# Extracts from my Notebooks

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**§1. Chauhan quadrilaterals?** Our house is being renovated these days. Our architect, Sandeep, has designed its new lounge as a non-rectangular quadrilateral. I noticed that, while calculating the steel reinforcement to be put in its roof and supporting beams, our structural engineer, Mr. C. S. Chauhan, had computed its area to be the product of the averages of its opposite sides. Which was good enough, because this particular quadrilateral – see Figure 0 – is not far from a rectangle.[1] However, in general – take, for example, a parallelogram with a very obtuse angle – this product of averages can be very different. So I started wondering if quadrilaterals, for which Chauhan's method gives the exact area, are necessarily rectangles? I suspected that the answer was yes, and e-mailed this problem to Dinesh, Keerti, and Vibhor on March 10. On March 13, I told them that I now had a roundabout proof – see § 2 for this – which was however distinctly post-Euclid, so I would reveal its details later, hoping that in the meantime they would find solutions that were more Euclidean.

Figure 0

**§2. My proof.** Consider first the case of a quadrilateral – with successive sides of lengths $a, b, c$ and $d$ – which is *cyclic.* Then its area is given by Brahmagupta's formula, $\sqrt{(s-a)(s-b)(s-c)(s-d)}$. Here, $s$ is its semi-perimeter, so $(s-b)(s-d) = \frac{(a-b+c+d)(a+b+c-d)}{4} = \frac{(a+c)^2-(b-d)^2}{4} \leq \frac{(a+c)^2}{4}$ with equality iff $b = d$; likewise, $(s-a)(s-c) \leq \frac{(b+d)^2}{4}$ with equality iff $a = c$. Therefore area is less than or equal to $\frac{(a+c)(b+d)}{4}$ with equality iff the opposite sides of the cyclic quadrilateral have equal lengths, that is, if and only if it is a rectangle. For the general case I now proved the following.

(2.1) *A quadrilateral, with sides $AB, BC, CD, DA$ of four prescribed positive lengths, encloses the maximum area if and only if it is cyclic.*

If $A = C$ and $B = D$, then the four sides have the same length, and $ABCD$ unfolds to a square. So assume that a pair of opposite vertices, say $A$ and $C$, are distinct. The line $AC$ bounds two closed half planes, 'lower' and 'upper'. By reflecting $B$ or $D$, if need be, in $AC$ – this does not decrease area – we can assume that $B$ is in the lower, and $D$ in the upper, closed half plane.

[1] Moreover, the fact the computed value was somewhat *higher* than the actual area, was all to the good from the viewpoint of engineering safety.

Figure 1

Let $\mu = \max(\frac{AC}{AB+BC}, \frac{AC}{CD+DA})$ and $\nu = \min(\frac{AC}{|AB-BC|}, \frac{AC}{|CD-DA|})$; so $0 < \mu \leq 1$ and $1 \leq \nu \leq \infty$. For any real $t$ such that $\mu \leq t \leq \nu$, one has unique points $B_t$ and $D_t$ – see Figure 1 – in the lower and upper closed half planes respectively, such that $AB_t = t.AB, B_tC = t.BC$ and $CD_t = t.CD, D_tA = t.DA$. So $B_t$ is on the lower half of the circle formed by all points whose distances from $A$ and $C$ are in the ratio $AB/BC$ – if $AB = BC$ this 'circle' is the bisector of $AC$ – that is, the circle having as diameter the portion of the line $AC$ intercepted by the internal and external bisectors of $\angle ABC$. Likewise, $D_t$ is on the upper half of the circle of points whose distances from $A$ and $C$ are in the ratio $AD/DC$. Thus, as $t$ increases from $\mu$ towards $\nu$, $(B_t, D_t)$ describes a pair of directed arcs lying on portions of these two semi-circles.

*Rigid case,* $\mu = \nu$. Since $t \equiv 1$, the arc pair $(B_t, D_t)$ reduces to just the point pair $(B, D)$. This happens iff the vertices $A$, $B$, $C$ and $D$ occur in order on a directed 'circle through infinity', i.e, a straight line. For these 'degenerate' cyclic quadrilaterals, of each pair of opposite angles, one is 0 and the other is $\pi$; equivalently, one side is the sum of the other three sides (the reason why Brahmagupta's formula gives us the correct area 0).

*Flexible cases,* $\mu < \nu$. As $t$ increases, the sum of the monotonically decreasing angles $\theta(t) = \angle AB_tC$ and $\phi(t) = \angle CD_tA$ goes, from a value bigger than $\pi$ at $\mu$, to values less than $\pi$ near $\nu$. For, $B_\mu$ or $D_\mu$ or both are on the line $AC$, and then necessarily between $A$ and $C$: so $\pi < \theta(\mu) + \phi(\mu) \leq 2\pi$. Likewise, if $\nu < \infty$, $B_\nu$ or $D_\nu$ or both are on the line $AC$, and then necessarily outside the segment $AC$: so $0 \leq \theta(\nu) + \phi(\nu) < \pi$. If $\nu = \infty$, i.e., if $AB = BC$ and $CD = DA$, then $B_t$ and $D_t$ recede to infinity on the lower and upper half of the right bisector of $AC$: so $\theta(t) + \phi(t) \to 0+$ as $t \to \infty$.

It follows that there exists a unique $\tau$ between $\mu$ and $\nu$ such that $\theta(\tau)+\phi(\tau) = \pi$. Let $\mathcal{Q}_t$ be the quadrilateral similar to $AB_tCD_t$ with sides having the original four lengths, obtained by dilating it by $1/t$ with respect to the mid-point of $AC$. As $t$ runs from 1 to $\tau$, $\mathcal{Q}_t$ describes *a path of quadrilaterals from $ABCD$ to a cyclic quadrilateral $\mathcal{Q}_\tau$ with the same four sides.* So it shall suffice to verify that the area of $\mathcal{Q}_t$ is strictly increasing as $t$ increases from $\mu$ to $\tau$, and strictly decreasing as $t$ increases further from $\tau$ towards $\nu$.

One has $\text{Area}(\mathcal{Q}_t) = \frac{1}{t^2}\text{Area}(AB_tCD_t) = \frac{1}{2}(AB.BC.\sin\theta + CD.DA.\sin\phi)$. So its derivative equals $\frac{1}{2}(AB.BC.\cos\theta.d\theta/dt + CD.DA\cos\phi.d\phi/dt)$. Here $d\theta/dt$ can be found from $\cos\theta = \frac{t^2(AB)^2+t^2(BC)^2-(AC)^2}{2(tAB)(tBC)}$, which gives $-\sin\theta.d\theta/dt = \frac{(AC)^2}{t^3AB.BC}$; likewise $-\sin\phi.d\phi/dt = \frac{(AC)^2}{t^3CD.DA}$. So the derivative of $\text{Area}(\mathcal{Q}_t)$ is equal to $-\frac{(AC)^2}{2t^3}(\frac{\cos\theta}{\sin\theta} + \frac{\cos\phi}{\sin\phi}) = -\frac{(AC)^2}{2t^3}\frac{\sin(\theta+\phi)}{\sin\theta\sin\phi}$. Therefore it is zero only when $\theta+\phi = 0, \pi$ or $2\pi$. Angle sum is 0, resp. $2\pi$, only at $t = \nu$, resp. $\mu$, and only when $B_\nu, D_\nu$, resp. $B_\mu, D_\mu$, are both on the line $AC$; then $\text{Area}(\mathcal{Q}_t)$ attains its minimum value 0 at $t = \nu$, resp. $\mu$. When $\theta + \phi = \pi$, that is, at $t = \tau$, $\text{Area}(\mathcal{Q}_t)$ attains its maximum value. This follows because its derivative is positive iff $2\pi > \theta + \phi > \pi$, that is, as $t$ increases to $\tau$; and negative iff $0 < \theta + \phi < \pi$, that is, as $t$ increases further from $\tau$ towards $\nu$. *q.e.d.*

So, my proof used: (1) $\frac{(a+c)(b+d)}{4} \geq \sqrt{(s-a)(s-b)(s-c)(s-d)}$, with equality iff $a = c, b = d$, and (2) a converse of Brahmagupta's formula.

(2.2) *For any quadrilateral,* $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ *is at least as big as its area, and is equal to the area if and only if it is cyclic.*

This was in fact my original formulation of (2.1), but Brahmagupta's formula seemed, after a while, to be only a pretty but avoidable frill in the present context, while the wording (2.1) has the merit of suggesting, besides (2.2), many other isoparametric results. I was busy working out some of these – see §5 – when I received, on March 15, the following delightfully brief solution [1] of the original problem, which was decidedly pre-Euclid!

**§3. Dinesh's proof.** The product of two sides of a triangle is at least twice as big as its area, with equality if and only if the included angle is a right angle. So, $(AB.BC - 2\triangle ABC) + (BC.CD - 2\triangle BCD) + (CD.DA - 2\triangle CDA) + (DA.AB - 2\triangle DAB) \geq 0$ — i.e., $(AB + CD)(BC + DA) \geq 4.\text{Area}(ABCD)$ — with equality if and only if $ABCD$ is a rectangle. *q.e.d.*

As I had by now refreshed my memory on Dido's problem from [2], I soon realized that the point used above, namely that, *the area of a triangle $PQR$ with sides $PQ$ and $QR$ of prescribed lengths is maximum iff $\angle PQR$ is a right angle,* coincides with the central idea in Steiner's clever proof of the Isoperimetric Inequality. I'll return to this later in §7; now, I'll present another gem of a solution [3], of the original problem, that I received on March 19-20.

