DIFFERENTIAL GEOMETRY

§1. Isometries of $\mathbb{R}^3$. Elements (x_1,x_2,x_3) of $\mathbb{R}^3$ will be called vectors and written as $\vec{x}$. We equip $\mathbb{R}^3$ with the usual dot product $\vec{x}.\vec{y} = x_1y_1+x_2y_2+x_3y_3$; the norm is defined by $|\vec{x}| = \sqrt{\vec{x}.\vec{x}}$; the distance between 2 points of $\mathbb{R}^3$ by $|\vec{x}-\vec{y}|$. An isometry $f: \mathbb{R}^3 \to \mathbb{R}^3$ is any function which preserves distance (i.e. $|f(\vec{x})-f(\vec{y})| = |\vec{x}-\vec{y}|$ for all $\vec{x},\vec{y} \in \mathbb{R}^3$). We will prove that f is an isometry of $\mathbb{R}^3$ iff it is of the form $f(\vec{x}) = \vec{a} + L(\vec{x})$ where $\vec{a} \in \mathbb{R}^3$ and $L \in O(3)$; here $O(3)$ denotes the group of all linear automorphisms of $\mathbb{R}^3$ which preserve dot product. 'If' is trivial. To see 'only if' we take $\vec{a}=f(\vec{0})$ and show that the isometry g defined by $g(\vec{x})=f(\vec{x})-\vec{a}$ is an element of $O(3)$. Clearly g preserves terms on the right of $2\mathbf{x}.\mathbf{y} = |\mathbf{x}|^2 + |\mathbf{y}|^2 - |\mathbf{x} - \mathbf{y}|^2$, so g preserves dot product also. So g takes the canonical orthonormal basis $\vec{e}_1,\vec{e}_2,\vec{e}_3$ into an orthonormal basis. For any $\vec{x} = \sum_i(\vec{x}.\vec{e}_i)\vec{e}_i$ it follows that $g(\vec{x}) = \sum_i\big(g(\vec{x}).g(\vec{e}_i)\big)g(\vec{e}_i) = \sum_i(\vec{x}.\vec{e}_i)g(\vec{e}_i)$ (since g preserves dot products); thus g is also linear. In the following 2 sets $A,B \subseteq \mathbb{R}^3$ will be called congruent if there exists an isometry f of $\mathbb{R}^3$ such that $f(A)=B$.

§2. Curves in $\mathbb{R}^3$. We will assume that each scalar (or 'parameter') u takes its values over some interval $I_u \subseteq \mathbb{R}$. A smooth vector valued function $\vec{r}(u)$ of a scalar u will be called a curve provided $\dot{\vec{r}}(u)$ (the derivative) is never zero. If the scalar u is a smooth function of another scalar w with $\frac{du}{dw}$ always non-zero, then the curve $\vec{r}(w)=\vec{r}(u(w))$ is said to be obtained by a reparametrisation; the reparametrisation is called orientation preserving if $\frac{du}{dw} > 0$. Given a curve $\vec{r}(u)$ the tangent line at the point 'u' will be the line through $\vec{r}(u)$ parallel to the vector $\dot{\vec{r}}(u)$. Note that the tangent line at the corresponding point 'w' of a curve $\vec{r}(w)$ obtained by a reparametrisation is the same. The unit tangent vector at the point 'u' is defined to be $\frac{\dot{\vec{r}}(u)}{|\dot{\vec{r}}(u)|}$. Note that the unit tangent vector at the point 'w' of a curve $\vec{r}(w)$ obtained by an orientation preserving reparametrisation is the same. Most of the definitions to be given will have an analogous nature: they will be invariant under all (or at least all orientation preserving) reparametrisations.

Given a point 'u_0' on a curve $\vec{r}(u)$ the arc length from 'u_0' to any other point 'u' will be defined by $s(u) = \int_{u_0}^{u}|\dot{\vec{r}}(u)|\,du$. (This formula shows that $|\dot{\vec{r}}(u)| \equiv 1$ only if $s=u+\text{const.}$). Note that $s(u)$ is a smooth function of u with $\frac{ds}{du} > 0$; so the inverse function $u(s)$ is also of the same type. For the reparametrised curve $\vec r(s)=\vec r(u(s))$ differentiation w.r.t. s will be indicated by primes. Note that $\vec r\,' = \dot{\vec r}\frac{du}{ds} = \frac{\dot{\vec r}}{|\dot{\vec r}|}$ is precisely the unit tangent vector $\vec t$. One should note that if the curves $\vec r(u), \vec r(w)$ are related by an orientation preserving reparametrisation then $s(u)=s(v)$ upto an additive constant (if the base points correspond i.e. $w_0 = w(u_0)$ one has exact equality). Due to this fact it follows that definitions made by going over to the parameter s and using derivatives w.r.t. s are automatically invariant under orientation preserving reparametrisations (e.g. per this principle we could have defined the unit tangent vector by $\vec t = \vec r\,'$). If $\vec r(u), \vec r(w)$ are related by an orientation reversing parametrisation then $s(u)=-s(w)$ upto an additive constant: this fact shows that definitions involving second, fourth etc. derivatives w.r.t. s only are infact invariant under all reparametrisations. As an example one has the *curvature vector* $\vec r\,''$ i.e. $\vec t\,'$. Note (from $\vec t.\vec t = 1$) that this vector is perpendicular to the unit tangent vector. The length of this vector is denoted by κ, the *curvature* and (in case $\kappa \neq 0$) the *unit normal vector* $\vec n$ is then defined by $\vec t\,' = \kappa\vec n$. We note that *curvature is zero iff $\vec r(s) = \vec a s + \vec b$* for some constant vectors $\vec a, \vec b$; i.e. iff the curve is a straight line. Thus the curvature measures the curve's departure from rectilinearity. *From now on it will be assumed that $\kappa(s)$ is never zero*. The *unit binormal vector* $\vec b$ is defined to be $\vec t \times \vec n$. It is easy to see that its derivative w.r.t. s must be parallel to $\vec n$: $\vec b.\vec b = 1$ yields $\vec b\,'.\vec b = 0$ and $\vec b\,'.\vec t = 0$ follows from $0 = (\vec b.\vec t)' = \vec b\,'.\vec t + \vec b.\kappa\vec n = \vec b\,'.\vec t$. We define the *torsion* $\tau(s)$ by $\vec b\,' = -\tau\vec n$. Note that *torsion is zero iff $\vec r.\vec b$ = const. for some constant unit vector $\vec b$*. Infact if $\tau = 0$, the unit binormal vector $\vec b$ is constant and $\vec r\,'.\vec b = 0$ integrated yields reqd. result; converse is equally obvious. Thus torsion measures the departure of a curve from planarity. Next we note that we have the following *Frenet-Serret formulae: $\vec t\,'(s) = \kappa(s)\vec n(s)$, $\vec n\,'(s) = \tau(s)\vec b(s) - \kappa(s)\vec t(s)$ and $\vec b\,'(s) = -\tau(s)\vec n(s)$*. The first and the third have already been discussed. To prove the second we note that $\vec n = \vec b \times \vec t$; on differentiating this we get $\vec n\,' = \vec b\,' \times \vec t + \vec b \times \vec t\,' = -\tau\vec n \times \vec t + \vec b \times \kappa\vec n = \tau\vec b - \kappa\vec t$, the required formula.

