuous maps f can be found arbitrarily close to any given continuous map $g: \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)} \to \mathbb{R}^\ell$. So (1.2) also establishes a conjecture of Tverberg, 1978 (= Reay [7], Problem 15) which amounts to saying that Tverberg's Theorem generalizes to all continuous maps g. For primes q this result is due to Bárány-Shlosman-Szűcs [1]. A simpler proof was given subsequently in [10].

(1.3) <u>Remark</u>. If $g(\sigma_1) \cap \ldots \cap g(\sigma_q) = \phi$, for a continuous g and some pairwise disjoint σ_i, then also $f(\sigma_1) \cap \ldots \cap f(\sigma_q) = \phi$, for all sufficiently close f. This shows that <u>the words "general position" are redundant in (1.1), however (1.2) ~~they are~~ probably requires some such condition ~~necessary in (1.2)~~</u>. ~~But~~ The proof will show however that (1.2) holds for all continuous g with isolated separated q-uple points lying in top dimensional cells of $K^q_\#$, provided each one of these is counted as many times as the absolute value of the <u>local degree</u> of g^q near this point (i.e. degree of map given by g^q from a small $((q-1)\ell)-1$-dimensional sphere enclosing this point to $\mathbb{R}^q_\# \simeq S^{q\ell-\ell-1}$).

Van Kampen [18], 1932, showed that <u>a continuous map from σ^{2n+2}_n to R^{2n} must have a separated 2-uple point.</u> The proof given in §2 is inspired by van Kampen's, and has many other applications--see §3--besides (1.2), e.g. we will state a Sierksma-type generalization of van Kampen's result, which too can be proved by the method of §2.

§2. Proof of Theorem (1.2)

The underlying field of coefficients of our chains and cochains will be $\mathbb{C}$, the field of complex numbers. Also, though it is not necessary, we prefer to work throughout with simplices (instead of cells) and so will replace the products of §1 with the following

(2.1) JOINS. Recall that the q-fold join $X^{(q)}$ of a space X consists of all points of the type $t.x = t_1x_1 + \ldots + t_qx_q$, where $x = (x_1, \ldots, x_q) \in X^q$ and $t = (t_1, \ldots, t_q) \in \tau$, the convex hull of the canonical basis vectors of $\mathbb{R}^q$. Here, it is understood that $t.x = t'.x'$ whenever $t = t'$ and $x_i = x_i'$ $\forall$ i such that $t_i = t_i' \neq 0$. We will identify X^q with the subspace of $X^{(q)}$ consisting of all points of the type $\frac{1}{q}.x = \frac{1}{q}x_1 + \ldots + \frac{1}{q}x_q$. By deleting the diagonal -- i.e. all points of the type $\frac{1}{q}y + \ldots + \frac{1}{q}y$ -- from $X^{(q)}$ we obtain the qth deleted join $X_*^{(q)}$. If X is triangulable by a simplicial complex K then $X^{(q)}$ can be triangulated by the simplicial complex $K^{(q)}$, the q-fold join ${}^1K. \ldots .{}^qK$ of q disjoint copies of K. Its simplices ${}^1\sigma_1 \cup \ldots \cup {}^q\sigma_q$, $\sigma_i \in K$, will usually be denoted by ordered q-tuples $(\sigma_1, \ldots, \sigma_q)$ of simplices of K. (Thus K^q is a proper sub poset of $K^{(q)}$.) The qth join configuration $K_*^{(q)}$ of K is the subcomplex of $K^{(q)}$ consisting of all pairwise disjoint ordered q-tuples of simplices of K.

(2.2) If $K = \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)}$ and $f: K \to \mathbb{R}^\ell$ is a general position continuous map, then the induced map $f^{(q)}: K^{(q)} \to (\mathbb{R}^\ell)^{(q)}$ images the codimension one skeleton of $K_*^{(q)}$ to $(\mathbb{R}^\ell)_*^{(q)}$.

Here $f^{(q)}$ is defined by $f^{(q)}(t_1x_1 + \ldots + t_qx_q) = t_1f(x_1) + \ldots + t_qf(x_q)$. The assertion follows easily from the fact that both the dimension of $K_*^{(q)}$, and the codimension of the diagonal in $(\mathbb{R}^\ell)^{(q)}$, are equal to $(q-1)(\ell+1)$. Infact note also that under $f^{(q)}$ the images of the top dimensional simplices of $K_*^{(q)}$ can hit the diagonal only finitely many times.

It is time now to consider certain group actions.