**§4. Vibhor's proof.** The signed area of $ABCD$ is the sum of those of the triangles $ABC$ and $ACD$, i.e., $\frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BC} + \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{CD}$, i.e., $\frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$. Thus it is the same for any quadrilateral $AB'CD'$ obtained from $ABCD$ by sliding a diagonal vector, say $\overrightarrow{BD}$, to a new position $\overrightarrow{B'D'}$ with respect to the other diagonal vector $\overrightarrow{AC}$. If in the new position the diagonals bisect each other, i.e., if $AB'CD'$ is a parallelogram, then $\overrightarrow{CD} + \overrightarrow{BA} = (\overrightarrow{CD'} + \overrightarrow{D'D}) + (\overrightarrow{BB'} + \overrightarrow{B'A}) = 2\overrightarrow{B'A}$, because $\overrightarrow{CD'} = \overrightarrow{B'A}$ and $\overrightarrow{D'D} = \overrightarrow{B'B}$. Using the triangle inequality we get $CD + BA \geq 2B'A$; likewise, $DA + CB \geq 2CB'$. Both are equalities only if $CD \| BA$ and $DA \| CB$, i.e., if the given $ABCD$ was already a parallelogram, and no sliding was needed. Otherwise, the product of the averages of opposite pairs of sides is lesser for the parallelogram. Yet, unless the parallelogram is a rectangle, even this new product exceeds the area. *q.e.d.*

Figure 2

Here *signed area* is positive – i.e., $\frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$ points towards the reader – or negative, depending on whether it lies to the left or right as we walk once around $ABCD$ in this sense. For some positions – see Figure 2 – of the sliding diagonal vector, the new quadrilateral is not only non-convex, but even self-intersecting; then it is the algebraical sum of its enclosed positive and negative areas which equals $\frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$. For the further spin-offs of this proof see §§8, ...

**§5. Isoperimetric Inequality** Though my proof of (2.1) was a bit long, not much more was needed to generalize this result to all polygons.

(5.1) *A closed $n$-gon, with sides $A_1A_2, \ldots, A_nA_1$ of $n$ prescribed positive lengths, encloses the maximum area if and only if it is cyclic.*

First we show the *existence* of an area maximizing $n$-gon. Fix $A_1$, and note that the $(n-1)$-tuples $(A_2, \ldots, A_n)$, formed by the remaining vertices, of all closed $n$-gons with sides of the prescribed $n$ lengths, determine a bounded and closed set $X \subset (\mathbb{R}^2)^{n-1}$. Being a real-valued and continuous function on it, area must attain its maximum value at some point(s) of this compact space $X$.[2]

Such an area maximizing $n$-gon has to be convex: otherwise, by reflecting a part of it in a supporting chord, we can increase its area. Moreover, (2.1) tells us that any four consecutive vertices occur in order on a directed circle (which may possibly be through infinity, i.e., a straight line). For, if say $A_1A_2A_3A_4$ is not cyclic, then by replacing it by the cyclic $A_1A'_2A'_3A_4$ having the same sides, we get an $n$-gon $A_1A'_2A'_3A_4 \ldots A_n$ with sides of the same $n$ lengths, but bigger area. Finally, we note that these directed circles are in fact all the same. For, the two circles on which $A_1, A_2, A_3, A_4$ and $A_2, A_3, A_4, A_5$ lie are both the same as the unique circle determined by the three points $A_2, A_3, A_4$, etc. *q.e.d.*

Now, by relaxing the constraints on the sides, I was led to the following further generalization, of which (5.1) corresponds to the special case when the parts $S_\alpha$ are singletons. Note also that (5.2) generalizes the original problem, for, applied to the partition $\{1, 2, 3, 4\} = \{1, 3\} \bigcup \{2, 4\}$, it tells us that Chauhan quadrilaterals are necessarily rectangles.

(5.2) *For any partition of $\{1, \ldots, n\}$ into disjoint nonempty sets $S_\alpha$, suppose there exist closed $n$-gons $A_1A_2 \cdots A_nA_1$ with average lengths of subsets of sides $\{A_iA_{i+1}, i \in S_\alpha\}$ equal to prescribed positive numbers $s_\alpha$. Then there exists a cyclic $n$-gon for which all sides belonging to the same subset have the same length $s_\alpha$, and the enclosed area is maximized at this, and only this, closed $n$-gon.*

The existence of a maximizing $n$-gon follows by a compactness argument (note that we now allow sides to have zero length) similar to the one given above, and, from (5.1), we already know that this must be cyclic. Since equal chords subtend equal angles at the center, and the area of a cyclic polygon is the signed – 'signed' is needed here because (at most) one of the chords might be subtending a reflex angle at the center – sum of the isosceles triangular areas subtended by its chords at the center, we can, without loss of generality, assume that each subset $\{A_iA_{i+1}, i \in S_\alpha\}$ consists of consecutive sides. If two of these, say $A_jA_{j+1}$ and $A_{j+1}A_{j+2}$, were to have different lengths, then, replacing $A_jA_{j+1}A_{j+2}$ with the isosceles triangle $A_jA'_{j+1}A_{j+2}$ having the same height – see Figure 3a – keeps the area same, but strictly decreases the average length of this subset of sides. Then further, by replacing this isosceles triangle by a bigger isosceles triangle $A_jA_{j+1}"A_{j+2}$ we can keep all the averages $s_\alpha$ same and strictly increase area. It follows that, for this area maximizing cyclic $n$-gon,

[2]For more on the topology of $X$ and other spaces of polygons see §  .

sides belonging to the same subset must have the same length. *q.e.d.*

Figure 3a Figure 3b

In the above argument – inspired by [2], which I was now re-reading – I've used a result (Figure 3b recalls its elegant proof) from Heron's lost treatise on the geometry of mirrors: *the shortest path via a mirror between two points is the one which light takes,* i.e., that obeying 'angle of incidence equals angle of reflection.' Thus I'd used calculus only in my proof of (2.1). So, naturally, I tried to interest Dinesh, Keerti and Vibhor in the problem of finding a nice geometric proof of (2.1), but this time it was to no avail.

As remarks made in the proof of (5.2) show, *the area of a cyclic $n$-gon is a symmetric function of $n$ variables,* viz., the lengths of its $n$ sides. Heron (a.k.a. Hero) found the algebraic formula $\sqrt{(s(s-a)(s-b)(s-c)}$ for the case $n = 3$. About seven centuries later, the next case $n = 4$ was disposed off by Brahmagupta's $\sqrt{(s-a)(s-b)(s-c)(s-d)}$. However, as $n$ increases, it soon becomes very hard to give similar algebraical formulae. Nevertheless, some amazing results – see § – have been recently obtained.

The case of (5.2) when the partition has only one part goes back at least to Zenodorus, who lived about three centuries before Heron.

(5.3) *A closed $n$-gon having a prescribed perimeter encloses the maximum area if and only if it is a regular $n$-gon* (i.e., cyclic with all sides equal).

There is a direct geometric proof of (5.3) in Tikhomirov [2], with existence of area minimizing $n$-gon assumed. This gap can be filled as before: our $n$-gons, with $A_1$ fixed, form a compact subspace $X \subset (\mathbb{R}^2)^{n-1}$, viz., that consisting of all $(A_2, \ldots, A_n)$ such that the lengths $A_1A_2, \ldots, A_{n-1}A_n, A_nA_1$ are non-negative with the prescribed sum, and area is a continuous function on $X$. The argument used in (5.2) now shows that this area-maximizing $n$-gon is convex with equal sides. It remains to prove that all its angles are equal. This Tikhomirov accomplishes by another clever use of Heron's mirror theorem.

A straightforward computation shows that if a regular $n$-gon has perimeter $L$, then its area is $\frac{L^2}{4\pi}(\frac{\pi/n}{\tan(\pi/n)})$. Thus its area is less than $\frac{L^2}{4\pi}$, the area of a circle of perimeter $L$, and approaches this value as $n$ becomes infinite. So, for simple closed curves, which heuristically are nothing but "polygons with infinitely many vertices," (5.3) obviously implies – cf. [2] – the following conclusion.

(5.4) Isoperimetric Inequality. *If a simple closed curve has length $L$, then its enclosed area is at most $\frac{L^2}{4\pi}$, the area of a circle of perimeter $L$.*

To firm up our heuristics we need exact definitions, which I'll now give, and make some routine verifications, which I'll omit. A *simple closed curve* is a subspace $C$ of the plane which is homeomorphic to the circle. A non self-intersecting polygon is an example, and for these, the meanings of 'enclosed', 'area' and 'perimeter' are clear enough. For any simple closed curve $C$, the Jordan Curve Theorem tells us that $\mathbb{R}^2 \setminus C$ has two path components, one of which is bounded, this is said to be the region *enclosed* by the curve. Consider polygons $P$ whose $n(P)$ vertices $A_1, A_2, \ldots A_{n(P)}$ occur in (either clockwise or anticlockwise) order on $C$. If $\delta(P)$, the maximum length of a side of $P$, is sufficiently small, then $P$ has no self-intersections. Thinking of the areas $S_P$ enclosed by these polygons $P$ as suitable approximations, we define the *area* $S_C$ of the region enclosed by the curve to be their limit as $n(P)$ approaches infinity and $\delta(P)$ approaches zero. The standard arguments of integration theory show that this limit exists, and that this area has the expected properties. Likewise, the *length* or perimeter $L_C$ of the curve is the limit of the polygonal perimeters $L_P$ as $n(P)$ approaches infinity and $\delta(P)$ approaches zero. This limit may not exist, i.e., *length is not defined for all simple closed curve.* But in (5.4) of course we are talking of a curve with a well-defined length $L$. For such curves, which are often called 'rectifiable,' this length has the expected properties.