§3. *Congruence of curves*. Let $\vec r(s)$ be a curve parametrised by arc length. A curve of the form $\vec r_1(s) = \vec a + L(\vec r(s))$ where $\vec a$ is some constant vector and $L \in O(3)$ will be said to be congruent (see §1) to the first curve. We note that s is also arc length parameter for this new curve; this follows from $|\vec r_1\,'(s)| = |L(\vec r\,'(s))| = |\vec r\,'(s)| = 1$. Differentiation shows that the unit tangent vectors are related by $\vec t_1(s) = L(\vec t(s))$ and the curvatures by

$\kappa_1(s)\vec{n}_1(s) = \kappa(s)L(\vec{n}(s))$: this last eqn. is possible (take norms) only if $\kappa_1(s) = \kappa(s)$ and so $\vec{n}_1(s) = L(\vec{n}(s))$. Thus $\vec{b}_1(s) = L(\vec{b}(s))$; differentiating this and looking at magnitudes of the 2 vectors we get $\tau_1(s)=\tau(s)$. Briefly we have checked that <u>congruent curves have same curvature and torsion</u>. What is much more striking is that the converse of this result is also tru<u>let the scalar $s \in I_s$ be arc length parameter for 2 curves $\vec{r}(s)$ and $\vec{r}_1(s)$ a</u> <u>let $\kappa(s)\equiv\kappa_1(s)$ and $\tau(s)\equiv\tau_1(s)$; then the 2 curves are congruent</u>. Infact we will show that $\vec{r}_1(s) = \vec{r}_1(0) + L(\vec{r}(s) - \vec{r}(0))$ where the orthogonal transformatic L is fixed by the requirement that $\vec{t}_1(0)=L(\vec{t}(0))$, $\vec{n}_1(0)=L(\vec{n}(0))$ and $\vec{b}_1(0)= L(\vec{b}(0))$. Since the curve $\vec{r}_1(0) + L(\vec{r}(s)-r(0))$ is congruent to $\vec{r}(s)$ we use the previous result to see that our problem is reduced to the following: 2 curves $\vec{r}(s),\vec{r}_1(s)$ parametrised by arc length $s \in I_s$ are given and $\vec{r}(0)=\vec{r}_1(0)$, $\vec{t}(0)=\vec{t}_1(0)$, $\vec{n}(0)=\vec{n}_1(0)$, $\vec{b}(0)=\vec{b}_1(0)$; we need to prove $\vec{r}(s)\equiv\vec{r}_1(s)$. Compute $(\vec{t}.\vec{t}_1+\vec{n}.\vec{n}_1+\vec{b}.\vec{b}$ by Frenet's formulae: one gets $\kappa\vec{n}.\vec{t}_1+\vec{t}.\kappa\vec{n}_1+(\tau\vec{b}-\kappa\vec{t}).\vec{n}_1+\vec{n}.(\tau\vec{b}_1-\kappa\vec{t}_1)+(-\tau\vec{n}).\vec{b}_1+\vec{b}.(-\tau\vec{n}_1) = 0$. So each of the cosines $\vec{t}.\vec{t}_1, \vec{n}.\vec{n}_1, \vec{b}.\vec{b}_1$ is constant equal to 1 and one gets $\vec{t}\equiv\vec{t}_1, \vec{n}\equiv\vec{n}_1, \vec{b}\equiv\vec{b}_1$. In particular $\vec{t}\equiv\vec{t}_1$ i.e. $\vec{r}'(s)\equiv\vec{r}_1'(s)$ combined with $\vec{r}(0)\equiv\vec{r}_1(0)$ gives $\vec{r}(s)\equiv\vec{r}_1(s)\ \forall s$. We now close this circle of ideas by proving the following existence theorem. <u>Let $\kappa(s)>0$ and $\tau(s)$ be two smooth functions of a scalar $s \in I_s$; then there exists a curve $r(s)$ parametrised by arc length whose curvature is $\kappa(s)$ and torsion is $\tau(s)$</u>. Infact we will show the existence of such a curve with in addition $\vec{r}(0)=\vec{0}$, $\vec{t}(0)=\vec{e}_1$, $\vec{n}(0)=\vec{e}_2$ and $\vec{b}(0)=\vec{e}_3$. Frenet-Serret formulas tell us that if $\vec{t}(s)=(\alpha_1(s),\alpha_2(s),\alpha_3(s))$, $\vec{n}(s)=(\beta_1(s),\beta_2(s),\beta_3(s))$ and $\vec{b}(s)=(\gamma_1(s),\gamma_2(s),\gamma_3(s))$ then $(\alpha_i(s),\beta_i(s),\gamma_i(s))$, i= 1,2,3, is a solution of the system of differential equations $\frac{d\alpha}{ds} = \kappa(s)\beta(s)$, $\frac{d\beta}{ds} = \tau(s)\gamma(s) - \kappa(s)\alpha(s)$, $\frac{d\gamma}{ds} = -\tau(s)\beta(s)$ with initial conditions at s=0 prescribed to be (1,0,0), (0,1,0) and (0,0,1). By the existence theorem for differential equations such solutions indeed exist. We now define our curve by $\vec{r}(s) = \int_0 \vec{t}(s)\,ds$. The verification that the given $\kappa(s)$ and $\tau(s)$ are indeed the curvature and torsion can be carried out in a straightforward way once we verify that for all s, $\vec{t}(s)$, $\vec{n}(s)$ and $\vec{b}(s)$ indeed form an orthonormal triad. This amounts to checking that the matrix $\begin{bmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{bmatrix}$ is orthogonal.

One can check the orthonormality of the rows by computing $(\alpha_1^2+\beta_1^2+\gamma_1^2)'$ $= 2\alpha_1\kappa\beta_1 + 2\beta_1(\tau\gamma_1-\kappa\alpha_1) + 2\gamma_1(-\tau\beta_1)=0$; so $\alpha_1^2+\beta_1^2+\gamma_1^2$ is constant =1; likewise $\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2$ is constant =0 etc. etc. This completes the proof.