(2.3) EQUIVARIANCE. We will denote by $\mathbb{Z}_q = \{Id, \nu, \nu^2, \ldots, \nu^{q-1}\}$ a cyclic group of q elements. It acts on $X^{(q)}$ via $\nu(t_1x_1 + \ldots + t_{q-1}x_{q-1} + t_qx_q) =$

$t_2x_2 + \ldots + t_qx_q + t_1x_1$. Note that the restriction of this action to $X_*^{(q)}$ is <u>fixed point free</u>, i.e. the orbit of each point has a length equal to some divisor of q bigger than 1. Analogously on $K^{(q)}$ the group Z_q acts via $\nu(\sigma_1, \ldots, \sigma_{q-1}, \sigma_q) = (\sigma_2, \ldots, \sigma_q, \sigma_1)$. The restriction of this action to $K_{\#}^{(q)}$ is <u>free</u>, i.e. each simplex has an orbit of length q. Lastly, note that the map $f^{(q)}$ of (2.2) commutes with these group actions.

(2.4) <u>The qth deleted join $(\mathbb{R}^\ell)_*^{(q)}$ of $\mathbb{R}^\ell$ has the Z_q-homotopy type of a fixed point free Z_q-sphere $S^{(q-1)(\ell+1)-1}$. Furthermore, the order q homeomorphism ν of $S^{(q-1)(\ell+1)-1}$ has degree $(-1)^{(q-1)(\ell+1)}$</u>.

To see this note that the space $(\mathbb{R}^\ell)_*^{(q)}$ is the join of $(\mathbb{R}^\ell)_{\#}^q$ and the Z_q-subspace Y of $(\mathbb{R}^\ell)^{(q)}$ consisting of all points $t_1x_1 + \ldots + t_qx_q$ with at least some $t_i = 0$. But Y has the Z_q-homotopy type of the sphere $S^{q-2} = \partial\tau$, as follows by symmetrically using some contraction of $\mathbb{R}^\ell$ to a point. Under $(t_1, \ldots, t_{q-1}, t_q) \mapsto (t_2, \ldots, t_q, t_1)$, S^{q-2} undergoes a change of orientation $(-1)^{q-1}$. And, by projecting $(\mathbb{R}^\ell)^q$ on the orthogonal complement $\Delta^\perp$ of the diagonal subspace Δ, and then normalising, we see that $(\mathbb{R}^\ell)_{\#}^q$ has the Z_q-homotopy type of the unit sphere $S^{(q-1)\ell-1}$ of $\Delta^\perp$. Since the diagonal undergoes no change of orientation under $(x_1, \ldots, x_{q-1}, x_q) \mapsto (x_2, \ldots, x_q, x_1)$ -- here each $x_i \in \mathbb{R}^\ell$ and so is an ℓ-tuple of numbers -- it follows that the change of orientation of $S^{(q-1)\ell-1}$ is same as that of $\mathbb{R}^{\ell q}$, i.e. it is $(-1)^{(q-1)\ell}$. So $(\mathbb{R}^\ell)_*^{(q)} \cong (\mathbb{R}^\ell)_{\#}^q \cdot Y$ has the Z_q-homotopy type of $S^{(q-1)(\ell+1)} \cong S^{(q-1)\ell-1} \cdot S^{q-2}$, and under ν the change of orientation of this sphere, i.e. the degree of ν, is $(-1)^{(q-1)(\ell+1)}$.

This enables us to define in a well known way -- cf. [16] -- the following top dimensional integral cochains of $K_{\#}^{(q)}$, $K = \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)}$.

(2.5) OBSTRUCTION COCYCLES. Choose any general position continuous map $f : K \to \mathbb{R}^\ell$. By (2.2) and (2.4) it induces a continuous Z_q-map F from the codimension one skeleton of $K_{\#}^{(q)}$ to a Z_q-sphere $S^{(q-1)(\ell+1)-1}$. We will fix an orientation of this sphere. Then, the associated <u>obstruction cocycle</u> υ_f, is the top dimensional cochain of $K_{\#}^{(q)}$ which assigns to each top dimensional oriented simplex

$\theta^{(q-1)(\ell+1)}$ of $K_{*}^{(q)}$ the degree of the restricted map $F\colon \partial\theta \to S^{(q-1)(\ell+1)-1}$. Here it is understood that the orientation of the sphere $\partial\theta$ is the one induced by that of θ.

