Oberwolfach Feb 5, 1995

(0) **Discrete $\leftrightarrow$ Continuous.** As we all know it is a better understanding of this interaction which is chiefly responsible for the mighty acheivements of this century. What makes the understanding of all this easier is that at heart only a very humble tool is really used in all this

Simplicial Complex A finite set K of finite sets{of} closed w.r.t. $\subseteq$

The key use which is made of this tool is to compute (co)homologies (singularity/ambiguity measures of) of a locally unambiguous thing e.g. $\int$ integration). (an indefinite $\int$ of a closed generic form)

So I want to look at a few of the numerous simplicial (co)homologies (standard variations: co, rel, reduced, local, coeffs; ...)

(1) **Usual cohomology** associative monomials K_{assoc}. Functions $K_{assoc} \xrightarrow{f} \mathbb{F}$ (field / even charac. zero in most cases). assumed alternating i.e.

$$f(v_0 v_1 \cdots v_q) = (-1)^{\pi} f(v_{\pi 0} v_{\pi 1} \cdots v_{\pi q}) \qquad \text{(i)}$$

for all $\pi \in S_{q+1}$. In this $C^*(K)$ a $\delta: C^*(K) \longrightarrow C^{*+1}(K)$ is defined by $(\delta f)(v_0 \cdots v_q) = \sum_{i \geq 0} (-1)^i f(v_0 \cdots \hat{v}_i \cdots v_q)$ and then one sets $H^*(K) = \ker\delta / \operatorname{im}\delta$, well defined because $\delta \circ \delta = 0$ in $C^*(K_{assoc})$ and alternating f's are closed w.r.t. δ.

(2) **Other subcomplexes of $C^*(K_{assoc}) \circlearrowright \delta$** Are there other sequences $G_{q+1} \subset S_{q+1}$ of permutation gps s.t. asking (i) for all $\pi \in G_{q+1}$ gives subcomplex, and if so what are the cohomologies?

Theorem 1: $G_{q+1} = \begin{cases} id \\ \mathbb{Z}/2 \text{ (reversals)} \\ C_{q+1} \text{ (rotations)} \\ D_{q+1} \text{ (dihedral)} \\ S_{q+1} \text{ (all)} \end{cases}$ are the only such seq^s of gps. And cohomologies are then

$$\begin{cases} H^*(K_{assoc}) \cong H^*(K) \\ H^*_{rev}(K_{assoc}) \cong H^*(K) \\ H^*_{cycl}(K_{assoc}) \cong \bigoplus_{k \geq 0} H^{*-2k}(K) \\ H^*_{dih}(K_{assoc}) \cong \bigoplus H^{*-4k}(K) \end{cases}$$

$$H^*_{alt}(K_{assoc}) \overset{def}{=} H^*(K)$$

③ **Subcomplexes of $C_*(K_{comm}) \circlearrowleft \partial$** The usual way of equipping K_{comm}, the commutative monomials of vertices supported on K, is to use a <u>total order</u> on vertices to identify them with increasing vertex sequences: so $C_*(K_{comm})$ becomes a subcomplex of $C_*(K_{assoc}) \circlearrowleft \partial$. There are other interesting subcomplexes: $K_{comm,r}$ = all monomials with each vertex of degree $\leq r$.

<u>**Theorem 2 (Bier):**</u> <u>For r odd $H_*(K_{comm,r}) \cong H_*(K)$</u>

<u>but for r ~~odd~~ even</u> $\underline{\tilde{H}_*(K_{comm,r})} \cong \bigoplus_{\sigma \in K} \tilde{H}_{*\,-\,r.\mathrm{Card}\,\sigma}(Lk_K \sigma)$.

<u>**Link:**</u> where of course $Lk_K\sigma$ is defined by the quotient Q.

$$K = \sigma Q + R \qquad (\text{in } K_{comm,1})$$

Note that $\tilde{H}_*(Lk_K\sigma) \cong H_{*+1}(K, \{\sigma\}^c)$ = local homology of K at σ and these homologies figure in combinatorial situations:–

④ **Deleted joins of K.** Attached to any K is its deleted join $K_* K \subset K \cdot \bar{K}$: all (σ, θ) s.t. $\sigma \cap \theta = \phi$. What is its homology? I don't know in general but there is a result very much like Bier's thm. for a simpler construction

$$K \bullet K \subset K * K \subset K \cdot K \quad : \text{all } (\sigma, \theta) \text{ s.t. } \sigma \cap \theta = \phi$$

and $\sigma \cup \theta \in K$.

<u>**Theorem 3**</u> $\tilde{H}_*(K \bullet K) \cong \bigoplus_{\sigma \in K} \tilde{H}_{*-\dim\sigma-1}(Lk_K \sigma)$

To understand this result conceptually (though a direct combin. proof is available) we know look at the 'continuous' side of simplicial complexes a bit. (Such a conceptual pf. of Brion's??)

⑤ The spaces of a simplicial complex K.