§4. Surfaces in $\mathbb{R}^3$. Let us say that a connected set $S \subseteq \mathbb{R}^3$ is a surface if for each point P of S we can choose some smooth function $\vec{r}(u,v)$, with $\frac{\partial\vec{r}}{\partial u}$ and $\frac{\partial\vec{r}}{\partial v}$ always linearly independent, which maps some region of $\mathbb{R}^2$ in a 1-1 way onto a neighbourhood of P. Such a function $\vec{r}(u,v)$ is said to be a parametrisation of a nhbd. of P. Given 2 surfaces a smooth map $f: S \to S^*$ is one which carries each such parametrisation $\vec{r}(u,v)$ to a smooth function $\vec{r}^*(u,v) = f(\vec{r}(u,v))$. Henceforth we will denote differentiation with respect to u by the suffix 1 and with respect to v by the suffix 2. The tangent space T_P at a point P with parameter values u,v is defined to be the subspace of $\mathbb{R}^3$ spanned by $\vec{r}_1(u,v)$ and $\vec{r}_2(u,v)$; it is easy to verify that this definition is independent of the manner in which a neighbourhood of P has been parametrised. (This notion is to be distinguished from 'tangent plane' at P which is the parallel affine plane through P: if no confusion is possible this too will be denoted by T_P). Given a smooth map $f: S \to S^*$ we define, for each $P \in S$, its linearization at P to be the linear map $f_P: T_P \to T_{P^*}$ (here $P^* = f(P)$) such that $\vec{r}_1 \mapsto \vec{r}_1^*$ and $\vec{r}_2 \mapsto \vec{r}_2^*$ (; again it is trivial to verify that this definition is independent of the parametrisation chosen for a nhbd. of P). If a unit vector $\vec{N}(P)$, normal to the tangent space T_P, has been assigned to each point $P \in S$, then it determines a Gauss map $f: S \to S^2$ by $f(P) = \vec{N}(P)$; here S^2 denotes the unit sphere $\{\vec{v} : |\vec{v}| = 1\}$. We will deal only with orientable surfaces i.e. those for which a smooth Gauss map can be fixed. (An open Möbius strip is an example of a non-orientable surface. It is known that all compact(=closed) surfaces in $\mathbb{R}^3$ are orientable). Note that a smooth Gauss map is uniquely defined upto sign; now f will denote such a map. The tangent plane to S^2 at $\vec{N}(P) = P^*$ is $\perp$ to $\vec{N}(P)$ and so $\parallel$ to the tgt. plane to S at P: thus the tangent space to S at P coincides with the tangent space to S^2 at P_*. Given $P \in S$ let $\vec{r}(u,v)$ be a parametrisation of a nhbd. of P and let $f(\vec{r}(u,v)) = \vec{N}(u,v)$; then the linearization f_P: $T_P \to T_P$ of the Gauss map is given by $\vec{r}_1 \mapsto \vec{N}_1$, $\vec{r}_2 \mapsto \vec{N}_2$. For each $P \in S$ the Gaussian curvature K_P is defined to be $\det f_P$ and the mean curvature μ_P to be $-1/2$ trace f_P. We will equip each tangent space T_P with the first fundamental form I defined by $I(\vec{v},\vec{w}) = \vec{v}.\vec{w}\ \forall\, \vec{v},\vec{w} \in T_P$. In terms of this +ve definite symmetric bilinear form the linearization f_P of the Gauss map can be interpreted as a bilinear form to be called the second fundamental form II: one has $II(\vec{v},\vec{w}) = (-f_P\vec{v}).\vec{w}$. It is important to note that this form is also symmetric. This follows because $-f_P(\vec{r}_i).\vec{r}_j = -\vec{N}_i.\vec{r}_j = \vec{N}.\vec{r}_{ji}$ for i, j=1,2. We now recall the standard way in which one can 'diagonalise' this symmetric bilinear form. For each $P \in S$ we choose $\vec{e}_a$ to be a unit vector $\in T_P$ at which the continuous function $II(\vec{v},\vec{v})$, restricted to unit vectors $\vec{v} \in T_P$, attains its maximum value κ_a; then choose $\vec{e}_b$ to be the unit vector $\perp \vec{e}_a$ such that $\vec{e}_a \times \vec{e}_b = \vec{N}(P)$ and put $\kappa_b = II(\vec{e}_b,\vec{e}_b)$. For any unit vector $\vec{v} = \cos\theta\,\vec{e}_a + \sin\theta\,\vec{e}_b$ one has <u>Euler's formula</u> $II(\vec{v},\vec{v}) = \kappa_a\cos^2\theta + \kappa_b\sin^2\theta$: this follows by noting that $II(v,v) = \kappa_a\cos^2\theta + 2II(\vec{e}_a,\vec{e}_b)\cos\theta\sin\theta + \kappa_b\sin^2\theta$ could attain its maximum at $\theta=0$ only if the derivative w.r.t. θ at $\theta=0$ (i.e. $2II(\vec{e}_a,\vec{e}_b)$) vanishes. Euler's formula shows that $II(\vec{v},\vec{v}) \geq \kappa_b$ for all unit vectors $\vec{v}$; thus κ_b is the minimum value of $II(\vec{v},\vec{v})$ as $\vec{v}$ runs over unit vectors. Note that $-f_P(\vec{e}_a) = \kappa_a\vec{e}_a$ and $-f_P(\vec{e}_b) = \kappa_b\vec{e}_b$. (Proof. Let $-f_P(\vec{e}_a) = x\vec{e}_a + y\vec{e}_b$; take dot products with respect to $\vec{e}_a$ and $\vec{e}_b$ to get $\kappa_a = x$ and $0 = y$; etc.). Hence Gaussian (resp. mean) curvature equals $\kappa_a\kappa_b$ (resp. $1/2(\kappa_a+\kappa_b)$). (One should note that the sign of μ depends on the "orientation"-i.e. the choice of the smooth Gauss map- with which S has been equipped; K is independent of orientation).

§5. <u>Theorema Egregium</u>. The <u>fundamental coefficients</u>, with respect to a local parametrisation $\vec{r}(u^1,u^2)$, are $g_{\alpha\beta} = I(\vec{r}_\alpha,\vec{r}_\beta)$ and $\Omega_{\alpha\beta} = II(\vec{r}_\alpha,\vec{r}_\beta)$. (Here and in the following each Greek suffix will have the possible values 1,2; further we will use the "dummy suffix convention": it will be understood that a summation has been performed over each Greek suffix which occurs more than once in any term). Alternatively we will sometimes write $u^1=u$, $u^2=v$ and use the symbols <u>E,F,G and L,M,N</u> for the fundamental coefficients g_{11}, g_{12}, g_{22} and $\Omega_{11}, \Omega_{12}, \Omega_{22}$ respectively. Note that $E>0$, $G>0$ and $\det[g_{\alpha\beta}] = EG-F^2>0$; the last follows by noting that $|\vec{r}_1\times\vec{r}_2|^2 = |\vec{r}_1|^2|\vec{r}_2|^2\sin^2\theta = |\vec{r}_1|^2|\vec{r}_2|^2(1-\cos^2\theta) = |\vec{r}_1|^2|\vec{r}_2|^2 - (\vec{r}_1.\vec{r}_2)^2 = EG-F^2$. The entries of the inverse matrix $[g^{\alpha\beta}]$ obey $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$. The <u>Christoffel symbols</u> $\Gamma^\alpha_{\beta\gamma}$ and $\Gamma_{\alpha\beta\gamma}$ are now defined by

(1) $$\vec{r}_{\alpha\beta} = \Gamma^\delta_{\alpha\beta}\vec{r}_\delta + \Omega_{\alpha\beta}\vec{N}$$