(2.6) <u>The obstruction cocycle v_f of f is zero iff F extends to a continuous $\mathbf{Z}_q$-map $K_{*}^{(q)} \longrightarrow S^{(q-1)(\ell+1)-1}$. Furthermore, v_f is</u> symmetric <u>, i.e. $\nu^* v_f = (-1)^{(q-1)(\ell+1)} v_f$, and, upto coboundary of a symmetric cochain, v_f is independent of f.</u>

The first part follows because $v_f(\theta) = \deg(F|\partial\theta)$ is zero iff F extends to θ. To see the transformation formula note that $(\nu^* v_f)(\theta) = v_f(\nu\theta)$ $= \deg(F|\partial\nu\theta) = \deg(\nu F|\partial\theta)$ because F commutes with the $\mathbf{Z}_q$-action, and this equals $(-1)^{(q-1)(\ell+1)} \deg(F|\partial\theta)$, i.e. $(-1)^{(q-1)(\ell+1)} v_f(\theta)$, by (2.4). For the last part let $g\colon K \to R^{\ell}$ be any general position continuous map, and let G be the corresponding continuous $\mathbf{Z}_q$-map from the codimension one skeleton of $K_{*}^{(q)}$ to $S^{(q-1)(\ell+1)-1}$. The connectivity of $S^{(q-1)(\ell+1)-1}$ ensures that, upto a $\mathbf{Z}_q$-homotopy, G will coincide with F on the codimension two skeleton of $K_{*}^{(q)}$. However, on any oriented codimension one simplex $\varphi^{(q-1)(\ell+1)-1}$ of $K_{*}^{(q)}$, G can differ from F by a degree amount $\mathscr{G}_{g,f}(\varphi)$ (i.e. the degree of the map furnished by F and G, from the sphere formed by identifying boundaries of 2 copies of φ, to $S^{(q-1)(\ell+1)-1}$). The coboundary $\delta(\mathscr{G}_{g,f})$ of $\mathscr{G}_{g,f}$ equals $v_g - v_f$. Further $\nu^*(\mathscr{G}_{g,f}) = (-1)^{(q-1)(\ell+1)} \mathscr{G}_{g,f}$ by a calculation similar to the one made for v_f.

We now use complex coefficients to define a top dimensional chain of $K_{*}^{(q)}$, $K = \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)}$.

(2.7) FUNDAMENTAL CYCLE. Let ω be a <u>qth root of unity</u> other than 1. Also, let the vertices of $\sigma^{(q-1)(\ell+1)}$ be named $1, 2, \ldots, (q-1)(\ell+1)+1$. Each top dimensional simplex θ of $K_{*}^{(q)}$, $K = \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)}$, is of the type $(\theta_1, \ldots, \theta_q) \equiv {}^1\theta_1 \cup \ldots \cup {}^q\theta_q$, $|\theta_1| + \ldots + |\theta_q| = (q-1)(\ell+1)+1$, where θ_i are pairwise disjoint subsets of these integers. We assign to any such θ the orientation prescribed by the natural order of the integers, and the complex coefficient $\omega_\theta = \prod_{1\le i\le q} \omega^{i|\theta_i|}$. (So, e.g., $\{{}^1 2, {}^1 5\} \cup \{{}^2 1, {}^2 3\} \cup \{{}^3 4\}$ is assigned the orientation $[{}^2 1, {}^1 2, {}^2 3, {}^3 4, {}^1 5]$.) The resultin complex linear combination $\Omega = \sum_\theta \omega_\theta \theta$ of oriented top dimensional simplices of $K^{(q)}_{\phi}$ will be called a <u>fundamental cycle</u> of $K^{(q)}_{\phi}$.

(2.8) <u>Ω is indeed a cycle of $K^{(q)}_{\phi}$, i.e. has boundary $\partial\Omega = 0$. Furthermore $\nu_*\Omega = \omega^{-\ell}\Omega$.</u>