We'll now sharpen (5.4) – unlike Tikhomirov [2], most authors refer to this sharpened result as the "Isoparametric Inequality" – by showing that the circle is the only area-maximizing curve.

(5.5) *If a simple closed curve has length $L$, its enclosed area is at most $\frac{L^2}{4\pi}$, and this maximum is attained only by the circle of perimeter $L$.*

An area-maximizing curve is convex. Otherwise, by reflecting a portion in a supporting line, we can get a new simple closed curve having the same length, and a bigger enclosed region. We assert next that any four points, $A_1, A_2, A_3, A_4$, taken in order on this curve, must form a cyclic quadrilateral, i.e., they must occur in order on some circle. For, if $A_1A_2A_3A_4$ is non-cyclic, we can, as in the proof of (2.1), increase its area by 'flexing' it ever so slightly around its vertices, without changing the lengths of its sides $A_1A_2, A_2A_3, A_3A_4, A_4A_1$. So we can just let the four portions $\widehat{A_1A_2}, \widehat{A_2A_3}, \widehat{A_3A_4}, \widehat{A_4A_1}$ of the convex curve 'ride' on these segments to obtain a new simple – for it is easy to verify that no self-intersections are introduced if the flexion is small enough – closed curve having the same length as before, but enclosing a bigger area. Finally we note, as in the proof of (5.1), that all 4-tuples of points on the curve determine the same circle, i.e. that, the curve must be a circle. *q.e.d.*

**§6. A poetical interlude.** Eudoxus had made precise (Dedekind type) definitions for adding and multiplying segments (real numbers), and Archimedes worked out with absolute rigour many complicated areas and volumes (integrals) using exhaustion (limits). So it is not in the least surprising that Archimedes and Zenodorus had essentially proved the Isoperimetric Inequality, *but why on earth was this result called Dido's Problem?*

It is highly unlikely that Archimedes himself called it by this name. For Queen Dido, you see, is mostly a love interest in the peregrinations of a Trojan survivor of the Battle of Troy, whose mythical exploits were invented and set in immortal verse by a Roman poet, who was still more than two centuries in the future! So who coined this name and when? *Here's what happened.*

The Roman contribution to mathematics is negligible, but it is to their everlasting credit that they let pockets of Greek mathematics flourish, for a few centuries more, in some parts of their sprawling empire. Heron was once visiting

Rome to get funding for that library in Alexandria. When his hosts requested the distinguished visitor to give a talk on his work, he readily agreed.

Heron was in fact eager to lecture, and share with these hospitable Romans an uncanny thought that had popped up after his work on mirrors. It seemed that rays of light could somehow figure out – and that too in advance – the shortest path to their intended destination! For, how else could one explain that, of all possible paths, they infallibly took this, and only this, shortest path? Moreover, basing himself on this purely physical law, he had found a new and elegant demonstration of a purely geometric theorem, namely, the celebrated Isoperimetric Inequality of the great Archimedes and Zenodorus.

Heron had no qualms either on account of the language in which he would be giving his colloquium talk. Why, his knowledge of Latin was better than that of most Romans! Good enough, for example, to thoroughly relish and savour the poetry of this Virgil that all of Rome was currently raving about.

However, in the days leading up to the lecture, an element of worry crept into, and steadily grew, in his conscientious mind. It dawned on him that – excepting, of course, the two Egyptian post-docs who had so loyally accompanied him in that trireme across the Mediterranean – there was virtually no one in Rome who had the background necessary to understand the delicate geometric reasoning that was the centerpiece of his planned talk.

Heron was not one of those carefree souls, who don't mind losing their audience five minutes into a talk. He belonged to that unfortunate minority which perpetually agonizes about whether they would be understood or not. So he decided that he needed to lighten up his talk, and somehow connect its subject matter with something that the Romans were already connected with.

It was then that those verses from Virgil's "Aeneid" (Book I, lines 365-368) that he had been reading the night before in bed when he dozed off, floated back to him; those lines – my poetic translation below was inspired by the translation into English prose given in [4] – about Dido and others fleeing her tyrant brother Pygmalion[3], the king of Tyre.

*Devenere locos, ubi nunc ingentia cernes*
*Moenia surgentemque novae Karthaginis arcem;*
*Mercatique solum, facti de nomine Byrsam,*
*Taurino quantum possent circumdare tergo.*

Landed at loci, where now you well perceive,
'Mid its ramparts the citadel of new Carthage rise;
Bought they land, called thereafter the Hide,
Just as much as that of a bull would circumscribe.

This is *all* that this epic has on this important event. So, naturally, Heron had to devote the first twenty minutes or so of his talk, "On Dido's Problem," in elaborating how that haloed hide of yore had been carefully cut into thin

[3]Not the one of *Pygmalion and Galatea.* These two were transformed in Shaw's play, and the movie, *My Fair Lady,* into Professor Henry Higgins and Eliza Doolittle, respectively. However, some confusion persists, because Dido is also called Elissa, the wanderer.

strips, which, joined one after another to make a long string, were then laid on the ground to form a perfect circle, showing thus that clever Dido knew – even way back then! – that a circle enclosed the maximum area.

The talk was a roaring success, and Heron got the funding he sought. What the Romans got from the remaining thirty minutes of his talk is unknown, but certainly, no evidence of any increased mathematical activity on their part has survived. However, the two disciples who had witnessed their master's triumph in Imperial Rome, took good care that we mathematicians would never ever forget the intimate link that was pointed out that memorable day between Queen Dido and the Isoperimetric Inequality.

**§7. Steiner's version.** The next result can be proved by applying (5.1) to a certain closed $2(n-1)$-gon, but § 3 suggests a quick direct proof.

(7.1) *A closed $n$-gon, with $n-1$ of its sides, say $A_1A_2, \ldots, A_{n-1}A_n$, of $n-1$ prescribed positive lengths, encloses the maximum area if and only if it is cyclic, with the $n$th side, $A_nA_1$, a diameter of the circumscribing circle.*

These $n$-gons with $A_1$ fixed form the compact subspace $Y \subset (\mathbb{R}^2)^{n-1}$ of all $(A_2, \ldots, A_{n-1})$ such that the $n-1$ distances $A_1A_2, \ldots, A_{n-1}A_n$ have the $n-1$ prescribed values, and area is continuous on $Y$.[4] So an area maximizing $n$-gon exists. It is, as before, convex. Further, each angle $A_1A_iA_n$, $1 < i < n$, must be a right angle. Otherwise, by replacing the triangle $A_1A_iA_n$ by the right-angled triangle $A_1'A_iA_n'$, with $A_1'A_i = A_1A_i$ and $A_iA_n' = A_iA_n$, we can obtain another such closed $n$-gon, with a strictly bigger area. *q.e.d.*

(7.2) In other words, maximum area is enclosed between a straight line and a *broken line* $A_1A_2 \cdots A_n$ having $n-1$ *links* of prescribed lengths and both ends on the line, if and only if the vertices occur in order on a semi-circle of diameter $A_1A_n$. More generally, an argument similar to that used in (5.2) shows that if we partition the $n-1$ links into disjoint subsets, and only constrain the average length of each part, then the maximizing position is semi-cyclic with links of each part equal. So, in particular, one has the following.

(7.3) *Maximum area is enclosed between a straight line and a broken line $A_1A_2 \cdots A_n$ of a prescribed length with both ends on line iff the vertices occur in order, at equal distances, on a semi-circle of diameter $A_1A_n$.*

We now move on to Steiner's version of the Isoparametric Inequality, where *arc* means a subset of the plane homeomorphic to [0,1], and since we are speaking of its length, the arc is understood to be rectifiable.

(7.4) *Maximum area is enclosed between a straight line and an arc of length $L$ with both ends on the line if and only if the arc is a semi-circle.*

By (7.3) the area enclosed by a broken line of length $L$ having $n-1$ links and both ends on line is at most $\frac{L^2}{2\pi}(\frac{\pi/2(n-1)}{\tan(\pi/2(n-1))})$. Thus it is less than $\frac{L^2}{2\pi}$, the area enclosed by a semi-circle of perimeter $L$, and approaches this value as

[4] $Y$ is clearly *an $(n-1)$-dimensional torus;* using this, and some Morse theory, we'll later work out the possible topologies of the space $X$ of closed $n$-gons with prescribed edges.