(-this makes sense because $\Omega_{\alpha\beta} = II(\vec{r}_\alpha,\vec{r}_\beta)$ equals $\vec{r}_{\alpha\beta}.\vec{N}$-) and $\Gamma_{\alpha\beta\gamma} = g_{\alpha\delta}\Gamma^\delta_{\beta\gamma}$. We can compute the matrix $[b^\beta_\alpha]$ of the linear map $+f_P : T_P \to T_P$ w.r.t. the basis $\vec{r}_1, \vec{r}_2$ by taking dot product of $\vec{N}_\alpha = b^\beta_\alpha\vec{r}_\beta$ with respect to $\vec{r}_\gamma$; this gives $-\Omega_{\alpha\gamma} = b^\beta_\alpha g_{\beta\gamma}$ i.e. $b^\beta_\alpha = -g^{\beta\gamma}\Omega_{\gamma\alpha}$. The 2 equations

(2) $$\vec{N}_\alpha = -g^{\beta\gamma}\Omega_{\gamma\alpha}\vec{r}_\beta$$

are called <u>Weingarten's equations</u>; they imply that $K = \det(-f_P) = |-g^{\beta\gamma}|.|\Omega_{\gamma\alpha}|$ i.e.

(3) $$K = \frac{LN-M^2}{EG-F^2}$$

Differentiating (1) w.r.t. u^γ and using Weingarten's equations (2) we can compute the triple derivatives

(4) $$\vec{r}_{\alpha\beta\gamma} = \vec{r}_\mu\{\Gamma^\mu_{\alpha\beta\cdot\gamma} + \Gamma^\delta_{\alpha\beta}\Gamma^\mu_{\delta\gamma} - g^{\lambda\mu}\Omega_{\alpha\beta}\Omega_{\gamma\lambda}\} + \vec{N}\{\Omega_{\alpha\beta\cdot\gamma} + \Gamma^\delta_{\alpha\beta}\Omega_{\delta\gamma}\}.$$

Now we make the simple but crucial observation that $\vec{r}_{\alpha\beta\gamma} = \vec{r}_{\alpha\gamma\beta}$. This yields

(5) $$\left(R^\mu_{\alpha\gamma\beta} \overset{def.}{=}\right) \Gamma^\mu_{\alpha\beta\cdot\gamma} - \Gamma^\mu_{\alpha\gamma\cdot\beta} + \Gamma^\delta_{\alpha\beta}\Gamma^\mu_{\delta\gamma} - \Gamma^\delta_{\alpha\gamma}\Gamma^\mu_{\delta\beta} = g^{\lambda\mu}(\Omega_{\alpha\beta}\Omega_{\gamma\lambda} - \Omega_{\alpha\gamma}\Omega_{\beta\lambda}),$$

the <u>Theorema Egregium</u> (=Celebrated Theorem) of Gauss; and also the <u>Mainardi-Codazzi equations</u>

(6) $$\Omega_{\alpha\beta\cdot\gamma} - \Omega_{\alpha\gamma\cdot\beta} = \Gamma^\delta_{\alpha\gamma}\Omega_{\delta\beta} - \Gamma^\delta_{\alpha\beta}\Omega_{\gamma\delta}.$$

We define $R_{\mu\alpha\beta\gamma} = g_{\mu\varepsilon} R^\varepsilon_{\alpha\beta\gamma}$; so (5) can also be written as

(5)' $$R_{\alpha\varepsilon\beta\gamma} = (\Omega_{\alpha\beta}\Omega_{\gamma\varepsilon} - \Omega_{\alpha\gamma}\Omega_{\beta\varepsilon}).$$

This shows in particular that $R_{\alpha\varepsilon\beta\gamma}$ is skewsymmetric in the first 2 and the last 2 indices and is unchanged if the first and third and second and last indices are interchanged (;alternatively these symmetry conditions follow directly from def. (5) and symmetry of $\Gamma^\alpha_{\beta\gamma}$ in the lower indices). So (5)' is essentially equivalent to the following single equation

(5)'' $$R_{1212} = LN - M^2.$$

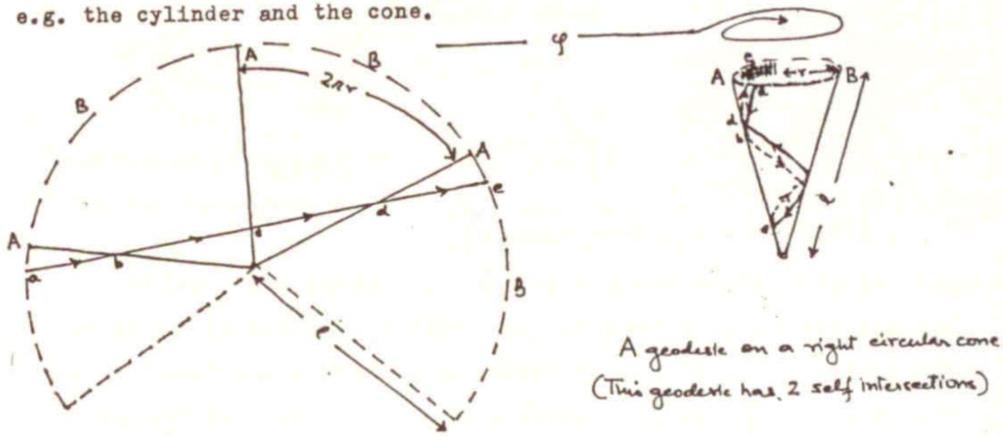
The geometrical significance of (5)'' stems from the fact that the LHS can be computed from a knowledge of the coefficients $g_{\alpha\beta}$ in a nhbd. of P. To see this it suffices to show that the Christoffel symbols can be so computed. To prove this note (from (1)) that $\vec{r}_\gamma \cdot \vec{r}_{\alpha\beta} = g_{\gamma\delta}\Gamma^\delta_{\alpha\beta} = \Gamma_{\gamma\alpha\beta}$; next differentiate $\vec{r}_\alpha \cdot \vec{r}_\gamma = g_{\alpha\gamma}$ w.r.t. u^β to get $\Gamma_{\gamma\alpha\beta} + \Gamma_{\alpha\gamma\beta} = g_{\alpha\gamma\cdot\beta}$; subtract this equation from the sum of the 2 similar ones obtained by permuting α,β,γ cyclically; this will give

(7) $$\Gamma_{\beta\alpha\gamma} = \tfrac{1}{2}\{g_{\alpha\beta\cdot\gamma} + g_{\beta\gamma\cdot\alpha} - g_{\alpha\gamma\cdot\beta}\}.$$

Combined with (3) the above remarks establish the remarkable fact that <u>the Gaussian curvature can be computed from the coefficients $g_{\alpha\beta}$ of the first fundamental form only</u> by a formula involving partial derivatives of order ≤ 2. Surfaces $S, S^* \subseteq \mathbb{R}^3$ will be called <u>isometric surfaces</u> if there exists a 1-1 onto function $S \xrightarrow{\varphi} S^*$, with φ and φ^{-1} both smooth, such that each linearization $T_P \xrightarrow{\varphi_P} T_{P^*}$ (here $P^* = \varphi(P)$) preserves the first fundamental forms. <u>At corresponding points P, P^* of 2 isometric surfaces, the Gaussian curvatures K_P and K_{P^*} are same</u>;infact to prove $K_P = K_{P^*}$ it suffices that some nhbd. of P in S be isometric to some nhbd. of P^* in S^* under an isometry φ carrying P to P^*. To see this let $\vec{r}(u,v)$ parametrize such a nhbd.; then $\vec{r}^*(u,v) = \varphi(\vec{r}(u,v))$ is a parametrisation of a nhbd. of P^* and one has $E(u,v) = E^*(u,v)$, $F(u,v) = F^*(u,v)$ and $G(u,v) = G^*(u,v)$.