The usual one $|K|$ in which each σ is replaced by $\mathrm{Conv}(\sigma)$: convex hull of the unit vectors of $\mathbb{R}^{\mathrm{vert} K}$ given by vertices of σ is better written

$$\mathbf{Conv\,K} = \bigcup_{\sigma \in K} \mathrm{Conv}(\sigma)$$

But there is nothing ~~sacred~~ about functor $K \xrightarrow{\mathrm{conv}}$ Spaces. Equally natural are $K \xrightarrow{\mathrm{lin}}$ Spaces, $K \xrightarrow{\mathrm{aff}}$ Spaces: in fact even more so, make sense over <u>any field</u> instead of $\mathbb{R}$! So

$$\mathbf{Aff\,K} = \bigcup_{\sigma \in K} \mathrm{Aff}(\sigma)$$

$$\mathbf{Lin\,K} = \bigcup_{\sigma \in K} \mathrm{Lin}(\sigma)$$

We note that latter is contractible while former $\simeq K$. But latter has informative singularity at 0 so its local homology there should be interesting. So for normed fields like $\mathbb{R}$ we consider $K \xrightarrow{\mathrm{sph}}$ Spaces also & use

$$\mathbf{Sph\,K} = \bigcup_{\sigma \in K} \mathrm{Sph}(\sigma)$$

which is link of Lin K at origin.

Again note that on $\mathrm{Lin}K - \{0\}$ (or on $\mathrm{Sph}\,K$) we have action of invertible elts of field (for $\mathbb{R}$ of $\mathbb{Z}/2$ on $\mathrm{Sph}_{\mathbb{R}}K$, for $\mathbb{C}$ of S^1 on $\mathrm{Sph}_{\mathbb{C}}K$, etc.). Dividing out — i.e. using $\sigma \longmapsto \mathrm{Proj}\,K$ — we get

$$\mathbf{Proj\,K} = \bigcup_{\sigma \in K} \mathrm{Proj}(\sigma).$$

We note now that

K•K is a triangulation of Sph(K)

with the switching action $(\sigma,\theta) \leftrightarrow (\theta,\sigma)$ corresponding to the antipodal $\mathbb{Z}/2$-action. So

<u>$H_*(K \bullet K)$ is the singular homology of Sph K</u>

while <u>equivariant homology $H_*^{equiv}(K \bullet K)$ is the homology of Proj K.</u>

Remarks :- Goresky - MacPherson.
Ziegler - Zivaljevic
Cohen - Macaulayness.
Kalai's shifting operation.

Above I looked at "singular homology".

⑥ ~~Semi-simplicial complexes~~ Non-abelian chain complexes.

$\partial^2 = 0$ holds $\because$ of $\partial_i \partial_j = \partial_{j-1}\partial_i \quad \forall\ i<j$

One has

$$K = K_{comm,1} \hookrightarrow K_{comm,r} \hookrightarrow K_{comm} \hookrightarrow K_{assoc} \hookrightarrow K_{pe} \hookrightarrow K_{sing}$$

One can think of these $\longrightarrow$ as <u>complete</u> functors from $\mathbb{N}$ to sets, the others are various "normalizations" of these. The last two are Kan complexes from which one can compute the homotopy groups of a space. For example one has

Moore homology of K :- Look at free non abelian groups.

$F_*^{*}(K_{comm})$. Consider $\ker\partial_1 \cap \ker\partial_2 \cap \cdots = F_*^{Moore}(K_{comm})$

Then it can be checked that $\ker\partial_0$ is always a normal subgroup of $\ker\partial_0$. Call this $\ker \frac{\ker\partial_0}{\mathrm{im}\,\partial_0} = H_*^{Moore}(FK)$. One has

Theorem 4 (Milnor). $\underline{H_*^{Moore}(FK) \cong \pi_{*+1}(S|K|)}$

⑦ **Non-trivial characters.** The cyclotomic example.

$$\partial(v_0 v_1 \cdots) = \sum_r \omega^r (v_0 v_1 \cdots \hat{v}_r \cdots)$$

$\omega = \exp\left(\frac{2\pi i}{p}\right)$, $p \geq 2$

Now $\partial^p = 0$ $\qquad H_{*;r,s} = \frac{\ker\partial^r}{\ker\partial^s} \qquad r+s=p.$

Theorem 5 $\quad H_{kp+r-1;r,s}(K) \cong H_{kp+s-1;s,r}(K) \cong H_{2k}(K)$

$$H_{kp-1;r,s}(K) \cong H_{kp-1;s,r}(K) \cong H_{2k-1}(K)$$

(zero for other possibilities)

If time available proof of Thm., esp. second part.

(Mayer Annals 1942 - Spanier B.A.M.S. 1949 defined similar with $\omega = 1$ using mod p coeffts. and one gets sim. results)

$$\begin{tikzcd} \Omega K \arrow[r, hook] & \text{contractible} \arrow[d] \\ & K \end{tikzcd}$$

$\pi_{i-1}(\Omega K) \cong \pi_i(K)$.

$\pi_{i-1}(\Omega S(K)) \cong \pi_i S(K) \cong \pi_{i-1}(K)$

$\boxed{\text{for } * \text{ large}}$