Bénard Polygons

By K. S. Sarkaria

We prove that if a neighborhood of a polygonal region admits a two-dimensional eigenfunction of the Laplacian, having a nonzero eigenvalue and such that its normal derivative vanishes on all the bounding edges, then the polygonal region must be a union of complete pieces of a tiling of the plane by congruent rectangles, or by congruent $(45^\circ, 45^\circ, 90^\circ)$ or $(30^\circ, 60^\circ, 90^\circ)$ triangles. Hydrodynamically, this means that during critical convection in a horizontal fluid layer uniformally heated from below, the mere occurrence of *one* arbitrary closed vertical polygonal fluid surface across which there is no transportation of fluid automatically guarantees the presence of one of the usual special convection patterns. In addition it shows that linear convection theory *seldom* predicts a regular fluid pattern: e.g., for the case of a triangular container having angles substantially different from $(45^\circ, 45^\circ, 90^\circ)$, $(30^\circ, 60^\circ, 90^\circ)$, $(60^\circ, 60^\circ, 60^\circ)$ or $(30^\circ, 30^\circ, 120^\circ)$, it predicts that the convection cells not touching the boundary, if any, should be noticeably non-polygonal. We also consider a nonlinear generalization and the noneuclidean analogues of such polygons.

1. Introduction

The well-known "turbulence problem" asks whether experimentally observed chaotic motions are logical consequences of the partial differential equa-

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tions of fluid mechanics (and the given initial and boundary conditions). This article makes *no* contribution to this very difficult problem. Rather, we will consider another, similar-sounding, but certainly much easier, open problem of rational hydrodynamics, and give a partial solution of the same.

When a thin fluid layer, parallel to the horizontal x–y plane, is heated uniformly from below, the hotter, lighter liquid at the bottom tends to rise to the top, but is prevented from doing so by the internal friction of the fluid, until the temperature just exceeds a *critical value*, at which the fluid layer is observed to partition off into polygonal regions. Within each of these regions the fluid starts flowing slowly and steadily, generally up the middle and down the vertical *walls* of these polygonal cells. (If the temperature is raised further, the flow becomes faster and eventually turbulent.) Such experiments go back to Bénard (1900) and it has been found that the critical temperature and the geometry of the observed flow *pattern* depend on many factors, such as boundary conditions, surface tension, solute content, electrical conductivity of the fluid, and presence of a magnetic field.

Are all these experimental observations deducible from the underlying partial differential equations? This question is certainly easier than the turbulence problem, because, the speed of the flow being small, these equations are now merely the linearized partial differential equations of fluid mechanics. Indeed, extensive work (for comprehensive and readable overviews see [1, 2]) based on this strategy has been done over the years, but only criticality seems to be thoroughly understood.

Most strikingly it is unknown *why does polygonal convection pattern should appear at all as a strictly logical consequence of the partial differential equations of the problem.* As our contribution to this tantalizing question, we show that these equations do at least imply that *if one closed polygonal vertical wall becomes visible within the convecting fluid, then, indeed, the entire pattern must already be visible* (moreover, we classify all the possible patterns). Here, the assumed "closed polygonal" wall can have *any* finite number of straight sides, and the region enclosed by it can be *nonconvex*; we demand only that it be "within," in the sense that there be fluid *both* in this polygonal region and in the region immediately outside it, with no transportation of fluid between these two regions. This heuristically enunciated proposition mirrors the more formally stated theorem of Section 2; e.g., to see this for the purely buoyancy-driven case, the points of which we must keep in mind are the following.

Following Boussinesq (1903), convection is supposed to be governed by the *equation of continuity*, $\nabla \cdot \mathbf{q} = 0$; the *Navier–Stokes equations of motion* $\rho d\mathbf{q}/dt = -\nabla p + \rho g(1 - \alpha T)\mathbf{k} + \mu \nabla^2 \mathbf{q}$; and the *equation of heat conduction* $dT/dt = \chi \nabla^2 \mathbf{T}$. Here, the "density" ρ, the gravitational acceleration g, the expansion coefficient α in the term $\rho g(1 - \alpha T)\mathbf{k}$ chosen to model buoyancy, the dynamic viscosity μ, and the thermal conductivity χ, are all treated as constants; while the velocity $\mathbf{q} = [u, v, w]$, the pressure p, and the temperature T, are functions of x, y, z, and t. These equations are clearly satisfied if we put $\mathbf{q} = \mathbf{0}$, $T = -cz$, and $p = \rho g(z - \alpha cz^2/2)$. This describes *rest*; i.e., the situation in which buoyancy is unable to overcome viscous resistance.

Rayleigh, 1916, sought the smallest c for which one also has a nearby *stationary* solution $\mathbf{q} = \mathbf{q}(x, y, z) \neq \mathbf{0}, T = -cz + \theta(x, y, z)$, and $p = \rho g(z - \alpha cz^2/2) + \rho(x, y, z)$. On substituting these into the equations, and using $d\mathbf{q}/dt = \boldsymbol{\partial}\mathbf{q}/\boldsymbol{\partial}t + (\mathbf{q}\cdot\boldsymbol{\nabla})\mathbf{q}$ and $dT/dt = \boldsymbol{\partial}T/\boldsymbol{\partial}t + (\mathbf{q}\cdot\boldsymbol{\nabla})T$, we obtain $\boldsymbol{\nabla}\cdot\mathbf{q} = 0, (\mathbf{q}\cdot\boldsymbol{\nabla})\mathbf{q} = -\boldsymbol{\nabla}p - \rho g\alpha\theta\mathbf{k} + \mu\boldsymbol{\nabla}^2\mathbf{q}$, and $-cw + (\mathbf{q}\cdot\boldsymbol{\nabla})\theta = \chi\boldsymbol{\nabla}^2\theta$. However, for $\mathbf{q}$ and θ small, $(\mathbf{q}\cdot\boldsymbol{\nabla})\mathbf{q}$ and $(\mathbf{q}\cdot\boldsymbol{\nabla})\theta$ are of a higher order of smallness, and thus, we assume from here on that the underlying differential equations of the problem are their *linearizations*:

$$\boldsymbol{\nabla}\cdot\mathbf{q} = \theta, \tag{1}$$

$$\mathbf{0} = -\boldsymbol{\nabla}p - \rho g\alpha\theta\mathbf{k} + \mu\boldsymbol{\nabla}^2\mathbf{q}, \tag{2}$$

