Cacti and Mathematics

by

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When I was asked to contribute a piece to this Souvenir, which is being brought out by the National Cactus and Succulent Society of India, on the occasion of its Twenty-Ninth Annual Cactus and Succulent Plants Show, I was quite reluctant. Frankly, I thought that this would merely amount to my usurping some valuable magazine space from some botanist or cactus-hobbyist. Yes, I do take great pride in being a son of Dr. J. S. Sarkaria, a founder and the moving spirit of this society for thirty years[1], as well as the explorer of many Indian succulent plants[2], and the creator of the famed National Cactus and Succulent Garden and Research Center[3] in Sector 5, Panchkula, but the simple fact of the matter is that the focus of my own interests lies elsewhere, in Mathematics.

However, once I had agreed to make a contribution, I decided my effort should be to try to bridge this difference, by pointing out some commonalties between these two apparently distant disciplines. Most cactophiles will tell you that what attracted them to Cacti in the first place was their ***form***, the fascinating patterns that these beautiful plants present to our eyes. A fascination for form is also what attracts geometers to Geometry; and, as you know from school, Geometry is about one-half of Mathematics already!

Forms and patterns are all around us, for example, in those *hexagonal tilings* that adorn many a floor. In Cacti, one sees tilings too, but these are now not tilings of the plane, but of the *closed* surfaces of these plants. Moreover, they are not *as* symmetric as the plane example that I just mentioned. Depending on the genus and species, one notices varying degrees of symmetries in the shapes and sizes of the tiles. You will be gratified to know that, for long, there have been mathematicians who have pondered over the nature, and the possible physical reasons for these forms. One of the current leaders in this esoteric field is Alan Newell of – very appropriately! – the University of Arizona. The SIAM (= Society for Industrial and Applied Mathematics) NEWS, "Plant Patterns and Phyllotaxis," October 26, 2004, at http://www.siam.org/news/news.php?id=263, reports on some researches of Newell with a younger colleague, and has some interesting photographs, for example, one of Arizona's well-known Saguaro cactus, but the picture that intrigued me the most was the following, which was captioned, '*hexagons on a cactus of the genus Matucana*' ? [4]

Fig. 1. *Matucana* ??

Leaving its full botanical identification for later, let me turn now to a mathematical question suggested by this plant: ***is it possible to arrange finitely many six-sided tiles in such a way that there are three tiles at each corner?*** Note that here a *tile* need not be flat, nor its *sides* straight or of equal lengths, all we insist is that they have no holes, and we assume the same for the closed surface as a whole.[5] The picture suggests that the answer should be 'yes', moreover we know for a fact already (those floor tilings again!) that *infinitely* many hexagons can be arranged in this fashion. Despite all this evidence, I'll prove below that the answer to this question is in fact **'no'!**

I'll however not start from scratch, but invoke a beautiful theorem of Euler[6] which applies to *any tiling* of any closed surface without holes. It says that one *always* has V – E + F = 2, where F denotes the number of tiles, E the number of sides, and V the number of corners. For example, for the tiling shown below one has V = 9 (the nine corners a, b, c, d, e, f, g, h and k), E = 14 (the four sides of the top, the five of the bottom, and five vertical sides) and F = 7 (the 5-cornered bottom tile, five 4-cornered tiles including the one on the top, and a triangular tile at the back), and sure enough, one does have 9 – 14 + 7 = 2.