A codimension one simplex $\varphi = (\varphi_1, \ldots, \varphi_q)$ of $K^{(q)}_{\phi}$, oriented as above by the natural order of the integers, has the same incidence number $[\theta^{[i]}:\varphi] = (-1)^{\hat{\varphi}}$ with respect to any of the q top dimensional oriented simplices $\theta^{[i]} = (\varphi_1, \ldots, \varphi_i \cup \{\hat{\varphi}\}, \ldots, \varphi_q)$ incident to φ. Here $1 \leq \hat{\varphi} \leq (q-1)(\ell+1)+1$ is the integer missing from all the φ_i, $1 \leq i \leq q$. Since $\partial\Omega = \sum_\theta \omega_\theta(\sum_{\varphi\subset\theta} [\theta:\varphi]\varphi)$ it follows that, in $\partial\Omega$, the simplex φ occurs with coefficient $(-1)^{\hat{\varphi}} \sum_{1\leq i\leq q} \omega_\theta[i] = (-1)^{\hat{\varphi}} \omega_\theta[1]\,(1 + \omega + \ldots + \omega^{q-1})$, which is zero because ω is a qth root of unity other than 1. To see the second part note that the simplicial isomorphism $\nu : K^{(q)}_{\phi} \to K^{(q)}_{\phi}$ maps each vertex ${}^{r}t$ to ${}^{\pi(r)}t$, where π denotes the cyclic permutation $(2, \ldots, q, 1)$ of $(1, \ldots, q-1, q)$. So ν preserves the orientation given to the top dimensional simplices. Since $\pi(i)-i = 1 \bmod q$ $\forall$ i, we also have $\omega_{\nu(\theta)} = \prod_{1\leq i\leq q} \omega^{i|(\nu(\theta))_i|} = \prod_{1\leq i\leq q} \omega^{i|\theta_{\pi(i)}|} = \omega_\theta \prod_{1\leq i\leq q} \omega^{-(\pi(i)-i)|\theta_{\pi(i)}|} = \omega_\theta(\omega^{-1})^{\sum_{1\leq i\leq q}|\theta_{\pi(i)}|} = \omega_\theta(\omega^{-1})^{(q-1)(\ell+1)+1} = \omega_\theta\omega^{\ell}$.

(2.9) DUALITY. Consider the subspace of chains $c = \sum_\theta c_\theta \theta$ -- i.e. complex linear combination of oriented simplices -- of $K^{(q)}_{\phi}$ which transform under $\mathbf{Z}_q$ according to $\nu_* c = \lambda c$: note that λ has to be a qth root of unity, that $c_\theta = \lambda c_{\nu(\theta)}$, and that the subspace is preserved by the boundary operator ∂. Likewise the subspace, of the dual space, consisting of cochains a transforming according to $\nu^* a = \lambda a$, is preserved by the dual coboundary operator δ. Let $K^{(q)}_{\phi}/\mathbf{Z}_q$ denote the set of orbits Θ -- each of these has cardinality q -- of simplices of $K^{(q)}_{\phi}$ under the given free action of $\mathbf{Z}_q$. We define $\langle a, c\rangle_\lambda = \sum_{\Theta\in K^{(q)}_{\phi}/\mathbf{Z}_q} \{c_\theta a(\theta) : \theta \in \Theta\}$.

(2.10) Stokes' formula. <u>For any qth root of unity, $\langle a, c\rangle_\lambda$ is well-defined and $\langle \delta a, c\rangle_\lambda = \langle a, \partial c\rangle_\lambda$.</u>

Since $c_{\nu(\theta)} a(\nu(\theta)) = c_{\nu(\theta)}(\nu^* a)(\theta) = c_{\nu(\theta)} \lambda a(\theta) = c_\theta a(\theta)$, the choice of $\theta \in \Theta$ is unimportant, and $\langle a, c\rangle_\lambda$ is well-defined. To verify the formula it obviously suffices to check the case $c = \Theta + \lambda^{-1}\nu(\ ) + \dots + \lambda^{1-q}\ \nu^{q-1}(\Theta)$: now $\langle \bar{\delta} a , c \rangle = \bar{\delta} a(\Theta) = a(\partial\Theta) = \langle a , \partial c \rangle_\lambda$, by choosing, for the orbit of each simplex occuring in ∂c , the corresponding representative in $\partial\Theta$.

The next argument will use a linear map $h: \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)} \to \mathbb{R}^\ell$ which has exactly $((q-1)!)^\ell$ intersecting unordered q-tuples of pairwise disjoint simplices and enable us to calculate $\langle \upsilon_f , \Omega \rangle_\lambda$ for some cases when υ_f and Ω transform compatibly.

(2.11) If q and ℓ are not both even, $(q,\ell) = k > 1$, and $\omega \neq 1$ is a kth root of unity, then $\nu^* \upsilon_f = \upsilon_f$, $\nu_* \Omega = \Omega$, and $|\langle \upsilon_f , \Omega \rangle_{+1}| = ((q-1)!)^{\ell+1}$. Again, if $q = 2^{r+1}$, $r \geq 1$, and $\ell = 2^r.t$, where t is odd, and $\omega = \exp(2\pi\sqrt{-1})/q)$, then $\nu^* \upsilon_f = -\upsilon_f$, $\nu_* \Omega = -\Omega$, and $|\langle \upsilon_f , \Omega \rangle_{-1}| = ((q-1)!)^{\ell+1}$.