$$-cw = \chi\boldsymbol{\nabla}^2\theta. \tag{3}$$

Note that $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}p) = 0, \boldsymbol{\nabla}\times\theta\mathbf{k} = [\boldsymbol{\partial}\theta/\boldsymbol{\partial}y, -\boldsymbol{\partial}\theta/\boldsymbol{\partial}x, 0], \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\theta\mathbf{k}) = [\boldsymbol{\partial}^2\theta/\boldsymbol{\partial}z\boldsymbol{\partial}x, \boldsymbol{\partial}^2\theta/\boldsymbol{\partial}z\boldsymbol{\partial}y, -\boldsymbol{\partial}^2\theta/\boldsymbol{\partial}x^2 - \boldsymbol{\partial}^2\theta/\boldsymbol{\partial}y^2]$, and $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{q}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{q}) - \boldsymbol{\nabla}^2\mathbf{q}$. So, on taking the curl of (2) twice, and using (1), we get $\mathbf{0} = \rho g\alpha[-\boldsymbol{\partial}^2\theta/\boldsymbol{\partial}z\boldsymbol{\partial}x, -\boldsymbol{\partial}^2\theta/\boldsymbol{\partial}z\boldsymbol{\partial}y, \boldsymbol{\partial}^2\theta/\boldsymbol{\partial}x^2 + \boldsymbol{\partial}^2\theta/\boldsymbol{\partial}y^2] + \mu\boldsymbol{\nabla}^4\mathbf{q}$. By applying $\mathbf{x}\boldsymbol{\nabla}^2$, and using (3), this gives $\mathbf{0} = -c\rho g\alpha[-\boldsymbol{\partial}^2 w/\boldsymbol{\partial}z\boldsymbol{\partial}x, -\boldsymbol{\partial}^2 w/\boldsymbol{\partial}z\boldsymbol{\partial}y, \boldsymbol{\partial}^2 w/\boldsymbol{\partial}x^2 + \boldsymbol{\partial}^2 w/\boldsymbol{\partial}y^2] + \chi\mu\boldsymbol{\nabla}^6\mathbf{q}$. So the $\mathbf{k}$ component of $\mathbf{q}$, $w(x, y, z)$, satisfies the following *eigenvalue equation* for c,

$$0 = -c\rho g\alpha(\boldsymbol{\partial}^2 f/\boldsymbol{\partial}x^2 + \boldsymbol{\partial}^2 f/\boldsymbol{\partial}y^2) + \chi\mu\boldsymbol{\nabla}^\sigma f. \tag{4}$$

As is easily seen, $\theta(x, y, z)$ also satisfies (4). Indeed, *the vertical component* $\zeta(x, y, z) = \boldsymbol{\partial}v/\boldsymbol{\partial}x - \boldsymbol{\partial}u/\boldsymbol{\partial}y$ *of the vorticity* curl($\mathbf{q}$) *is identically zero*, and so, because $-\boldsymbol{\partial}^2 w/\boldsymbol{\partial}z\boldsymbol{\partial}x = \boldsymbol{\partial}/\boldsymbol{\partial}x[\boldsymbol{\partial}u/\boldsymbol{\partial}x + \boldsymbol{\partial}v/\boldsymbol{\partial}y] = \boldsymbol{\partial}^2 u/\boldsymbol{\partial}x^2 + \boldsymbol{\partial}^2 v/\boldsymbol{\partial}y\boldsymbol{\partial}x = \boldsymbol{\partial}^2 u/\boldsymbol{\partial}x^2 + \boldsymbol{\partial}^2 u/\boldsymbol{\partial}y^2$ and likewise $-\boldsymbol{\partial}^2 w/\boldsymbol{\partial}z\boldsymbol{\partial}y = \boldsymbol{\partial}^2 v/\boldsymbol{\partial}x^2 + \boldsymbol{\partial}^2 v/\boldsymbol{\partial}y^2$, u and v are also eigenfunctions of (4). The assertion just made regarding ζ follows (cf. [1], pp. 22, 32) because the third component of the curl of (2) gives $0 = \boldsymbol{\nabla}^2\zeta$, but the natural boundary conditions for ζ on top and bottom are either $\zeta = 0$ (rigid) or $\boldsymbol{\partial}\zeta/\boldsymbol{\partial}z = 0$ (free), and, in either case, by applying the well-known uniqueness theorems for $\boldsymbol{\nabla}^2$, we must have $\zeta = 0$.

Let $\mathscr{V}$ denote the vector space of all smooth functions $f(x, y, z)$, and let $\mathscr{V}_{\boldsymbol{\lambda}} \subset \mathscr{V}$, $\lambda \geq 0$, be the subspace of all those for which $\boldsymbol{\partial}^2 f/\boldsymbol{\partial}x^2 + \boldsymbol{\partial}^2 f/\boldsymbol{\partial}y^2 = -\lambda^2 f$. If f belongs to $\mathscr{V}_{\boldsymbol{\lambda}}$, then all partial derivatives of f also belong to $\mathscr{V}_{\boldsymbol{\lambda}}$: also recall that $\mathscr{V}$ is the topological direct sum of these subspaces $\mathscr{V}_{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda} \geq 0$. So, *without loss of generality we may only consider the case when all our functions f are in the same* $\mathscr{V}_{\boldsymbol{\lambda}}$, $\lambda \geq 0$. With this understood, the above eigenvalue Equation (4) becomes

$$0 = c\rho g\alpha\lambda f + \chi\mu(-\lambda^2 + \boldsymbol{\partial}^2/\boldsymbol{\partial}z^2)^3 f. \tag{5}$$

Obviously, we need only those $\lambda \geq 0$ for which $\mathscr{Z}_\lambda \neq 0$; i.e., $\lambda^2 \in \operatorname{spec}(\mathscr{L})$, the *spectrum* (an important Euclidean geometric invariant) of the fluid layer $\mathscr{L}$. Thinking of x and y as fixed, (5) is an ordinary linear differential equation with constant coefficients and can be easily solved by using the given boundary conditions at the top and bottom of $\mathscr{L}$. For example, for the free–free case, $w = \partial^2 w/\partial z^2 = \partial^4 w/\partial z^4 = 0$ at both top and bottom, so (5) *has no nonzero solution* $w(x, y, z)$ *for* $\boldsymbol{\lambda} \neq 0$, but for all other $\boldsymbol{\lambda}^2 \in \operatorname{Spec}(\mathscr{L})$, there is at least one $c = c_\lambda$, necessarily positive, for which it has a nonzero solution, and the sought-for *critical temperature gradient* is their infimum,

$$\inf[c_\lambda : 0 \neq \lambda^2 \in \operatorname{Spec}(\mathscr{L})]. \tag{6}$$

Because $u, v \in \mathscr{Z}_\lambda$, $\lambda \neq 0$, there exists a unique pair $f, g \in \mathscr{Z}_\lambda$ such that $u = \partial f/\partial x - \partial g/\partial y$ and $v = \partial f/\partial y + \partial g/\partial x$; namely, $f = \lambda^{-2}(\partial u/\partial x + \partial v/\partial y) = \lambda^{-2}\partial w/\partial z$ and $g = \lambda^{-2}(\partial v/\partial x - \partial u/\partial y) = 0$. So $\mathbf{q} \in (\mathscr{Z}_\lambda)^3$ *is entirely determined by its vertical component* w, and we have

$$\mathbf{q} = [\lambda^{-2}\partial^2 w/\partial z\partial x, \lambda^{-2}\partial^2 w/\partial z\partial y, w]. \tag{7}$$

Note finally that there is no transportation of fluid across $lx + my + n = 0$ iff $lu + mv = 0$ on this plane. Using (7), we see equivalently that *there is no transportation of fluid across a vertical plane* $lx + my + n = 0$ *if and only if* $l\partial w/\partial x + m\partial w/\partial y = 0$ *on this plane*. This motivates the formal definition that follows, and shows that the theorem below implies the proposition enunciated above.