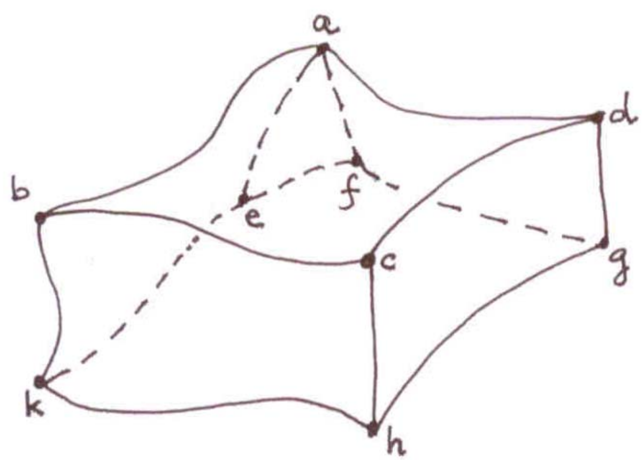

Fig. 2

Likewise, you can draw as many, and as complicated, tilings as you like, and you will always find like magic that V – E + F = 2 holds in each and every case! Since this is so easy to do, you should certainly make some more verifications. If you do, you will soon be convinced that Euler's theorem is true, but I emphasize that this alone will not prove it – *in mathematics even a million swallows may not a summer make!* – in full generality. Who knows, there might be another tiling for which the formula fails? Anyway, now that we have at least understood the statement of this remarkable theorem, I will return to the mathematical question suggested by the unidentified ‘Matucana’.

Assume that F hexagonal tiles can be arranged so that there are three tiles at each corner, and let E and V denote the number of sides and corners of this tiling. Each of the F tiles has 6 sides, and each of the E sides belongs to 2 tiles (the two on either side of it in the assumed tiling) so we have $6F = 2E$. Moreover, since each of the V corners belongs to 3 sides of the assumed tiling, and each of the E sides has 2 corners, we also have $3V = 2E$. Therefore, we must have $6V – 6E + 6F = 4E – 6E + 2E = 0$, that is, $V – E + F = 0$. On the other hand, Euler’s theorem tells us that we should have $V – E + F = 2$. This contradiction shows that our initial assumption is untenable, i.e., no such tiling is possible.

This argument gives in fact more: **we can find all *uniform* tilings**, where by uniform I mean that each tile has the same number p – where p is at least 3 – of sides, and at each corner of the tiling one has the same number q – where q is at least 3 – of tiles. Arguing just as before we have $pF = 2E$ and $qV = 2E$ in addition to (our closed surface has no holes) Euler’s formula $V – E + F = 2$. So $(2/q)E – E + (2/p)E = 2$, which gives $E = 2pq/(2p – pq + 2q)$, which shows, since E is positive, that $2p – pq + 2q = 4 – (p – 2)(q – 2)$ must be positive, i.e., the product of $p – 2$ and $q – 2$ must be less than 4. A very strong condition indeed, for obviously, there are *only five possibilities*: $\{p, q\} = \{3, 3\}, \{3, 4\}, \{4, 3\}, \{3, 5\}$ or $\{5, 3\}$! The free-hand drawings of Fig. 3 show that *all these five possibilities occur*, for example, when $p = 5$ and $q = 3$, then $E = 2.5.3/(2.5 – 5.3 + 2.3) = 30$, $F = 2.30/5 = 12$ and $V = 2.30/3 = 20$, and the ‘dodecahedron’ shown is such a tiling. (Had we allowed p and q to be arbitrary, i.e., 2 or bigger, then we would also have obtained a ‘uniform tiling’ for each {p,q} such that p or q is equal to 2.) These five tilings were known to Plato; indeed, it is shown in Euclid[7] that here one can insist that the tiles be regular p-gons, for example, his dodecahedron is made from 12 regular pentagons.