That υ_f and Ω transform as indicated follows from (2.6) and (2.8) (e.g. in the second case $\omega^{2^r} = \exp(\pi\sqrt{-1}) = -1$, so $\omega^\ell = (-1)^t = -1$). Also, since by (2.6), υ_f depends on f only upto an additive coboundary, Stokes' formula (2.10) shows that the value of $\langle \upsilon_f , \Omega \rangle_\lambda$ does not depend on the general position map f. We will use an f very close to the linear map $h: \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)} \to \mathbb{R}^\ell$ defined as follows.

Let $\bar{1}, \dots , \overline{\ell+1}$, be the vertices, and $\hat{\xi}$ the barycentre, of an ℓ-simplex $\xi^\ell \subset \mathbb{R}^\ell$. Then the map h -- which is determined by what it does to the vertices 1,2, ...,$(q-1)(\ell+1)$,$(q-1)(\ell+1)+1$ -- images the first q-1 vertices to $\bar{1}$, the next q-1 to $\bar{2}$, and so on, with the very last vertex, $(q-1)(\ell+1)+1$, being imaged to the barycentre $\hat{\xi}$.

One has $h(\Theta_1) \cap \dots \cap h(\Theta_q) \neq \phi$, for a pairwise disjoint ordered q-tuple $\Theta = (\Theta_1, \dots , \Theta_q)$ of simplices, iff one of the Θ_i's is equal to $\{(q-1)(\ell+1)+1\}$ (so has h-image $\hat{\xi}$), and all others are ℓ-simplices with h-images equal to ξ^ℓ. For the purpose of computing the number of $\mathbf{Z}_q$-orbits Θ arising from such q-tuples Θ , we will count the number of such Θ's with $\Theta_q = \{(q-1)(\ell+1)+1\}$. The jth, $1 \leq j \leq \ell+1$, vertices of Θ_i , $1 \leq i \leq q-1$, determine, and are determined by any of the (q-1)! permutations of $\{(q-1)(j-1)+1, \dots ,(q-1)j\}$. So there are in all $((q-1)!)^\ell$

such Z_q-orbits Θ and thus $((q-1)!)^{\ell}$ such cardinality q sets $\{\theta_1, \ldots, \theta_q\}$.

For any of the $((q-1)!)^{\ell+1}$ such θ's, $\omega_\theta = \omega^{(\ell+1)(1+\ldots+(q-1))} = \omega^{(\ell+1)(q(q-1)/2)} = \bar{\bar{\omega}}$ say. So the definition of $\langle v_f, \Omega \rangle_{\pm 1}$ shows that it equals $\bar{\bar{\omega}} . \sum_{\Theta} \deg\{F: \partial\theta \longrightarrow S^{(q-1)(\ell+1)-1}\} = \bar{\bar{\omega}} . \sum_{\Theta} \deg\{H: \partial\theta \longrightarrow S^{(q-1)(\ell+1)-1}\}$ where H is determined by $h^{(q)}$ and the homotopy equivalence $(\mathbb{R}^\ell)^{(q)}_* \simeq S^{(q-1)(\ell+1)-1}$ discussed while proving (2.4). Under the linear map $h^{(q)}: K^{(q)}_* \longrightarrow (\mathbb{R}^\ell)^{(q)}$ the images $h^{(q)}(\theta)$ of top dimensional oriented simplices θ of the above type all coincide, and constitute an oriented $((q-1)(\ell+1))$-dimensional simplex which cuts the ℓ-dimensional diagonal of $(\mathbb{R}^\ell)^{(q)}$ transversely in an interior point. Hence the nonzero degrees in question are either all +1 , or else all -1 , and $\langle v_f, \Omega \rangle_{\pm 1} = \pm \bar{\bar{\omega}} . ((q-1)!)^{\ell+1}$.

So, to complete the proof of (1.2) for the above cases, it suffices to check the following.