2. Bénard polygon

A polygon $P \subset \mathbb{R}^2$ will be called a *Bénard polygon* if there exists a nonzero function ϕ and a nonzero real number λ, such that

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\lambda^2\phi, \tag{8}$$

in some neighborhood of the closed polygonal region $\mathbf{P} \subset \mathbb{R}^2$ bounded by P, and such that the normal derivative of ϕ satisfies

$$\frac{\partial\phi}{\partial\mathbf{n}} = 0, \tag{9}$$

on all the edges of the boundary P. If, moreover, we can find a ϕ such that $\partial\phi/\partial\mathbf{v}$ is not identically zero for some unit vector $\mathbf{v}$, then P will be called *nontrivial* (note that a nontrivial P might also admit a "trivial" ϕ with $\partial\phi/\partial\mathbf{v} \equiv 0$ in some direction $\mathbf{v}$). The following result says that these "eigenpolygons" of the Laplacian are quite special.

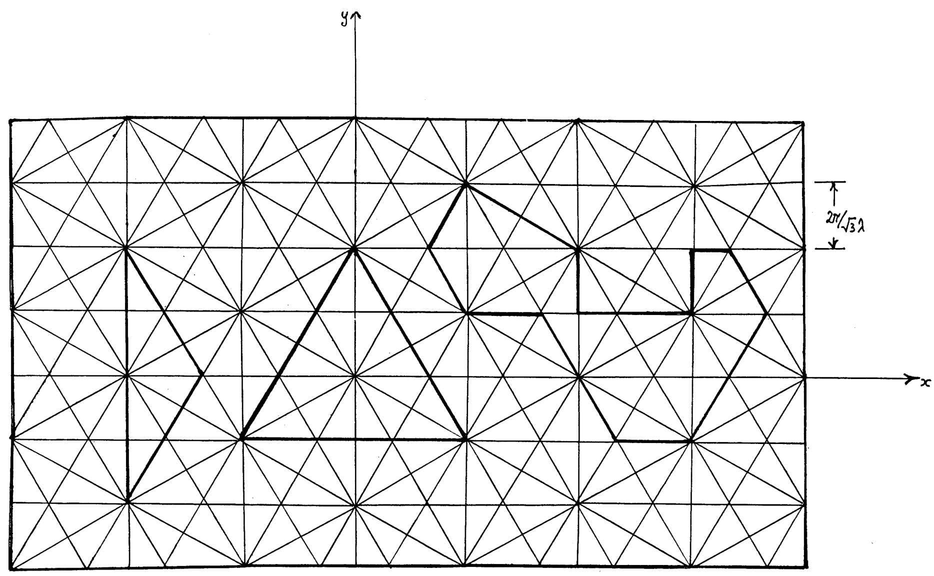

Figure 1.

THEOREM 1. *A polygon $P \subset \mathbb{R}^2$ is a nontrivial Bénard polygon iff it is formed by the edges of a tiling of $\mathbb{R}^2$ by congruent rectangles, or by congruent triangles having angles* (45°, 45°, 90°) *or* (30°, 60°, 90°).

Note that in this statement, "rectangles" can be assumed to be not squares, for the diagonals of a square subdivide it into congruent (45°, 45°, 90°) tiles. Also, hexagonal and equilateral triangular tilings are covered by this statement, because the diagonals of a hexagon cut it up into (60°, 60°, 60°) tiles, whose medians subdivide them further into (30°, 60°, 90°) tiles. [Figure 1 shows three Bénard polygons drawn in the (30°, 60°, 90°) tiling of $\mathbb{R}^2$.] Note also the immediate corollary that *a Bénard triangle must necessarily have angles* (45°, 45°, 90°), (30°, 60°, 90°), (60°, 60°, 60°), *or* (30°, 30°, 120°). (Trivial Bénard polygons are easy to classify: see Section 4.)

The proof occupies Sections 3–5 and is essentially split up into three propositions. In Section 6, we discuss further the implications of the above result vis-à-vis convection patterns, and in Sections 7 and 8 we sketch some interesting generalizations and analogues.

3. Walls

We shall use the fact that ϕ, being a solution of (8) not merely on **P** but on an open set of $\mathbb{R}^2$ containing **P**, is necessarily *real analytic* at all points of **P**. Such results go back at least to Holmgren; indeed, one has the definitive theorem of Petrovsky (see, e.g., [3], Ch. IV, especially pp. 96 and 97 and Cor. 4.4.1 on p. 114), which tells us that on an open set of $\mathbb{R}^n$, all classical (in fact, even distributional) solutions of a partial differential equation with constant coefficients are real analytic, if and only if the equation is elliptic.

Also, we shall use the fact that the Laplacian commutes with the map μ^* on functions—$\mu^*(\phi) = \phi^\circ\mu$—induced by any *isometry* μ of $\mathbb{R}^2$; and so, if $\boldsymbol{\phi}$ is a solution of (8) on $\Omega \subseteq \mathbb{R}^2$, then $\mu^*(\phi)$ must be a solution of (8) on $\mu^{-1}(\Omega) \subseteq \mathbb{R}^2$. This commutativity is easily verified by a computation based on the fact that $\mu : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry of $\mathbb{R}^2$ if and only if it is a translation followed by an orthogonal linear transformation. The key point for us will be that any solution of (8) is symmetric with respect to reflection in its *walls*, i.e., the straight lines on which (9) holds; e.g., one has the following.

PROPOSITION 1. *Let $\Omega \subset \mathbb{R}^2$ be an open ball with L a straight line passing through its center. If a solution $\phi : \Omega \to \mathbb{R}$ of* (8) *satisfies* (9) *on a subinterval of $L \cap \Omega$, then it satisfies* (9) *identically on $L \cap \Omega$, and one has $\phi = \mu^*\phi$ where μ denotes reflection in L.*

Proof: By using a suitable isometry μ of $\mathbb{R}^2$ we can assume, without loss of generality, that $\Omega = [(x, y) : 0 \leq (x^2 + y^2)^{\mathbf{1/2}} < r]$ and that the straight line L is the x-axis $y = 0$. Next, using the principle of unique analytic continuation, it is straightforward to reduce further to the case when $\phi(x, y)$ has a convergent power series representation $\boldsymbol{\Sigma}_{i\geq 0} a_{ij} x^i y^j$ valid throughout this open ball Ω.

