EXTERIOR SHIFTING

by

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ABSTRACT

We give a new proof of Kalai's result that exterior shifting preserves additive cohomology of a complex.

§1. Introduction

In his thesis [2] Kalai introduced a very interesting construction which associates, to each simplicial set K, a combinatorially much simpler simplicial set $\Delta(K)$ of the same size, which retains many of the properties of K. This construction employs an exterior algebra over some field $\mathbb{F}$, so we will call it *exterior shifting* in the following.

The most striking fact about this construction was established subsequently by Kalai in [3], with a proof finally appearing in print in Bjorner-Kalai [1], viz, that *exterior shifting preserves additive (co)homology* with coefficients $\mathbb{F}$.

The object of this note is to give, after recalling the definition of exterior shifting in §2, a new proof of Kalai's result in §3. This we do by equipping the shifting process with a certain *seiving isomorphism* which is shown to commute with coboundaries.

In sequels to this note we'll construct analogous equivariant seiving isomorphisms for the equivariant generalization of exterior shifting introduced in [4], and relate these to the combinatorics of simplicial complexes embeddable in a given euclidean space.

§2. Definitions

(2.1) We'll work with a totally ordered set **U** of N *vertices*,

$$v_1, v_2, \ldots, v_N.$$

Any set of vertices will be called a *simplex* and will be equipped with the induced total order. We will denote by U the *universe* of all simplices.

By a *simplicial set* we mean any subset K of U. We denote by $L(K,\mathbb{F})$, or just $L(K)$, the $\mathbb{F}$-vector space spanned by any simplicial set $K \subseteq U$. The simplices of K prescribe the *canonical basis* of $L(K)$. We will first identify $L(U)$ with the underlying vector space of a certain algebra.

(2.2). Namely, consider the *exterior algebra* $(\Lambda,\wedge)$ generated by the N vertices, i.e. the associative algebra with unity over $\mathbb{F}$ generated by the vertices subject to the relations

$$v\wedge w = -w\wedge v$$

for all vertices v and w, with relations $v\wedge v = 0$ specified separately in char $(\mathbb{F}) = 2$ case. Recall that this algebra is *graded* by the non-negative integers, and that it is graded-commutative.

Identifying each (ordered) t-dimensional simplex $\sigma = v_0 v_1 \cdots v_t$ with the degree t+1 monomial $v_0\wedge v_1 \cdots \wedge v_t$ we identify $L(U)$ with the underlying graded vector space of the algebra Λ.

(2. 3). Next, we need, in our field $\mathbb{F}$, a sequence of N *transcendentals* $\xi_1, \xi_2, \ldots, \xi_N$ i.e. field elements which are algebraically independent over some subfield. To ensure that such elements can be chosen, we'll assume that our field $\mathbb{F}$ is "big enough", i.e. has transcendence degree N or more over some subfield. For example we can take the field $\mathbf{F} = \mathbf{F}_p(\xi_1, \xi_2, \ldots, \xi_N)$ of all rational functions in N indeterminates over the prime field $\mathbf{F}_p$ of characteristic p.

Using these transcendentals, we define N degree one elements x_j $L_1(U)$, called *letters*, by

$$x_j = \Sigma_i (\xi_j)^i v_i.$$

Each sequence of letters determines under the product $\wedge$ an element of the algebra Λ. Since we have, in our exterior algebra $x \wedge y = -y \wedge x$ for *all* degree one elements x and y, it follows that, in case a letter repeats, this element is zero, and more generally it can at most undergo a sign change if the letters are permuted. We rule out these degeneracies by restricting only to strictly increasing sequence of letters, which will henceforth be called ***words***.

Because of the algebraic independence of the ξ_j's it follows that the elements of $L(U)$, determined as above by these words, are linearly independent and so determine a graded vector space basis U_x of $L(U)$.

Remark. In [1], Bjorner and Kalai use N^2 transcendentals ξ_{ji}, the letters being now defined by $x_j = \Sigma_i \xi_{ji} v_i$. However, the above van der Monde description, using just N transcendentals, suffices, as we'll see below in (2.5), to ensure that our $\Delta(K)$'s are also "shifted".

(2.4) As just explained, the set of all words specifies a new vector space basis $\Delta(U) = U_x$ of $L(U)$.