§6. <u>Curves on a surface</u>. By a <u>curve on S</u> we shall mean a curve $\vec{r}(t)$ (-in the sense of §2-) which takes its values on the surface $S \subseteq \mathbb{R}^3$; for the portion of S parametrised by $\vec{r}(u,v)$ one will thus have $\vec{r}(t) = \vec{r}(u(t),v(t))$ where $u(t)$ and $v(t)$ are smooth functions of t. Using the path connectedness of S it is an easy matter to establish that for any 2 points $P,Q \in S$ one can find a curve on S which passes through both P and Q. We define the <u>distance</u> $d_S(P,Q)$ to be the infimum of the distances from P to Q measured over all such curves; it is easy to verify the usual triangle inequality. Any isometry (§5) clearly (- $\int|\dot{\vec{r}}(t)|dt = \int|\vec{r}_1\dot{u}+\vec{r}_2\dot{v}|dt = \int\sqrt{E\dot{u}^2+2F\dot{u}\dot{v}+G\dot{v}^2}\,dt$ -) preserves this distance; the converse proposition (-i.e. that any distance preserving map $S \xrightarrow{f} S^*$ is an isometry in the sense of §5 -) seems much harder. By a <u>vector field along the curve $\vec{r}(t)$</u> on S we mean that for each t we are given a tangent vector $\vec{w}(t) \in T_{\vec{r}(t)}$ such that in any parametrisation $\vec{r}(u,v)$ one has $\vec{w}(t)=w_1(t)\vec{r}_1(u(t),v(t)) + w_2(t)\vec{r}_2(u(t),v(t))$ with the components $w_1(t)$, $w_2(t)$ smooth functions of t. An important example is the <u>velocity vector field</u> $\dot{\vec{r}}(t)$ of the curve. Given a vector field $\vec{w}(t)$ along the curve $\vec{r}(t)$ its <u>covariant derivative</u> $\frac{D\vec{w}}{dt}$ is also a vector field along $\vec{r}(t)$ defined by projecting $\frac{d\vec{w}}{dt}$ orthogonally on each tangent space $T_{\vec{r}(t)}$. It is important to note that <u>the calculation of a covariant derivative can be carried out entirely in terms of the first fundamental form</u>; infact in terms of the Christoffel symbols (§5), which we know (-eqn. (7) of §5-) how to compute in terms of the first fundamental form. (Proof. Differentate $\vec{w} = w_1\vec{r}_1 + w_2\vec{r}_2$ with respect to t to get $\dot{\vec{w}} = \dot{w}_1\vec{r}_1 + \dot{w}_2\vec{r}_2 + w_1\vec{r}_{11}\dot{u} + w_1\vec{r}_{12}\dot{v} + w_2\vec{r}_{21}\dot{u} + w_2\vec{r}_{22}\dot{v}$; take its dot product with respect to $\vec{r}_1$ and $\vec{r}_2$ to get the 2 eqns. $aE + bF = \dot{w}_1 E + \dot{w}_2 F + w_1\Gamma_{111}\dot{u} + w_1\Gamma_{112}\dot{v} + w_2\Gamma_{121}\dot{u} + w_2\Gamma_{122}\dot{v}$, $aF + bG =$ etc. for a and b where $a\vec{r}_1 + b\vec{r}_2 = \frac{D\vec{w}}{dt}$, the tgt. component of $\dot{\vec{w}}$). The vector field $\vec{w}(t)$ is called a <u>parallel vector field</u> along $\vec{r}(t)$ if $\frac{D\vec{w}}{dt}$ is identically zero. Note (using $\frac{d}{dt}\langle\vec{w},\vec{w}\rangle = 2\langle\vec{w},\frac{D\vec{w}}{dt}\rangle$) that such a vector field always has constant length. Ofcourse a constant vector field along $\vec{r}(t)$ need not be parallel; for such a vector field $\vec{w}(t)$ note that $\dot{\vec{w}}(t)$ and so $\frac{D\vec{w}}{dt}$ is $\perp$ to $\vec{w}(t)$: thus we can define a scalar $\left[\frac{D\vec{w}}{dt}\right]$ by $\frac{D\vec{w}}{dt} = \left[\frac{D\vec{w}}{dt}\right](\vec{N}\times\vec{w}(t))$. A curve $\vec{r}(t)$ on S is called a <u>geodesic</u> if the velocity vector field $\dot{\vec{r}}(t)$ is parallel along $\vec{r}(t)$. It is clear that if $\vec{r}(t)$ is a geodesic then all the reparametrised curves $\vec{r}(at)$ (-here a is a non-zero constant-) are also geodesics and that no other reparametrisation of

$\vec{r}(t)$ is a geodesic. Choosing a suitable a, $|\dot{\vec{r}}(at)|\equiv 1$; so we observe that a curve $\vec{r}(t)$ is a geodesic only if its reparametrisation by arc length $\vec{r}(s)$ is a geodesic. On the other hand (by using a proposition proved above it follows that) if $\varphi: S \to S^*$ is a <u>local isometry</u> (-i.e. an isometry when restricted to any sufficiently small portion of S-) then for each geodesic $\vec{r}(s)$ on S one gets a geodesic $\vec{r}^{\,*}(s) = \varphi(\vec{r}(s))$ on S^*; this fact is useful for visualising geodesics on some simple surfaces e.g. the cylinder and the cone.

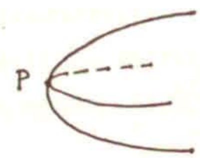


A geodesic on a right circular cone. (This geodesic has 2 self intersections)

