Extracts from my Notebooks (II)*

K. S. Sarkaria

§12. Euclidean motions.

(12.1) A rigid or Euclidean *motion* (or an isometry) $P \mapsto P'$ of a Euclidean space[1] *preserves distances between its points*: $|PQ| = |P'Q'|$. It is a *homeomorphism* because any sphere around P is imaged continuously and injectively, so also surjectively, on the sphere with the same radius around P'. We note next that any *triangle PQR is congruent to the triangle $P'Q'R'$.* In particular, if PQR is degenerate, with $R = tP + (1-t)Q$, then so is $P'Q'R'$, with $R' = tP' + (1-t)Q'$. Therefore, any (directed) *line λ is imaged to a line* $\lambda' = f(\lambda)$. Since corresponding angles of congruent triangles are equal, a Euclidean motion also *preserves angles*[2] *between lines*: $\text{angle}(\lambda, \mu) = \text{angle}(\lambda', \mu')$. A Euclidean motion with a fixed (or invariant) point P is called a (possibly orientation reversing) *rotation* around P. If we choose P as our origin, the Euclidean space becomes a vector space with a positive definite inner product, and it follows from above that a rotation around P is an isomorphism of this algebraic structure. A key fact about a general Euclidean motion is that 'it is not too far' from being a rotation.

(12.1.1) *Any Euclidean motion $f : V \to V$ has an invariant point $P = f(P)$ or an invariant line $\lambda = f(\lambda)$.*

The choice of an origin O equips our Euclidean space V with a vector space structure, and we can write $f = R + T$, where T denotes (addition of, or *translation* by) the vector $\overrightarrow{Of(O)}$, and R rotates each position vector $\overrightarrow{OP}$ to the position vector (at the origin O) which is equal and parallel to $\overrightarrow{f(O)f(P)}$.

The further choice of an orthonormal basis identifies V with $\mathbb{R}^n, n = \dim(V)$, and the motion $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by $P = (p_i) \mapsto (\sum_j R_{ij}p_j + t_i) = f(P)$,

*July 18, 2009 (the final footnote was added in April 2016).

[1] Euclidean 1-space (line) was defined by Eudoxus. If an $(n-1)$-subspace W of a Euclidean n-space V, $n \geq 2$, does not contain a point P, then Euclid's parallel postulate tells us that there is a unique disjoint $(n-1)$-subspace W_P which contains P. The distance on V is defined to be Pythagorean. Any ordered pair of points determines a vector $\overrightarrow{PQ}$ and a line $\{tP + (1-t)Q : t \in \mathbb{R}\}$. The choice of an origin O identifies V with a vector space with operations $\mathbf{p} + \mathbf{q}$, $t\mathbf{p}$ and dot product $\mathbf{p} \cdot \mathbf{q}$ if we put $\mathbf{p} = \overrightarrow{OP}$. Then the choice of an orthonormal basis identifies this vector space with $\mathbb{R}^n$ with the usual operations and dot product if we put $(p_1, \ldots, p_n)$ = components of $\mathbf{p}$ with respect to this basis.

[2] Angle is the distance, on the unit sphere around any O, between the points in which the rays from O parallel to the directed lines λ and μ cut this sphere: so $\text{angle}(\lambda, \mu) \in [0, \pi]$.

where $[R_{ij}]$ is the orthogonal matrix of the rotation R with respect to the chosen basis, and t_i is the ith component of the vector T.[3] Thus f can be represented by the $n \times (n+1)$ matrix $\begin{pmatrix} R & T \end{pmatrix}$.[4]

Let $W = \{P : R(P) = P\}$ be the *fixed subspace* of the rotation R. Note that W and its orthogonal complement $W^\perp$ are invariant subspaces of R. So, if the orthonormal basis is compatible with $V = W^\perp \oplus W$, f is represented by the matrix $\begin{pmatrix} A & O & T_1 \\ O & I & T_2 \end{pmatrix}$, where T_1 and T_2 are the components of T in $W^\perp$ and W respectively, and A denotes the orthogonal matrix of the restriction $A : W^\perp \to W^\perp$ of R. It does not have a nonzero fixed point.

So there is a unique $u \in W^\perp$ such that $A(u) - u = -T_1$. If we translate the above orthonormal basis from O to the new origin u, the matrix of f becomes $\begin{pmatrix} A & O & O \\ O & I & T_2 \end{pmatrix}$.[5] Therefore, unless $W = \{O\}$ when f is a rotation with unique fixed point u, $u + W$ contains a line λ parallel to T_2 which is invariant. *q.e.d.*

(12.1.2) The Euclidean subspace $u + W$ is called the *axis* of f. If f is a rotation, it is its fixed subspace. Otherwise, it is easy to check that *the invariant lines of f are all parallel to each other,* axis(f) *is the union of all the invariant lines of f, and the points of all the invariant lines of f are translated in the same direction by the same amount* $\mathrm{trans}(f) = |T_2|$ (the magnitude of the W-component of $T = \overrightarrow{Of(O)}$). Thus the Euclidean motion f has a fixed point iff $\mathrm{trans}(f) = 0$, and if the last member of the orthonormal basis of W is parallel to T_2, then the matrix of f simplifies further to $\begin{pmatrix} A & O & O & O \\ O & I & O & O \\ O & O & 1 & \mathrm{trans}(f) \end{pmatrix}$.

The *rotational part* R does not depend on the choice of the origin O. Indeed, its restriction $S^{n-1} \to S^{n-1}$ to the unit sphere around O (which determines it) is the same as the *induced map* $[\lambda] \mapsto [\lambda'],\ \lambda' = f(\lambda)$, if we identify each $P \in S^{n-1}$ with the *pencil* $[\lambda]$ of all directed lines λ parallel to $\overrightarrow{OP}$.[6] The *angle function* $S^{n-1} \to [0, \pi],\ [\lambda] \mapsto \mathrm{angle}(\lambda, \lambda')$ is well-defined and continuous. It takes only positive values iff $W = \{O\}$, so we have established the following.

(12.1.3) *A Euclidean motion has a unique fixed point iff its angle function is always positive; otherwise, it has an invariant line.*

[3]Cf. page 1 of *Differential Geometry Notes.*

[4]Or better, by $\begin{pmatrix} R & T \\ O & 1 \end{pmatrix}$, which converts composition to matrix multiplication, and exhibits the group of all Euclidean motions of n-space as a Lie subgroup of $GL(n+1, \mathbb{R})$.

[5]Because $(p_i) \mapsto (\sum_j R_{ij} p_j + t_i)$, i.e. $(p_i^* + u_i) \mapsto (\sum_j R_{ij}(p_j^* + u_j) + t_i)$, is the same as $(p_i^*) \mapsto (\sum_j R_{ij}(p_j^* + u_j) - u_i + t_i)$ in the new translated coordinates.

[6]Moreover, the *tangent bundle* $T(S^{n-1})$ can be viewed as the *space of all* (directed) *lines* of the Euclidean n-space by identifying each $\lambda \in P$ with its intersection with the tangent space T_P. There is also an induced map $\lambda \mapsto f(\lambda)$ in TS^{n-1}, and the two induced maps commute with the bundle projection $TS^{n-1} \to S^{n-1},\ \lambda \mapsto [\lambda]$. However, the zero section depends on O, and this induced $TS^{n-1} \to TS^{n-1}$ is seldom a vector bundle isomorphism. Similarly, *the space of all undirected lines* $\{\pm\lambda\}$ *of Euclidean n-space* can be identified with $T(\mathbb{R}P^{n-1})$, that of their parallel pencils with $\mathbb{R}P^{n-1}$, and the motion f induces natural maps in these, whose 2-fold covers are precisely the aforementioned induced spherical maps.

Here is another proof. If angle$(\lambda, \lambda') > 0$, and $O \in \lambda$, the *distance function* $P \mapsto |PP'|, P' = f(P)$, takes arbitrarily large values on all $P \in \lambda$ which are sufficiently far from O. So, in a sufficiently large ball around any O, this function takes values lower than at points outside ball. So there exists a P at which $|PP'|$ is minimum. We must have $P' = P$, otherwise $|MM'| = \cos(\theta/2).|PP'| < |PP'|$ – Fig. 11a – where M is the mid-point of PP'. There is no other fixed point, for then the line through the two fixed points would be fixed.

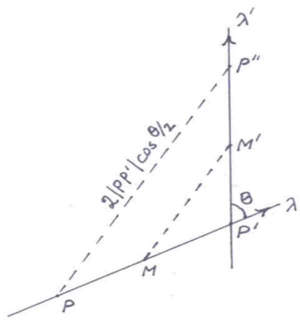

Figure 11a

When $\dim(V) = 1$ an isometry is either a translation or a reflection in a point. When $\dim(V) = 2$, and there is a pencil on which the angle function takes the value 0, i.e., a pencil whose lines are mapped to each other by f, then there is a *quotient isometry* on the 1-dimensional Euclidean space whose points are the lines of this pencil. If this has a fixed point, the line of the pencil – i.e., one of the vertical lines of Fig. 11b – corresponding to this fixed point is invariant. Otherwise, take any P and its f-image P', and observe that for its f-image P'', there are only two possibilities. For the first, P, P' and P'' are collinear, and this sloping line is invariant; for the second, the mid-point M of PP' is collinear with M' and M'', and this horizontal line is invariant.

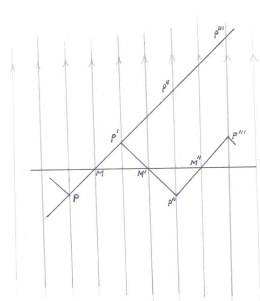

Figure 11b

For $\dim(V) = n \geq 3$, if there is a parallel pencil which is preserved by f (i.e., one on which the angle function takes the value 0) then the quotient isometry is on an $(n-1)$-dimensional Euclidean space. If it has an invariant point, the corresponding line of the parallel pencil is invariant; if it has an invariant line, the corresponding line of parallel lines gives us an invariant plane of f, and the restriction of f to this plane has an invariant line. *q.e.d.*

Two motions g and h have the same geometry if there is a motion k such that $kg = hk$, i.e., if $h = kgk^{-1}$. One then says that h is *conjugate* to g under k. A full list of conjugacy invariants of a motion is known, and it would be quite natural at this point to embark on this classification, but we'll postpone it to a later section. In the next section, using only the aforementioned facts about motions, *we'll show that* (8.2) *does generalize to higher dimensions* – in the sense that the finiteness problem posed in the first paragraph of (11.E) has a positive answer – *provided the tiling is isohedral.* In the course of this long argument we'll also use the following simple observation.

(12.1.4) *Commuting motions have intersecting axes.*

If g commutes with h, and λ is an invariant line of h, then so is $g(\lambda)$: $h(g(\lambda)) = g(h(\lambda)) = g(\lambda)$. So g maps $\mathrm{axis}(h)$ into itself, and there must be an invariant point or invariant line of g in $\mathrm{axis}(h)$, but this invariant point or line of g is also in $\mathrm{axis}(g)$. *q.e.d.*

§13. "Barlow's proof" of Bieberbach.

(13.1). We return again to (face-to-face and convex polyhedral) *tilings* of euclidean space. In §9 a tiling was called *isohedral* if it admitted a group of euclidean motions which acted transitively on its tiles; but from now on we'll prefer the synonym, *crystallographic,* and the same adjective shall also be used for any such group and its constituent euclidean motions. These crystallographic tilings, groups and motions are very special, and their classification (§11.F) in dimensions two and three remains one of the outstanding, and still inadequately understood, achievements of nineteenth century mathematics.

A translation is obviously a crystallographic motion. Generalizing a 3-dimensional argument of Schoenflies[7], Bieberbach[8] showed conversely that, *if a crystallographic motion of n-space has a rotational part sufficiently close to the identity, then it is a translation.* We'll show later that this original – and allegedly[9] intricate – proof hinges only on Euclid's geometric definition of irrationality. In this section however we'll employ a modification of the proof in Vince [13], which transforms it into a generalization of a 2-dimensional argument of Barlow, and yields the following result, which is *best possible,* because rotation by 60° preserves an equilateral tiling of the plane.

[7] Also given later by him as §5, *Unmöglichkeit irrationaler Drehungswinkel,* pp. 230-236, in his text-book: Artur Schoenflies, *Theorie der Kristallstruktur,* Berlin (1923).

[8] See especially §8, *Gruppen mit irrationaler Drehwinkeln,* of Ludwig Bieberbach, *Über die Bewegungsgruppen der Euklidischen Räume,* Math. Ann. 70 (1911), 297-336.

[9] See, for example, [10], p. 41, and [13], p. 27.

(13.2) A crystallographic restriction. *A Euclidean motion with a nonzero angle function less than $\pi/3$ cannot be crystallographic.*

Let $\mathfrak{T}$ be a face-to-face convex polyhedral tiling of n-space V, $\mathcal{G}$ a group of euclidean motions acting transitively on the tiles of $\mathfrak{T}$, and $g \in \mathcal{G}$ a motion such that $\|g\| := \sup(\text{angle}(g)) < \pi/3$, our job is to show that $\|g\| = 0$.

If this is not so, i.e., if g is not a translation, then $\text{axis}(g) \subsetneq V$; we choose any $P \in \text{axis}(g)$ as origin, and let W be the the orthogonal complement of the proper subspace $\text{axis}(g)$. The axis of any conjugate $g^k := kgk^{-1}$ is given by $\text{axis}(g^k) = k(\text{axis}(g))$. So, in case k is a translation, $\text{axis}(g^k)$ is parallel to $\text{axis}(g)$ and cuts W in a single point; and this last statement, namely that, $\text{axis}(g^k) \bigcap W = \text{point}$, remains true even if $\|k\| < \pi/2$.

Let $\mathcal{L}_{g,\epsilon}$ denote the set of all these intersection points $P^k := \text{axis}(g^k) \bigcap W$, as k runs over all members of the group $\mathcal{G}$ such that $\|k\| \leq \epsilon < \pi/2$.