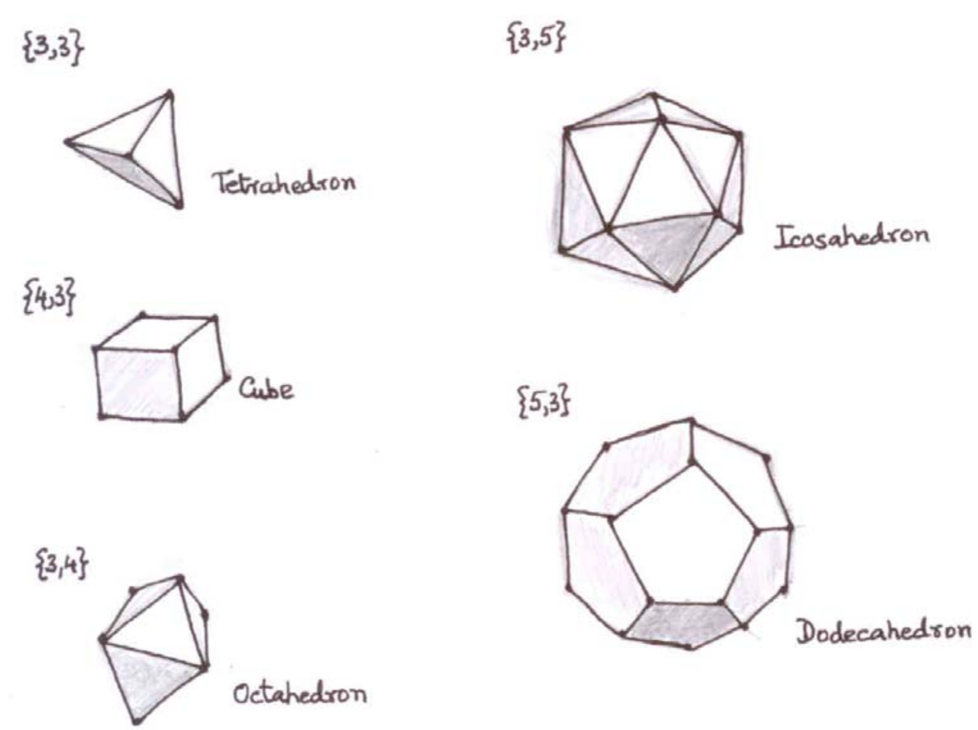

Fig. 3. Platonic Solids

So, starting with a question that the 'Matucana' of Figure 1 had provoked, we have learnt some pretty mathematics rather quickly (and more is mentioned in some Notes). Its time to go back to the botanical identification. Here too the answer is unexpected: *Figure 1 shows hexagons on a* ***pine-cone****, not on any cactus!* Apparently, even the very highbrow SIAM website had goofed, and put a wrong caption on a picture provided by the two researchers. The mystery cleared up when I saw a subsequent paper[8] by them where the same picture is given with the right caption. The moral of this teeny tale is *only* that one should double-check just about everything one sees on the web, *not* that there are no cacti with nice polygonal patterns. There are in fact quite a few, including the following cute species, which is named oh so aptly!

Fig. 4. *Tephrocactus geometricus*

Notes

1. For the inception of this society, see the "Editorial" on page 1 of the innaugural issue Number 1 of the *Journal of National Cactus & Succulent Society of India,* Volume I (1981). Dr. J. S. Sarkaria was the editor and the main contributor to this journal for many years. (Another founder, Prof. S. P. Bhandari's article, "Cactus Hobby—Reminiscences," is on pages 44-49 of this innaugural issue.) The first Annual Show was organized almost single-handedly by my father in the Panchayat Bhavan in 1979, and was the turning point in this society's history. Its roaring success proved the pessimists wrong and lent wings to the fledgling society! No account of this event seems available in the society's papers, but one does have "Glimpses of the 9th Cactus and Succulent Plants Show, 1987," on pp. 20-26 of Volume VII of this journal.

2. See Dr. J. S. Sarkaria, "Indian Succulent Log—Parts I, II, III, IV, V, VI, VII, VIII, IX, X," *Journal N.C.S.S.I.,* Volume I (1981) 64-67, Volume II (1982) 31-37, 62-66, Volume III (1983) 17-19, 76-82, Volume IV (1984) 35-40, 74-80, Volume V (1985) 66-79, Volume VI (1986) 44-64, Volume VII (1987) 47-60. This series of ten papers gives an account of these field explorations, in particular, from Part VIII, p. 67, and Part

X, p. 51, we learn that the new succulent species, *Caralluma sarkariae* and *Caralluma bhupinderana* – named after my mother Bhupinder Kaur – were discovered by him on 5.1.1976 and 11.10.1978 respectively.