$n$ becomes infinite. So, amongst all arcs –"broken lines with infinitely many vertices" – of length $L$ having ends on line, enclosed area attains its maximum value when the arc is a semi-circle. Conversely, if this maximum is attained at arc $\widehat{ACB}$, then $\angle ACB$ is necessarily a right angle. For otherwise, by flexing the triangle $ACB$ ever so slightly at $C$ – and letting the portions $\widehat{AC}$ and $\widehat{CB}$ of the arc ride on the moving segments $AC$ and $CB$ – we can get a new arc that encloses a strictly bigger area. *q.e.d.*

In fact, this version is equivalent to (5.4). If possible, suppose there is a simple closed curve of length $2L$, which encloses an area bigger than $L^2/\pi$. Without loss of generality we can assume that this curve is convex, for, if need be, we can look at its convex hull, and magnify it so that the bounding curve still has length $2L$. Now take any two points $A$ on $B$ on this curve which bisect it in two equal arcs. Then one of these length $L$ arcs, together with the chord $AB$, encloses an area bigger than $L^2/2\pi$, which contradicts (6.4), etc.

*Steiner's version of Dido's Problem.* I've a hunch that Jacob S. was led to (6.4) mulling over the fact that Phoenicians were sea-going people. As such, they must have built Carthage on the sea. So, assuming its coastline straight for the sake of simplicity, their Queen Dido must have had that cowhide string laid, not in a circle as Heron had wrongly assumed, but in a semi-circle.[5]

**§8. Congruent tilings.** There is no difficulty in adjusting the proof of §4 so that it becomes strictly *à la Euclid.* For example, here is a cutting and pasting argument, which shows that two copies of a quadrilateral have the same area as a parallelogram with sides equal and parallel to its diagonals.

Lay the two copies $ABCD$ and $A'B'C'D'$ with their diagonals $AC$ and $A'C'$ on the same straight line, and with just one common point $C = A'$. Cut partially along these diagonals – see Figure 4 – and fully along the other diagonals to snip off the four triangles $\{AED, E'C'D'\}$ and $\{AEB, E'C'B'\}$. It is easily seen that the first pair can be pasted to make the triangle $DCD'$, and the second to make $BB'C$, so the two quadrilaterals together yield the parallelogram $BB'D'D$.

As a bonus, this argument also gives the next, and at first flush somewhat surprising result, which is also depicted in the same figure.

Figure 4

(8.1) *The plane can be tiled by congruent copies of any quadrilateral.*

For this, lay not two, but an infinite row of tiles, each sharing a vertex with the previous, and diagonals subdividing a line into equal segments. Then, starting with each of these, lay a column of tiles with other diagonals on a line. Now verify – this is the same as the 'easily seen' above – that each vacant quadrilateral is also congruent to the given tile. *q.e.d.*

[5] On a more serious note, I recall that Rome had completely destroyed Carthage, and massacred all Carthagians, more than a century before Virgil wrote his poem. Nevertheless, hi-tech archeology has managed to uncover some evidence about its layout, so much so that, fairly detailed maps of Carthage are now posted on the web. These show that it was indeed on the sea, but its coastline was far from straight, and its terrain far from flat.

So here was I once again in tilings – where I had been when Mr. Chauhan had intruded into my thoughts – but this time the above picture prompted me to look at tilings in a less topological, and a more geometric way. That is, though I confined myself as before to tilings in which any two tiles intersect in a (possibly empty) common face, the primary additional requirement now was that the tiles be geometrically congruent to each other.

As some of my ex-students from Panjab University might tell you[6], I have this thing about an Egyptian called Euclid; in particular that, I used to tell them that his first two books were aimed at just one big theorem: *by cutting and pasting, an arbitrary polygon can be changed into a square.*

So, the mad thought momentarily came to my mind: maybe the plane can be tiled by congruent copies of any polygon? Certainly, any triangle will do too, because, putting two congruent triangles together we can make a parallelogram, and any parallelogram tiles. However, sanity was soon restored as counterexamples were found, and cogitation revealed many obstructions.

In fact, *for $n \geq 5$, congruent copies of a generic convex $n$-gon cannot tile the plane.* The sum of the internal angles of a convex $n$-gon is $(n-2)\pi$, and, by perturbing slightly, we can also ensure, conversely, that an integral linear combination of these angles is a multiple of $\pi$ only if the coefficients are equal. Congruent copies of such a generic $n$-gon with $n \geq 5$ cannot tile the plane, because the angles at a vertex of a putative tiling cannot add up to the requisite value $2\pi$. Indeed, for $n \geq 7$, they can all be ruled out!

(8.2) *The plane cannot be tiled by congruent copies of a convex $n$-gon with $n > 6$, if only its extreme points are considered as its vertices.* The proviso is necessary: by putting $n-3$ additional equidistant vertices on an edge of a triangle we obtain, for any $n \geq 3$, a non-strictly convex $n$-gon whose congruent copies obviously tile the plane.[7]

If there were such a tiling, the angles at each vertex would add up to $2\pi$, with each tile having contributed $(n-2)\pi$ worth of these angles. So the ratio, (number of tiles)$\div$(number of vertices), for the portion inside a large circle, would be roughly $2/(n-2)$. Thus, on an average, each vertex would be incident to $2n/(n-2)$ tiles. But $n > 6$ is the same as $2n/(n-2) < 3$, so it follows that some vertex would be incident to just two tiles, which is absurd. *q.e.d.*

The pendulum had thus swung to the other end, and now it even seemed likely to me that one should be able to characterize the—from above, necessarily very special—strictly convex hexagons and pentagons whose congruent copies can tile the plane. However, somewhat surprisingly, from the review article [5], and more recent postings regarding this topic on the web, I have learnt that *this classification problem remains very much open for pentagonal tiles!* The story thus far is quite interesting, so can bear repetition.

[6]However I'm not too sure: most students in this temple of higher learning are opposed to acquiring any, so quite naturally (and with my wholehearted approval) they 'bunked' my classes, but even of the handful that attended, I am afraid most were mentally absent.

[7]Appropriately serrating the edge instead, one can likewise get numerous non-convex $n$-gonal tiles, for all $n \geq 5$ (see also the drawings of Escher).

**§9. From David Hilbert to Marjorie Rice.** Convex hexagonal tiles were classified by Reinhardt, an assistant of Hilbert, in his 1918 Ph.D. thesis, and it was assumed for fifty years that he had also classified convex pentagonal tiles. The incompleteness of his list was pointed out in 1968 by Kershner in the *American Mathematical Monthly.* Moreover, the three new types of tiles that Kershner discovered were solutions to the second part of *Hilbert's Eighteenth Problem:* they cannot tile the plane *isohedrally,* i.e., in such a way that the symmetries of the tiling act transitively on the tiles. (Actually, Hilbert had only posed the analogous 3-dimensional problem, and Reinhardt had settled it in 1928; prisms on these Kershner pentagons give simpler examples.) Wider publicity was given to this work by Martin Gardner in the *Scientific American* of 1975, and this fresh airing had quite unforeseen consequences.

A rank amateur, Richard James, found another convex pentagonal tile, showing that Kershner was wrong in claiming completeness for his extended list. Agony was piled on by yet another amateur, Marjorie Rice, who produced as many as four new types of pentagonal tiles. (If memory serves me right, some of her discoveries now decorate a floor of the math tower of Ohio State University?) This was the position when [5] appeared in 1980, and from the web I learn that yet another convex pentagonal tile was found later, in 1985, by Rolf Stern. However no one has apparently had, after 1975, the temerity to claim that the extant list of pentagonal tiles is now complete!

On this matter of amateurs and professionals, let me remark that I personally avoid using the (unfortunate but hackneyed) word 'professionalism' to praise, and even more, 'amateurish' to criticize. Good mathematics has seldom been born out of purely pecuniary considerations, and seldom without a goodly dose of amateurish enthusiasm.

*The quadrilateral tiling of Figure 4 is isohedral:* given any tile, translations by its diagonal vectors, and half-turns about the mid-points of its edges, map tiles to tiles, and a suitable composition – which moreover is unique if the quadrilateral is generic – of these isometries takes the given tile to any desired tile. As against this, *Figure 5 shows another quadrilateral tiling, by congruent* $\{60°, 120°\}$*-rhombi, which is not isohedral:* the shaded tile is mapped to itself by any symmetry – there are only four of these – because it is the only tile with two vertices of valence 3 and two of valence 5. Elaborating further on this same idea, the reader can check that *there are uncountably many distinct tilings by such congruent rhombi – or, for that matter, by congruent right isosceles triangles – having no symmetry other than the identity map!* Hilbert however was apparently sure that, if the congruent copies of a (convex) polygon could tile the plane, then they must also be able to tile it isohedrally.

Figure 5

The next Figure 6, from Marjorie Rice's website, shows one of her pentagonal tilings. It uses congruent copies of *a generic convex pentagon* $ABCDE$ *obeying the conditions* $EA = AB = BC = CD$, $2D + C = 360°$ *and* $2E + B = 360°$. In *any* tiling by such tiles, the tile sharing the unequal edge $DE$ with $ABCDE$ must be its mirror image in this edge. Further, this reflection is the sole congruence between these two tiles. However, this reflection cannot be a symmetry of the tiling, because it does not map the third tile at $D$ to itself. This follows because $ABCDE$ is not symmetric with respect to the bisector of the angle $C$. Thus, *no tiling by congruent copies of $ABCDE$ is isohedral,* and this pentagonal tile is a solution of the second part of Hilbert's Eighteenth Problem.

Figure 6

**§10. A belated preface.** Last Baisakhi[8] I made a resolution: *"just enjoy math writing up things for website only."* My intention was to post independent short stories from my notebooks, starting with 'I. Chauhan quadrilaterals,' which I was already in the process of writing up.