We continue with the policy (§2) of defining invariants for a curve $\vec{r}(t)$ by first switching to an arc-length parameter s and then differentiating with respect to s (primes). For a curve on a surface S we define the <u>normal curvature</u> κ_n to be $\vec{r}''.\vec{N}$ and the <u>geodesic curvature</u> κ_g to be $\left[\frac{D\vec{r}'}{ds}\right]$. Note that $\vec{r}(s)$ is a geodesic iff $\kappa_g \equiv 0$. In §7 we will establish a remarkable connection between geodesic curvature and the Gaussian curvature of S. The relation between normal curvature and Gaussian curvature is much more transparent. Note first that for each vector $w \in T_P$ one can find a curve $\vec{r}(t)$ on S such that $\vec{r}(0) = P$ and $\dot{\vec{r}}(0) = w$ (: if in a local parametrisation $\vec{r}(u,v)$ P has parameter values u_0, v_0 and $\vec{w} = w_1\vec{r}_1(u_0,v_0) + w_2\vec{r}_2(u_0,v_0)$ take $\vec{r}(t) = \vec{r}(u_0 + w_1 t, v_0 + w_2 t)$). One has <u>Meusnier's theorem</u>: if $\vec{w}$ is a unit vector in T then $II(\vec{w},\vec{w})$ equals the normal curvature at P of any curve through P whose velocity vector at P is proportional to $\vec{w}$. (Choose a local parametrisation; then $\vec{r}(s) = \vec{r}(u(s),v(s))$ gives $\vec{r}'' = \vec{r}_1 u'' + \vec{r}_2 v'' + \vec{r}_{11}u'^2 + 2\vec{r}_{12}u'v' + \vec{r}_{22}v'^2$ from which it follows that $\kappa_n = \vec{r}''.\vec{N} = Lu'^2 + 2Mu'v' + Nv'^2 = II(\vec{w},\vec{w})$ because $\pm\vec{w} = u'\vec{r}_1 + v'\vec{r}_2$). We see thus that the <u>principal curvatures</u> κ_1 and κ_2

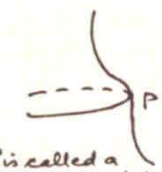
(see §4) are the maximum and minimum values for the normal curvature at P of a curve on S passing through P and that the Gaussian curvature K_P is the product of these 2 values. This interpretation of K_P enables us to have a rough idea of what the surface looks like near P.

$K_P > 0$ (i.e. K_a, K_b have same sign)



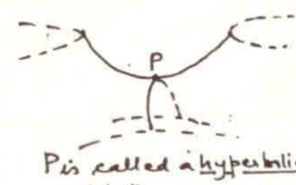
P is called an elliptic point (or, if $K_a = K_b$, an umbilic)

$K_P = 0$ and P non-planar (i.e. exactly one of K_a, K_b is zero)



P is called a parabolic point

$K_P < 0$ (i.e. K_a, K_b have opposite sign)



P is called a hyperbolic point

An important way in which curves arise on surfaces is as integral curves of vector fields. A vector field on S (or on a portion thereof) assigns to each P a tangent vector $\vec{w}(P) \in T_P$ in such a way that in each local parametrisation $\vec{r}(u,v)$ one has $\vec{w}(u,v) = w_1(u,v)\vec{r}_1(u,v) + w_2(u,v)\vec{r}_2(u,v)$ with the component functions $w_1(u,v)$ and $w_2(u,v)$ smooth. An integral curve $\vec{r}(t) = \vec{r}(u(t),v(t))$ of this vector field is one for which $u(t),v(t)$ is a solution for the ODEs $\frac{du}{dt} = w_1(u,v)$, $\frac{dv}{dt} = w_2(u,v)$. Every point of the surface has a nhbd. which admits an orthogonal parametrisation $\vec{r}(u,v)$ i.e. one for which $\vec{r}_1 \perp \vec{r}_2$ at all points. To prove this one starts first with any $\vec{r}(\tilde{u},\tilde{v})$ parametrising a nhbd. of P so small that over it one can fix 2 mutually $\perp$ vector fields $\vec{u}, \vec{v}$. By the theory of ODEs one has a non-trivial smooth function $u(\tilde{u},\tilde{v})$ which is constant over the integral curves of $\vec{v}$; and likewise a smooth function $v(\tilde{u},\tilde{v})$ constant over the integral curves of $\vec{u}$. One takes $\vec{r}(u,v) = \vec{r}(\tilde{u}(u,v),\tilde{v}(u,v))$ etc.

§7. Gauss-Bonnet Formula. Given a surface $S \subseteq \mathbb{R}^3$ and a region $\Omega \subseteq S$ parametrised by $\vec{r}(u,v)$ we define the area of Ω to be $\iint |\vec{r}_1 \times \vec{r}_2|\, du\, dv$. The parametrisation used is immaterial because $\left|\frac{\partial \vec{r}}{\partial \tilde{u}} \times \frac{\partial \vec{r}}{\partial \tilde{v}}\right| = \left|\left(\vec{r}_1 \frac{\partial u}{\partial \tilde{u}} + \vec{r}_2 \frac{\partial v}{\partial \tilde{u}}\right) \times \left(\vec{r}_1 \frac{\partial u}{\partial \tilde{v}} + \vec{r}_2 \frac{\partial v}{\partial \tilde{v}}\right)\right| = |\vec{r}_1 \times \vec{r}_2| \left|\frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})}\right|$ and so, by the change of variables formula, we see that $\iint \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|(u,v)\, du\, dv = \iint \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|(\tilde{u},\tilde{v}) \left|\frac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})}\right| d\tilde{u}\, d\tilde{v}$ $= \iint \left|\frac{\partial \vec{r}}{\partial \tilde{u}} \times \frac{\partial \vec{r}}{\partial \tilde{v}}\right| d\tilde{u}\, d\tilde{v}$. (That this definition coincides with the usual notion of "area" can be seen by noting that a small parallelogram bounded by two u=const. curves and two v=const. curves has approximately the area

$|\vec{r}_1 \delta u \times \vec{r}_2 \delta v|$). We extend the definition of area to regions $\Omega \subseteq S$ which can be partitioned into a finite number of subregions of the above kind by taking the sum of the areas of these smaller regions (;it is easy to verify that the partition of Ω used won't matter). Still more generally for any smooth function $\varphi : \Omega \to \mathbb{R}$, $\iint_\Omega \varphi \, dA$ will be defined analogously by considering the integral $\iint \varphi |\vec{r}_1 \times \vec{r}_2| \, du\, dv$ (;which can be seen, by a calculation similar to the one above, to be independent of the parametrisation used). As a first step towards calculating $\iint_\Omega K \, dA$ we prove the following lemma. If $\vec{w}(t)$, $\vec{v}(t)$ are 2 unit vector fields along a curve on S with angle from $\vec{v}(t)$ to $\vec{w}(t)$ being $\varphi(t)$, then

(1)
$$\left[\frac{D\vec{w}}{dt}\right] = \left[\frac{D\vec{v}}{dt}\right] + \frac{d\varphi}{dt}$$

(; angles in T_p are measured compatibly with the orientation of S). To prove this denote $\vec{N} \times \vec{w}$ (resp. $\vec{N} \times \vec{v}$) by $\vec{w}_\perp$ (resp. $\vec{v}_\perp$) and note that $\left[\frac{D\vec{w}}{dt}\right] = \vec{w}' . \vec{w}_\perp = (\cos\varphi \, \vec{v} + \sin\varphi \, \vec{v}_\perp)' . (\vec{N} \times (\cos\varphi \, \vec{v} + \sin\varphi \, \vec{v}_\perp)) = (-\sin\varphi \frac{d\varphi}{dt} \vec{v} + \cos\varphi \, \vec{v}' + \cos\varphi \frac{d\varphi}{dt} \vec{v}_\perp + \sin\varphi \, \vec{v}_\perp') . (\cos\varphi \, \vec{v}_\perp - \sin\varphi \, \vec{v}) = \sin^2\varphi \frac{d\varphi}{dt} + \cos^2\varphi \, \vec{v}' . \vec{v}_\perp + \cos^2\varphi \frac{d\varphi}{dt} - \sin^2\varphi \, \vec{v} . \vec{v}_\perp'$ which equals RHS because $\vec{v} . \vec{v}_\perp = 0$ yields $-\vec{v} . \vec{v}_\perp' = \vec{v}' . v_\perp = \left[\frac{D\vec{v}}{dt}\right]$. As a corollary it follows that if in an orthogonal parametrisation a curve $\vec{r}(u(s), v(s))$ makes an angle $\varphi(s)$ with $\vec{r}_1(u(s), v(s))$, then