More generally, for any $K \subseteq U$, we will now replace the canonical basis K of $L(K)$ with a ***generic basis*** specified by a set $\Delta(K)$ of words. Thus $\Delta(K)$ is a subset of U_x, the ***universe of all words***, rather than of the previous universe of all simplices $U = U_v$. However, if need be, one can of course identify the two universes under $v_i = x_i$.

Thinking of $L(K)$ as the graded quotient $L(U)/L(U\backslash K)$ of $L(U)$, the words determine a graded spanning set of $L(K)$. To obtain a graded basis from this, we now put the words in *lexicographic order* and *seive out* those which, as elements of the quotient space $L(K)$, are linearly dependent on the lexicographically preceding words. What remains is the required graded basis $\Delta(K) \subseteq U_x$.

(2.5). By a *shifted simplicial set* we mean a subset K of U whicl is closed under the *product partial order* $\leq$. We recall that $\sigma \leq \theta$ iff σ and θ have the same cardinality, and the first vertex of σ is less than or equal to the first vertex of θ , and likewise for the second, third, ... vertices.

We check below that the $\Delta(K)$ of (2.4) is always a shifted simplicial set. For this purpose we'll make essential use of the fact that the field elements ξ_j were algebraically independent over a subfield: so far we have only used the much weaker fact that the words determined linearly independent elements of $L(U)$.

Towards this end we note first that the *permutations of the field elements* $\{\xi_1, \xi_2, \ldots, \xi_N\}$ extend, because of their algebraic independence over some subfield, to *field automorphisms of* $\mathbf{F}$, and thus to $\mathbf{F}_p$*-linear algebra automorphisms of* Λ. Here $p = char(\mathbf{F})$, and $\mathbf{F}_p \subset \mathbf{F}$ denotes the smallest subfield of $\mathbf{F}$. The definition (2.3) of the letters x_i now shows that these algebra automorphisms effect the corresponding *permutations of the letters* $\{x_1, x_2, \ldots, x_N\}$.

Proposition 1. *For any simplicial set K, the simplicial set $\Delta(K)$ is a shifted simplicial set.*

Proof. Let $\sigma \leq \theta$ with σ not in $\Delta(K)$. Consider the permutation π of the N letters which maps the letters of σ in order on the letters of θ, and the letters not in σ in order on the letters not in θ. Note that under this permutation a letter x not in σ

is mapped to a letter $y \le x$. So if a word τ is lexicographically less than σ, then $\pi(\tau)$ is lexicographically less than θ. Applying the $\mathbf{F}_p$-algebra automorphism π to the lexicographic dependency which seived out σ during the seiving process of (2.4), we see that θ too must have been seived out *q.e.d.*

§3. Cohomology

(3.1). We equip $U \times U$ with the usual *incidence relation*, i.e. two simplices σ and θ of U are said to be incident to each other if one of them is a face (subset) of the other. The *incidence number* $[\sigma:\theta]$ will be defined to be zero unless $\theta \in U$ is a codimension one face of $\sigma \in U$, in which case it is $(-1)^i$ if deletion of the $(i+1)$th vertex of σ yields θ.

The *boundary* ∂ and *coboundary* δ of U are the dual linear maps $\partial, \delta : L(U) \longrightarrow L(U)$ defined by

$$\partial(\sigma) = \sum_{\theta} [\sigma:\theta]\ \theta \text{ and } \delta(\theta) = \sum_{\sigma} [\sigma:\theta]\ \sigma.$$

Likewise, for any $K \subseteq U$ we will equip $K \times K$ with *restricated incidence relation*, and then define the dual linear maps $\partial, \delta : L(K) \longrightarrow L(K)$ exactly as above using this restricted incidence.

Note that the (co)boundary of K is thus obtainable from the restriction $\partial, \delta : L(K) \longrightarrow L(U)$ of the (co)boundary of U by composing it with the *projection map* $[..]_K : L(U) \longrightarrow L(K)$ provided by the canonical basis K of L(K).

From now on we'll confine ourselves to coboundaries δ, for which we have the following alternative definition in terms of the exterior product of $L(U)$.