However this first story was refusing to remain short: it had grown so, and almost on its own! It would have been an act of violence on my part to give only the bare bones – say, just the statement and the three proofs, §§ 1-4 – and omit entirely the extra flesh and shape that this baby was putting on virtually every day. So much of interest had happened just in this first month! Most of this is now in §§ 5-9 above, but certainly not all. For example, I have still to tell you how Vibhor's proof generalizes to $n$-dimensional octahedra with diagonals parallel and equal to $n$ fixed vectors. Besides, shortly after Baisakhi, I started seeing glimmers of a Morse theoretic argument that would yield the possible topological types for the space $X$ of all closed $n$-gons with sides of $n$ prescribed lengths, and the details of this argument were in my hand by May 15.

Now my intention became to push this first story till this application of Morse theory, and then wrap it up – before it got totally out of hand – with only a brief mention of the recent generalizations of Heron's formula.

This resolve was not easy to keep. After re-reading Osserman's review article [6], I was by now fully aware that the story also led naturally into some beautiful and basic differential geometry and analysis. However the straw that finally broke this resolve was another paper. This I shall disclose in the next section, § 11, which should also give you an idea of the nature of my notebooks.

So, by June 4, 2008, the date on which I'd typed up § 9, I was decided that *this first story shall essentially be the only one, and shall be pushed along without any planned ending in mind.*[9] Besides the sheer fun of it, I hope thus to exemplify a magical fact about mathematics: it is, in its complete entirety, an almost logical consequence of very small parts of it! Yes, it is very vast, but 'unity of mathematics' is not an empty cliché: it is very real, almost organic. Almost as surely as the germ of an analytic function determines all of it, *the tiniest living bit of mathematics is enough to clone back the entire beast!*[10]

[8]Punjab's New Year Day used to fall always on April 13, but p. 81 of my current notebook reminds me that, for some reason, last Baisakhi was on April 12, 2008.

[9]However I do hope that its posted installments shall be found to be somewhat more interesting than those of the usual soap opera!

[10]I must emphasize that there is nothing so very special or unique about 'Chauhan quadrilaterals' in this respect. This elementary problem is no more, and no less, fertile than many others, that could equally well have served to seed this process.

**§11. Twenty-six pages.** Despite the din, I had been working peacefully enough during the daytime in the main house only, in a couple of rooms so far spared by the renovation, but now their turn had come. So, by May 15, I was obliged to pack much of my stuff for long storage, and bring the remaining books and papers to my present and smaller room in the annexe.

Perhaps most people procrastinate over paperwork, but in my case it has been way beyond normal! This became clear to me as I tried, quite vainly, to bring some sort of order to material that has accumulated over the years. The week or two before the move, that I had grudgingly set aside only for this onerous activity, turned out in the end to be grossly inadequate. So basically, I just packed the stuff as such in boxes, and have – once again! – postponed its organization to when I unpack these boxes in my new office.

From the piles of photocopied papers, I could keep only a handful. I decided to choose only those papers which had, in some way or the other, at some time or the other, struck some chord in me. This choosing turned out to be fun, and, since the choices I made are indicative of what, in mathematics, has worked for me, and what hasn't, I have even made a list of these favourites.

I also made an inventory of all my notebooks, and was surprised that there were more than a hundred items. Some are probably worthless, but browsing through others, I was reminded often enough of results which should have been written up. I have a feeling that, as this story unfolds, I shall get some opportunities to make at least partial amends in this respect.

Coming back to my current notebook, the following is a summary of its twenty-six pages – pp. 99-124, May 27 to July 4 – immediately after the list of favourites, This should give you an idea about how I work, also it previews some topics that I plan to deal with in more detail in later sections.

**(11.A) Möbius balcony.** There is a sketch of what the new rear balcony of my office shall look like, and the remark that its frame only appears to be one-sided, the surface of any material body is necessarily two-sided.[11]

**(11.B) Notes on Heron theory.** In §2 we used the fact that a quadrilateral can be flexed. In sharp contrast to this, Euclid postulated in Book XI that convex 3-dimensional polytopes are rigid! The flexing quadrilateral encloses the maximum area $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ iff it is cyclic. When $d = 0$ this reduces to Heron's $\sqrt{s(s-a)(s-b)(s-c)}$ for the area of a triangle, which was probably known to Brahmagupta.[12] Some years ago someone sent me a moving newspaper article [7] about terminally-ill David Robbins trying to develop a general formula for the area of any cyclic $n$-gon in terms of its $n$ sides.[13] Surprisingly, *the best route to Robbins' purely* 2*-dimensional generalizations is via an elegant* higher-dimensional *generalization of Heron's formula!* Possibly,

[11]In fact in *"Cacti and Mathematics,"* it was convenient to *define* a closed orientable surface to be one which bounds a material body! I plan to give more, about the mathematical motifs to be used in the renovated house, in *"213, 16A."*

[12]There is quite a bit about Brahmagupta on the web, but I was unable to find the original Sanskrit enunciation of his theorem.

[13]I had found the note of surprise in this article misplaced: wouldn't a painter, similarly stricken but still physically and mentally strong enough to paint, not carry on painting?

Cayley had begun by noticing the following cute identity,

$$s(s-a)(s-b)(s-c) = -\frac{1}{16}\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 \\ 1 & a^2 & 0 & c^2 \\ 1 & b^2 & c^2 & 0 \end{vmatrix},$$

and wondered if bigger determinants of the same type had a similar geometric interpration? They do, a simple wedge product calculation shows that *the squared $n$-dimensional volume of an $n$-simplex equals* $\frac{(-1)^{n+1}}{n!2^n}|d_{ij}^2 : 0 \le i,j \le n+1|$, where $d_{ij}$ is the distance between the $i$th and the $j$th vertex, with the fictitious 0th vertex at the same distance 1 from all others.

Consider now a closed triangulated 2-manifold $M^2$ (simplex-wise) linearly embedded in $\mathbb{R}^3$, and extend $M^2$ to a triangulation $V^3$ of the enclosed volume without using any additional vertices. Applying Cayley's formula to each tetrahedron of $V^3$ and adding we can compute this volume, but this involves the interior edges also. However, by applying Cayley's formula to the degenerate volume zero 4-simplices determined by any 5 vertices we also get plenty of relations. Sabitov showed that for a generic $M^2$ – and so for a convexly embedded triangulated 2-sphere – these relations determine the interior edges in terms of the edges of $M^2$. So, *a generic linearly embedded triangulated 2-manifold $M^2 \subset \mathbb{R}^3$ is rigid,* a generalization of Euclid's rigidity postulate.[14]

Also, even for non-generic positions, Sabitov could eliminate the interior edges from the formula for volume, to show that, 48 *times the squared volume, enclosed by a linearly embedded triangulated 2-manifold $M^2 \subset \mathbb{R}^3$, satisfies a monic polynomial over the ring over $\mathbb{Z}$ generated by the squared edges of $M^2$.* Showing – in striking contrast to the 2-dimensional situation – that *the volume enclosed by a flexing $M^2 \subset \mathbb{R}^3$ is constant.*[15]

It seems that, by applying Sabitov's results to a long bipyramid, with a given cyclic $n$-gonal normal central section, one should be able to obtain practically all the results proved or conjectured by Robbins, for example that, $-16$ *times the squared area, enclosed by a cyclic $n$-gon, satisfies a monic polynomial over the ring over $\mathbb{Z}$ generated by its squared edges,* etc.

**(11.C) Why not Egyptian?** It is funny how Euclid and Heron of Alexandria are always 'Greek' – but maybe not in Egyptian school textbooks? – while Cleopatra, of much the same stock and place, is usually 'Egyptian'.

**(11.D) "Towards the Poincaré conjecture . . . "!** Having come (§9) to the second part of Hilbert's Problem 18, it was natural to peep at its first. I recall that in Book I of Euclid – and in school – congruence $ABC \equiv A'B'C'$ of triangles is defined thus: one should be able to 'move' $ABC$ so that it exactly covers $A'B'C'$. Now we perhaps prefer: there should exist a distance preserving

[14]The postulate itself had been proven long before by Cauchy; thus the fate of Euclid's rigidity postulate was quite different than that of his parallel postulate.