(2)
$$k_g(s) = \frac{1}{2\sqrt{EG}} (G_1 v' - E_2 u') + \varphi' .$$

(Apply last lemma to the unit vector fields $\vec{r}\,'$ and $\vec{r}_1/\sqrt{E}$ noting that $\left[D\left(\frac{\vec{r}_1}{\sqrt{E}}\right)/ds\right] = \left(\frac{\vec{r}_1}{\sqrt{E}}\right)' . \left(\frac{\vec{r}_2}{\sqrt{G}}\right) = \left(\left(\frac{\vec{r}_1}{\sqrt{E}}\right)_1 u' + \left(\frac{\vec{r}_1}{\sqrt{E}}\right)_2 v'\right) . \left(\frac{\vec{r}_2}{\sqrt{G}}\right) = \frac{\vec{r}_{11} . \vec{r}_2}{\sqrt{EG}} u' + \frac{\vec{r}_{12} . \vec{r}_2}{\sqrt{EG}} v'$ because $\vec{r}_1 . \vec{r}_2 = 0$; further $\vec{r}_1 . \vec{r}_2 = 0$ also shows $\vec{r}_{12} . \vec{r}_2 = -\vec{r}_1 . \vec{r}_{12} = -\frac{1}{2}(\vec{r}_1 . \vec{r}_1)_2 = -\frac{1}{2} E_2$ etc.). On the other hand we note that in an orthogonal parametrisation the Gaussian curvature is given by

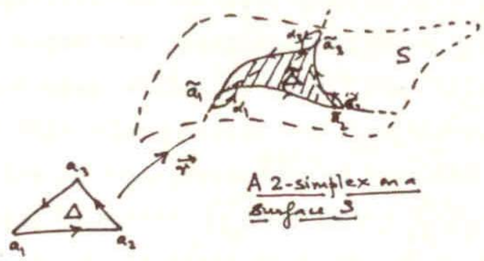
(3)
$$K = -\frac{1}{2\sqrt{EG}} \left(\left(\frac{E_2}{\sqrt{EG}}\right)_2 + \left(\frac{G_1}{\sqrt{EG}}\right)_1 \right).$$

(Proof: Take dot product of (1), §5 with $\vec{r}_1$, $\vec{r}_2$ to compute the Christoffel symbols: $\Gamma^1_{22} = -\frac{1}{2}\frac{G_1}{E}$, $\Gamma^1_{21} = \frac{1}{2}\frac{E_2}{E}$, $\Gamma^1_{11} = \frac{1}{2}\frac{E_1}{E}$, $\Gamma^2_{22} = \frac{1}{2}\frac{G_2}{G}$, $\Gamma^2_{21} = \frac{1}{2}\frac{G_1}{G}$; substitute these values in $K = \frac{R_{1212}}{EG} = \frac{E R^1_{212}}{EG}$ $\frac{1}{G}(\Gamma^1_{22\cdot1} - \Gamma^1_{21\cdot2} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{21} - \Gamma^1_{21}\Gamma^1_{12} - \Gamma^2_{21}\Gamma^1_{22})$ to get $\frac{1}{G}\left(-\left(\frac{1}{2}\frac{E_2}{E}\right)_2 + \left(\frac{1}{2}\frac{G_2}{G}\right)\left(\frac{1}{2}\frac{E_2}{E}\right) - \left(\frac{1}{2}\frac{E_2}{E}\right)^2\right) + \frac{1}{G}\left(\left(-\frac{1}{2}\frac{G_1}{E}\right)_1 + \left(-\frac{1}{2}\frac{G_1}{E}\right)\left(\frac{1}{2}\frac{E_1}{E}\right) - \left(\frac{1}{2}\frac{G_1}{G}\right)\left(-\frac{1}{2}\frac{G_1}{E}\right)\right) = -\frac{1}{2\sqrt{EG}}\left(\frac{E_2}{\sqrt{EG}}\right)_2 - \frac{1}{2\sqrt{EG}}\left(\frac{G_1}{\sqrt{EG}}\right)_1$ i.e. RHS of (3)). A 2-simplex $\tilde{\Delta} = \widehat{a_1 a_2 a_3} \subseteq S$ is the image of a triangular region $\Delta = a_1 a_2 a_3$ lying in the domain of some local parametrisation $\vec{r}(u,v)$; $\widehat{a_1 a_2}$, $\widehat{a_2 a_3}$ and $\widehat{a_3 a_1}$ are said to be its 3 "sides" or "faces" or incident 1-simplices (: they constitute the 'boundary' $\partial\tilde{\Delta}$ of $\tilde{\Delta}$);

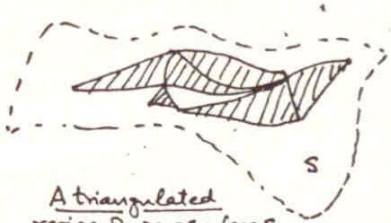
and $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ are its 3 vertices or incident <u>0-simplices</u>. We have the <u>Gauss-Bonnet formula</u>

(4) $$\iint_{\tilde{\Delta}} K\,dA = 2\pi - (\beta_1+\beta_2+\beta_3) - \int_{\partial\tilde{\Delta}} \kappa_g(s)\,ds.$$

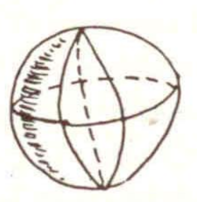
A priori the line integral and the β_i in the RHS of (4) are defined in terms of a smooth unit normal vector $\vec{N}$ on the 2-simplex $\tilde{\Delta}$: the line integral is done w.r.t. the compatible cyclic orientation of $\partial\tilde{\Delta}$ and β_i, the angle through which the tangent to $\partial\tilde{\Delta}$ turns at the vertex $\tilde{a}_i$, is measured compatibly. Note however that the value of each term in (4) is independent of the choice of $\vec{N}$ (:e.g. if $\vec{N}$ is replaced by $-\vec{N}$ tgt. vector $\vec{t}$ to $\partial\tilde{\Delta}$ is also replaced by $-\vec{t}$ and so $\vec{N}\times\vec{t}$ and thus $\kappa_g(s)$ is unaltered).