(2.12) <u>If v_f and Ω transform compatibly, then the general position map $f: \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)} \to \mathbb{R}^\ell$ has at least $|\langle v_f, \Omega \rangle_{\pm 1}| \div (q-1)!$ separated q-uple points.</u>

By definition $\langle v_f, \Omega \rangle_{\pm 1} = \sum_{\Theta \in K^{(q)}_*/Z_q} \{\omega_\theta . \deg(F: \partial\theta \to S^{(q-1)(\ell+1)-1}) : \theta \in \Theta\}$ where F is determined by $f^{(q)}$ and the homotopy equivalence $(\mathbb{R})^{(q)}_* \simeq S^{(q-1)(\ell+1)-1}$ of (2.4). Since f is in general position, we can choose a sufficiently fine simplicial Z_q-subdivision E of $K^{(q)}_*$, such that $f^{(q)}$ is linear on E, and the images $f^{(q)}(\beta)$ of top dimensional simplices β of E are either disjoint from the diagonal, or else intersect it transversely in a single interior point. Note that such intersections are in one-one correspondence with the permutations of the separated q-uple points of f. If, amongst the β's subdividing a top dimensional $\theta \in K^{(q)}_*$ there are r, say $\beta_1, \ldots, \beta_r$, of the intersecting type, then obviously $\deg(F: \partial\theta \to S^{(q-1)(\ell+1)-1})$ is the sum of r integers $n_i = \pm 1$, with sign depending on the orientation of $f^{(q)}(\beta_i)$, $1 \leq i \leq r$. Thus $\langle v_f, \Omega \rangle_{\pm 1}$ is the sum of N complex numbers of modulus 1, where N = number of Z_q-orbits arising from separated q-uple points, i.e. $(q-1)!$ times the number of (unordered) separated q-uple points of f. So the result follows because $N \geq |\langle v_f, \Omega \rangle_{\pm 1}|$.

For the (known) case $q = 2$, (2.6) and (2.8) (now $\omega = -1$) show that under ν, υ_f and Ω transform alike, except for a sign difference, so $\langle \upsilon_f, \Omega \rangle$ is well defined mod 2 , and one can prove (1.2) exactly as above with all calculations now mod 2.

Note that, for any pair (q, ℓ) of natural numbers, we now know the truth of (1.2 for some pair (q, ℓ'), $\ell' \geq \ell$. So the following assertion suffices to complete the proof of (1.2).

* (2.13) If there exists a general position continuous map $\Theta^{(q-1)\ell}_{(q-1)\ell} \xrightarrow{\varphi} \mathbb{R}^{\ell-1}$ with exactly N separated q-uple points, then there must also exist a general position map $\sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)} \xrightarrow{f} \mathbb{R}^{\ell}$ with exactly $N.((q-1)!)$ separated q-uple points.

Let $\sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)} = \tau^{q-2}_{q-2} \cdot \Theta^{(q-1)\ell}_{(q-1)\ell}$, and let $\mathbb{R}^{\ell}_{+}$ and $\mathbb{R}^{\ell}_{-}$ denote the 2 components of $\mathbb{R}^{\ell} \setminus \mathbb{R}^{\ell-1}$. We can assume that there is a simplicial subdivision E of $\Theta^{(q-1)\ell}_{(q-1)\ell}$ such that φ is a general position linear map with respect to E, and the Nq coordinates x of the N separated q-uple points of φ are contained in the (relative) interiors of Nq pairwise disjoint simplices β of E. Consider a linear map $\bar{f}: \tau^{q-2}_{q-2} \cdot E \to \mathbb{R}^{\ell}$ which coincides with φ on E , and which images the vertices of τ into $\mathbb{R}^{\ell}_{+}$. Since there are only q-1 such vertices it follows that, for any q pairwise disjoint simplices $\{\alpha_1, \ldots, \alpha_q\}$ of $\tau^{q-2}_{q-2} \cdot E$ the q-fold intersection $\bar{f}(\alpha_1) \cap \ldots \cap \bar{f}(\alpha_q)$ is nonempty only if $\varphi(\alpha_1 \setminus \tau) \cap \ldots \cap \varphi(\alpha_q \setminus \tau)$ is nonempty. And, the same must also remain true for any general position linear map $f: \tau^{q-2}_{q-2} \cdot E \to \mathbb{R}^{\ell}$ sufficiently close to $\bar{f}$.