[15]The story goes that this had already been verified by Sullivan, for the flexible $M^2$ in the tea-room of I.H.E.S., by blowing pipe-smoke into the circular window provided in a triangle of this model, and observing that the smoke did not come back out when it was flexed!

map $f : \mathbb{R}^2 \to \mathbb{R}^2$, such that $f(A) = A', f(B) = B', f(C) = C'$. The first part asks if there are only finitely many groups of Euclidean motions or isometries of $\mathbb{R}^n$ with 'crystals' (bounded fundamental domains)? The answer was shown by Bieberbach[16]– it was already known for $n \leq 3$ – to be 'yes': *there are, up to conjugation with an affine motion of* $\mathbb{R}^n$, *only finitely many groups of Euclidean motions of* $\mathbb{R}^n$, *which are discrete and with a compact orbit space.*

As is my wont in such matters, I had searched my *jail-book*[17] first thing after making the 'Vibhor tiling' of Figure 4. Sure enough, the generic quadrilateral tiling was in it – see [8], p. 56 – and I had soon learnt that crystallographers call the group of its symmetries **p2**, which is one of 17 such groups, while in 3-dimensional space there were exactly 230 of these crystallographic groups. Playing on with this tiling, I now observed the cute fact that, *the quotient map* $\mathbb{R}^2 \to \mathbb{R}^2/\mathbf{p2}$ *is a covering of the* 2*-sphere branched over four points.* This because, the half-turns around the mid-points $\{A, B, C, D\}$ of the edges of any tile are in **p2**, so the quotient is the 2-sphere obtained from tile after making the boundary identifications shown in Figure 7, and the map is two-fold branched at the pre-images of the four points $\{\overline{A}, \overline{B}, \overline{C}, \overline{D}\}$ of the 2-sphere.[18]

Figure 7

Has someone written quick and neat proofs of these 19th century results, that there are 17 crystallographic groups for $n = 2$, and 230 for $n = 3$? I went to the web on June 2 to find out, and out popped, to my surprise, a paper by Milnor entitled, *"Towards the Poincaré Conjecture and the Classification of* 3*-Manifolds"*! Here is why: these crystallographers – Fedorov in Russia (1890), Schönflies in Germany (1891), and Barlow in England (1894)– had implicitly classified all closed 3-dimensional manifolds which are *flat,* i.e., which can be equipped with a Riemannian metric whose sectional curvatures are zero. For, an $n$-manifold is flat iff it is the orbit space of a crystallographic group *whose non-constant motions have no fixed points,* and when $n = 3$, only 10 of the 230 groups satisfy this condition, so *there are ten flat closed* 3*-manifolds.* The symmetry group **p2** of a 'Vibhor tiling' (or of the parallelogram tiling determined by the mid-points of its edges) does not satisfy this condition, and its orbit space $S^2$ is not flat.[19] For $n = 2$, *there are two flat closed* 2*-manifolds* : $\mathbb{R}^2/\mathbf{p1} \cong \mathbb{T}^2$ and

[16]The thesis advisor of Reinhardt (§9) who solved the second part of "18" 18 years later. Talking of advisors, my advisor Phillips' advisor, Milnor, shall be entering this story presently, along with Charlap and Gromov, who also signed my 1974 Stony Brook thesis.

[17]I once told Keerti – who has not let me forget this joke since – that if I had to spend time in a jail that allowed just one book, then I'd take the book [8] by Coxeter (I hope I am not tempting fate, actual jails in India are not remotely as balmy as this fictitious one).

[18]I emphasize that my notebooks are workbooks, so with lots of mistakes. For example, on page 102, I used $5 - 4 + 1 = 0$ (??) – the identifications on the quadrilateral's boundary give a cell subdivision with 5 vertices, 4 edges and 1 face – to conclude that the quotient $\mathbb{R}^2/\mathbf{p2}$ must be a 2-torus. This mistake was corrected many days later on page 114.

[19]The four non-hyperbolic closed 2-manifolds – $S^2$, $\mathbb{R}P^2$, $\mathbb{T}^2$, and K.B. – are the ones which occur as such orbit spaces, but there are other possibilities, for example, for **p4g**, the group of symmetries of the square tiling, the orbit space is a 2-cell. Problem: *characterize crystallographic groups whose orbit spaces are closed* $n$*-manifolds!*

$\mathbb{R}^2/\mathbf{pg} \cong$ K.B., here **p1** is generated by two independent translations, and **pg** by a translation and a glide reflection. For $n \geq 3$, Bieberbach's theorem tells us that *there are finitely many flat closed $n$-manifolds,* but the exact number is unknown for $n \geq 5$ (for $n = 4$ there is a computer-aided result).

It was at about this time that this tale became open-ended (§10) and I decided that its very next section would be on Euclidean motions and this finiteness theorem about them. A truth so general ought to have a simple explanation, I felt, and indeed, many reasonably simple proofs of Bieberbach are available, for example, there is one in Charlap's book [10].

As Charlap stresses, the job is to show that a crystallographic group has *one* nonzero translation, from which it follows that it has $n$ linearly independent ones, etc. For the fixed point free case, this translates into the geometric statement that, *a closed $n$-manifold is flat iff it can be covered by the $n$-torus.* In 1978, Gromov proved more generally that, a closed manifold is *almost flat,* i.e., it can be equipped with a diameter one Riemannian metric with sectional curvatures arbitrarily close to zero, iff it can be covered by a *nilmanifold,* i.e., a compact coset space of a Lie group whose Lie algebra is nilpotent. From page 42 of [10], I learnt that Gromov was led to this work from an attempt to understand "what's really going" in the proof of Bieberbach's theorem, and from the page 41 before it, that *Bieberbach's own proof had used a "non-trivial result from number theory" about the approximation of irrationals by rational.*

Which rang a bell! While reading my thesis[20] in 1974, Gromov had been particularly interested in this example: *for a linear foliation $F$ on the 2-torus, $E_1(F)$ is finite dimensional iff its slope cannot be well-approximated by rationals.* He had remarked that it should be possible to extend this to nilmanifolds! Which suggests strongly to me that he was at that time reading Bieberbach's original proof of his theorem, and already knew where he might be heading.

However, in the modern proofs of Bieberbach that I saw – including those inspired by Gromov, for example, that of Vince[21] – diophantine approximation is *not* used. So, if I want to learn how diophantine approximation had been used by Bieberbach, I shall have to look up the original paper.

**(11.E) Saved from Seine!** What can we say about the strictly convex *crystals* (fundamental domains) of a crystallographic group? Yes, there are uncountably many[22] of them, but they have the same content, and must be fairly special, because the congruent copies of a crystal have to tile space (§8), and that too isohedrally (§9), under the action of the group. Using (8.2), which does not use isohedrality, we know that, *a planar strictly convex crystal has*

[20]My thesis had dealt mainly with the global analytical properties – finite dimensionality, Serre duality, etc. – of the terms of a spectral sequence $E_k(F)$, which converges to the de Rham cohomology of a smooth closed manifold equipped with the smooth foliation $F$.

[21]I've postponed a discussion of this almost canonical proof to §12 (Vince had also played a key rôle in the dénouement of *"Equivelar maps"*).

[22]For example, any quadrilateral whose sides have mid-points $\{A, B, C, D\}$ is a fundamental domain of the group **p2** of Fig. 7 (showing yet again why quadrilaterals with equal diagonal vectors have the same area) and there are many others too, for example, the hexagonal closure of all points closer to $A$ than to any other point in its orbit.

*at most six sides.* So I tried to generalize the argument of (8.2) to show that, *there are only finitely many combinatorial types of strictly convex $n$-dimensional polyhedra whose congruent copies can tile $n$-space?*[23] I did not succeed, but I learnt and rediscovered some interesting things in the process.

To obtain (8.2) I had used Euclid's $\sum \theta = (v-2)\pi$ for the sum of the internal angles of a convex $v$-gon[24], i.e. that, *the sum of the external angles $\theta^*$ of any convex polygon is $2\pi$,* where $\theta^* = \pi - \theta$.

A similar argument would have yielded the required result for polytopes if only it were true that the sum of the internal solid angles obeyed $\sum \theta > v\pi$ if the number $v$ of vertices is sufficiently large. Alas! *for any $\epsilon > 0$, there are convex polytopes with arbitrarily large number of vertices, and $\sum \theta \leq \epsilon v \pi$.* To see this, start with any convex polytope in which the dihedral angle of an edge is very small, and just put lots of new vertices on the interior of this edge! In particular, the sum of the "external solid angles" $2\pi - \theta$ is not bounded.[25]

Nevertheless, Euclid's result extends neatly – *the sum of the external solid angles of any convex $3$-polytope is $4\pi$* – if we define an external angle at a vertex to be the (measure of the) half-cone generated by exterior half-normals of its incident faces. For, convexity ensures that, by putting all these half-cones – see Fig. 8 – together at the origin, one obtains a conical subdivision of 3-space.

Figure 8

Also, *this spatial Euclid's theorem is equivalent to Euler's formula!* The external angle (above half-cone) of a vertex cuts the unit sphere in a spherical polygon bounded by great circle arcs cut by planes through origin perpendicular to the edges incident to the vertex. Therefore the exterior angles of this polygon equal the facial angles $A, B, C, \ldots$ at the vertex. So, the area of this spherical polygon, i.e., the exterior solid angle $\theta^*$ at this vertex, is equal to $2\pi - (A + B + C + \cdots)$.[26] Adding over all the vertices, we get $\sum \theta^* = 2\pi v -$ (sum of facial angles of polytope), i.e., $2\pi v - \sum_i (t_i - 2)\pi$ where $t_i$ is number of edges of the $i$th face, i.e., $2\pi v - (2e - 2f)\pi = (v - e + f)2\pi = 4\pi$ iff $v - e + f = 2$.

The above proof is – see the charming paper [11] of Samelson – from a handwritten copy made by Leibniz in 1660 (but published only in 1860) of a now-lost paper of Descartes, that had been salvaged from a ship (bringing his belongings back to France after his death in Sweden) which sank in the Seine in 1650! That is, more than a hundred years before Euler found $v - e + f = 2$.

[23] From p. 966 of [5] it was clear that this problem was open (in 1980), but from p. 960 not so clear if Delone (= Delaunay) had shown that the answer was 'yes' under isohedrality. If he has, it lends credence to this conjectural picture: *the strictly convex crystals of a crystallographic group can be organized into a finitely stratified space, with the combinatorial type of the crystal constant in each strata?* I did not assume isohedrality in my attempt.