(13.2.1) Lemma. *If $\epsilon > 0$, then $\mathcal{L}_{g,\epsilon}$ contains points other than P.*

Let S be a tile of $\mathfrak{T}$ containing P. Take any ray in W starting from P, and let $g_i(S), g_i \in \mathcal{G}$, be an infinite sequence of distinct tiles of $\mathfrak{T}$ intersected by this ray as we travel along on it. Using compactness, we choose a subsequence $g_{i(j)}$ whose rotational parts converge. So, for j_1, j_2 and $j_2 - j_1$ sufficiently large, $k := g_{i(j_2)}(g_{i(j_1)})^{-1} \in \mathcal{G}$ has an arbitrarily small angle function, and $\overrightarrow{Pg_{i(j_1)}(P)}$ and $\overrightarrow{g_{i(j_1)}(P)g_{i(j_2)}(P)}$ make arbitrarily small angles with the ray. So the k-image of the first reversed, i.e., $\overrightarrow{g_{i(j_2)}(P)k(P)}$, also makes an arbitrarily small angle with the ray, and the same is true also for the nonzero sum $\overrightarrow{Pk(P)}$ of these three vectors. This rules out that $P^k = k(\text{axis}(g)) \bigcap W$ is equal to P, for then, the line $Pk(P)$ would be on $k(\text{axis}(g))$, implying that its pre-image under k, a line on $\text{axis}(g)$ through P, has swung under k by almost ninety degrees. $\square$

(13.2.2) Lemma. *We can choose an $\epsilon > 0$* – independent of the choice of the origin $P \in \text{axis}(g)$ – *such that $P^{k_1} = P^{k_2}$ if and only if $g^{k_1} = g^{k_2}$ and $\mathcal{L}_{g,\epsilon}$ is a discrete subset of W.*

Since the tiles of $\mathfrak{T}$ are congruent convex polytopes, there are only finitely many at each vertex, and we can find a $\delta_{\mathfrak{T}} > 0$ such that, if the distance between any two points is less than this constant, then they lie in adjacent (i.e. non-disjoint) tiles; also we take care that $\delta_{\mathfrak{T}}$ is less than the magnitude of the, at most finitely many up to conjugacy, non-identity translations of $\mathcal{G}$ which throw a tile to an adjacent tile. Then we choose $\epsilon > 0$ such that $\delta_{\mathfrak{T}} - 2\epsilon|\text{trans}(g)| > 0$, and we also take care that 4ϵ is less than the minimum nonzero angle of the, at most finitely many up to conjugacy, non-translations of $\mathcal{G}$ which throw a tile on an adjacent tile. We assert that the conclusion holds for such an ϵ.

Otherwise, we can find $g^{k_1} \neq g^{k_2}$ with P^{k_1} and P^{k_2} within a distance $\frac{1}{2}(\delta_{\mathfrak{T}} - 2\epsilon|\text{trans}(g)|)$ of each other. So their translates $g^{k_1}(P^{k_1})$ and $g^{k_2}(P^{k_2})$, under g^{k_1} and g^{k_2} respectively, are within a distance $\frac{1}{2}(\delta_{\mathfrak{T}} + 2\epsilon|\text{trans}(g)|)$ of each other: this because the translations vectors of g^{k_1} and g^{k_2} are rotations of the same vector, $\text{trans}(g)$, by angles less than ϵ. On the other hand, the distance between $g^{k_1}(P^{k_2})$ and $g^{k_1}(P^{k_1})$, being the same as the distance between P^{k_2}

and P^{k_1}, is less than $\frac{1}{2}(\delta_{\mathfrak{T}} - 2\epsilon|\mathrm{trans}(g)|)$. It follows that the distance between $g^{k_1}(P^{k_2})$ and $g^{k_2}(P^{k_2})$ is less than $\delta_{\mathfrak{T}}$, and so they are in adjacent tiles of $\mathfrak{T}$. So $g^{k_1}(g^{k_2})^{-1}$, a motion whose angle function is bounded by 4ϵ – because $g^{k_1}(g^{k_2})^{-1} = k_1(gk_1^{-1}k_2g^{-1})k_2^{-1}$ – and which takes $g^{k_2}(P^{k_2})$ to $g^{k_1}(P^{k_2})$ at distance less than $\delta_{\mathfrak{T}}$ from it, and which maps a tile to an adjacent tile, must in fact be the identity map. The contradictory assumption $g^{k_1} \neq g^{k_2}$ is thus false, and the stated assertion is correct. □

(13.2.3) Lemma. *There exists a $C > 0$ such that, if g obeys $\|g\| < C$, then $\|kgk^{-1}g^{-1}\| \leq \|k\|$ for all k with $\|k\|$ sufficiently small.*

Taking rotational parts is a homomorphism, so we need only consider orientation preserving rotations of $\mathbb{R}^n$ about its origin, which form the smooth connected group manifold $SO(n)$. Now, $SO(n) \ni k \mapsto k \in SO(n)$ has the same and *positive* rate of change at $1 \in SO(n)$ in any radial direction. For each $g \in SO(n)$, we also have the smooth map $SO(n) \ni k \mapsto kgk^{-1}g^{-1} \in SO(n)$ which takes 1 to 1. Furthermore, this map varies smoothly with g. When $g = 1$, it coincides with the constant map, $k \mapsto 1$, which has *zero* rate of change at 1 in any radial direction. So, for g close enough to 1, the radial rates of change at 1 of $k \mapsto kgk^{-1}g^{-1}$ shall be strictly less than those of $k \mapsto k$, therefore we can find a $C > 0$ having the stated property. □

We'll first prove the result under the extra hypothesis $\|g\| < C$ on g.[10]

As $\epsilon > 0$ approaches zero, the set $\mathcal{L}_{g,\epsilon}$ stays put or becomes thinner, with $\mathrm{axis}(g^k)$ through its points P^k approaching parallelism with $\mathrm{axis}(g)$ through P. Using (13.2.1) and (13.2.2) we can choose a $Q = P^k \in \mathcal{L}_{g,\epsilon}$ which is at the minimum positive distance from P. (This nearest point can, à priori, move farther from P as our discrete set becomes thinner.)

Let $h = kgk^{-1}, \|k\| < \epsilon$, denote the conjugate of g whose axis cuts W in this nearest point Q. The conjugate of g with h is also of the type $g^k, \|k\| \leq \epsilon$, if ϵ is small enough: this because $hgh^{-1} = (kgk^{-1}g^{-1})g(kgk^{-1}g^{-1})^{-1}$ and $\|kgk^{-1}g^{-1}\| \leq \|k\| \leq \epsilon$ by (13.2.3). So $R = \mathrm{axis}(hgh^{-1}) \cap W$ is in $\mathcal{L}_{g,\epsilon}$.

This point R cannot be P. For, by (13.2.2) this is possible only if $hgh^{-1} = g$, i.e., if h and g commute. But then by (12.1.4) their axes, which are distinct, intersect at some point P', which contradicts (13.2.2) applied to P' as origin.

Also R is nearer to P than Q. To see this note that the axis of hgh^{-1} is obtained by applying h to $\mathrm{axis}(g)$. The translational vector of h is, for ϵ small, almost the same as that of g, so this part hardly moves $\mathrm{axis}(g)$. On the other hand, its rotational part is quantitatively the same as that of g, but about its own axis through Q. So the triangle PQR is almost isosceles, $QP \approx QR$, with angle at Q less than $\pi/3$. It follows that the third side PR is lesser than the other two almost equal sides. This contradiction, that $R \neq P$ is a point of $\mathcal{L}_{g,\epsilon}$ which is even nearer to P than Q, shows that our initial hypothesis was invalid: a $g \in \mathcal{G}$ satisfying $\|g\| < \min\{\pi/3, C\}$ must be a translation.

[10] We'll see later that *the conclusion of* (13.2.3) *is true iff* $0 < C \leq \pi/3$, so this step would conclude our proof of (13.2) if this stronger result is used instead of (13.2.3).

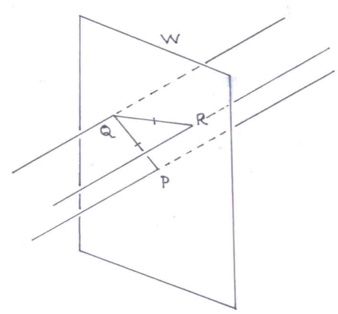

Figure 12a

To conclude, we repeat this argument without the extra hypothesis on g.

We now know that, *for $\epsilon > 0$ small, the k's are translations, so the sets $\mathcal{L}_{g,\epsilon}$ are the same, $\mathcal{L}_g$.* Conjugating g by the conjugate h corresponding to the chosen $Q \in \mathcal{L}_g$ at least positive distance from P is the same as conjugating by $kgk^{-1}g^{-1}$, a translation ($\|k\| = 0$ implies $\|kgk^{-1}g^{-1}\| = 0$), so R is in $\mathcal{L}_g$, is not P, but is nearer – see Fig. 12a – to it than Q because PQR is isosceles with angle at Q between equal sides less than $\pi/3$. This contradiction shows that a $g \in \mathcal{G}$ satisfying $\|g\| < \pi/3$ is a translation. *q.e.d.*

This restriction (13.2) is more than sufficient to deduce the following.

(13.3) Bieberbach's Theorem. *A crystallographic tiling of n-space is preserved by n linearly independent translations.*

Let a tiling $\mathfrak{T}$ of n-space V admit a group of euclidean motions $\mathcal{G}$ acting transitively on its tiles. For any ray in V with initial point P, the argument of (13.2.1) supplies us with a $k \in \mathcal{G}$ with angle less than $\pi/3$ and $\overrightarrow{Pk(P)}$ arbitrarily close to this ray. By (13.2) any such k is a translation. *q.e.d.*

Let $\mathcal{L}$ denote the subgroup of $\mathcal{G}$ consisting of all translations. It is abelian and normal – if k is a translation then so is any conjugate gkg^{-1} – indeed, *$\mathcal{L}$ is the unique maximal abelian and normal subgroup of $\mathcal{G}$.* Because, if h is a non-translation, then a conjugate khk^{-1} by a translation $k \in G$ transverse to axis$(h) \subsetneqq V$ has parallel disjoint axis $k(\text{axis}(h))$, so by (12.1.4), h cannot commute with this conjugate. *q.e.d.*

(13.4) It were pages 60-61 of the jail-book [8] that had led me to Figure 12a. Coxeter attributes the 2-dimensional argument on these pages to Barlow, 1890, the last of the triumvirate (after Fedorov and Schoenflies) who independently completed the classification of crystallographic groups for $n = 2, 3$.[11]

In fact, Barlow's full 2-dimensional argument, as given in [8], also yields an interesting addendum to (13.2). *If a crystallographic motion g has an axis*

[11] Much was known already, notably, Camille Jordan had found all the crystallographic groups of orientation preserving motions of 3-space about 25 years before! I was unable to locate Barlow's (or Fedorov's) original proof of the 3-dimensional case of (13.3).

of codimension two (note that codimension is $0, 1$ iff g is a translation, glide reflection in a hyperplane) *then it must rotate the pencil* $V_g \cong W$ *of codimension* 2 *subspaces parallel to* $\mathrm{axis}(g)$ *through* $\theta = 60°, 90°, 120°$ *or* $180°$.

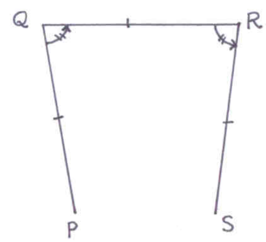

Figure 12b

This planar isometry has no fixed point other than P – and so rotates all points of W by a constant nonzero θ – otherwise, $\mathrm{axis}(g)$ is bigger. As before, choose a $Q \in \mathcal{L}_g \subset W$ at least positive distance from P. We cannot have θ between $0°$ and $60°$ because then R, the point of $\mathcal{L}_g$ obtained by rotating QP by θ about Q, is not P but is nearer to P than Q. Again, if θ were between $60°$ and $90°$, then rotating RQ about R through θ – see Fig. 12b – would give us a point S of $\mathcal{L}_g$ which is not P but is at a lesser distance from it than Q. Obtuse values of θ other than $120°$ and $180°$ can be similarly ruled out by rotating in the other direction through $\pi - \theta$, i.e., we now use, instead of R, the equidistant point on the line QR on the other side of Q, which is also in $\mathcal{L}_g$, etc. *q.e.d.*

Regarding the last step in the above argument note that in fact *all* points of the line QR whose distances from Q are integral multiples of $|QR|$ are in $\mathcal{L}_g$. Indeed, $\mathcal{L}_g$ is a *lattice,* i.e., a discrete maximal dimensional additive subgroup, of the vector space V_g. For it is the orbit of P under the translations induced in V_g by the, by (13.3) n-dimensional, abelian subgroup $\mathcal{L}$ of all translations of $\mathcal{G}$. *The rotation induced by g in the pencil V_g of subspaces parallel to its axes preserves this lattice $\mathcal{L}_g$.* For, any $k(\mathrm{axis}(g)) \in V_g$ goes to $g(k(\mathrm{axis}(g))) = (gkg^{-1})(\mathrm{axis}(g)) \in V_g$ under this induced rotation, and when translation k is in $\mathcal{G}$, then so is the translation gkg^{-1}.

These remarks indicate that an inductive classification of crystallographic motions is very much on the cards, however we'll postpone this till later, and turn now to our immediate goal (as declared towards the end of §12).

§14. Delone finiteness.

In this section we'll complete the proof of a finiteness theorem for crystallographic groups which was sketched by Delone. Also we'll show how the Bieberbach finiteness theorem is a particular case of this.