Because $\boldsymbol{\partial\phi/\partial} y(x, 0) = \boldsymbol{\Sigma}_{i\geq 0} a_{i1} x^i$ for $-r < x < +r$, this function of x is analytic in $(-r, +r)$. Because we are given that it vanishes on a subinterval of $(-r, +r)$ it must vanish identically, which happens iff $a_{i1} = 0$ for all $i \geq 0$. Next, on substituting $\boldsymbol{\phi}(x, y) = \Sigma_{i\geq 0} a_{ij} x^i y^j$ in the differential equation (8), we get the recurrent relations

$$(i + 2)(i + 1)a_{i+2,\, j} + (j + 2)(j + 1)a_{i,\, j+2} = -\lambda^2 a_{ij} \tag{10}$$

for all $i \geq 0$ and $j \geq 0$. These relations (10) show that $a_{ij} = 0$ for all j odd, and so we have the asserted symmetry $\phi(x, -y) = \phi(x, y)$. ■

4. Primitive case

We first deal with *primitive Bénard polygons*; i.e., those that admit a nontrivial ϕ satisfying the conditions of Section 2, and the additional condition that no wall of ϕ should pass through the interior of the polygonal region $\mathbf{P}$. (However, we remark that for the direct part of the following, we shall only use the fact that no wall, *through a vertex of P*, should pass through the interior of $\mathbf{P}$.)

PROPOSITION 2. *A primitive Bénard polygon must be a rectangle, or a triangle with angles* $(45^\circ, 45^\circ, 90^\circ)$, $(30^\circ, 60^\circ, 90^\circ)$, *or* $(60^\circ, 60^\circ, 60^\circ)$, *and, conversely, with the possible exception of* $(60^\circ, 60^\circ, 60^\circ)$ *triangles, all these are, indeed, primitive Bénard polygons.*

Proof: Let L_1 and L_2 denote the two walls of ϕ that bound an internal angle θ of $\mathbf{P}$ at some vertex v (so $\theta \neq \pi$) of P. This angle is shown shaded in Figure 2, which also shows an Ω (cf. Proposition 1) with v as center, in which the solution ϕ of (8) is known to exist.

We cannot have $\theta > \pi$, because then some parts of these walls L_1, L_2 pass through the interior of $\mathbf{P}$. Also, if $\pi/2 < \theta < \pi$, then Proposition 1 gives a wall $\mu_2(L_1)$ that passes through the interior of $\mathbf{P}$. So $0 < \theta \leq \pi/2$, and, by using Proposition 1, repeatedly (see Figure 2) we obtain, on reflecting L_1 in L_2, a wall L_3, and then on reflecting L_2 in L_3, we obtain a wall L_4, and so on. Because the angle between any two consecutive walls of this sequence $L_1, L_2, L_3, L_4 \ldots$ is θ, after some steps the wall L_1 will recur; otherwise we would get a wall with some part in the interior of $\mathbf{P}$. Thus, we have shown that *any internal angle* θ *of* $\mathbf{P}$ *must be an integral divisor of* π; i.e., that $\theta = \pi/m$ for some integer $m \geq 2$.

All internal angles being $\leq \pi/2$, it follows that P cannot have more than four sides, because the sum of the t internal angles of a t-gon is $(t-2)\pi$. If

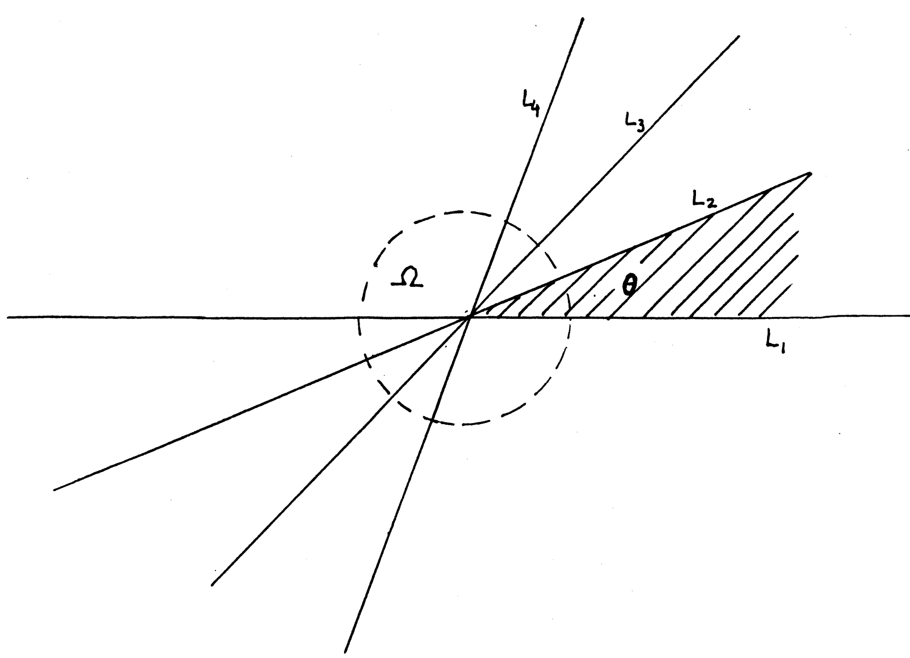

Figure 2.

P is a quadrilateral, then the sum of its four angles being 2π, we must have $m = 2$ for all the four angles; i.e., P must be a rectangle. Otherwise, P is a triangle. So, the sum of the three angles π/m, π/n, and π/p being π, we seek integers $m, n, p \geq 2$ such that

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = 1. \tag{11}$$

These are easily found: one has $\{m, n, p\} = \{3, 3, 3\}, \{2, 4, 4\}$ or $\{2, 3, 6\}$, which give the three triangular possibilities.

Conversely, because $\cos(\lambda \cdot x) + \cos(\lambda \cdot y)$ solves (8) over $\mathbb{R}^2$, and all its walls are as in Figure 3(a), it follows that (45°, 45°, 90°) triangles are primitive. Likewise, all the walls of the solution $\cos(\lambda x) + \cos(\lambda \cdot \cos(2\pi/3) \cdot x + \lambda \cdot \sin(2\pi/3) \cdot y) + \cos(\lambda \cdot \cos(4\pi/3) \cdot x + \lambda \cdot \sin(4\pi/3) \cdot y)$ are as shown in Figure 1, so (30°, 60°, 90°) triangles are primitive. Finally, for any $A \neq 0, 1$, all the walls of the solution $A \cdot \cos(\lambda \cdot x) + \cos(\lambda \cdot y)$ form a square pattern, and, for a and b nonzero, distinct, and such that $\lambda^2 = a^2 + b^2$, the walls of the solution $\cos(a \cdot x) \cdot \cos(b \cdot y)$ form a rectangular pattern, Figure 3(b). So rectangles are also primitive. ∎