[24] It follows from the case $v = 3$, which is logically equivalent to *Euclid's fifth postulate.*

[25] These now-obvious facts took quite some time to sink in! Misguided by the smooth case, when all internal solid angles are $2\pi$, I had thought the needed lower bound likely, overlooking the fact that almost all the vertices could be on a lower stratum of positive dimension.

[26] This Harriot-Girard formula is easy: note that the *lune* between two great semicircles enclosing angle $\alpha$ has area $2\alpha$, then cut the unit sphere up into some lunes – cf. [8], p. 95 – to see that a spherical triangle with internal angles $\alpha, \beta, \gamma$ has area $\alpha + \beta + \gamma - \pi$, etc.

As Samelson points out, "Gaussian curvature" of polytopes is concentrated at their vertices, and is measured by these spherical polygons. So Descartes has given us *the Gauss-Bonnet theorem for any polytopal surface,* convex or not! Now the unit normal (Gauss) "map" of Fig. 8 may blow up a vertex to a spherical polygon with self-intersections, but its signed area $\theta^*$ can be computed as before. So, in almost perfect analogy with the smooth case, the signed area $\sum \theta^*$ is $2\pi$ times the Euler characteristic of the surface.[27]

Alternatively, $2\pi(v-e+f) = \sum \theta - 2\sum \phi + 2\pi f$, *the alternating sum of the internal solid angles of* all *the cells of the polytopal surface,* where $\phi$ denotes the dihedral angle of an edge.[28] For, each $\theta$ is the area of a spherical polygon on the unit sphere with internal angles equal to the dihedral angles of the $\delta$ edges incident to the vertex. So $\theta = 2\pi - \delta\pi +$(sum of the dihedral angles of these $\delta$ edges). Summing over all vertices we get $\sum \theta = 2\pi v - 2\pi e + 2\sum \phi$.

This formula came up during my search for that impossible lower bound for $\sum \theta$. Soon afterwards I checked from Grünbaum [12] that it was the 3-dimensional case of the *angle-sum or "Gram" relation.* It seems that this rudimentary polyhedral theory has now been developed by Cheeger and others to practically the same level as the index theory of smooth manifolds.

**(11.F) Crystallographic coverings.** To work out $\mathbb{R}^n \to \mathbb{R}^n/G$ we only need to know what the crystallographic group $G$ does to one crystal. The 17 planar groups are conveniently depicted in this way in Chapter 1 of Berger [13] (and it would be nice to have similar pictures for all the 230 spatial ones). So, continuing as in Figure 7, where $G = \mathbf{p2}$, it was fun working out this branched covering for all the 17 planar groups! *There are only* 7 *orbit spaces* $\mathbb{R}^2/G$, *viz., the four non-hyperbolic closed* 2*-manifolds, the cylinder, the Möbius strip, and the* 2*-disk.* In particular, of the 5 groups, **p1**, **p2**, **p3**, **p4** and **p6**, which contain only orientation-preserving motions, the first had orbit space $\mathbb{T}^2$, and the other four $S^2$, but the four coverings $\mathbb{R}^2 \to S^2$ were branched differently; more generally, the 17 branched coverings were all topologically distinct.

This was however only to be expected (but I realized this only later!) because it is easy to see in general that, *distinct crystallographic groups $G$ give rise to topologically distinct branched coverings* $\mathbb{R}^n \to \mathbb{R}^n/G$. For, an isomorphism of two such branched covering spaces, restricted to a generic un-branched fiber, would give us an isomorphism of the two groups.

These branched coverings are very special – for one, they factor through the $n$-torus – but I don't know if they have been characterized topologically? Or whether, it is known when their base spaces are manifolds – is the Poincaré 3-manifold amongst them? – and whether these manifolds can always be equipped with a Riemannian metric having constant non-negative curvature?

**(11.G) Jagatgarh?** On page 116 it is recorded how I found by chance on June 17 on the web a detailed map, circa 1955, in which Banasar Garhi – see

[27] That is, $4\pi$ times the Euler characteristic of the region enclosed by the surface; in this form the polytopal Gauss-Bonnet formula extends to all dimensions.

[28] The solid angle of an edge, being the area of a lune of angle $\phi$ on the unit sphere, is $2\phi$; the solid angle of any face is $2\pi$.

Fig. 9 – is called "Jagatgarh Fort" and the whole surrounding area shown to be in P.E.P.S.U. (Patiala and East Punjab States Union). Thus I now have (at long last!) some documentary support for the history of this fortress as sketched in a *Note* of the concluding part of *"The Forgotten Shaheeds of Dagshai."*

Figure 9

**(11.H) Rook-boards.** On June 25-26, I typed up a paper[29] on crosswords, and towards its end, posed this problem: "find all grids of a given size such that the types and lengths of its words form a given sequence." I had demanded that the white squares of the grid be *rook-connected,* i.e. that a single "rook" like that of chess, except that it remains always on white squares, can visit them all. *If one drops this requirement, grids of large enough size can always be found,* for any sequence of types and lengths; so there is a well-defined *minimal size* for each sequence; but there seems no reason why a grid of this minimal size should be either unique or rook-connected? However, actual crossword grids are not merely centrally symmetric; they are, as a rule, quite tight with respect to their sequence of words: they are minimal, unique, rook-connected and every black square is indispensable (making it white would change the sequence).

In working out an actual grid, it helps to exploit its central symmetry, paying special attention to the central row, and use stick patterns, which are much faster to draw, as one looks for the right grid. These points are illustrated in Figs. 10a, b, c which show how I found the minimal sized grid for the sequence $(h5, v8, v8, v6, v5, v5, v5,\ h8, h5, h8,\ h5, \{h3, v3\},\ v3, v8, v8,\ h6, h6, h3, v6, v5, v5,$ $v5, h5, h8, h5, h8, h5)$ to solve the cryptic Crossword 1313 for which the *Indian Express* of June 23, 2008 had 'forgotten' to give the (correct) grid.

Figures 10a, 10b, 10c

Consider positionings of "rooks" on any *rook-board,* i.e., white portion of any grid, such that no "rook" can capture any other. Treating these as simplices one obtains a simplicial complex. For the case of a rectangular rook-board, the topology of this simplicial complex has been worked out, and put, via the Borsuk-Ulam theorem, to good combinatorial use. See, for example, Matousek [14] p. 163, and my old paper, *"A generalized Kneser conjecture."* Are there interesting generalizations of these results for other rook-boards?

**(11.I) Fake tori!** Flat, and more generally, almost flat manifolds are examples of *aspherical manifolds,* i.e., those which can be covered by contractible manifolds (so their higher homotopy groups vanish). The Bieberbach-Gromov theory is thus only a step towards the *topological space form problem*: closed aspherical manifolds are homeomorphic if and only if their fundamental groups are isomorphic? This suggests that one should consider crystallographic-like groups of homeomorphisms; and, for crystallographic groups themselves, *not* exclude

[29] *"Fly, getting sad on a fair amount of alcohol?(10)."* Writing it helped dispel blues that had become bottled up due to lack of progress in understanding Bieberbach's theorem.

from consideration, 'Escher-like' or even wild crystals. I hope to show, by using Weierstrass-type (continuous but nowhere differentiable) functions on $n$-space, why an $n$-torus can admit *exotic smooth* and *fake piecewise-linear* structures. As it is, the statements in this field – see, for example, W.-C. Hsiang[30] and Shaneson [15] – are clear enough, but only a tiny cognoscenti can make head or tails of the published proofs. Which is a sin, for these striking results are, in my humble opinion, *the* most beautiful achievements of 20th century mathematics![31]

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[30] The twin brother of W.-Y. Hsiang, who made a valiant effort to settle the third and final part of Hilbert's Eighteenth Problem, i.e., Kepler's Sphere Packing Problem.

[31] *They drew a circle that put me out,*
*Heretic, rebel, a thing to flout;*
*But love and me had the wit to win,*
*We drew a circle that took them in!*