(13.5) Consider any (face-to-face) tiling $\mathfrak{T}$ of n-space V, with congruent copies of a strictly convex polyhedron S, which is crystallographic. That is, by (13.3), the group $\mathcal{L}$ of translations mapping its tiles to tiles contains n linearly independent members. Delone [17] showed that there are only finitely many combinatorial possibilities for $\mathfrak{T}$, in particular, that the number of facets of S is bounded in terms of n. His insight[12] was a simple argument which yields this lemma: *if distinct tiles T and $k^2(T)$, $k \in \mathcal{L}$, have facets in common with tile S, then $k(T) = S$.* Choose a point A in the relative interior of the common facet of S and T, and a point $k^2(B)$ in the relative interior of the common facet of S and $k^2(T)$. Since k is a translation, the points $\{A, B, k^2(B), k^2(A)\}$ are the four vertices – see Fig. 13 – of a parallelogram. The mid-point C of its diagonal $Ak^2(B)$ is an interior point of S, but, being also the mid-point of its median $k(A)k(B)$, it is also an interior point of $k(T)$: so $k(T) = S$. *q.e.d.*

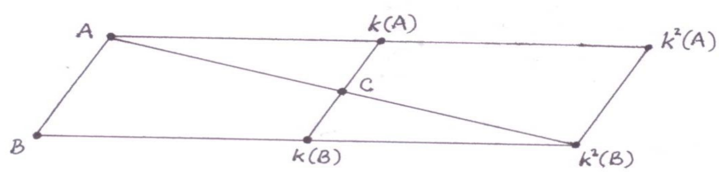

Figure 13

The case $\mathcal{L}$ transitive on tiles[13] of this lemma is due to Minkowski, 1897 (using special geometric properties[14] of tiles which hold in this case) who extracted from it this finiteness theorem: *a convex fundamental domain S of $\mathcal{L}$ has at most* $2(2^n - 1)$ *facets* (see Lehrsatz VI of [18]). Label each tile T by the necessarily unique $k \in \mathcal{L}$ such that $T = k(S)$. No tile other than S has a common facet with S and a label in the subgroup $\mathcal{L}^2$ of squares; for, if $k^2(S)$ were such a tile, then, by the lemma, $k(S) = S$, a contradiction because $k \neq \mathrm{id}$. So if a tile $T = t(S)$ other than S has a facet in common with S, then t is in some other coset of $\mathcal{L}^2$. Any other tile with label in the same coset is of the type $k^2(T), k \in \mathcal{L}$, and by the lemma, only one of these could possibly have a facet in common with S, because $k(T) = S$ if and only if $kt = \mathrm{id}$, i.e., $k = t^{-1}$. Therefore the number of facets of S is no more than twice the number of non-identity elements of $\mathcal{L}/\mathcal{L}^2 \cong \mathbb{Z}^n/2\mathbb{Z}^n \cong (\mathbb{Z}/2\mathbb{Z})^n$, i.e., at most $2(2^n - 1)$. *q.e.d.*

[12] Delaunay (the other spelling of his name) was past seventy then, and continued doing mathematics and mountaineering – his other love – well into his eighties. *Delone's Peak* in the Altai range is named after him. Likewise, *Alexander's Chimney* in the Rockies honours a famous (but shy: he used to go into his upper storey office by climbing through its window) topologist. Whitney, de Rham, Milnor, Herbrand, Paley and Hurewicz (the last three lost their lives in mountaineering accidents) are some other examples of this duality.

[13] The basic and, *by far,* the most important case: Number Theory – which, post-Minkowski, has returned to its geometric Euclidean roots – is largely the study of this case only!

[14] Notably, it is shown in [18] that a convex fundamental domain is centrally symmetric in this case; A. M. Macbeath, *On convex fundamental regions for a lattice,* Can. J. Math. 13 (1961) 177-178, deduces the same as a corollary of the Brunn-Minkowski inequality.

In fact, since $t^{-1} = (t^{-1})^2 t$ is in the coset of t, and $t(S) \cap S$ is a facet of S together with its translate $S \cap t^{-1}(S)$, *each translational coset of* $(\mathcal{L})^2$ *contributes* 0 *or* 2 *facets to* S*;* the number of facets of S is thus *even,* and obviously at least $2n$, in this $\mathcal{L}$ transitive case. Choose an interior point O of S as origin, and any basis $e_1, \ldots, e_n$ for the lattice $\mathcal{L}(O)$. Then each tile contains a unique point with integral coordinates, and the result can be restated thus: if tiles containing the integral points $(t'_1, \ldots, t'_n)$ and $(t_1, \ldots, t_n)$ share a facet with S and $(t'_1, \ldots, t'_n) \equiv (t_1, \ldots, t_n) \bmod 2$, then $(t'_1, \ldots, t'_n) = -(t_1, \ldots, t_n)$. Two two-dimensional examples are shown in Fig. 14, the integral points belonging to the 4 cosets of $\mathcal{L}^2(O) = 2(\mathcal{L}(O))$ have been assigned 4 distinct symbols.

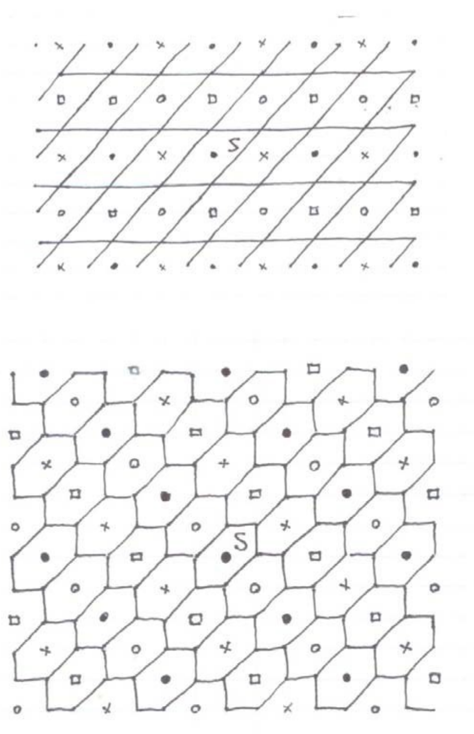

Figure 14

When $\mathcal{L}$ is not transitive on tiles, we can and shall uniquely label the tiles of the crystallographic $\mathfrak{T}$ by throwing in some additional, say δ, right[15] cosets of $\mathcal{L}^2$. For example, "Vibhor's" tiling requires 4 more cosets which are shown coloured in Fig. 15, note that each of these contains a unique tile obtained from S by a half-rotation around the mid-point of a side (in this example the 4 uncoloured translational cosets don't contribute any side to S).

[15]Since $(\mathcal{L})^2 g = g(\mathcal{L})^2$ for any motion g, we'll usually drop this adjective.

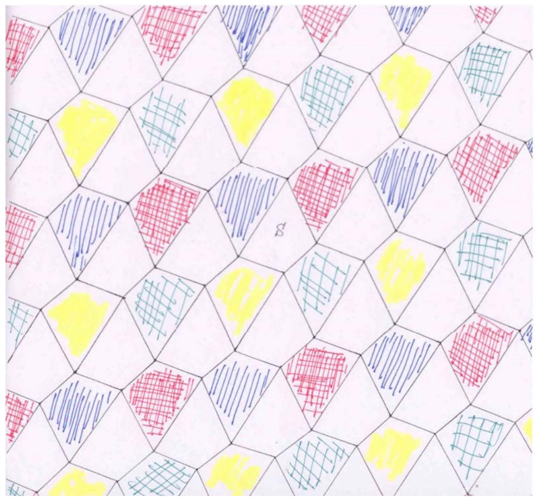

Figure 15

Quite generally, *each non-translational coset can contain the label t of at most one tile T having a facet in common with S.* Otherwise, as we saw above using our lemma, $t = k^{-1}$, a translation, which is now impossible, *q.e.d.* So it follows that, *S has at most $2(2^n - 1) + \delta$ facets,* we show next that δ is finite, and bounded by a function of n.

The above cosets are subsets of $\mathcal{G}$, the group of all motions which preserve $\mathfrak{T}$. Any $g \in \mathcal{G}$ maps the lattice $\mathcal{L}(O)$ on the translated — by $\overrightarrow{Og(O)}$ — lattice $\mathcal{L}(g(O))$, for $g(k(O)) = (gkg^{-1})(g(O))$ and $k \in \mathcal{L} \Rightarrow gkg^{-1} \in \mathcal{L}$. So g maps D_O, the set of all points no further from O than any other point of $\mathcal{L}(O)$, on the translated set of all points no further from $g(O)$ than any other point of $\mathcal{L}(g(O))$. The rotation induced by g, in the lines through O, therefore preserves the subset of lines that pass through the finitely many vertices of the (Dirichlet) fundamental domain D_O of $\mathcal{L}$, a polytope with at most $2(2^n - 1)$ facets. The number of these vertices, so these rotations, is bounded in terms of n. Since two motions are in the same coset of $\mathcal{L}$ iff they induce the same rotation, *the quotient $\mathcal{G}/\mathcal{L}$ is finite, with cardinality bounded by a function of n.* The total number of cosets of $(\mathcal{L})^2$ is 2^n times this cardinality $|\mathcal{G}/\mathcal{L}|$, and the number required to uniquely label the tiles is $2^n(|\mathcal{G}/\mathcal{L}| \div |\mathcal{H}|)$, where $\mathcal{H}$ denotes the finite subgroup of $\mathcal{G}$ consisting of motions which map S on itself.[16] Exactly 2^n of these are translational and constitute $\mathcal{L}$, so $\delta = 2^n(|\mathcal{G}/\mathcal{L}| \div |\mathcal{H}|) - 2^n$.

The group $\mathcal{G}$ was not required to act transitively on the vertices of $\mathfrak{T}$, and these may even have different valences, but, *each vertex of the tiling $\mathfrak{T}$ is incident to at most $2^n(|\mathcal{G}/\mathcal{L}| \div |\mathcal{H}|)$ tiles.* We shall—Delone argues differently—again use Figure 13, this time with $S = T$, and A any point in the interior of T, to see

[16]In [17], Delone mentions that, unlike regular spherical tilings – the symmetry group A_5 of the icosahedron, which has 20 tiles, has no subgroup with 20 elements – *no example of a crystallographic tiling $\mathfrak{T}$ is known for which $\mathcal{G}$* does not *contain a subgroup which acts simply transitively on the tiles:* is such an example known now?

that, *a tile T cannot share any face with $k^2(T)$, $k \neq \mathrm{id}$, $k \in \mathcal{L}$:* otherwise, if $k^2(B) \in T \cap k^2(T)$, the point C is both in the interior of T, as well as in another tile $k(T)$, which is not possible. It follows that the number of tiles at any vertex is at most equal to the number of $\mathcal{L}^2$ cosets used in the labelling. *q.e.d.*

Thus there are finitely many possibilities for the local combinatorics of the crystallographic $\mathfrak{T}$, however, the argument sketched in [17] (it does not even invoke the simple connectedness of n-space) is inadequate for its main theorem, viz., *there are only finitely many topologically distinct crystallographic tilings of n-space by congruent copies of a strictly convex polytope.*[17] These inadequacies are addressed in the next subsection.

(13.6) There is still some juice left in Figure 13, it shows also that, *if tiles T and $k^2(T)$, $k \neq \mathrm{id}$, $k \in \mathcal{L}$, share faces with a tile S, then these shared faces are incident to a bigger dimensional face of S which is also a face of $k(T)$.* For, if A and $k^2(B)$ are in the relative interiors of the, by above necessarily disjoint, faces $S \cap T$ and $S \cap k^2(T)$ of S, then C is in the relative interior of such a bigger dimensional face of S, as well as in $k(T)$. *q.e.d.*

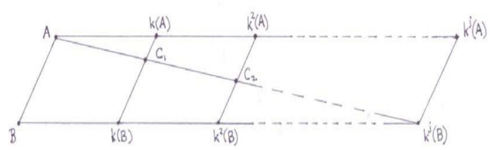

Figure 16

Further, the longer Figure 16 above shows that, *if T shares a face with S, then, for $j \geq 3$, no $k^j(T)$, $k \neq \mathrm{id}$, $k \in \mathcal{L}$, can share a face with S.* Take A in the relative interior of $S \cap T$, and, if possible, $k^j(B)$ in the relative interior of $S \cap k^j(T), j \geq 3$. Then C_2 is in the relative interior of a face of S containing $S \cap T$, as well as in $k^2(T)$. So T and $k^2(T)$ have $S \cap T$ as a common face, which is not possible, since we saw above that these tiles are disjoint. *q.e.d.*

In other words, for $j \geq 3$, an $\mathcal{L}^j$ coset contains the label t of *at most one* tile T which shares a face with S. Therefore, *the number of tiles sharing a face with S is at most $|\mathcal{G}/\mathcal{L}^3| \div |\mathcal{H}|$.* In particular, for the number-theoretical case $\mathcal{G} = \mathcal{L}$, we obtain the bound n^3 because $\mathcal{L}/\mathcal{L}^3 \cong \mathbb{Z}^n/3\mathbb{Z}^n \cong (\mathbb{Z}/3\mathbb{Z})^n$, and the ordinary tiling by n-cubes shows that this bound is the *best possible.*

For the sake of simplicity, we'll first do the case when a group of motions acts simply transitively on tiles, and now use $\mathcal{G}$ to denote this group. Also, it is convenient to reason on a *Poincaré dual* $\mathfrak{V}$ of the cell-subdivision $\mathfrak{T}$, i.e.,

[17] I was unable to obtain the more detailed paper, B. N. Delone and N. N. Sandakova, *Theory of Stereohedra* (Russian), Trudy Mat. Inst. Steklov 64 (1961) 28-51, but its review by Schwerdtfeger, MR0137029 (25#487), also indicates similar misgivings.

a cell-subdivision of n-space whose codimension j cells have as vertices some chosen "barycenters" $v_S, v_T, v_U, \ldots$ of the tiles $S, T, U, \ldots$ incident to the relative interior of the corresponding j dimensional cell of $\mathfrak{T}$. As before, we choose a *base tile* S, and give to any tile T of $\mathfrak{T}$ the *label* $t \in \mathcal{G}$ such that $t(S) = T$, and the same label t shall also be given to the dual vertex v_T of $\mathfrak{V}$. On the other hand, the coset $[t] \in \mathcal{G}/\mathcal{L}^3$ shall be called the *colour* of T or v_T. We note that the cardinality of this finite *group of colours* $\mathcal{C} = \mathcal{G}/\mathcal{L}^3$, so the number of its isomorphism types, is bounded in terms of n. Since the action of $\mathcal{G}$ on $\mathfrak{V}$ is transitive on its vertices, their closed *stars* $\mathrm{St}_{\mathfrak{V}}(v_T)$ are isomorphic under it. So the last result shows that, *the vertices of each star carry distinct colours, and there is a unique incidence preserving isomorphism* $\mathrm{St}_{\mathfrak{V}}(v_S) \to \mathrm{St}_{\mathfrak{V}}(v_T)$*, which left multiplies the colour of each vertex by* $[t] \in \mathcal{C}$*.*

We'll say that two tilings $\mathfrak{T}$ and $\mathfrak{T}'$ have the same *coloured star type* if there is a finite group isomorphism/identification, $\mathcal{C} \cong \mathcal{C}'$, plus an incidence preserving isomorphism $\mathrm{St}_{\mathfrak{V}}(v_S) \to \mathrm{St}_{\mathfrak{V}'}(v_{S'})$, which takes each vertex to a vertex of the 'same' colour. Since the number of coloured star types of tilings is bounded in terms of n, it suffices to show that, *if two tilings have the same coloured star type, then they have the same topological type.* More precisely, we'll show that, the given colour and incidence preserving isomorphism of basic stars *extends in a unique way* to an incidence preserving isomorphism of cell-subdivisions, $\mathfrak{V} \to \mathfrak{V}'$, which takes each vertex v to a vertex v' of the same colour.