3. Much earlier, in 1979-1980, Dr. J. S. Sarkaria had created a smaller cactus garden in Jammu, a brief account is given in P. K. Wattal, "A New Cactus Garden – Jammu University New Campus," *Journal N.C.S.S.I.*, Volume II (1982) 68-69. In "Progress Report on National Cactus, Succulent Gardens and Research Centre, Sector-5, Panchkula, India," *Journal N.C.C.S.I.*, Volume VIII (1990) 35-44, Dr. J. S. Sarkaria had sketched a description of the Panchkula cactus garden, for the benefit of the society-members, even before this garden was thrown open to the general public. Later on, he developed this preliminary sketch into a full-length book, that was almost complete when he passed away on August 15, 2004. This scientific book, entitled *National Cactus and Succulent Garden and Research Center* (2005), viii + 248 pp., is available on-line at http://www.geocities.com/sarkaria_2000. It gives valuable information about this garden, including in particular, the botanical names of almost all of the hundreds of species that the garden has in its vast collection, and the precise mounds and/or glasshouses where each species can be seen. (An earlier booklet of Dr. J. S. Sarkaria, *Stapelieae Collection Record* (1991), 23 pp., lists – with precise field location for each specimen – all the species of the genera of this family, also called Asclepiadaceae, that were in this garden by 15.7.91.) Shortly after his demise, the society and his family members had requested H.U.D.A. in writing that the Panchkula cactus garden be re-named to honour the memory of its eminent creator Dr. J. S. Sarkaria; regrettably, no action has been taken on this request so far, despite many reminders.

4. This SIAM NEWS discusses the paper of P.D. Shipman and A.C. Newell, "Phyllotactic patterns on plants," *Physical Review Letters*, 92:168102 (2004) 1-4. On p. 4 of this paper, the authors cite the 1991 book, *Cacti*, by Innes and Glass, and write, 'Fine examples of hexagonal configurations are given by the species *Matucana krahnii* (p. 186) and *Tephrocactus geometricus* (p. 287)'. This profusely illustrated book is available in Panchkula cactus garden's small library – also donated by Dr. J. S. Sarkaria – so I have been able to see these two pages. Each has one picture with a polygonal pattern, but on page 186, it is *Matucana madisoniorum* on the top right of the page which is such (not *krahnii* on the bottom right) and both are very different from this curious Figure 1. Unfortunately, there is no plant of the genus *Matucana* of Cactacae in the Panchkula garden; however, there are some species of the genus *Tephrocactus*, but not *geometricus*.

5. Let me clarify this *vital hypothesis*: any closed surface has the two-dimensional hole occupied by the material body bounded by it, what we want is that there be *no one-dimensional hole*, i.e., it should be possible to shrink any loop on the surface to a point without leaving it (the tiles however should have no holes at all). The following **surfaces of a doughnut, a coffee cup, or a lump of plasticine with three holes**, are examples of closed surfaces *not* satisfying this vital hypothesis, because the indicated loops in Figs. 5a-5c cannot be so shrunk. For any F three or bigger, the surface of a doughnut *can* be tiled by F hexagonal tiles in such a way that there are three tiles at each corner. This is left as an exercise, Fig. 5d is a generous hint for the case F = 3, the three tiles being A, B and C (after you have solved this problem you should muse over Fig. 5e also).