Note that the solution $\cos(\lambda \cdot x)$ used above is, up to an isometry μ of $\mathbb{R}^2$, and a multiplicative constant A, the only *trivial solution* of (8). Its walls are all horizontal lines $y = C$, and all vertical lines $x = m\pi/\lambda$, with m an integer. Therefore, it follows that *a polygon P is Bénard with respect to a trivial solution of* (8) *iff all its angles are $\pi/2$ or $3\pi/2$, and all ratios of sides, in one of the two directions, are rational.*

For some of the examples used in the above proof we symmetrized this trivial solution over a small rotation group. As against these, we note that *for all rotation groups G with order bigger than* 8*, all the walls of the solution $\phi(x, y) = \Sigma_{\theta \in G} \cos(\lambda \cdot \cos\theta \cdot x + \lambda \cdot \sin\theta \cdot y)$ of* (8) *must pass through the origin.* Furthermore, because $\partial\phi/\partial x = 0$ on $x = 0$, and $\partial\phi/\partial y = 0$ on $y = 0$,

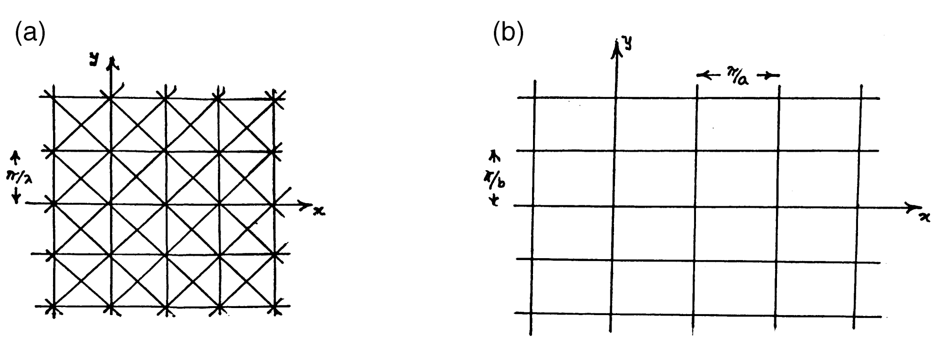

Figure 3.

the two axes, and the lines obtained by rotating them by angles $\theta \in G$, are walls. Thus, the angle between adjacent walls through the origin is an integral divisor of both $\pi/2$ and $2\pi/|G|$.

5. Finiteness

Given a $\boldsymbol{\phi}$ by the *degree* m of a point of $\mathbb{R}^2$ we mean the number of walls of $\boldsymbol{\phi}$ passing through it, and a point O will be called a *crossing* of $\boldsymbol{\phi}$ if its degree is at least two. In the following, we use the fact (see the proof of Proposition 2) that if the degree m is finite, then the angular distance between adjacent walls through O is π/m.

PROPOSITION 3. *Given that P is any nontrivial Bénard polygon with ϕ as in Section 2, then there are only finitely many walls of ϕ that meet* **P**.

Proof: Assume, if possible, that the nontrivial $\boldsymbol{\phi}$ has an infinite number of walls passing through **P**. Then there must be an infinite number of crossings of ϕ in **P**, indeed, each wall has at least one on the boundary P; viz, where it exits from **P**.

If the degree of a crossing O were infinite, then symmetry with respect to walls through O implies that there are infinitely many of these walls in directions arbitrarily close to any given direction. Therefore, all lines through O must be walls; i.e., $\boldsymbol{\phi}$ is constant on each circle around O and depends only on the distance r from v. Moreover, this dependence is real analytic, because $\boldsymbol{\phi}(r)$, $r > 0$ satisfies an ordinary linear differential equation with real analytic coefficients (namely, $r\partial^2\phi/\partial r^2 + \partial\phi/\partial r = -\lambda^2 r\phi$, so such a $\phi(r)$ would, in fact, be a Bessel function of the zero-th order). Consider a ray from O, which exits **P** at a distance r_0 from O, at an interior point v of an edge of P. We have $\text{grad}(\phi) = 0$ at all points w of this edge sufficiently close to v. The distances from 0 to these points w gives a subinterval $(r_0 - \varepsilon, r_0 + \varepsilon)$ on which $\phi'(r)$ is zero. So, by real analyticity, $\phi(r)$ is a constant function, which gives the contradiction $\lambda = 0$.

Therefore, all crossings have finite degree. Furthermore, these degrees must be bounded above. If not, we can find an O such that arbitrarily close to it there is a crossing having an arbitrarily high finite degree. Consider now any point p on a line with normal vector $\mathbf{n}$ passing through O. If $\partial\phi/\partial\mathbf{n} \neq 0$ at p, then we must have $\partial\phi/\partial\mathbf{m} \neq 0$ for all q in a sufficiently small neighborhood U of p and all $\mathbf{m}$ in a sufficiently small neighborhood V of $\mathbf{n}$. However, from the definition of O, it is clear that we can find a wall with normal vector $\mathbf{m} \in V$, which passes through U. So all lines through O must be walls, but we already ruled this out in the last paragraph.

The boundary edges of P are in finitely many directions, and all walls meet them. Therefore, from the boundedness of the degrees, it follows that each of the infinitely many walls through **P** must be in one of finitely many directions only. One of these, say, that perpendicular to unit vector **v**, will be the direction of infinitely many walls. Because **P** is compact, given any $\delta > 0$, we can find two members of this parallel pencil of lines that are at distance $< \delta$ from each other. By symmetry it follows that between any two members of this pencil there must be another. Therefore, we must have $\partial\phi/\partial\mathbf{v} = 0$ identically, which contradicts the fact that ϕ is a nontrivial solution of (8). ■

We now can complete the *proof of the Theorem*. Note that the *parts of* **P**—obtained by subdividing it by using the (by Proposition 3 finitely many) walls of $\boldsymbol{\phi}$ in **P**—have no walls passing through their interior; i.e., their boundaries are what we called primitive Bénard polygons. Therefore, each of these parts must be one of the four possibilities given by Proposition 2. To see that any two *parts are congruent*, it clearly suffices to look at the case when they share a common edge: the symmetry of ϕ with respect to this common wall shows that, were they not congruent, by reflection in this wall, we should have additional walls in them. Last, we note that, by successive reflections in edges, any *part generates a tiling* of all of $\mathbb{R}^2$; namely, the tilings picturized in Figures 1 and 3, or, possibly, the equilateral triangular tiling, which we can subdivide by using the medians into that of Figure 1.

Conversely, each of these tilings is given by all the walls of a nontrivial solution ϕ of (8) defined throughout $\mathbb{R}^2$; namely, the examples given while proving Proposition 2. So any P, having edges in such a tiling, is a nontrivial Bénard polygon as per the definition of Section 2. ■

For any polygon $P \subset \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, let $\boldsymbol{\gamma}_{P,\lambda}$ denote the vector space of all solutions $\boldsymbol{\phi}$s of (8), on some open set containing **P**, which satisfy the boundary condition (9) on P. We have characterized all Ps for which $\dim \mathscr{V}_{P,\lambda} > 0$ for some $\boldsymbol{\lambda} = 0$. It remains to find these numbers $\dim(\mathscr{V}_{P,\lambda})$ exactly, and to exhibit nice bases for these solution spaces.