For each of the separated q-uple points $\{x_1, \ldots, x_q\}$ of φ one has the q simplic $\{\beta_1, \ldots, \beta_q\}$, lying within pairwise disjoint simplices of $\Theta^{(q-1)\ell}_{(q-1)\ell}$, such that $x_1 \in \operatorname{int}\beta_1$, $\ldots, x_q \in \operatorname{int}\beta_q$. We observe that $\varphi(\beta_1) \cap \ldots \cap \varphi(\beta_q)$ is a point, and that the same statement is true for any linear map $E \to \mathbb{R}^{\ell-1}$ sufficiently close to φ . We now replace $\bar{f}$ by a general position linear map $f: \tau^{q-2}_{q-2} \cdot E \to \mathbb{R}^{\ell}$ in which the vertices of $\beta_1, \ldots, \beta_{q-1}$, are perturbed slightly into $\mathbb{R}^{\ell}_{-}$, and that of β_q slightly into $\mathbb{R}^{\ell}_{+}$. Using the observation just made, and, once again, the fact that τ has only q-1 vertices, we now see further that if $\alpha_1 \setminus \tau = \beta_1, \ldots, \alpha_q \setminus \tau = \beta_q$, then $f(\alpha_1) \cap \ldots \cap f(\alpha_q)$ is nonempty --and then equal to a point-- iff $\alpha_q = \beta_q$, and each α_i , $1 \leq i \leq q-1$, contains exactly one of the $q-1$ vertices of τ. Each $\{\beta_1, \ldots, \beta_q\}$ gives rise to $(q-1)!$ such $\{\alpha_1, \ldots, \alpha_q\}$, thus such a general position continuous map $f: \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)} \to \mathbb{R}^\ell$ has exactly $N.((q-1)!)$ separated q-uple points.

§3. Further applications

A simplicial complex will be called a pseudo-q-manifold if each codimension one simplex is incident to precisely q top dimensional simplices (so pseudo-2-manifold = pseudomanifold). It is easily seen that, amongst the skeletons of simplices $K = \sigma^i_j$, $j \leq i$, the only ones for which the qth join configuration, $K^{(q)}_{\not\ast}$, is a pseodo-q-manifold, are those of the type σ^t_t (which were considered above) and σ^{qs+q-2}_{s-1}. For these latter, a proof similar to that of §2, with complex fundamental cycles defined analogously to those of (2.7), establishes the following

(3.1) Generalized "van Kampen-Sierksma" Theorem. A general position continuous map $f: \sigma^{qs+q-2}_{s-1} \to \mathbb{R}^\ell$, where $\ell(q-1) \leq q(s-1)$, must have at least $((q-1)!)^s$ separated q-uple points.

Note that once again this implies a weaker linear version analogous to (1.1).

The case $q = 2$ of (3.1) was proved independently by van Kampen [18], 1932, and Flores [4], 1933. Flores used the Borsuk-Ulam Theorem [2], 1933 ($\cong$ Lusternik-Schnirelman Theorem [5], 1930) and the fact that the pseudomanifold $(\sigma^{2s}_{s-1})^{(2)}_{\not\ast}$ is indeed an antipodal (2s-1)-sphere. Also, a more general Borsuk-Ulam result was used in [10] to prove a weaker version of (3.1) for all primes p, viz., that a continuous map $g: \sigma^{ps+p-2}_{s-1} \to \mathbb{R}^\ell$, $\ell(p-1) \leq p(s-1)$, must have at least one separated p-uple point.

As in (2.5) one can define a cocycle v_G for any $\mathbf{Z}_q$-map G from the codimension one skeleton of $K^{(q)}_{\not\ast}$, $K = \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)}$, to the fixed-point-free $\mathbf{Z}_q$-sphere $S^{(q-1)(\ell+1)-1}$ which is zero iff G extends to a $\mathbf{Z}_q$-map $K^{(q)}_{\not\ast} \to S^{(q-1)(\ell+1)-1}$. It is easily verified that, upto coboundary of a symmetric cochain, v_G is same as v_f. So the non vanishing of $\langle v_f, \Omega \rangle$ yields the first part of the following

(3.2) Generalized Borsuk-Ulam Theorem. Let $q \geq 2$, and let $S^{(q-1)(\ell+1)-1}$ denote the fixed-point-free $\mathbf{Z}_q$-sphere of (2.4). Then there exists no continuous $\mathbf{Z}_q$-map from

$K^{(q)}_{*}$, $K = \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)}$, to $S^{(q-1)(\ell+1)-1}$. This implies that there exists no continuous $\mathbb{Z}_q$-map from the (t+1)-fold join of {q points} to the t-fold join of {q points}.

The proof of the second part involves using the join formula, $(L.M)^{(q)}_{*} \cong L^{(q)}_{*}.M^{(q)}_{*}$, which shows that $(\sigma^t_t)^{(q)}_{*}$ is the (t+1)-fold join of {q points}. Note that $(\sigma^t_t)^{(2)}_{*}$ = (t+1)-fold join of {2 points} = octahedral t-sphere, so (3.2) contains the classical Borsuk-Ulam Theorem [2], viz., that there exists no continuous $\mathbb{Z}_2$-map from the antipodal sphere S^t to S^{t-1}. See also Dold [3] for some other Borsuk-Ulam results which too can be proved by van Kampen's method.