To see that there is a *unique* colour and incidence preserving extension, note that the vertex v_S has to go to $v_{S'}$, and, because n-space is connected, any two vertices v and w of $\mathfrak{V}$ can be joined to each other by an *edge path* $vv_1, v_1v_2, \ldots, v_{r-1}w$. So it suffices to check that the image w' of any vertex $w \in \mathrm{St}_{\mathfrak{V}}(v)$ is uniquely determined by the image v' of v. This is true because v and v' have the same colour c, so there is a unique colour and incidence preserving isomorphism, $\mathrm{St}_{\mathfrak{V}}(v) \to \mathrm{St}_{\mathfrak{V}}(v_S) \to \mathrm{St}_{\mathfrak{V}'}(v_{S'}) \to \mathrm{St}_{\mathfrak{V}'}(v')$, where the first and third arrows left multiply colours of vertices by c^{-1} and c.

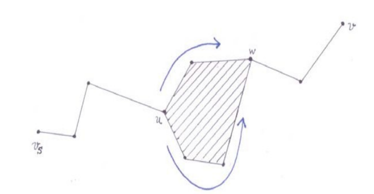

Figure 17

So, using an edge path $v_Sv_1, v_1v_2, \ldots, v_{r-1}v$ of length r from v_S to v, we can compute the image v' of v in r steps as follows: $(v_1)'$ is the image of v_1 under $\mathrm{St}_{\mathfrak{V}}(v_S) \to \mathrm{St}_{\mathfrak{V}'}(v_{S'})$, then $(v_2)'$ is the image of v_2 under (the unique colour and index preserving isomorphism) $\mathrm{St}_{\mathfrak{V}}(v_1) \to \mathrm{St}_{\mathfrak{V}'}(v_1)'$, and so on, finally v' is the image of v under $\mathrm{St}_{\mathfrak{V}}(v_{r-1}) \to \mathrm{St}_{\mathfrak{V}'}(v_{r-1})'$. We now turn things around, and make this recipe our *definition of* v' because it gives the same answer for any edge path from v_S to v: n-space is simply connected, so any two edge paths of $\mathfrak{V}$ from v_S to v are related by finitely many moves – see Figure 17 – in which a portion $\widetilde{uw}$ of the edge path sweeps over a single 2-cell, but, since a cell is in the stars of all its vertices, we obtain the same u' and w' by either route. This colour preserving vertex map $v \mapsto v'$ extends uniquely to an incidence preserving cellular map $\mathfrak{V} \to \mathfrak{V}'$ because the set of vertices $\{a, b, \dots\}$ of any cell is mapped bijectively to the subset $\{a', b', \dots\}$ of vertices of $\mathrm{St}_{\mathfrak{V}'}(a')$ having the same colours, which is the set of vertices of a cell of the same dimension. Finally, we note that, the inverse isomorphism $\mathrm{St}_{\mathfrak{V}'}(v_{S'}) \to \mathrm{St}_{\mathfrak{V}}(v_S)$ of basic stars, extends similarly to a unique colour and incidence preserving cellular map $\mathfrak{V}' \to \mathfrak{V}$, which is the two-sided inverse of $\mathfrak{V} \to \mathfrak{V}'$. *q.e.d.*

For an arbitrary $\mathcal{G}$ transitive on tiles, the isotropy subgroup $\mathcal{H}$ of S is no longer trivial, but it is finite and shares only the identity with $\mathcal{L}$. We'll *label* each tile $T = t(S)$ and its dual vertex v_T by the left coset $t\mathcal{H}$, i.e., by *all* motions of $\mathcal{G}$ that map S to T; its left translate $g(t\mathcal{H})$ is thus the label of $g(T)$. The double coset $\mathcal{L}^3 t\mathcal{H}$, i.e., the left coset $[t]\mathcal{H}$ of $\mathcal{H}$ in the finite quotient group $\mathcal{C} = \mathcal{G}/\mathcal{L}^3$, shall be the *colour* of T or v_T; its left translate $[g]([t]\mathcal{H}) = (\mathcal{L}^3 g)(\mathcal{L}^3 t\mathcal{H}) = \mathcal{L}^3 gt\mathcal{H}$ is thus the colour of $g(T)$. We saw that, of these $|\mathcal{G}/\mathcal{L}^3| \div |\mathcal{H}|$ colours, the ones occurring in the closed star of v_S are all distinct. So there is a unique incidence preserving isomorphism $\mathrm{St}_{\mathfrak{V}}(v_S) \to \mathrm{St}_{\mathfrak{V}}(v_T)$ which left multiplies the colour of each vertex by $[t] \in \mathcal{C}$. Two tilings $\mathfrak{T}$ and $\mathfrak{T}'$ shall have the same *coloured star type* iff there is an isomorphism/identification of finite group pairs, $(\mathcal{C}, \mathcal{H}) \cong (\mathcal{C}', \mathcal{H}')$, plus an incidence preserving isomorphism $\mathrm{St}_{\mathfrak{V}}(v_S) \to \mathrm{St}_{\mathfrak{V}'}(v_{S'})$, which takes each vertex to a vertex of the 'same' colour. The number of these types is bounded in terms of n, and, just as above, we can see that such an isomorphism of basic stars *extends in a unique way* to an incidence preserving isomorphism of cell-subdivisions, $\mathfrak{V} \to \mathfrak{V}'$, which takes each vertex v to a vertex v' of the same colour. So the number of topological types of crystallographic tilings of n-space by congruent copies of a convex tile is bounded in terms of n. *q.e.d.*

This suggested a rather sweeping topological generalization.

(13.7) *Any simply connected manifold V can admit, for each N, only finitely many topological types of cell-subdivisions $\mathfrak{V}$ having isomorphic vertex-stars with at most N vertices.*

We note that, $N^2 - 2N + 2$ *colours can be given to the vertices of $\mathfrak{V}$ in such a way that vertices of each star have distinct colours.*[18] For, if a coloring is 'bad' at v, i.e., if v has the same colour as one of the, at most $N-1$, other vertices in $\mathrm{St}_{\mathfrak{V}}(v)$, then we can re-colour (only) v differently from the, at most $(N-1)^2 = N^2 - 2N + 1$, vertices other than v in the stars of these $N-1$ vertices; this operation keeps the 'good' vertices good, and reforms v.

Further, in a $\mathfrak{V}$ equipped with a vertex colouring $v \mapsto \mathrm{col}(v)$ by M colours such that the vertices of each star have distinct colours, we can choose a base

[18] This bound can be improved, but, even in the planar (e.g., *non-euclidean crystallographic*) case, the vertices in a star, so the minimum number of colours required, can be arbitrarily large. As against this, the *Four Colour Theorem* implies that 4 colours can be assigned to the vertices of a planar $\mathfrak{V}$ in such a way that the vertices of any *edge* have distinct colours.

vertex v_0, and then, for each vertex v, a permutation $\mathrm{per}(v)$ of the M colours inducing a combinatorial isomorphism $\mathrm{St}_{\mathfrak{V}}(v_0) \to \mathrm{St}_{\mathfrak{V}}(v)$, taking care to choose for v_0 the identity permutation. Then $v \mapsto (\mathrm{col}(v), \mathrm{per}(v))$ is a finer colouring by at most $M \times M!$ colours, having the additional property that, for any pair of vertices (v, w), a combinatorial isomorphism $\mathrm{St}_{\mathfrak{V}}(v) \to \mathrm{St}_{\mathfrak{V}}(v_0) \to \mathrm{St}_{\mathfrak{V}}(w)$ is uniquely specified by the finer colours of v and w.

So a *colour box* $\mathcal{C}$ shall be a finite set which is equipped[19] for each ordered pair (p, q) in it, with a unique permutation $p \to q$ taking p to q, such that $p \to p = \mathrm{id}$, and $p \to q \to r = p \to r$, and we'll only consider cell-subdivisions $\mathfrak{V}$ with a vertex colouring $v \mapsto \mathrm{col}(v) \in \mathcal{C}$, which is one-one on each star, and such that the permutation of colours $p \to q$ defines a combinatorial isomorphism $\mathrm{St}_{\mathfrak{V}}(v) \to \mathrm{St}_{\mathfrak{V}}(w)$ whenever $\mathrm{col}(v) = p$ and $\mathrm{col}(w) = q$.

Two such coloured cell-subdivisions, $\mathfrak{V}$ of V, and $\mathfrak{V}'$ of V, shall have the same *coloured star type* if there is an isomorphism/identification, $\mathcal{C} \cong \mathcal{C}'$ of their colour boxes, plus an incidence preserving isomorphism $\mathrm{St}_{\mathfrak{V}}(v) \to \mathrm{St}_{\mathfrak{V}'}(v')$ of two stars, which takes each vertex to a vertex of the 'same' colour.

We note that the number of coloured star types is bounded in terms of N. Also, since V and V' are (connected and) simply connected[20], repeating the previous argument we see that this isomorphism between basic stars $\mathrm{St}_{\mathfrak{V}}(v) \to \mathrm{St}_{\mathfrak{V}'}(v')$ extends to a unique colour and incidence preserving isomorphism of cell-complexes $\mathfrak{V} \to \mathfrak{V}'$. This establishes the required finiteness. *q.e.d.*

This finiteness does not hold under the weaker hypothesis that $\mathfrak{T}$ has isomorphic tiles: for example, using Figure 5 of (7.2), it is easy to show that, *the plane admits uncountably many combinatorially distinct quadrilateral tilings.*

Our proof of *Delone's finiteness theorem* (13.6) also suggested a direct and simple proof of *Bieberbach's finiteness theorem* (13.8) (posed by Hilbert in 1900 as the first part of his Problem XVIII) which, thanks to (13.4), is equivalent to the statement that $\mathbb{Z}^n$ *has only finitely many* extensions *by finite groups of a bounded order;* this enables us to postpone group cohomology till later.

(13.8) *There are, up to isomorphism, only finitely many crystallographic groups $\mathcal{G}$ of motions of n-space.*

For each $\mathcal{G}$, choose a tiling $\mathfrak{T}$ – the Dirichlet-Voronoi tiling of any orbit will do – on which this group of motions acts simply transitively. As before, we choose a base tile S, and label and colour the tiles of $\mathfrak{T}$ – so the dual vertices of its Poincaré dual $\mathfrak{V}$ – by $\mathcal{G}$ and $\mathcal{C} = \mathcal{G}/\mathcal{L}^3$. *The coloured star type of $\mathfrak{T}$ determines its simply transitive group $\mathcal{G}$!* For, if a tile T has colour $c \in \mathcal{C}$, left multiplication by c defines an isomorphism, $\mathrm{St}_{\mathfrak{V}}(v_S) \to \mathrm{St}_{\mathfrak{V}}(v_T)$, which we can extend – using edge paths, and invoking the simple-connectedness of V, like the argument in (13.6) – to a unique combinatorial isomorphism $\mathfrak{V} \to \mathfrak{V}$ which left multiplies the colour of each vertex by c, and which must coincide, by uniqueness, with *the* unique group element t such that $T = t(S)$ and $[t] = c$. *q.e.d.*

[19] According to a Wikipedia article, similar *groupoid* structures occur in *quasicrystallography,* which made me wonder: does (13.7) apply to such tilings?

[20] *Simple-connectedness is necessary:* the planar square tiling has the same local combinatorics as any disjoint union of its copies, or, any square tiling on a torus.

When $\mathcal{G}$ is (merely) transitive, the colour of T is a left coset $c\mathcal{H}$ of the isotropy subgroup $\mathcal{H} \subset \mathcal{C}$ of the base tile, and, for any $h \in \mathcal{H}$, left multiplication by $ch \in \mathcal{C}$ defines a combinatorial isomorphism $\mathrm{St}_{\mathfrak{V}}(v_S) \to \mathrm{St}_{\mathfrak{V}}(v_T)$ which we can extend to a unique combinatorial isomorphism $\mathfrak{V} \to \mathfrak{V}$ left multiplying the colour of each vertex by ch; thus we again recover, from the coloured star type, all the group elements t such that $T = t(S)$ and $[t] \in c\mathcal{H}$.

Let us call a manifold cell-subdivision $\mathfrak{T}$ a *combinatorially crystallographic tiling* if the group $\mathcal{G}$ of its combinatorial automorphisms is transitive on its highest dimensional cells. Then $\mathcal{G}$ acts transitively on the vertices of its Poincaré dual $\mathfrak{V}$, which therefore has isomorphic vertex-stars. So (13.7) implies in particular that, *a simply connected manifold V possesses only finitely many combinatorially crystallographic tilings with at most N vertices in its dual vertex stars,* but the colouring used in that proof had nothing to do with $\mathcal{G}$, so we cannot hope to recover $\mathcal{G}$ from that coloured star type. However, if $\mathcal{G}$ has a normal subgroup $\mathcal{N}$ of bounded index, such that the natural colouring by $\mathcal{C} = \mathcal{G}/\mathcal{N}$ is one-one on each vertex star[21], and if the coloured star type is defined as before using this colouring, then we can recover $\mathcal{G}$ in a similar way from it.