Proposition 2. *For any $K \subseteq U$, the coboundary $\delta : L(K) \longrightarrow L(K)$ satisfies*

$$\delta(\omega) = [Y \wedge \omega]_K \quad ,$$

for all $\omega \in L(K)$, *where* Y *denotes the sum of all the* N *vertices.*

Proof. If $\theta = v_0 v_1 \ldots . \in K$, then $\delta(\theta) \in L(U)$ is given by

$$\delta(\theta) = \sum_i (-1)^i \delta_i(\theta) ,$$

where $\delta_i(\theta)$ is the sum of all simplices $v_0 v_1 \ldots v_{i-1} v v_i \ldots$ as v runs over all vertices beyween v_{i-1} and v_i. The result folows because, as an exterior monomial, such a simplex equals $(-1)^i v \wedge v_0 \wedge v_1 \wedge \ldots$ *q.e.d.*

The above *coboundary formula* is the reason why exterior shifting will turn out to be well behaved with respect to δ.

(3.2) We denote by $D : \Lambda \longrightarrow \Lambda$ the exterior algebra automorphism which multiplies the ith vertex v_i by the ith power of the first indeterminate ξ_1. We now examine what happens to the coboundary after conjugation with this "diagonal" automorphism.

Proposition 3. *For any simplicial set* $K \subseteq U$ *the map* $D.\delta.D^{-1}$ *:* $L(K) \longrightarrow L(K)$ *is well-defined and obeys*

$$(D.\delta.D^{-1})\ (\omega) = [x_1 \wedge \omega]_K$$

Proof. The point to note is that D simply multiplies each simplex by some nonzero scalar, so there is available, for any $K \subseteq U$, an induced vector space automorphism $D : L(K) \longrightarrow L(K)$. The result now follows from Proposition 2 because $D(Y) = x_1$. *q.e.d.*

(3.3) For any $K \subseteq U_v$, let $\Delta(K) \subseteq U_x$ be defined by the seiving process of (2.4) above.

We will now define a graded linear isomorphism $\mathcal{L} : L(K) \longrightarrow$

$L(\Delta(K))$. This definition will be in terms of the *generic basis* $[\sigma]_K$, $\sigma \in \Delta(K)$, of $L(K)$, and the *canonical basis* $[\sigma]_\Delta$, $\sigma \in \Delta(K)$, of $L(\Delta(K))$.

We define

$$\mathcal{L}\ [\sigma]_K = [\sigma]_\Delta \text{ if } x_1\sigma \in \Delta(K).$$

In case $x_1\sigma$ is not in $\Delta(K)$, then we have, in $L(K)$, the lexicographic dependency, $[x_1\sigma]_K = \sum_\theta c_\theta\ [x_1\theta]_K$, with the θ's lexicographically less than σ and in $\Delta(K)$, and we define

$$\mathcal{L}\ [\sigma]_K = [\sigma]_\Delta + \sum_\theta c_\theta\ [\theta]_\Delta \text{ if } x_1\sigma \in \Delta(K).$$

Proposition 4. *For any* $K \subseteq U$, $\mathcal{L} : L(K) \longrightarrow L(\Delta(K))$ *is a linear isomorphism obeying*

$$\mathcal{L}\ [x_1 \wedge \omega]_K = [x_1 \wedge \mathcal{L}(\omega)]_\Delta.$$

for all $\omega \in L(K)$.

Proof. The fact that $\mathcal{L}$ is indeed an isomorphism follows because its matrix, with respect to the above two defining bases, is "lower triangular" with 1's on the diagonal.

It is enough to check the formula when $\omega = [\sigma]_K$ with $\sigma \in \Delta(K)$. In case $x_1\sigma \in \Delta(K)$, then both sides equal $[x_1\sigma]_\Delta$. Otherwise both sides equal $\sum_\theta c_\theta\ [x_1\theta]_\Delta$. *q.e.d.*

(3.4). Consider now the exterior algebra $\Lambda_x = L(U_x)$. We denote by $u : \Lambda_x \longrightarrow \Lambda_x$ the algebra automorphism which maps each letter to itself except the first letter x_1 which is mapped to the sum Y_x of all the N letters.

By Proposition 1 we know that $\Delta(K) \subseteq U_x$ is always shifted, thus the following result shows what happens to its coboundary upon conjugation with this automorphism.