(3.3) Remark. Although the notion of degree was introduced by L.E.J.Brouwer, most of its properties (used in §2) are due to H.Hopf, who also gave some of the first degree theoretic proofs of the Borsuk-Ulam Theorem (see [2], footnote 6).

Similar results hold for simplicial complexes K other than skeletons of simplices.

(3.4) A continuous map from $S^{(q-1)(\ell+1)-1}$ to $\mathbb{R}^\ell$ must have a separated q-uple point with respect to any triangulation K of $S^{(q-1)(\ell+1)-1}$.

Note that (1.2) implies this immediately for the minimal spherical triangulation $K = \sigma^{(q-1)(\ell+1)}_{(q-1)(\ell+1)-1}$, and it is not hard to deduce the general case from this by means of a direct argument. We remark that (3.4) extends still further to many spherical cell subdivisions K, e.g., improving on Roudneff [8], to all those associated to oriented matroids. We hope to discuss this and other aspects of the role of matroids in the theory characteristic classes elsewhere. (See also [12], proof of (2.5).)

The pseudo-q-manifolds $K^{(q)}_{*}$, $K = \sigma^t_t$, σ^{qs+q-2}_{s-1}, seem to play an important role vis-à-vis the combinatorics of the characteristic classes of Pontrjagin et al. Infact it seems that the complex fundamental cycles supported on them -- and some additional quaternionic cycles constructed by using finite multiplicative subgroups of $\mathbb{H}$ -- probably suffice to detect, in some generic (e.g. as in [13]) sense, the non-vanishing of all known characteristic classes. This is indicated by the fact that the pseudo-q-manifolds occuring in the image of the functor $K \rightsquigarrow K^{(q)}_{*}$ are all apparently given by the following

(3.5) Classification Theorem. The qth join configuration, $K^{(q)}_{*}$, of a simplicial complex, K, is a pseudo-q-manifold iff K is a join of some simplicial complexes of the type σ_t^t and σ_{s-1}^{qs+q-2}.

A complete proof of this result has so far been given only for the case when $q = 2$ and $\dim(K_{\neq}^{(2)}) = 2(\dim K) + 1$: see [11].

The vanishing of the top dimensional symmetric cohomology classes $\tau = [\tau_f]$ considered here -- or, more pertinently, the existence of a suitable Z_q-map -- is, under suitable dimensional restrictions, not only necessary, but also sufficient, for the existence of a continuous map without any (separated or not) q-uple points. For instance, consider the case of a continuous map from an (s-1)-dimensional simplicial complex K to $\mathbb{R}^\ell$ when $\ell(q-1) = q(s-1)$. Now $K_{\neq}^{(q)}$ has dimension $\leq qs-1$ while $(\mathbb{R}^\ell)_*^{(q)}$ has by (2.4) the Z_q-homotopy type of a fixed point free sphere S^{qs-2}. The obstruction class τ is zero iff there is a Z_q-map from $K_{\neq}^{(q)}$ to S^{qs-2}. Further we have the following

(3.6) Generalized Van Kampen-Wu-Shapiro Theorem. For a given $q \geq 2$, and all s sufficiently big, the (qs-1)-dimensional symmetric cohomology class τ of an (s-1)-dimensional simplicial complex K is zero iff there exists a continuous map without q-uple points from K to $\mathbb{R}^\ell$ whenever $\ell(q-1) \geq q(s-1)$.

The proof of this result is inspired also by van Kampen [18] who gave an argument for the case $q = 2$ and $s > 3$, which exploits $\tau = 0$ to successively eliminate the isolated 2-uple points of a general position continuous map $K^{s-1} \to \mathbb{R}^{2(s-1)}$. Note however that van Kampen's argument contained an unproved lemma, viz., the p.ℓ. version of the (now) well known Whitney Trick. The first complete proofs of this case were given only in 1957 independently by Shapiro and Wu. (The result is true also for $q = 2$ and $s = 2$ -- the case of graphs -- by virtue of a separate argument using Kuratowski's planarity criterion.)

An exposition of the van Kampen-Whitney constructions, and their various generalizations, will be given in [14], which will also contain more details regarding some of the results stated here.

Finally we remark that the method of §2 can also be modified to compute the qth Tverberg number $N_q(X)$ -- i.e. the least N such that any continuous map from σ_N^N to X has a separated q-uple point -- for some spaces X other than $\mathbb{R}^\ell$ (cf. [7] for analogous linear problems, e.g. that of Eckhoff on p.169).

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