(13.9) For a group $\mathcal{G}$ of motions acting transitively on the tiles of a given tiling $\mathfrak{T}$ of n-space by convex congruent polygons, we know that a $g \in \mathcal{G}$ satisfying $\|g\| < \pi/3$ is a translation. To establish this point (13.2) we had used, in the proofs of (13.2.1) and (13.2.2), *not* $\mathfrak{T}$ itself, but *other* (Dirichlet-Voronoi) tilings of the group. We can avoid this detour by making these changes:-

For (13.2.1), choose, in succession, tiles D_i of $\mathfrak{T}$ encountered by the said ray in W, and then, for each i, a $g_i \in \mathcal{G}$ taking P to a point of D_i. The rest of the proof is as before.

For (13.2.2) "$\epsilon > 0$ small" is now chosen differently. There exists a $\delta_{\mathfrak{T}} > 0$ such that, if the distance between any two points is less than this constant, then they must lie in adjacent (i.e. non-disjoint) tiles. Firstly, we'll ensure $\delta_{\mathfrak{T}} - 2\epsilon|\mathrm{trans}(g)| > 0$. So, P^{k_2} and P^{k_1} within a distance $\frac{1}{2}(\delta_{\mathfrak{T}} - 2\epsilon|\mathrm{trans}(g)|)$ of each other shall imply that the distance between any two of the points $\{g^{k_2}(P^{k_2}), g^{k_1}(P^{k_2}), g^{k_2}(P^{k_1}), g^{k_1}(P^{k_1})\}$ is less than $\delta_{\mathfrak{T}}$, and so they are in adjacent tiles of $\mathfrak{T}$. Secondly, we'll also ensure that 4ϵ is less than the minimum nonzero angle of a non-translation of $\mathcal{G}$ throwing a tile on an adjacent tile; and that $\delta_{\mathfrak{T}} > 0$ is also less than the magnitude of any non-identity translation of $\mathcal{G}$ throwing a tile to an adjacent tile (this is possible to arrange because there are only a finite number of motions throwing a tile S to an adjacent tile). So $(g^{k_2})^{-1}(g^{k_1})$, a motion whose angle function is bounded by 4ϵ, which takes $g^{k_2}(P^{k_2})$ to $g^{k_1}(P^{k_2})$ at distance less than $\delta_{\mathfrak{T}}$ from it, and which maps a tile to an adjacent tile, must in fact be the identity map. The contradictory assumption $g^{k_2} \neq g^{k_1}$ is thus false, and the stated assertion is correct.

§14. Affine crystallography. Euclid's geometry already involves plenty

[21] I don't know if an $\mathcal{N}$ always exists if the number of vertices in a star is bounded by N; but, *if $\mathfrak{T}$ is a euclidean (non-convex) crystallographic tiling, some $\mathcal{L}^{q(N)}$ will do the job.*

of spaces which are only *locally* Euclidean, that is, which are *manifolds.*[22] For example, there is the connected and closed manifold $SO(n)$ of orientation preserving rigid motions of n-space V around a fixed origin, and the bigger open manifold $\mathcal{M}_n$ of all rigid motions of n-space which has two components (of all orientation preserving and all orientation reversing motions).

The quickest way of checking this – following essentially Descartes! – is once again to use, as in (12.1.1), an orthonormal basis at the origin, which puts motions $g \in \mathcal{M}_n$ in one-to-one correspondence with matrices $\begin{pmatrix} R & T \\ O & 1 \end{pmatrix}$ where R is an orthogonal $n \times n$ matrix (i.e., each column has norm 1 and each pair of distinct columns dot product 0) and T is any n-vector. The implicit function theorem then exhibits $\mathcal{M}_n$ as an open (and smooth, even algebraic) submanifold of $(n+1)^2$-space; and $SO(n)$ as its closed submanifold determined by the further conditions, $\det(R) = 1$ and $T = 0$.

This cartesian or algebraic method has the advantage that composition of motions becomes matrix multiplication which is so very mechanical![23] So it is the method of choice in text-books, e.g., in Charlap [10], and we'll use below his abbreviated notation $[R\ \ T]$ for $\begin{pmatrix} R & T \\ O & 1 \end{pmatrix}$. However our proof of the following result of Bieberbach and Frobenius is less matricial than Charlap's, and the subsequent comments (14.1.1) make it almost entirely geometric.

This result involves the bigger manifold group $\mathcal{A}_n$ of all affine motions of V – i.e., we allow R to be any non-singular $n \times n$ matrix – and implies that, up to conjugacy in this bigger group, there are only finitely many discrete subgroups $\mathcal{G}$ of $\mathcal{M}_n$ with a compact orbit space $V/\mathcal{G}$. Which raises like questions about other discrete subgroups of $\mathcal{A}_n$, i.e., leads into affine crystallography[24] which

[22] All our manifolds shall be Hausdorff unless otherwise mentioned, with special attention given to those which are usually connected and compact, i.e., to *closed manifolds.* To me these have been, from the day I learnt about them, *obviously natural (= God-given) objects*, and it is not in the least surprising that Mathematics of the last 100 years has virtually revolved around them! This compelling mental construct is to my mind, *far more 'real', 'tangible', and central for an understanding of Nature* (as a whole) than, say, the mostly vague things like 'elementary particles' that have been so successfully sold to the lay public.

[23] Algebra is said to have begun only when some propositions in the *Elements* were converted into symbolic mechanical rules by later Central Asian mathematicians. (Perhaps because they too had to teach students who wanted to get the answer without thinking?) However this assertion glosses over the fact that, *Euclid himself was possibly more an algebraist than a geometer:* with his axiomatic method, he had in fact tried to convert *all* the geometry of his predecessors into formal language! Admittedly, he fell short – complete formalizations of his, and other related geometries, appeared only in the nineteenth century – but it is remarkable how often he had put his finger on the key point. For example, it is uncanny how the simple (with hindsight) device of just discarding his fifth postulate – note that we are only making his geometry simpler – immediately puts into our kitty all closed 2-manifolds! Continuing this game – I am referring here to Thurston's geometrization theorem, now proved in full by Perelman – we also get all closed 3-manifolds: a great result that Euclid would doubtless be able to relish, and relate to intimately, were he to re-visit us today.

[24] The 'great game' of the last footnote was launched by Poincaré's insight that, *one obtains* all *closed* 2*-manifolds as similar orbit spaces* $V/\mathcal{G}$*, provided one discards Euclid's fifth postulate.* Affine crystallography is another part of the 'great game': this time we discard the metrical postulates of that venerable 'algebraist' Euclid, but retain his fifth postulate.

involves – as we'll show by examples – new and interesting features.

(14.1) *For any abstract isomorphism $f : \mathcal{G} \to \mathcal{G}'$ of crystallographic groups of motions of V, there exists an affine isomorphism $\phi : V \to V$ such that $\phi \circ g \circ \phi^{-1} = f(g)$ for all $g \in \mathcal{G}$.*

Proof. From (13.3), the translation subgroups $\mathcal{L}$ and $\mathcal{L}'$ contain $\dim(V)$ linearly independent translations, and are the maximal abelian normal subgroups of $\mathcal{G}$ and $\mathcal{G}'$. So f restricts to an isomorphism $\mathcal{L} \to \mathcal{L}'$, $k \mapsto k'$, and if k and k' translate an origin O by $\overrightarrow{OP}$ and $\overrightarrow{OP'}$, there is a unique linear isomorphism $\psi : V \to V$ such that $\psi(P) \equiv P'$, i.e., $\psi \circ k \circ \psi^{-1} = k' = f(k)$ for all $k \in \mathcal{L}$.

Replacing $\mathcal{G}$ by $\psi(\mathcal{G})$ the problem reduces to *the case when $\mathcal{G}$ and $\mathcal{G}'$ have the same translation subgroup $\mathcal{L} = \mathcal{L}'$ on which f is the identity.* We identify $V \sim \mathbb{R}^n$ by choosing an orthonormal basis at the origin, so any motion $g \sim [R \;\; T]$, the matrix such that $g(v) = Av + T \;\forall\; v \in \mathbb{R}^n$. Regarding the isomorphism $f : \mathcal{G} \to \mathcal{G}'$, $g \sim [R \;\; T] \mapsto [R' \;\; T'] \sim g'$, we note that, though T' depends on both R and T, *the rotational part R' of $g' = f(g)$ is equal to the rotational part R of g.* For, $[R \;\; T][I \;\; U][R \;\; T]^{-1} = [I \;\; RU]$ (i.e., $g \circ (\;\;) \circ g^{-1}$ is a linear isomorphism of the vector space of all translations, and coincides with the rotational part of g), and on applying f to this equation, its left side becomes $[R' \;\; T'][I \;\; U][R' \;\; T']^{-1} = [I \;\; R'U]$, while the right side stays $[I \;\; RU]$.

It remains to find a translation $\phi \sim [I \;\; U]$ such that $\phi \circ g \circ \phi^{-1} \equiv g'$, i.e., $[I \;\; U][R \;\; T][I \;\; -U] \equiv [R \;\; T']$, i.e., $[R \;\; U - RU + T] \equiv [R \;\; T']$, i.e., $[R \;\; U - RU + T - T'] \equiv [R \;\; O]$, i.e., $[I \;\; U][R \;\; T - T'][I \;\; -U] \equiv [R \;\; O]$. But, *the matrices $[R \;\; T - T']$ form a finite group $\mathcal{H}$:* 'group' because $\mathcal{H}$ is the image of $\mathcal{G}$ under the homomorphism $[R \;\; T] \mapsto [T \;\; T - T']$, 'finite' because $[R \;\; T - T'] \mapsto [R \;\; O]$ is a monomorphism from $\mathcal{H}$ into the rotation group (of $\mathcal{G}$ or $\mathcal{G}'$) which, from (13.5), is finite. Since the *average* $\frac{1}{|\mathcal{H}|}\sum\{h(v) : h \in \mathcal{H}\}$ is, for any $v \in V$, a fixed point of any $h \in \mathcal{H}$, it follows that *the motions of $\mathcal{H}$ rotate the space V around a common fixed affine subspace W.* Any translation ϕ which shifts the origin O to a point $O' \in W$ shall do the needful. *q.e.d.*

(14.1.1) In the above proof we first reduced the problem to the case when $\mathcal{G}$ and $\mathcal{G}'$ have the same translation subgroup $\mathcal{L}$ with f identity on it. Then we checked that $g' = f(g)$ has the same rotation group as g, that is, $g' \circ g^{-1}$, $g \in \mathcal{G}$, *is a translation* (by the vector $T' - T$). Then that $\{g' \circ g^{-1} : g \in \mathcal{G}\}$ *is a finite set, so* $\phi = \text{average}\{g' \circ g^{-1} : g \in \mathcal{G}\}$ *is well-defined, and one has* $\phi \circ h \circ \phi^{-1} = h' \;\forall\; h \in \mathcal{G}$. This last can be verified thus: note that $h' \circ (\;\;) \circ h^{-1}$ is an affine isomorphism of the vector space of all translations, so it commutes with averaging, but $\{h' \circ g' \circ g^{-1} \circ h^{-1} : g \in \mathcal{G}\} = \{(h' \circ g') \circ (h \circ g)^{-1} : g \in \mathcal{G}\}$ is the same finite set, so $h' \circ \phi \circ h^{-1} = \phi$. *q.e.d.*

In fact we found *all* such translations: $\{g' \circ (\;\;) \circ g^{-1} : g \in \mathcal{G}\}$ *is a finite group of affine motions of the vector space of translations, and, for any translation k,* $\phi = \text{average}\{g' \circ k \circ g^{-1} : g \in \mathcal{G}\}$ *satisfies* $h' \circ \phi \circ h^{-1} = \phi \;\forall\; h \in \mathcal{G}$, as can be verified by a computation like the one above for $k = \text{id}$.

Moral: *identify V with the vector space of all translations!* Then $f : \mathcal{L} \cong \mathcal{L}'$ defines the linear part ψ of the affine motions $\phi = \text{average}\{f(g) \circ k \circ (\psi(g))^{-1} :$

$g \in \mathcal{G}\} \circ \psi$ of V (here $\psi(g) := \psi \circ g \circ \psi^{-1}$) which satisfy $f(h) \circ \phi \circ h^{-1} = \phi \; \forall \; h \in \mathcal{G}$ – by a calculation like the one above because $f(h) \circ (\quad) \circ (\psi(h))^{-1}$ is an affine isomorphism of V – and this recipe gives all such ϕ's.

Given $\phi_1 \circ h \circ \; {\phi_1}^{-1} = f(h) \; \forall \; h \in \mathcal{G}$, one has $\phi_2 \circ h \circ \; {\phi_2}^{-1} = f(h) \; \forall \; h \in \mathcal{G}$ iff $h \circ ({\phi_1}^{-1} \circ \phi_2) = ({\phi_1}^{-1} \circ \phi_2) \circ h \; \forall \; h \in \mathcal{G}$, therefore, *there are as many ϕ's as translations which commute with all members of $\mathcal{G}$.* Since a motion commutes with a translation parallel to its axis, and only with these translations – for, its conjugates with others have parallel, but not the same, axis – *these translations constitute the fixed subspace of the rotational group of $\mathcal{G}$,* and there is a unique solution ϕ iff this subspace is null. For instance, if $\mathcal{G}$ – like our good ol' "Vibhor group"! – contains a rotation g with a *unique* fixed point, then ϕ must, of course, map this fixed point to the unique fixed point of $f(g) \in \mathcal{G}'$.