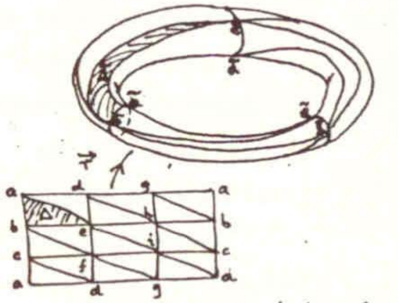
A 2-simplex on a surface S



A triangulated region Ω on a surface S
Euler characteristic of $\Omega = 0$



A closed surface triangulated with 10 2-simplices; Euler characteristic is 2.



A closed triangulated surface with 18 2-simplices; Euler characteristic is zero. (One 2-simplex shown shaded).

We will see later that while proving (4) there is no loss of generality in assuming that the above parametrisation $\vec{r}(u,v)$ can be chosen to be orthogonal; the smooth unit normal $\vec{N}$ on $\tilde{\Omega}$ shall be in the direction of $\vec{r}_1\times\vec{r}_2$. First we use (2) to note that $\int_{\partial\tilde{\Delta}} \kappa_g\,ds = \int_{\partial\Delta}(P\,du + Q\,dv) + \int_{\partial\tilde{\Delta}} \varphi'\,ds$ where $P = -\frac{E_2}{2\sqrt{EG}}$ and $Q = \frac{G_1}{2\sqrt{EG}}$. Next we note that $\int_{\partial\tilde{\Delta}} \varphi'\,ds$ is the angle through which the tangent turns as one describes the 3 sides of $\partial\tilde{\Delta}$; so it is intuitively clear (-but not altogether trivial to prove-) that it should equal $2\pi - (\beta_1+\beta_2+\beta_3)$. On the other hand by Green's theorem $\int_{\partial\Delta}(Pdu+Qdv)$ equals $\iint_{\Delta}(\frac{\partial Q}{\partial u}-\frac{\partial P}{\partial v})du\,dv$ i.e. $-\iint_{\tilde{\Delta}} K dA$ by using (3). We will now proceed to the computation of $\iint_{\tilde{\Delta}} K dA$ over regions more complicated than a "triangle". A region Ω of S is said to be triangulable if it is a finite union of some 2-simplices which are either pairwise disjoint or else intersect in a common edge or vertex: these 2-simplices, together with their sides and vertices, constitute a triangulation of Ω. Its boundary $\partial\Omega$ is made up of those 1-simplices which are incident to precisely one 2-simplex of the triangulation. We note that $\partial\Omega$ is empty only if S is compact (="closed") and $\Omega = S$. It is known that all closed surfaces are triangulable. To compute $\iint_S K dA$ for a closed surface we select some triangulation of S; let us suppose that this triangulation has α_0 vertices, α_1 edges and α_2 triangles. For each of the α_2 2-simplices $\tilde{\Delta}_i$ we have the equation (4); we add these equations to obtain $\iint_S K dA = 2\pi\alpha_2 - \sum_{i=1}^{\alpha_2}(\beta_{1i}+\beta_{2i}+\beta_{3i})$. This follows because each edge is incident to 2 triangles and it makes two opposite contributions to the line integrals (: this is because either the chosen surface normals on the 2 triangles are in same direction but tgt. vector to edge is reversed or else the tgt. vector to edge is same vis-à-vis the 2 triangles but $\vec{N}$ is reversed; in either case $\vec{N}\times\vec{t}$ and so sign of $K_g(s)$ is reversed in the 2 contributions). We now substitute for each external angle in terms of the corresponding internal angle $\gamma = \pi - \beta$ and see that the sum of all the angles γ is $2\pi\alpha_0$ ($\because$ the sum of all the angles at a vertex is 2π). So $\iint_S K dA = 2\pi\alpha_2 - 3\pi\alpha_2 + 2\pi\alpha_0$. Now we note that each triangle has 3 sides and each side is incident to 2 triangles to get $3\alpha_2 = 2\alpha_1$. Hence if a closed surface S can be triangulated with α_0 vertices, α_1 sides and α_2 triangles then

(5) $$\iint_S K dA = 2\pi(\alpha_0 - \alpha_1 + \alpha_2).$$

In particular this remarkable formula shows that irrespective of what triangulation is chosen the number $\alpha_0 - \alpha_1 + \alpha_2$ is the same. This is infact true for all triangulable regions Ω (and not very hard to prove directly): $\alpha_0 - \alpha_1 + \alpha_2$ is called the Euler characteristic and is denoted by $\chi(\Omega)$. It is at once clear that 2 diffeomorphic surfaces S, S^* (-i.e. those for which there exists a 1-1 onto smooth map $S \xrightarrow{\varphi} S^*$ with φ^{-1} also smooth-) have same Euler characteristic. Thus (5) shows that $\iint_S K dA$ is a diffeomorphism invariant of the closed surface S. There is no difficulty in generalising the calculation to triangulable regions Ω with $\partial\Omega \neq 0$. Let us triangulate Ω with α_0 vertices, α_1 edges and α_2 triangles and let us suppose α_0' of these vertices and α_1' of these edges lie on $\partial\Omega$. We now proceed as above writing eqn. (4) for each triangle and adding the ensuing α_2 equations. This time the line integrals $\int_{\partial\Omega}$ don't cancel. Furthermore at vertices $\tilde{a}_j$ on $\partial\Omega$ sum of all the internal angles is not 2π but less: say $2\pi-\theta_j$. Furthermore now instead of $3\alpha_2=2\alpha_1$ one has $3\alpha_2=2\alpha_1-\alpha_1'$. Thus we see that for any triangulated region Ω of a surface $S \subseteq \mathbb{R}^3$

(6)
$$\iint_{\Omega} K\,dA = 2\pi\chi(\Omega) + \pi\alpha_1' - \sum_{j=1}^{\alpha_0'} \theta_j - \int_{\partial\Omega} \kappa_g\,ds .$$

It is worthwhile to note that we have proved (6) without assuming that Ω is orientable: we equipped each 2-simpex with a normal but these normals need not match. Formula (4) is a special case of (6): now $\alpha_1'=3$ and $\theta_j = \pi+\beta_j$. Formula (6) also tells us why there was no loss of generality when we assumed (while proving (4)) that $\vec{r}(u,v)$ is an orthogonal parametrisation: we can always cut up our 2-simplex into such small triangles that each one of them (-see p.9-) can be covered by an orthogonal parametrisation; then (6) applied to this triangulation can be easily seen to be nothing but (4). Note, from (4), that if each side of the triangle $\partial\tilde{\Delta}$ is a geodesic then

(7)
$$\iint_{\tilde{\Delta}} K\,dA = -\pi + (\gamma_1+\gamma_2+\gamma_3)$$

here $\gamma_i = \pi-\beta_i$ are the internal angles of $\tilde{\Delta}$. Thus Gauss-Bonnet theorem clarifies the geometrical significance of Theorema Egregium by showing how the Gaussian curvature can be measured from the sum of the angles of a small geodesic triangle: $K(P) = \lim_{\tilde{\Delta}\to P} \frac{(\gamma_1+\gamma_2+\gamma_3)-\pi}{\text{Area}(\tilde{\Delta})}$.