6. Physical interpretation

Resuming the discussion of Section 1, we recall that the boundary conditions at the top and bottom of $\mathscr{L}$ are needed to work out the critical temperature gradient (6). Note, however, that the boundedness of $\mathscr{L}$ also plays a role; e.g., they make Spec($\mathscr{L}$) discrete, so the *critical wave number* $\boldsymbol{\lambda}$ at which this infimum occurs usually differs from that based on the assumption $\mathscr{L} = \mathbb{R}^2$, thus explaining (cf. [2], pp. 71, 73–75) why *the observed critical temperature gradient is usually more than that predicted by assuming* $\mathscr{L} = \mathbb{R}^2$.

It seems that we should next use the conditions prevailing on the lateral boundary $\partial\mathscr{L}$ to decide whether or not a polygonal pattern will occur.

Assuming no transportation of fluid across $\partial\mathscr{L}$, we have on it $\mathbf{q}\cdot\mathbf{n} = 0$; i.e., $u\ \partial\beta/\partial x + v\ \partial\beta/\partial y = 0$ if $\partial\mathscr{L}$ is given by $\beta(x, y) = 0$; i.e., $\partial w/\partial\mathbf{n} = 0$ by using (7). Thus, we are led to the following *boundary value problem* for $\phi(x, y) = w(x, y, C)$.

$$\boldsymbol{\nabla}^2\boldsymbol{\phi} - \boldsymbol{\lambda}^2\boldsymbol{\phi} = 0 \quad \text{on} \quad \mathscr{L}, \quad \boldsymbol{\partial\phi/\partial\mathbf{n}} = 0 \quad \text{on} \quad \boldsymbol{\partial}\mathscr{L}. \tag{12}$$

We confine ourselves to the *case of a polygonal container*. Now, of course, *any* nonzero solution of (12) gives rise to a polygonal fluid pattern. However, all its polygonal cells might be touching $\partial\mathscr{L}$. Indeed, it might even have no interior walls; i.e., it might have just one polygonal cell, namely, $\mathscr{L}$ itself. Let us agree that a *regular pattern* of convection is one in which at least one cell, which is strictly polygonal, is contained entirely in the interior $\mathrm{int}(\mathscr{L}) = \mathscr{L} - \partial\mathscr{L}$. (Note that a priori this definition is much broader than what "regular pattern" usually connotes in Bénard theory.) Thus, our theorem has the immediate, but remarkable, corollary that *a regular pattern is possible in a polygonal container only if its lateral boundary $\partial\mathscr{L}$ is a Bénard polygon in the sense of Section 2*. Because such polygons are very special, it follows—on the basis of the linear theory being used—that regular fluid patterns are very special indeed. For example, we predict that *if one observes critical fluid convection in a triangular container with angles substantially different from* (45°, 45°, 90°), (30°, 60°, 90°), (60°, 60°, 60°), *or* (30°, 30°, 120°), *the pattern will be always noticeably irregular*.

Surprisingly, the above obvious line of attack is not pursued in Bénard theory (neither could we find anything relevant on the above boundary value problem in the partial differential equation literature). Instead, almost always, it is assumed that $\mathscr{L} = \mathbb{R}^2$, and, more seriously, the basic question posed in Section 1; i.e., why any polygonal pattern should appear at all, is simply *evaded*; e.g., by imposing some ad hoc symmetry requirement, which forces one to choose for ϕ one of the examples used in Section 4! Often it is just laid down by fiat (see, e.g., [4], pp. 143, 144) that only those $\boldsymbol{\phi}$s that have hexagonal or square symmetry will be considered, or else (see [1], p. 43) it is argued that (there being no points or directions in the layer that are preferred) the visible walls of the flow pattern must (why?) form an absolutely symmetric tesselation of the plane. Indeed, (see, e.g., [2], p. 86) *the prevailing consensus seems to be that linear theory always predicts a regular fluid pattern; as we have shown above, this is far from the truth*. It seldom (because Bénard $\partial\mathscr{L}$s are so rare) predicts a regular fluid pattern: these are the exceptions, not the rule.

However, returning to one of these exceptional situations, let us note that an edge of the tiling formed by the walls of ϕ is *visible* to an experimenter observing the flow (cf. [2], p. 27) if and only if one has on it either $\boldsymbol{\phi}(x, y) > 0$ always, or else $\phi(x, y) < 0$ always. For a solution ϕ whose walls form a (30°, 60°, 90°) tiling, the subset of visible edges determines a *hexagonal* tiling, and for a ϕ whose walls form a (45°, 45°, 90°) tiling, this subset gives a *square* tiling. The visible walls $\lambda x = n\pi$ of the trivial solution $\phi(x, y) = \cos(\lambda x)$ partition off the plane into infinite strips or *rolls*, and that of $\cos(ax)+\cos(by)$ into *rectangles*. Thus, as mentioned already, our theorem shows that if just one closed polygon of walls is visible, then, in fact, an entire convective flow pattern of one of these kinds must be visible.

We emphasize that in this result, just the presence in $\mathrm{int}(\mathscr{L})$ of any closed curvilinear vertical surface with $\partial w/\partial \mathbf{n} = 0$ on it won't do; we made essential use of the fact it comes discretized into finitely many straight walls; i.e., that it is polygonal. We remark also that for the general case of a curved vertical $\partial\mathbf{x}$, our theorem does give some strong necessary conditions for the existence of a regular pattern, but because these are more complicated, we will not go into this matter here.

Also note that mere *boundedness of ϕ at infinity does not imply the existence of a closed polygon of walls*; e.g., as mentioned before, by symmetrizing $\cos(\lambda \cdot x)$ by any rotation group of order >8, one gets a solution of (8) having lots of walls through the origin but (as follows by using the theorem, or by direct computation) none other. These and other special solutions may be of use for doing a comparative study of the stability of *all* the flow patterns mentioned above, within a *single* nice finite dimensional manifold of solutions of (8). Lacking this, only stability of rolls vs hexagons, or of rolls vs squares, is usually studied (cf. [4], p. 144) within the linear finite dimensional spaces of solutions having hexagonal, respectively square, symmetry.