(14.2) Since the compact connected space $X := V/\mathcal{G}$ of orbits of any crystallographic group of rigid motions is obtainable by identifying pairs of $(n-1)$-cells on the boundary of a fundamental region, it is an n-dimensional *polyhedron,* i.e., it is triangulable by a (finite) simplicial complex. Its intrinsic *stratification* $X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0 \supseteq \emptyset$ – here X_j is the subspace of points in the j-skeletons of all triangulations – partitions X into the $n+1$ (possibly empty j-dimensional) manifolds $X_j - X_{j-1}, 0 \leq j \leq n$; and the polyhedron is a manifold iff its lower dimensional strata are all empty, i.e., iff $X_{n-1} = \emptyset$.

In §11.D, we had posed the problem of characterizing *crystallographic groups $\mathcal{G}$ for which $V/\mathcal{G}$ is a closed n-manifold;* accordingly, we'll be looking a lot at such manifolds from euclidean, and other related, geometries.

(14.3) *If no non-identity element of a crystallographic group $\mathcal{G}$ has a fixed point, then its orbit space is a closed manifold with fundamental group $\mathcal{G}$ and cohomology groups $H^j(\mathcal{G})$.*

... Each orbit $[P]$ has as neighbourhood the set of all orbits $[Q]$, $Q \in \mathrm{int}(D_P)$. This subset is the bijective continuous image of $\mathrm{int}(D_P)$, in the orbit space which is Hausdorff, so it is homeomorphic to it. Therefore the orbit space is a closed n-manifold, and euclidean n-space V is an unbranched covering space of $V/\mathcal{G}$ with group of covering transformations $\mathcal{G}$. The assertion about its fundamental group and homology follows. *q.e.d.*

This f.p.f. hypothesis is the same as saying that $\mathcal{G}$ be torsion-free. For, if some finite power of a motion g is the identity, then g has only periodic orbits, and the averages of these orbits, constitute axis(g), on which g is the identity. On the other hand, if the powers of a rotation g are all distinct, they form a non-discrete cyclic subgroup of the compact rotation group, thus such a g is not crystallographic. *q.e.d.*

Torsion free crystallographic groups are called "Bieberbach Groups" in [10], their orbit spaces give us, in each dimension n, all the closed "flat n-manifolds" (i.e., n-manifolds that can be equipped with a riemannian metric with all sectional curvatures identically zero). Though much is known about them, even their classification is unknown for $n \geq 4$. Regarding classification the policy we'll adopt from here on is to not go much beyond finiteness theorem, i.e., just indicate maybe some crucial principle that is used, and instead work out interesting examples. For example, in the next article we'll look at a closed 3-dimensional orientable euclidean flat manifold – the Hantzsche Wendt manifold – that arises in a very natural way from the same lattice which figures in the Kepler sphere packing problem, i.e., the last part of Hilbert's Problem VIII. In the remaining subsections we'll turn to *some affinely flat manifolds,* which shall be defined analogously using other discrete subgroups of $\mathcal{A}_n$.

(14.4) This §15 will end with restatement of Goldman's result in terms of $E_1(\mathcal{F})$ of my 1974 thesis.

§16. " Best C".

Though my 1974 thesis was about global analysis on a foliated manifold, I have, over the years, drifted into more discrete mathematics. Now too, I kept on postponing writing the details of some calculus proofs which I had obtained shortly after the Chauhan episode. Admittedly, there were valid reasons for this delay: if one wants to understand the mathematical landscape of today from the perspective of Euclid – as, quixotically enough, I am attempting to do these days! – it does take a while to get to infinitesimals. However, at some point in time I realized that, by an over-insistence on such an exposition only, I was risking losing some ideas altogether, and so that, I ought to stop procrastinating and type up these proofs. What enabled me finally to start doing just this was that I got hooked on yet another problem involving calculus: *finding the best C that would make Lemma (13.2.3) work.* In trying to solve this concrete problem I re-acquainted myself – in a way at once pleasant and painful[25] – with the Lie group theory necessary to start talking with my own mathematical self of long ago. This reconnection is crucial, for this is after all an attempt to understand my own mathematical impulses, and I do hope I'll be able to bring it to a point where it can serve – at least for me before I pass on! – as a useful retrospective on my own entire previous, published and unpublished, work.

Before I go any further, let me signal an important *change of notation:* as before, $\text{angle}_g : S^{n-1} \to [0, \pi]$ is the great circle distance by which the motion g of $\mathbb{R}^n$ rotates the pencil parallel to each $\overrightarrow{x} \in S^{n-1}$; however, from now on, the maximum value of this function shall be denoted simply by $\text{angle}(g)$, instead of the previous $\|g\|$, which notation shall be reserved now for another, and more natural norm, that we shall define presently.

[25] *Painful,* because I made all sorts of mistakes in great abundance; *pleasant,* because it reinforced my belief that Lie group theory is practically the day after Euclid!

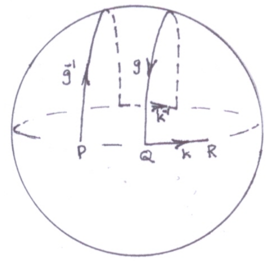

Figure 18

We begin by observing that a rotation of 3-space is doubled – Fig. 18 – when one takes its commutator with a half-turn about a perpendicular axis, which shows that *the trivial bound* $\mathrm{angle}(kgk^{-1}g^{-1}) \leq 2\mathrm{angle}(k)$ *is the best possible over* all *rotations* k *and* g.

On the other hand, using a qualitative argument invoking only the smoothness of the *group manifold*[26] of all orientation preserving rotations $SO(n)$, we had shown in (13.2.3) that taking commutators with a fixed rotation having maximum angle less than a certain $C > 0$ does not increase the maximum angles of small rotations.

Moreover, there is the following quantitative bound (16.1) which is independent of n, and quite easy to prove – and which we'll show subsequently in (16.2) to be surprisingly good for all $n \geq 4$ – by using $\mathrm{chord}_g : S^{n-1} \to [0, 2]$, the function which assigns to each $x \in S^{n-1}$ the length of the chord from x to $g(x)$ (presumably because of this, Vince [13] prefers to work with chords instead of angles); $\mathrm{chord}(g)$ shall denote the maximum value of this function.

(16.1) *If the rotation* g *of* n*-space is such that* $\mathrm{angle}(g) \leq 2\sin^{-1}(1/4)$ ($\approx 28.955°$) *then* $\mathrm{angle}(kgk^{-1}g^{-1}) \leq \mathrm{angle}(k)$ *for all* $k \in \mathrm{SO}(n)$.

The angle and chord functions are related to each other by the homeomorphism $[0, \pi] \ni t \to 2\sin(t/2) \in [0, 2]$ and its inverse $[0, 2] \ni t \to 2\sin^{-1}(t/2) \in [0, \pi]$. So the statement is equivalent to showing that, if $\mathrm{chord}(g) \leq 1/2$, then $\mathrm{chord}(kgk^{-1}g^{-1}) \leq \mathrm{chord}(k)$ for all k.

To see this note that, $\mathrm{chord}(kgk^{-1}g^{-1}) = \sup\{|(kgk^{-1}g^{-1} - 1)(x)| : x \in S^{n-1}\} = \sup\{|(kg - gk)(k^{-1}g^{-1}(x))| : x \in S^{n-1}\} = \sup\{|(kg - gk)(y)| : y \in S^{n-1}\} = \sup\{|((k-1)(g-1) - (g-1)(k-1))(y)| : y \in S^{n-1}\} = \sup\{|((k-1)(g-1)(y) - (g-1)(k-1)(y))| : y \in S^{n-1}\} \leq \sup\{|((k-1)(g-1)(y)| : y \in S^{n-1}\} + \{\sup|(g-1)(k-1)(y))| : y \in S^{n-1}\} \leq \mathrm{chord}(g)\sup\{|((k-1)(z)| : z \in \mathbb{R}^n, |z| \leq 1\} + \mathrm{chord}(k)\sup\{|(g-1)(w)| : w \in \mathbb{R}^n, |w| \leq 1\} = \mathrm{chord}(g)\mathrm{chord}(k) + \mathrm{chord}(k)\mathrm{chord}(g) = 2\mathrm{chord}(g)\mathrm{chord}(k)$, for all $g, k \in \mathrm{SO}(n)$. *q.e.d.*

[26] This is a useful synonym of *Lie group* for those to whom these are manifolds that happen to be groups, rather than groups that happen to be manifolds (smoothness is guaranteed, by a theorem of Gleason et al., from the mere continuity of the group operations).

(16.2) "Best C" = 30° for $n \geq 4$: *if* angle$(g) \leq \pi/6$, *then* angle$(kgk^{-1}g^{-1}) \leq$ angle(k) *for all* k, *but one has rotations* g *and* k, *with* angle$(g) - \pi/6$ *and* angle(k) *arbitrarily small, such that* angle$(kgk^{-1}g^{-1}) >$ angle(k).

(16.2.1) We take any $P \in S^{n-1}$, and let $Q = gk^{-1}g^{-1}(P)$ and $R = k(Q)$ – see Fig. 19 – and since the case angle$(k) = \pi$ is trivial, we'll assume $P, R \neq -Q$. In the concluding step of our proof we'll use this obvious *lemma:* if the angle at Q between the chords QP and QR is at most $\pi/3$, then PR cannot be longer than both of them. The following is also true, but we won't be using it.

A non-lemma[27] : *if the angle at* Q *between great circle arcs* $\widehat{QP}$ *and* $\widehat{QR}$ *is at most* $\pi/3$, *then* $\widehat{PR}$ *cannot be longer than both of them.*

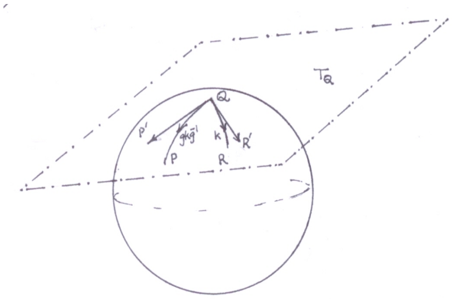

Figure 19

The tangents QP' and QR' to these arcs at Q are in a 2-dimensional subspace of T_QS^{n-1}, and P, Q and $\widehat{PQ}$ are all in the 2-sphere generated by the great circles through Q tangent to this subspace, so we can assume $n = 3$.

A great circle arc is bigger than another if and only if the corresponding chord is bigger, so it suffices to show that, if say $\widehat{QP} \geq \widehat{QR}$, then $PR^2 \leq PQ^2$ under the given hypothesis. To verify this, we choose the centre as our origin, and rectangular axes such that $Q = (0, 0, 1)$; $P = (\sin\phi_1, 0, \cos\phi_1)$, where $\phi_1 = \widehat{QP}$, is on the xz-plane; and $R = (\cos\theta\sin\phi_2, \sin\theta\sin\phi_2, \cos\phi_2)$, where $\phi_2 = \widehat{QR}$, is on the plane obtained by rotating the xz-plane about the z-axis through θ, the angle at Q between the great circle arcs $\widehat{QP}$.

Then $PR^2 \leq PQ^2$ is the same as $(\sin\phi_1 - \cos\theta\sin\phi_2)^2 + \sin^2\theta\sin^2\phi_2 + (\cos\phi_1 - \cos\phi_2)^2 \leq \sin^2\phi_1 + (1 - \cos\phi_1)^2$, i.e., $\cos\theta\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2 \geq \cos\phi_1$, which is an equality if $\phi_2 = 0$ and $\cos\theta > \frac{\cos\phi_1(1-\cos\phi_2)}{\sin\phi_1\sin\phi_2} = \frac{\tan(\phi_2/2)}{\tan\phi_1}$, where $0 < \phi_2/2 \leq \phi_1/2 < \pi/2$, otherwise. Since the right side of this inequality

[27]For some time I'd thought it would help, but this turned out not to be the case, principally because orbits in S^{n-1} of one parameter subgroups of SO(n) are seldom great circles; indeed, for $n \geq 4$, these orbits may not even be closed. I note also that the angle between chords QP and QR can be bigger than that between the great circle arcs $\widehat{QP}$ and $\widehat{QR}$ – even this had taken a little while to sink in! – so this result does require some work.

is at most $\frac{\tan(\phi_1/2)}{\tan\phi_1} = \frac{\tan(\phi_1/2)}{2\tan(\phi_1/2)}(1-\tan^2(\phi_1/2)) = \frac{1}{2}(1-\tan^2(\phi_1/2) < 1/2$, it holds for $\cos\theta \geq 1/2$, i.e., for $\theta \leq \pi/3$, the given hypothesis. □

(16.2.2) Let us pause and review our set up. Euclid's pristine space V has now been equipped – a great convenience, as the years since Descartes have taught us, but 'an act of violence' nevertheless, as Weyl remarked – with an origin and an orthonormal basis, so it is now $\mathbb{R}^n$ with its (column) vectors equipped with the usual dot product $v.w$ (i.e., v^*w where star denotes transpose) and the group of orientation preserving rotations of V around the chosen origin is now $SO(n)$, the group of orthogonal – i.e., with columns (so also rows) orthonormal – matrices with determinant 1. It sits in the ambient set $\mathcal{M}_{n,n}$ of all $n \times n$ matrices which is also a Euclidean vector space, this time of dimension n^2, in the obvious way, and from now on we'll use $\|\cdot\|$ to denote its norm, that is, $\|g\|$ *shall denote the square root of the sum of the squares of the entries of a matrix* g. Observe that orthogonal matrices are all at the same distance, viz. $\sqrt{n}$, from the zero matrix, and more generally, it can be verified that $\|g\|$ does not change if g is pre- or post-multiplied by an orthogonal matrix.

So $SO(n)$ is a connected and closed submanifold of a round sphere, with each tangent space equipped with the dot product of the ambient Euclidean space $\mathcal{M}_{n,n}$. In particular, this applies to $\mathfrak{o}(n)$, the tangent space at the identity matrix $1 \in SO(n)$, which consists of all $n\times n$ skewsymmetric matrices.[28] We'll now recall why there is a one-parameter subgroup $k(t)$ – that is, a homomorphism from the additive reals $t \in \mathbb{R}$ to $SO(n)$ – passing through any k at time 1, and work out the relationship between the norm $\|\mathfrak{k}\|$ of its tangent vector $\mathfrak{k} \in \mathfrak{o}(n)$ at time 0 and the function $\text{angle}_k : S^{n-1} \to [0,\pi]$.