Proposition 5. *For any shifted simplicial set* $\Delta \subseteq U_X$, *the map* $u^{-1}.\delta.u : L(\Delta) \longrightarrow L(\Delta)$ *is well-defined and obeys*

$$(u^{-1}.\delta.u)(\omega) = [x_1 \wedge \omega]_\Delta ,$$

for all $\omega \in L(\Delta)$.

Proof. For any shifted Δ, note that this "upper triangular" automorphism u maps $L(U_X \setminus \Delta)$ into itself, thus there is an induced linear isomorphism $u : L(\Delta) \longrightarrow L(\Delta)$ given by $[\tau]_\Delta \longrightarrow [u(\tau)]_\Delta$.

The required formula follows from proposition 2 because $u(x_1) = Y_X$. *q.e.d.*

(3.5). For any $K \subseteq U$, we define the *seiving isomorphism* $\Delta = \Delta_K$ $L(K) \longrightarrow L(\Delta(K))$ to be the composition $u .\mathcal{L}.D$ where $D : L(K) \longrightarrow L(K)$, $\mathcal{L} : L(K) \longrightarrow L(\Delta(K))$, and $u : L(\Delta(K)) \longrightarrow L(\Delta(K))$ are the linear isomorphisms considered in (3.2), (3.3) and (3.4).

Theorem. *For any* $K \subseteq U$, *the seiving isomorphism* $\Delta : L(K) \longrightarrow L(\Delta(K))$ *commutes with the coboundaries* δ *of* $L(K)$ *and* $l(\Delta(K))$.

Proof. This follows immediately from Propositions 3, 4, and 5 above *q.e.d.*

For any $K \subseteq U$, we have the subspace $Z(K) = \{\omega \in L(K) : \delta(\omega) = 0\}$ of *cocycles* and the still smaller subspace $B(K) = \{\omega \in Z(K) : \omega = \delta(\tau)\}$ of *cobounding cocycles*. The *cohomology* of K is defined to be the quotient space $H(K) = Z(K)/B(K)$.

Corollary 1. *Exterior shifting preserves the cohomology of any simplicial set* $K \subseteq U$.

Proof. In fact, since the linear isomorphism Δ commutes with the coboundaries, there are induced linear isomorphisms $\Delta : Z(K) \longrightarrow Z(\Delta(K))$ and $\Delta : B(K) \longrightarrow B(\Delta(K))$, and so also a linear isomorphism $\Delta : H(K) \longrightarrow H(\Delta(K))$. *q.e.d.*

A simplicial set $K \subseteq U$ is called a *complex* if the incidence numbers — see (3.1) — associated to the restricted incidence relation in K x K obey the identities

$$\sum_{\theta} [\sigma:\theta].[\theta:\tau] = 0$$

for all pairs of simplices $\sigma \in K$, $\tau \in K$.

Corollary 2. *If the simplicial set* $K \subseteq U$ *is a complex then so is its exterior shift* $\Delta(K)$.

Proof. The above incidence number identities are equivalent to asking that the coboundary $\delta : L(K) \longrightarrow L(K)$ be of order two, i.e. that $\delta.\delta = 0$. Using Δ it follows that the coboundary $\delta : L(\Delta(K)) \longrightarrow L(\Delta(K))$ must then also be of order two. *q.e.d.*

A sufficient (but by no means necessary) condition for a simplicial set $K \subseteq U$ to be a complex is that it be *closed* with respect to the incidence relation of U, i.e. if $\sigma \in K$ and $\theta \in U$ is such that $\theta \leq \sigma$, then we must have $\theta \in K$ also. Such a closed K is also called a ***simplicial complex.***

It is easily verfied — see [2] — that exterior shifting also preserves this closure property, i.e. if K is a simplicial complex, then so is $\Delta(K)$.

Another simple property of exterior shifting is that $K \geq M$ implies $\Delta(K) \geq \Delta(M)$. In this context we remark that the isomorphisms in cohomology induced by the seiving isomorphisms are *functorial* i.e. one has a commutative diagram

$$\begin{array}{ccc} H(K) & \xrightarrow{K} & H(\Delta(K)) \\ \downarrow & & \downarrow \\ H(M) & \xrightarrow{\Delta_M} & H(\Delta(M)) \end{array}$$

where the vertical homomorphisms are induced by the restriction maps.

References

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[4] K.S.Sarkaria, *Shifting and embeddability of simplicial complexes*, MPI preprint 92/51 (1992).