To correlate the above with experiment, one must, as usual, think of $\partial\mathscr{L}$, not as the actual lateral boundary, where one usually has $\mathbf{q} = 0$, but as a very thin boundary layer's parallel inner surface, on which only $\mathbf{q}\cdot\mathbf{n} = 0$ holds. (Under linear theory, the boundary condition $\mathbf{q} = 0$, would have yielded the zero solution only.) Because $\mathbf{q} \equiv \mathbf{0}$ at rest, maybe linearization is valid at criticality; i.e., when the "break" (see [2], p. 84) in the heat flux curve occurs because of the sudden supplementation of heat conduction by convection. However, visual observation occurs somewhat later, and it is moot if linearization is valid when the patterns become visible. Indeed, very close to the lateral boundary $\mathbf{q}$ is noticeably nonzero: therefore, the spatial variation of $\mathbf{q}$ cannot be assumed to be of the same order of smallness as $\mathbf{q}$. That is why one deletes this region by the hypothetical boundary displacement just mentioned. However, we might still be left with some (say, isolated for the sake of simplicity) points of $\mathrm{int}(\mathscr{L})$, where linearization is suspect; e.g., stagnation points, or random, but pointlike, disturbances that pop up (cf. [2] p. 30) in an otherwise uniform critical temperature field. With this as our motivation, we show next that the above conclusions are virtually unchanged, even under the *weaker linearization hypothesis* that the equations of Section 1 are valid only away from some, unspecified in advance, but isolated points.

7. Singular points

A polygon $P \subset \mathbb{R}^2$ will be called *weakly Bénard* if one can find a solution ϕ of (8), having possibly some *isolated singular points*, which is valid in a neighborhood of **P**, and which satisfies (9) at all the ordinary points of the edges of P. We now classify these.

Of course Proposition 1 is unchanged if all points of the ball Ω are ordinary points of ϕ. Furthermore, an analogue holds when the center of Ω is the sole singular point of ϕ in Ω, provided L is a *ray* through the center. Thus, *one still has symmetry of ϕ with respect to reflections in its walls*, except that one must now allow that some *walls may be rays or segments*; i.e., are terminating at one or both ends in singular points. (Here use is made also of the fact that after deleting the singular points, a connected open set remains connected, so any analytic function is still determined uniquely by its restriction to any open subset.) Applying this symmetry we see that for any $v \in \mathbb{R}^2$, either the number of walls through v is infinite, or *the angle between adjacent walls through v is an integral divisor of* 2π (instead of π before). Using this, we now obtain the following generalization of Proposition 2, where, again, *primitive* means that no walls of ϕ pass through the interior of **P**.

PROPOSITION 4. *A primitive weakly Bénard polygon must be one of the four Bénard triangles, or a rectangle, or one of the quadrilaterals and pentagons of Figure* 4 *or a regular hexagon.*

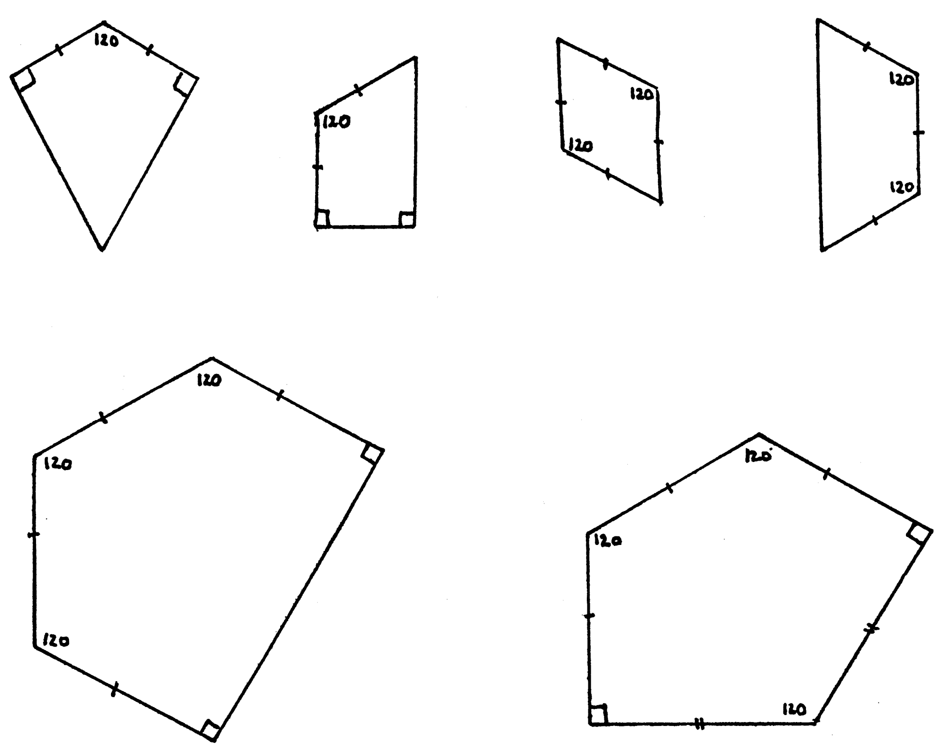

Figure 4.

Proof: Let $P \subset \mathbb{R}^2$ be weakly Bénard with ϕ a solution of (8), valid in a neighborhood of **P** except for some isolated points, which satisfies (9) on the ordinary points of the edges of P, and none of whose walls passes through the interior of **P**. We first check that if P has n sides, then $3 \leq n \leq 6$, and that the internal angles of P, arranged in nonincreasing order, are $(2\pi/m_1, \ldots, 2\pi/m_n)$, with $(m_1, \ldots, m_n)$ being one of the following nondecreasing n-tuples of positive integers (listed in "lexicographic" order for each n).

$n = 6 : (3, 3, 3, 3, 3, 3)$.

$n = 5 : (3, 3, 3, 3, 6)$ **or** $(3, 3, 3, 4, 4)$.

$n = 4 : (3, 3, 4, 12), (3, 3, 6, 6), (3, 4, 4, 6)$ **or** $(4, 4, 4, 4)$.

$n = 3 : (3, 7, 42), (3, 8, 24), (3, 9, 18), (3, 10, 15), (3, 12, 12), (4, 5, 20),$
$(4, 6, 12), (4, 8, 8), (5, 5, 10)$ **or** $(6, 6, 6)$.

To see this, recall that the sum of the n internal angles of P is $(n-2)\pi$. Because each of these internal angles is of the type $2\pi/m_i$, $m_i \geq 3$, the n-tuple $(m_1, \ldots, m_n)$ of these integers must satisfy

$$\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_n} = \frac{n-2}{2}. \tag{13}$$

Each of the fractions on the left side of (13) is at most 1/3, so we must have $n \cdot 1/3 \geq (n-2)/2$; i.e., $n \leq 6$; and for $n = 6$, all the six m_is must be 3. For $n = 5$, the right side of (13) is 3/2, and, because $2.1/3 + 3.1/4 < 3/2$, at least three of the five m_is should be 3. When exactly three are 3, the other two m_is must be both 4 in order for all the five fractions to add up to 3/2. In case four m_is are 3, the fifth must be the reciprocal of $3/2 - 4 \cdot (1/3)$; i.e., 6. Proceeding likewise for $n = 4$ and $n = 3$, one gets all the possibilities listed above.