The *conjugacy classification of rotations*[29] tells us that there is an orthonor-

[28] More generally, *the tangent space at $g \in SO(n)$ consists of matrices obtainable from skewsymmetric matrices by right (or left) multiplication by g:* for, if $a(t)a(t)^\star \equiv 1$ with $a(0) = g$, then differentiation at $t = 0$ gives $\mathfrak{a}g^\star + g\mathfrak{a}^\star = 0$ where $\mathfrak{a} = (\frac{da}{dt})_{t=0}$, so $\mathfrak{a}g^\star$ is skewsymmetric (likewise, by differentiating $a(t)^\star a(t) \equiv 1$, one sees that $g^\star\mathfrak{a}$ is skewsymmetric), and conversely, all such matrices do occur as tangent vectors as can be easily verified by using the next paragraph.

[29] As I've mentioned before, I forget even commonplace details, especially those of an algebraical nature, rather quickly, however Herstein's good old *Topics in Algebra* [19] quickly refreshed my memory on conjugacy (or similarity) classification of matrices. Over $\mathbb{C}$, an orthogonal matrix is an instance of a unitary matrix ($kk^* = 1 = k^*k$, where star now denotes *conjugate* transpose, i.e., a matrix preserving the dot product v^*w of $\mathbb{C}^n$), so à fortiori a *normal matrix* ($gg^* = g^*g$); likewise, any skewsymmetric matrix ($\mathfrak{k}^\star = -\mathfrak{k}$) is also normal. Theorem: *there is an orthonormal basis of $\mathbb{C}^n$ which diagonalizes g if and only if g is normal.* 'Only if' is obvious, for normality is preserved when we switch to another orthonormal basis, and $\text{diag}(\lambda_1, \lambda_2, \ldots)$ is normal. For 'if' note that these diagonal entries are the *roots of g,* i.e. of $\det(g-\lambda) = 0$, an nth degree polynomial equation over $\mathbb{C}$ invariant under *any* change of basis, and that $\mathbb{C}^n$ is the direct sum of the nonzero subspaces K_i – cf. (1.3) of *"When is the locally nilpotent part a direct summand?"* which however mainly discussed the infinite dimensional case when the K_i's may *not* be direct summands (and despite which long paper, here am I, a bare 18 months later, reminding myself once again about the rudiments of even the finite dimensional theory!) – on which some power of $g - \lambda_i$ vanishes. Lemma: *if g is normal, then* $K_i = \ker(g-\lambda_i) = \ker(g^* - \overline{\lambda_i})$. For, $g - \lambda_i$ and $g^* - \overline{\lambda_i}$ commute, and their product h obeys $h^* = h$, so $h^{2t}v = 0 \Rightarrow 0 = (h^{2t}v)^*v = (h^tv)^*(h^tv) \Rightarrow h^tv = 0 \Rightarrow \cdots \Rightarrow hv = 0$, i.e., $(g^* - \overline{\lambda_i})(g-\lambda_i)v = 0$ (likewise $(g-\lambda_i)(g^*-\overline{\lambda_i})v = 0) \Rightarrow 0 = ((g^*-\overline{\lambda_i})(g-\lambda_i)v)^*v = ((g-\lambda_i)v)^*((g-\lambda_i)v) \Rightarrow (g-\lambda_i)v = 0$ (likewise $(g^*-\overline{\lambda_i})v = 0$). □ So, if $v \in K_i$ and $w \in K_j$,

mal basis which 'almost-diagonalizes' k to $\mathrm{diag}(R_{\theta_1}, R_{\theta_2}, \ldots), \pi \geq \theta_1 \geq \theta_2 \geq \cdots \geq 0$, where $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ if $\theta > 0$, while $R_0 = 1$. So $x.k(x) = (x_1^2 + x_2^2)\cos\theta_1 + (x_3^2 + x_4^2)\cos\theta_2 + \cdots \geq \cos\theta_1$ for any unit vector x, therefore $\theta_1 = \mathrm{angle}(k)$, the maximum value of the function angle_k, and more generally, one can check that, *the θ_i's are the critical values of the function* angle_k.[30] Further, in this basis, $k(t) = \mathrm{diag}(R_{t\theta_1}, R_{t\theta_2}, \ldots)$, so its derivative at $t = 0$ is the almost-diagonal skewsymmetric matrix $\mathfrak{k} = \mathrm{diag}(r_{\theta_1}, r_{\theta_2}, \ldots)$, where $r_\theta = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$ if $\theta > 0$, while $r_0 = 0$, which gives $\|\mathfrak{k}\|^2 = 2\sum_i (\theta_i)^2$. [31]

(16.2.3) Using an orthonormal basis of V which almost-diagonalizes a given rotation g to $\mathrm{diag}(R_{\theta_1}, R_{\theta_2}, \ldots)$, as above, we shall now work out, *an orthogonal direct sum decomposition of the tangent space $\mathfrak{o}(n)$, into subspaces of dimensions ≤ 2, which are preserved by the automorphism $\mathfrak{k} \mapsto g\mathfrak{k}g^{-1}$.*

To describe these, we'll imagine each skewsymmetric matrix partitioned, by pairing each odd row/column with the next row/column – i.e., as shown in Figure 20 for $n = 3, 4, 5$ – into smaller submatrices or *blocks* $A_{p,q}$. For n even these blocks are all of size 2×2, but for n odd we'll also need to consider the size 1×2 and 2×1 blocks of the last row and column. Matrix multiplication shows that, if $A_{p,q}$ is a size 2×2, 1×2 or 2×1 block of $\mathfrak{k}$, then $R(\theta_p)A_{p,q}R(-\theta_q)$, $A_{p,q}R(-\theta_q)$ or $R(\theta_p)A_{p,q}$ is the corresponding block of $g\mathfrak{k}g^{-1}$.

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then $\lambda_i v^* w = (\overline{\lambda_i} v)^* w = (g^* v)^* w = v^* g w = v^* \lambda_j w = \lambda_j v^* w$, showing $v^* w = 0$, i.e. $v \perp w$, for $i \neq j$. *q.e.d.* For example, for $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $\begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$, the roots are $e^{\pm i\theta}$ and $\pm i\theta$, so they diagonalize over the complex numbers to $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$ and $\mathrm{diag}(i\theta, -i\theta)$ respectively, but are considered in canonical form over $\mathbb{R}$; and more generally – as follows easily from $\ker(g - \lambda_i) = \ker(g^* - \overline{\lambda_i})$ proved above – the roots of any orthogonal k are 1, -1 or pairs $e^{\pm i\theta}$, while those of any skewsymmetric matrix $\mathfrak{k}$ are 0 or pairs $\pm i\theta$, which gives us the almost-diagonal canonical forms of these types of matrices over $\mathbb{R}$.

[30] In (16.3) I'll give another proof of the classification of rotations via a direct analysis of the critical values of their angle functions (this is one of the two calculus proofs that I'd been postponing writing up, the other pertains to spaces of polygons with given arm lengths).

[31] I emphasize that *we're working under the assumption* $\mathrm{angle}(k) < \pi$, using which it is easy to check further that there is a unique one-parameter subgroup $k(t)$ passing through k at time 1, so a unique $\mathfrak{k} \in \mathfrak{o}(n)$, and in fact that, *$k \leftrightarrow \mathfrak{k}$ is a homeomorphism between the open subset $\{k : \mathrm{angle}(k) < \pi\}$ of $SO(n)$ and a bounded open neighbourhood of $0 \in \mathfrak{o}(n)$,* within which each $\mathfrak{k}$ can certainly be joined to 0 by a segment (and, for our strictly local purpose, we can safely work, if need be, in a yet smaller neighbourhood that can be assumed convex). However, the remaining part $\{k : \mathrm{angle}(k) = \pi\}$ of $SO(n)$ is crucial for understanding the global shape of $SO(n)$: to each such k, there correspond a sphere's worth of $\mathfrak{k}$'s on the boundary of this bounded open neighbourhood of $0 \in \mathfrak{o}(n)$, and $SO(n)$ is homeomorphic to the space obtained from its closure by identifying each such bunch of $\mathfrak{k}$'s to a single point.

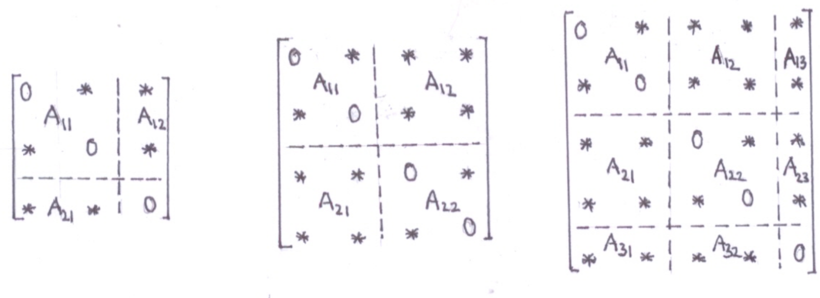

Figure 20

(a) The 1-dimensional subspaces U_p of matrices having all blocks other than $A_{p,p}$ zero; *U_p is a fixed subspace of* $\mathfrak{k} \mapsto g\mathfrak{k}g^{-1}$ because

$$\begin{bmatrix} \cos\theta_p & \sin\theta_p \\ -\sin\theta_p & \cos\theta_p \end{bmatrix} \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}.$$

(b) For n odd, the 2-dimensional subspaces V_p of matrices having blocks other than the pth 2×1 block $\begin{bmatrix} x \\ y \end{bmatrix}$ and its negative transpose zero; *V_p rotates by θ_p under $\mathfrak{k} \mapsto g\mathfrak{k}g^{-1}$*, because, restricted to this block, the map is $R(\theta_p)$.

(c) Finally, for $n \geq 4$, our map preserves the 4-dimensional subspaces $W_{p,q}$, $q > p$, of matrices with all blocks other than the 2×2 block $A_{p,q}$ and its negative transpose zero. Let $W^-_{p,q}$, resp. $W^+_{p,q}$, be the 2-dimensional subspace of $W_{p,q}$ for which $A_{p,q}$ is of the type $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$, resp. $\begin{bmatrix} x & y \\ y & -x \end{bmatrix}$. These subspaces are orthogonal to each other, because the dot product of any matrix of the first type with any matrix of the second type is zero, and they are also preserved, more precisely, *$W^-_{p,q}$, resp. $W^+_{p,q}$, rotates by $\theta_p - \theta_q$, resp. $\theta_p + \theta_q$, under $\mathfrak{k} \mapsto g\mathfrak{k}g^{-1}$*, because a straightforward computation shows that

$$R(\theta_p)\begin{bmatrix} x & y \\ \mp y & \pm x \end{bmatrix} R(-\theta_q) = \begin{bmatrix} x' & y' \\ \mp y' & \pm x' \end{bmatrix}, \text{where} \begin{bmatrix} x' \\ y' \end{bmatrix} = R(\theta_p \mp \theta_q)\begin{bmatrix} x \\ y \end{bmatrix}.$$

(16.2.4) When the tangent vector $\mathfrak{k}$ to the one-parameter subgroup $k(t)$ lies in one of the aforementioned subspaces of $\mathfrak{o}(n)$ which are invariant under $g(\cdot)g^{-1}$, then the relationship between $\|\mathfrak{k}\|$ and angle_k proved in(16.2.2) becomes much simpler, for example, *if $\mathfrak{k} \in W^+_{1,2}(g)$, then $\|\mathfrak{k}\| = 2\text{angle}(k)$.*

For, if $A_{1,2}(\mathfrak{k}) = \begin{bmatrix} x & y \\ y & -x \end{bmatrix}$, $\det(\mathfrak{k} - \lambda) = \begin{vmatrix} -\lambda & 0 & x & y \\ 0 & -\lambda & y & -x \\ -x & -y & -\lambda & 0 \\ -y & x & 0 & -\lambda \end{vmatrix} (-\lambda)^{n-4} =$ $(\lambda^2 + x^2 + y^2)^2(-\lambda)^{n-4}$, so the roots of $\mathfrak{k}$ are $\pm i\sqrt{x^2+y^2}$ repeated twice and

0 repeated $n-4$ times; so now $\mathfrak{k}$ and $k(t)$ almost-diagonalize (in some other orthonormal basis) to $\mathrm{diag}(r_{\theta_1}, r_{\theta_2}, \ldots)$ and $\mathrm{diag}(R_{\theta_1}, R_{\theta_2}, \ldots)$, where $\theta_1 = \theta_2 = \sqrt{x^2+y^2}$ and $\theta_i = 0$ for $i \geq 3$; so $\|\mathfrak{k}\|^2 = 4(\theta_1)^2$, i.e., $\|\mathfrak{k}\| = 2\mathrm{angle}(k)$.

(16.2.5) We'll show now that, *if* $\theta_1(g) + \theta_2(g) > \pi/3$, *then, for all* $\mathfrak{k} \in W^+_{1,2}(g)$ *sufficiently close to the origin, one has* $\mathrm{angle}(kgk^{-1}g^{-1}) > \mathrm{angle}(k)$, which implies the second part of (16.2), for the hypothesis applies in particular to any g with $\mathrm{angle}(g) > \pi/6$ and $\theta_2(g) \geq \pi/6$.

The one-parameter subgroup $k(t)$ corresponds to $\mathfrak{k}$, so $k(st)$ corresponds to $s\mathfrak{k}$, for example, $k(t)^{-1} = k(-t)$ corresponds to $-\mathfrak{k}$, and $gk(t)g^{-1}$ corresponds to $g\mathfrak{k}g^{-1}$; also, the curve $k(t)gk(-t)g^{-1}$ through $1 \in SO(n)$ (usually *not* a one-parameter subgroup) has the tangent vector $\mathfrak{k} - g\mathfrak{k}g^{-1}$ at $t = 0$ [32]; let $a(t)$ denote the one-parameter subgroup corresponding to this vector $\mathfrak{a} = \mathfrak{k} - g\mathfrak{k}g^{-1}$.