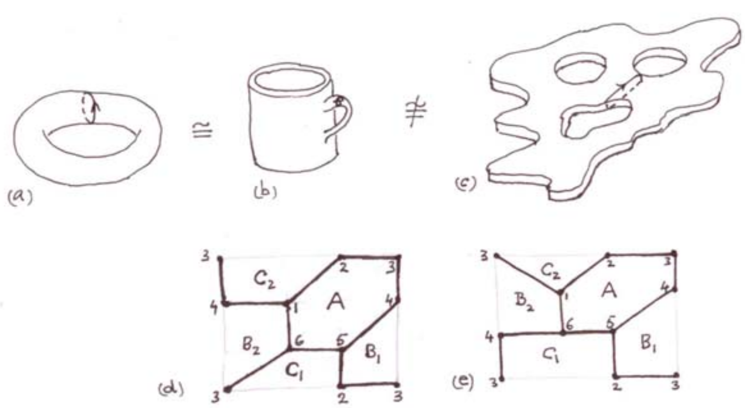

Fig. 5

6. For more on Euler (1707-1783, pronounced 'Oiler') see E. T. Bell, *Men of Mathematics*, a must-read for anyone into mathematics. Euler made immortal contributions to all of mathematics, in particular, his work led to the creation of topology in the 1800's, a subject that has been central in the mathematics of the last 100 years. **Question:** ***what is a topologist?*** **Answer:** ***a topologist is someone who can't tell between his doughnut and coffee-cup!*** So true: X and Y are called *topologically equivalent* if there is a one-one onto relation between the points x of X and the points y of Y which is continuous both ways, and topologists can indeed 'see' such a relation between a doughnut and coffee-cup at once. For others, it is a boon that a weaker Euler's theorem is true even without the vital hypothesis: V – E + F *is the same for all tilings of the same surface.* This number, called the **Euler characteristic** of the surface, is obviously the same for topologically equivalent surfaces. What is not at all obvious is that the converse is also true: **surfaces are topologically equivalent if and only if their Euler characteristics are the same!** This is very useful, for it is so very easy to compute the characteristic: just take *any* convenient tiling of the surface and count its tiles, sides and corners! Using, for example, the three hexagon tiling, now V = 6, E = 9 and F = 3, we see that the surface of the doughnut has characteristic 6 – 9 + 3 = 0; similar easy calculations, which we leave to the reader, show that the surface of the coffee-cup also has characteristic 0, but the surface of the plasticine blob with three holes has characteristic – 4. So the first two are topologically equivalent, while the last is topologically distinct. A more general, but still quite easy, calculation shows that the surface of a plasticine blob with g holes has characteristic 2 – 2g. What is not at all that easy is that, **any closed surface is topologically equivalent to the surface of a plasticine blob with some holes**. Thus we have a topological classification of the closed surfaces of three dimensional space; similar higher dimensional (example: Fig 5e represents a closed surface of 4-dimensional space!) problems is what pre-occupies modern-day topology the most.

7. "In Euclid" (pronounced 'You-klid') means "somewhere in *The Elements of Euclid*," a treatise (circa 250 B.C.) that has influenced rational thought more than anything else. As Euclid himself emphasized, almost all that he presented in it is due to *other* Greek mathematicians who preceded him, above all, Eudoxus (pronounced 'You-dox-us'). The logical rigour of this treatise is unrelenting and astonishing even by today's standards: that is why it is was so influential and is so admired, but that is also why it is *not* for the beginner! The construction of the Platonic bodies is in the last volume and you have to absorb a lot from the ones before it to absorb these proofs. A *much* better option is H.S.M. Coxeter, *Introduction to Geometry* (1969), a delightful book that contains a wealth of geometric information in its 450 odd pages. An incredible amount of mathematics is connected to the Platonic solids, indeed, a lot is tied to the regular pentagon! Define the **golden ratio** τ to be the diagonal of a regular pentagon of side 1, now take any **golden rectangle**, i.e., one with length/breadth = τ, and snip off a square so that you are left with a rectangle, then this residual rectangle is also golden! (Golden ratio and rectangle have a huge fan following amongst painters and architects, for example, the designers of 'City Beautiful' were completely hooked on them, and you can spot tons of golden rectangles in the facades of many Chandigarh buildings.) So this snipping process can be continued endlessly, which would definitely not be the case if τ were a ratio of two whole numbers, so τ *cannot be written as a finite fraction,* i.e., it is irrational. Symbolically, the snipping property reads $\tau = 1 + 1/\tau = 1 + 1/(1 + 1/\tau) = 1 + 1/((1 + 1/(1 + 1/\tau)) = \dots$, so τ *equals the infinite continued fraction that uses only* 1's! Computing its successive initial segments, we get 1/1, 2/1, 3/2, 5/3, 8/5, 13/8 … , i.e., the successive ratios of **Fibonacci's sequence** 1, 1, 2, 3, 5, 8, 13, … (remember this from the *Da Vinci Code*?) which nature just loves! For example, the **phyllotaxis** (literally "leaf arangement") of many plants follows rules encoded in this sequence, in particular, from pages 169-172 of Chapter 11, "The golden section and phyllotaxis," of Coxeter's book you can find out how *the hexagonal scales of a pineapple are geometrically arranged per simple mathematical rules that can be formulated in terms of Fibonnaci's sequence*! As is to be expected, some complications and new frills appear as one delves further into phyllotaxis – as Newell and Shipman do – and there remains much in this fascinating subject that is still not properly understood.

8. See P.D. Shipman and A.C. Newell, "Polygonal planforms and phyllotaxis on plants," *Journal of Theoretical Biology* 236 (2005) 154-197, the photograph in question (i.e. Fig. 1 of the present paper) is Fig. 1(b) on p. 155, and bears the caption 'hexagons on a pine cone'.

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