We note now that *the two arms of an internal angle of the type $2\pi/m$, with m odd, must be both of the same length*, because continuing ϕ by reflections around the vertex, we must be able to arrange an odd number of copies of this angle around the vertex symmetrically. This implies, for $n = 3$, that if the 3-tuple has one odd entry, then the other two entries should be equal. Therefore, of the triangles, only $(3, 12, 12)$, $(4, 6, 12)$, $(4, 8, 8)$, and $(6, 6, 6)$ remain; i.e., the four Bénard triangles. It is easily seen that there is no quadrilateral $(3, 3, 4, 12)$ with the two arms of both 120° angles of same length; there are two $(3, 3, 6, 6)$s and two $(3, 4, 4, 6)$s as drawn above; and, of course, the rectangles $(4, 4, 4, 4)$. There is no pentagon with four 120° angles, all with both arms equal, so no $(3, 3, 3, 3, 6)$; on the other hand, there are two pentagons $(3, 3, 3, 4, 4)$ satisfying this arms condition. Finally, the arms condition implies that a $(3, 3, 3, 3, 3, 3)$ must have all six arms equal. ■

To classify weakly Bénard fluid patterns it is necessary to decide which of the above cases actually occur as primitive weakly Bénard polygons. In any case, looking at Figure 1, it is easily seen that each polygon of Figure 4 is located within this $(30^\circ, 60^\circ, 90^\circ)$ tiling, and generates, by reflection, a coarser subtiling by congruent pieces.

COROLLARY. *All weakly Bénard polygons are Bénard.*

To see this, note that because there are only finitely many singular points of ϕ in a neighborhood of **P**, the argument of Proposition 3 generalizes almost verbatim, to show that any nontrival weakly Bénard polygon has only finitely many walls of the nontrival ϕ penetrating it, and so can be dissected, by means of these finitely many walls, into finitely many pieces. Being primitive, each of these pieces must be one of the possibilities given by Proposition 4, and also all pieces must be congruent to each other, because any two sharing an edge are symmetric with respect to reflection in this edge.

8. Noneuclidean polygons

Consider now a P on S^2, the unit sphere of $\mathbb{R}^3$, whose edges are great circle arcs. Using spherical coordinates $0 \le \psi < \pi$ and $0 \le \theta < 2\pi$, and the spherical Laplacian, we now replace (8) by

$$\frac{1}{\sin\psi}\frac{\partial}{\partial\psi}\left(\sin\psi\frac{\partial\phi}{\partial\psi}\right) + \frac{1}{\sin^2\psi}\frac{\partial^2\phi}{\partial\theta^2} = -\lambda^2\phi \tag{14}$$

and call P a *spherical Bénard polygon* if there exists, for some $\lambda \neq 0$, a solution ϕ of this partial differential equation, which is valid in a neighborhood of **P** (one of the two closed regions of S^2 bounded by P) and obeys the condition $\partial\phi/\partial\mathbf{n} = 0$ (now $\mathbf{n}$ is tangent vector normal to edge) identically on the edges of P. We have checked the "only if" part of the following: *P is a spherical Bénard polygon if and only if it is formed by the edges of a tiling of S^2 by congruent spherical triangles having angles* $(90^\circ, 60^\circ, 60^\circ)$*,* $(90^\circ, 60^\circ, 36^\circ)$*,* $(90^\circ, 45^\circ, 30^\circ)$*, or* $(90^\circ, 90^\circ, \pi/t)$*, where t is any integer* ≥ 2.

The point now is that the sum of the n internal angles of a spherical n-gon exceeds $(n-2)\pi$—the excess being the spherical area of the n-gon—and so, for P primitive, we are looking for n integers $m_i \geq 2$ such that the sum of their reciprocals exceeds $n-2$. This forces $n = 3$, and for this n, the only nondescending 3-tuples of integers obeying this condition are $(2, 2, t)$, $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$, which gives the triangles listed. In each case, the area of the triangle turns out to divide 4π, a necessary condition that it tile S^2. Indeed, a little reflection shows that all the triangles do tile the sphere; e.g., $(90^\circ, 60^\circ, 60^\circ)$ triangles result if we subdivide a "cubical subdivision" further by diagonals of all the 4-gons, and (90°, 60°, 36°) triangles result if we subdivide the 5-gons of a "dodecahedral subdivision" further by 10 rays through their barycenters, and so forth. To prove the "if" part, one needs to find explicit ϕs whose great spherical walls have one of these possible geometries. This should be possible; e.g., by looking at the "metaharmonic functions" of [5] (see Appendix II of [5]), but we have not yet done this checking.

For a hyperbolic geometry the situation is slightly different, because now the sum of the n internal angles falls short of $(n-2)\pi$ by the hyperbolic area. Therefore there are many more possibilities. However, with some additional conditions we can get finite lists of primitive *hyperbolic Bénard polygons*. For example, if we confine to tilings that subdivide a "Fuchsian tiling of the first kind of genus g" (the pairwise identification of the edges of the tilings gives a surface of genus g—recall $g \geq 2$—and the quotient map should not be branched), then the sum of the internal angles is precisely $(n-2g)\pi$, and we can make finite lists analogous to those above.

Using the open unit disk $\mathbb{H} = [(x, y) : r < 1] \subset \mathbb{R}^2$ with the Riemannian metric $16(1-r^2)^{-4}(dx^2+dy^2)$ as our model for this geometry we are looking for a polygon $P \subset \mathbb{H}$ whose edges are segments of circles normal to the bounding circle $\partial\mathbb{H}$, such that there is a solution ϕ of

$$\frac{1}{4}(1-r^2)^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = -\lambda^2\phi \tag{15}$$

valid in a neighborhood of the compact region $\mathbf{P} \subset \mathbb{H}$ enclosed by P, which has zero normal derivative on all the edges of P. It seems that these hyperbolic patterns should arise experimentally if one observes convection when the temperature gradient tapers off as a function of r. Likewise, the spherical polygons should be closely related to the convection in spherical layers studied by Busse et al. (see [4]).

The angular conditions indeed should hold if we move over to the general case of any Riemannian geometry $ds^2 = E(x, y)dx^2 + 2G(x, y)dxdy + F(x, y)dy^2$ as long as we confine ourselves, as in the two homogenous cases above, to *geodesic polygons*. This is so, because these conditions were derived using only some infinitesimal symmetries. However, because there are no global symmetries now, the assertions regarding tiling will become meaningless in this general context.

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References

1. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Oxford, New York, 1961, Dover, New York, 1981.
2. E. L. Koschmieder, *Bénard Cells and Taylor Vortices*, Cambridge, New York, 1993.
3. L. Hormander, *Linear Partial Differential Operators*, Springer, New York, 1969.
4. M. Golubitsky, I. Stewart, and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory, vol. II*, Springer, New York, 1988.
5. I. N. Vekua, *New Methods for Solving Elliptic Equations*, North-Holland, New York, 1967.

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