The included angle $\theta_1(g) + \theta_2(g)$ between the equal length vectors $\mathfrak{k}$ and $g\mathfrak{k}kg^{-1}$ of $W^+_{1,2}(g)$ is given to be bigger than $60°$, so it follows that their difference $\mathfrak{a} = \mathfrak{k} - g\mathfrak{k}g^{-1} \in W_{1,2}(g)$ has a bigger length, i.e., $\|\mathfrak{a}\| > \|\mathfrak{k}\|$, which, by (16.2.4), is equivalent to $\mathrm{angle}(a(t)) > \mathrm{angle}(k(t))\ \forall t$.

Since $a(t)$ and $k(t)gk(t)^{-1}g^{-1}$ are both of the form $1 + t\mathfrak{a} + o(t^2)$ as $t > 0$ approaches 0, it follows that $\mathrm{angle}(k(t)gk(t)^{-1}g^{-1}) > \mathrm{angle}(k(t))$ also holds, provided t is sufficiently close to 0.

(16.2.6) The automorphism $\mu \mapsto g\mu g^{-1}$ of the space $\mathcal{M}_{n,n}$ of all $n \times n$ matrices[33] has the identity matrix 1 as a fixed point, and in (16.2.3) above we worked out a decomposition, into orthogonal invariant subspaces of dimension ≤ 2, of its restriction to the tangent space at 1 to the invariant submanifold $SO(n)$, i.e., to the vector subspace $\mathfrak{o}(n)$ of all skewsymmetric $n \times n$ matrices. The same method gives an analogous decomposition of $\mathcal{M}_{n,n}$ using which we shall now prove that, *the included angle, between the matrices* μ *and* $g\mu g^{-1}$ *of equal length, is at most* $2\mathrm{angle}(g)$*,* and this upper bound is the best possible.

We work again in an orthonormal basis that almost-diagonalizes g, and imagine all matrices partitioned into blocks exactly as before. There being no condition on the blocks (previously $A_{q,p}$ was the negative transpose of $A_{p,q}$) it is simplest to consider the mutually orthogonal subspaces $\mathcal{X}_{p,q}$ of all $n \times n$ matrices having blocks other than $A_{p,q}$ zero.

For n odd and $q = (n+1)/2$, resp. $p = (n+1)/2$, $A_{p,q}$ is a 2×1, resp. 1×2 block of the last column, resp. last row, so $\mathcal{X}_{p,q}$ is 2-dimensional, and $\mu \mapsto g\mu g^{-1}$ left, resp. right, multiplies this block by R_{θ_p}, resp. R_{θ_q}.

In all other cases, $A_{p,q}$ is a 2×2 block, so $\mathcal{X}_{p,q}$ is 4-dimensional, but decomposes into orthogonal 2-dimensional subspaces $\mathcal{X}^-_{p,q}$, resp. $\mathcal{X}^+_{p,q}$, for which $A_{p,q}$ is of the type $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$, resp. $\begin{bmatrix} x & y \\ y & -x \end{bmatrix}$, and the same computations as before show that these subspaces are also invariant, more precisely, that $\mathcal{X}^-_{p,q}$, resp. $\mathcal{X}^+_{p,q}$, rotates by $|\theta_p - \theta_q|$, resp. $\theta_p + \theta_q$, under $\mu \mapsto g\mu g^{-1}$.

[32] All these assertions follow by computing derivatives at $t = 0$, for example, the product rule tells us that that of $k(t)gk(t)^{-1}g^{-1}$ is $(\frac{dk(t)}{dt})_{t=0} + (\frac{d(gk(t)^{-1}g^{-1})}{dt})_{t=0}$, etc.

[33] Intrinsically, $\mathrm{Hom}(V, V)$, equipped with the inner product $\mathrm{trace}(\mu\nu^*)$, where transpose of any map μ is defined by the requirement, $\mu^*(v).w = v.\mu(w)$, for all $v, w \in V$.

In particular, it follows that any matrix μ rotates by at most $2\theta_1$, i.e., $2\mathrm{angle}(g)$, and this bound is attained by the nonzero matrices in $\mathcal{X}^+_{1,1}$.

(16.2.7) So the automorphism $\mu \mapsto g\mu g^{-1}$ of $\mathcal{M}_{n,n}$ maps any chord $k-1$ (of the round sphere $\{\mu \subset \mathcal{M}_{n,n} : \|\mu\| = \sqrt{n}\}$ in which $SO(n)$ is contained) from the identity matrix 1 to a rotation k, to another such chord $gkg^{-1}-1$ of the same length, such that the angle at 1 between these two chords is at most $2\mathrm{angle}(g)$, therefore, at most $60°$ under the hypothesis of (16.2).

The evaluation map $\mathrm{Hom}(V,V) \ni \mu \mapsto \mu(v) \in V$ *is an orthogonal projection if* $v \in S^{n-1}$, for, in an orthonormal basis of V having the unit vector v as its first member, this is nothing but the 'first column map' $\mathcal{M}_{n,n} \to \mathbb{R}^n$.

Since angles do not increase under an orthogonal projection [?] the angle at v between the (possibly unequal) chords $k(v)-v$ and $gkg^{-1}(v)-v$ of S^{n-1} is therefore less than $60°$. So the 'third side', the chord from $k(v)$ to $gkg^{-1}(v)$, cannot be longer than both these chords, and, since it has the same length as the chord from v to $k^{-1}gkg^{-1}(v)$, this implies $\mathrm{angle}(kgk^{-1}g^{-1}) \leq \mathrm{angle}(k)$, *q.e.d.* This is how I had concluded the proof of (16.2) during a very pleasant 2-week stay in the hills ...

But alas! as in chess annotations, the inserted question mark above signals a (silly) mistake: *orthogonal projection can increase angle,* for example, for t large, the angle between $\mathbf{i}+t\mathbf{k}$ and $\mathbf{j}+t\mathbf{k}$ is almost 0, but they project orthogonally under $(x,y,z) \mapsto (x,y)$ to the perpendicular vectors $\mathbf{i}$ and $\mathbf{j}$. This mistake was discovered ... on June 18, 2009 ... It took me (as usual) a couple of days ... but then I got down to the task of trying to complete the proof of (16.2) ... I made mistake after mistake ... on July 8, 2009 ...

(16.2.8) Since $k \mapsto \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ preserves maximum angle, *we can w.l.o.g. assume that n is even,* and then, *we'll identify* $\mathbb{R}^n$ *with* $(\mathbb{R}^2)^{n/2}$. We note that the pth (2-dimensional) component of $\mu(v)$, $\mu \in \mathcal{M}_{n,n}$, $v \in S^{n-1}$, is the sum of the n two-dimensional vectors $A^{\mp}_{p,q}(\mu)(v_q)$, $1 \leq q \leq n/2$, obtained by evaluating the pth row of blocks $A^{\mp}_{p,q}(\mu)$ of μ on the components $v_q \in \mathbb{R}^2$ of $v \in \mathbb{R}^n$. Each of these blocks $A^{\mp}_{p,q}(\mu)$ is of the type $\begin{bmatrix} x & y \\ \mp y & \pm x \end{bmatrix}$ – that is, the second column is obtained by rotating the first through $\pm\pi/2$ – and the corresponding block of $g\mu g^{-1}$ is obtained by rotating both columns by $\theta_p(g) \mp \theta_q(g)$, so the same rotation takes their linear combination $A^{\mp}_{p,q}(\mu)(v_q)$ to $A^{\mp}_{p,q}(g\mu g^{-1})(v_q)$. *Assuming* $\mathrm{angle}(g) < \pi/6$, the absolute values of these angles are all less than $\pi/3$, which implies that, *if nonzero, the pth two-dimensional component of $g\mu g^{-1}(v)-\mu(v)$, i.e. $\sum(A^{\mp}_{p,q}(g\mu g^{-1})(v_q) - A^{\mp}_{p,q}(\mu)(v_q))$, has length strictly less than the sum $\sum l^{\mp}_{p,q}$ of the lengths of the n summands $A^{\mp}_{p,q}(\mu)(v_q)$, $1 \leq q \leq n/2$, of the pth component of $\mu(v)$.* For the case $\mu = 1-k$, $k \in SO(n)$, one has $g\mu g^{-1}(v)-\mu(v) = k(v)-gkg^{-1}(v) = kgk^{-1}g^{-1}(u)-u$, where $u = gkg^{-1}(v) \in S^{n-1}$.

We come now to the key idea: *assume also that no rational linear combination of the angles $\theta_i(g)$ is equal to 2π.* Since orbits of an "irrational rotation" are dense in S^1, this extra assumption ensures that, there is an N such that, for each p, the directions of the nonzero summands $A^{\mp}_{p,q}(g^N\mu g^{-N})(v_q)$ of the

pth component of $g^N\mu g^{-N}(v)$ are aligned arbitrarily closely to each other, i.e., the length of this component is arbitrarily close to the upper bound $\sum l^{\mp}_{p,q}$. Using the last paragraph, *there exists an N such that, if nonzero, the length of $g\mu g^{-1}(v)-\mu(v)$ is less than that of $g^N\mu g^{-N}(v)$.* For $\mu = 1-k$, $k\in SO(n)$, $g^N\mu g^{-N}(v) = g^N(1-k)g^{-N}(v) = g^N(w-k(w))$, where $w = g^{-N}v \in S^{n-1}$; therefore $g^N\mu g^{-N}(v)$ has the same length as the chord $w-k(w)$. Thus for $k\neq 1$ and any $u\in S^{n-1}$ we have found $w\in S^{n-1}$, such that the length of $u-kgk^{-1}g^{-1}(u)$ is less than that of $w-kw$, that is, *under the extra conditions imposed on g, we have* $\text{chord}(kgk^{-1}g^{-1}) < \text{chord}(k)$ *for all $k\neq 1$.*

These g's form a dense subset of the set of *all g's satisfying* $\text{angle}(g)\leq \pi/6$. Now, if $\text{chord}(kgk^{-1}g^{-1}) > \text{chord}(k)$ for any such g and some k, then the same would be true, by continuity, in a small neighbourhood of such a g, but this contains g's satisfying the extra hypotheses. So we must have $\text{chord}(kgk^{-1}g^{-1})\leq \text{chord}(k)$ for all $k\in SO(n)$ whenever g satisfies $\text{angle}(g)\leq\pi/6$. *q.e.d.*

(16.3) ... [34]

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[34] ... denotes, as it did on pages 19 and 27, deletion of material. In this original typescript, most figures had been inserted—on March 6, 2009 someone had taught me how to do this in pdflatex—but curiously Figures 13 through 17 were not : these have been inserted now. Apparently, I'd been revising the typescript but had done so only till page 8 : the remaining §13 was supposed to become §14 and the present §14 the unfinished §15, unfinished because I got hooked on 'best C' and started §16 as well. Despite its auspicious date – 07/08/09 in the notation of the land in which we were then travelling from place to place – (16.2.8) also was flawed, and after some days I had put this problem and typescript aside never to return to it till now : to note that the following definitive theorem is nevertheless true for rotations of euclidean n-space for any $n\geq 3$, it has (16.2) as an immediate corollary.

We have $\text{angle}(kgk^{-1}g^{-1})\leq\text{angle}(k)$ *for all k if and only if $\theta_1(g)+\theta_2(g)\leq\pi/3$.* If the last inequality is strict $g(\ )g^{-1}$ rotates the orthogonal summands of $\mathfrak{k}\in\mathfrak{o}(n)$ in its invariant subspaces (16.2.3) by angles less than 60°, so the nonzero summands of $\mathfrak{k}$ are bigger than those of $\mathfrak{k}-g\mathfrak{k}g^{-1}$. If $\mathfrak{k}$ is tangent to the one-parameter subgroup $k(t)$ the curve $k(t)gk^{-1}(t)g^{-1}$ is tangent to $\mathfrak{k}-g\mathfrak{k}g^{-1}$ at $t=0$, using which it follows, for all k sufficiently close to the identity, that the chords $\{z,(kgk^{-1}g^{-1})(z)\}$ of S^{n-1} have lengths $\leq\text{angle}(k)$. Further, for any k with $\text{angle}(k)<\pi$ and N big, $l=k(\frac{1}{N})$ is close to the identity and the sum of the lengths of the chords $\{(l^{i-1}gl^{-i+1}g^{-1})(y),(l^igl^{-i}g^{-1})(y)\}, 1\leq i\leq N$ approaches the length of the spherical curve $(k(t)g(k(t))^{-1}g^{-1})(y), 0\leq t\leq 1$. Since l^{-i+1} is an isometry chord $\{(l^{i-1}gl^{-i+1}g^{-1})(y),(l^igl^{-i}g^{-1})(y)\}$ has the same length as $\{(gl^{-i+1}g^{-1})(y),(lgl^{-i}g^{-1})(y)\}$, i.e., the chord $\{z,(lgl^{-1}g^{-1})(z)\}$ where $z=(gl^{-i+1}g^{-1})(y)$. So the sum of these N chord lengths is bounded by $N.\text{angle}(l)=\text{angle}(k)$. The converse was proved for all $n\geq 4$ in (16.2.5); for $n=3$, $\theta_1(g)+\theta_2(g)\leq\pi/3$ is the same as saying $\text{angle}(g)\leq\pi/3$, and the necessity of this constraint on g can be shown similarly by using the invariant subspace $V_1(g)$ of $\mathfrak{o}(3)$, so now the 'best C' is 60°.

Talking of flaws, Baisakhi – see §10 – is not Punjab's New Year Day : the first month in a Bikrami calendar is Chait, this festival falls on the first day of the second month, Baisakh.

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Chandigarh 160015, INDIA.
E-mail: sarkaria_2000@yahoo.com
Website: http://www.geocities.com/sarkaria_2000/