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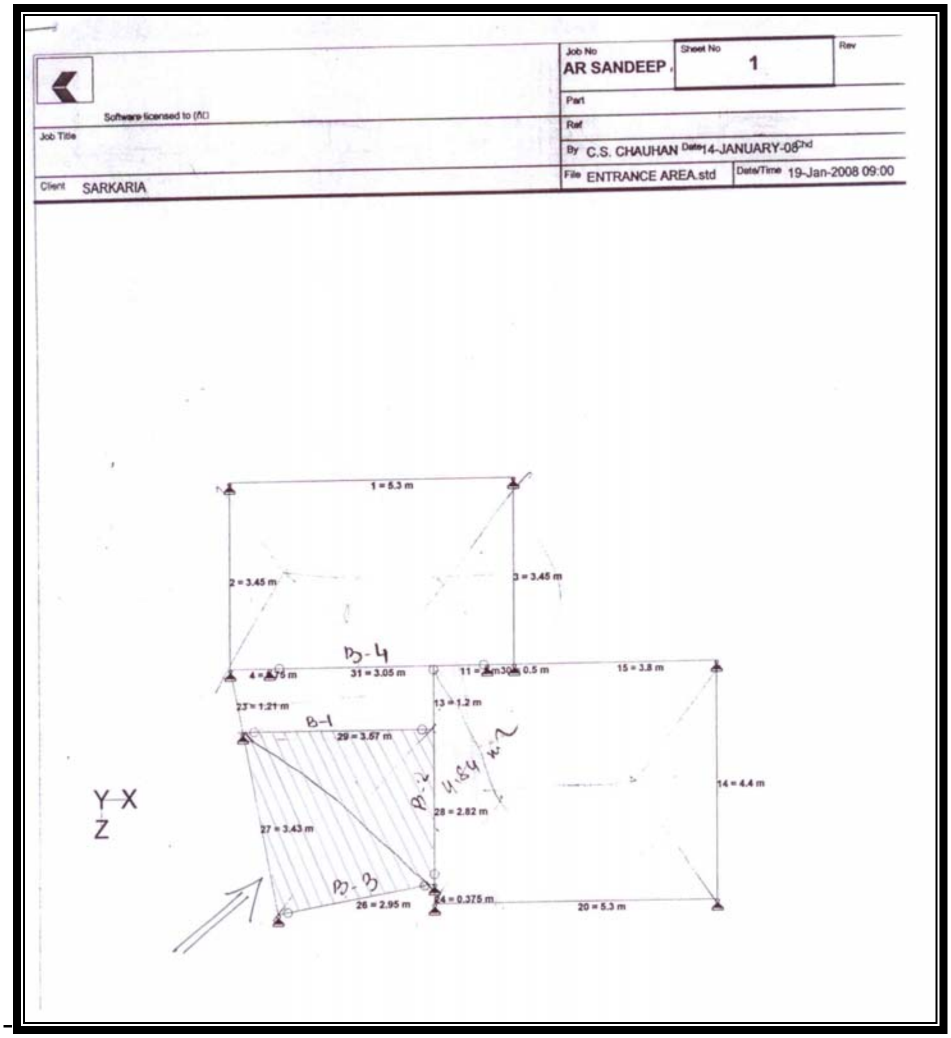
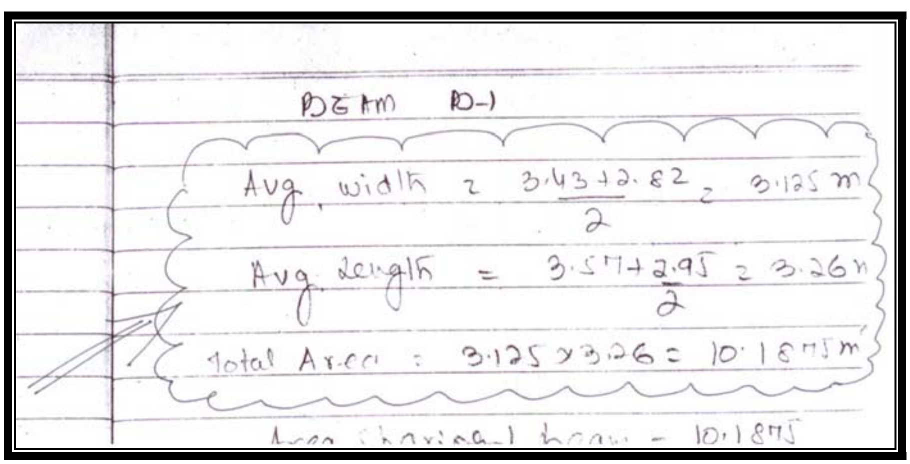


BEAM B-1

Avg. width = $\frac{3.43+2.82}{2}$ = 3.125 m

Avg. length = $\frac{3.57+2.95}{2}$ = 3.26 m

Total Area = 3.125 × 3.26 = 10.1875 m

Figure 0

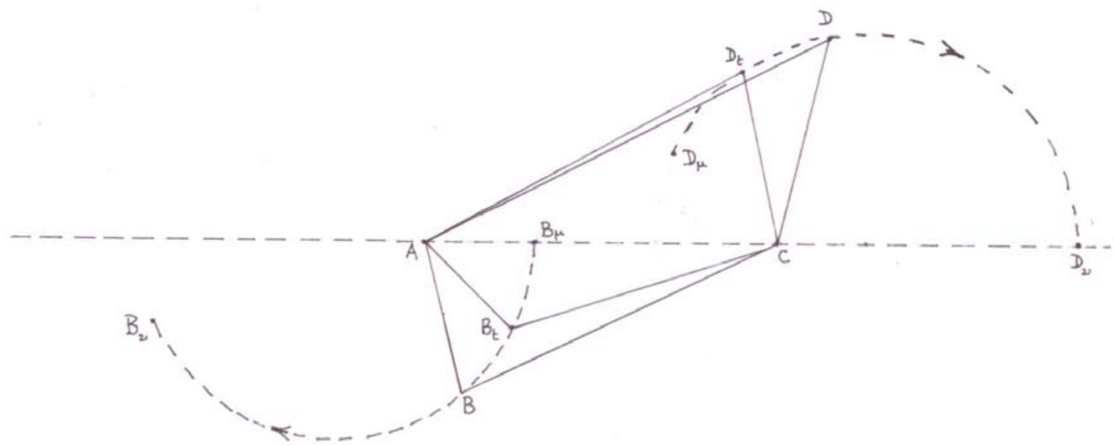



Figure 1

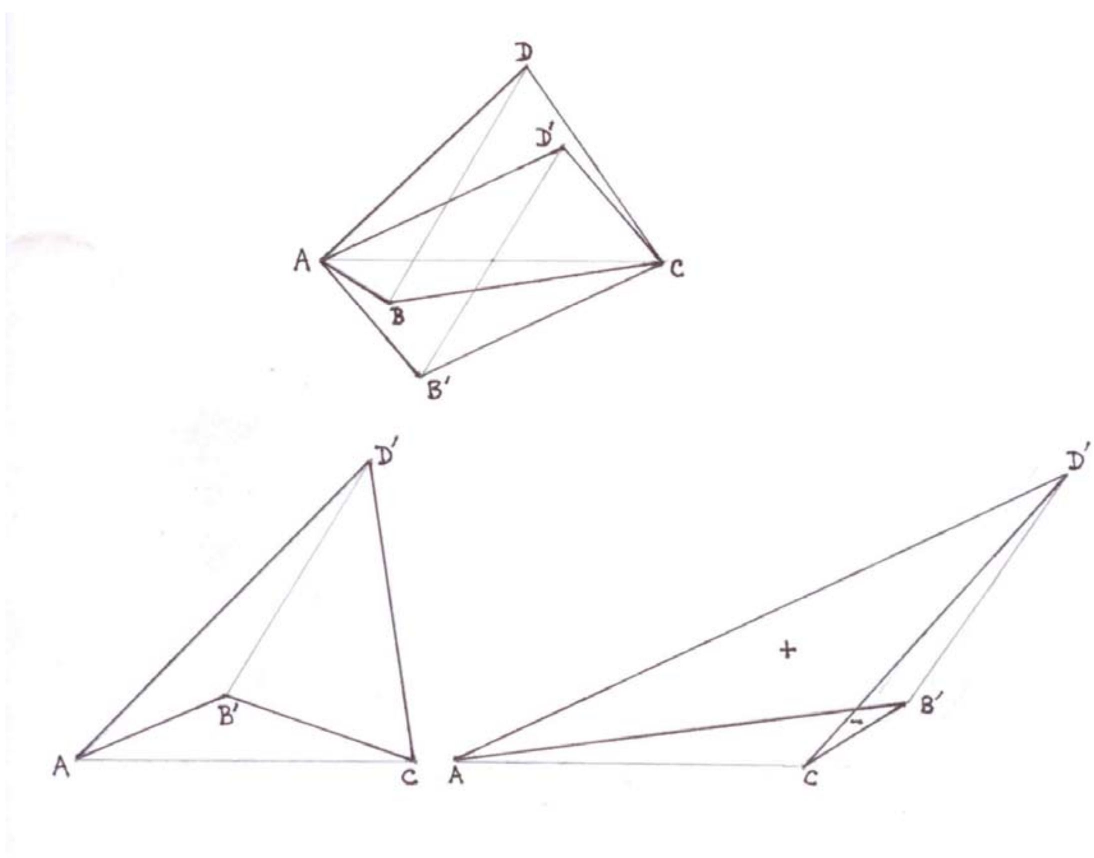



Figure 2

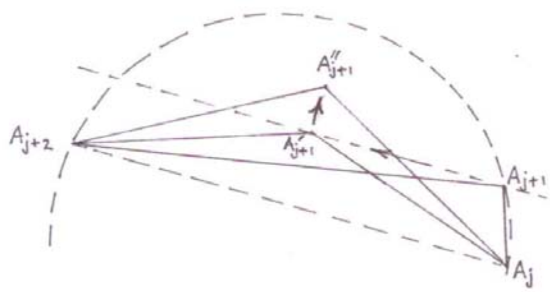



Figure 3a

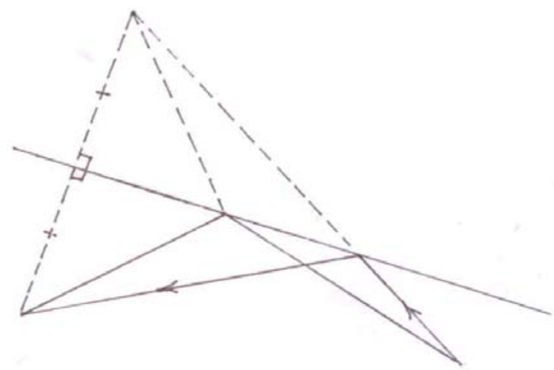


Figure 3b

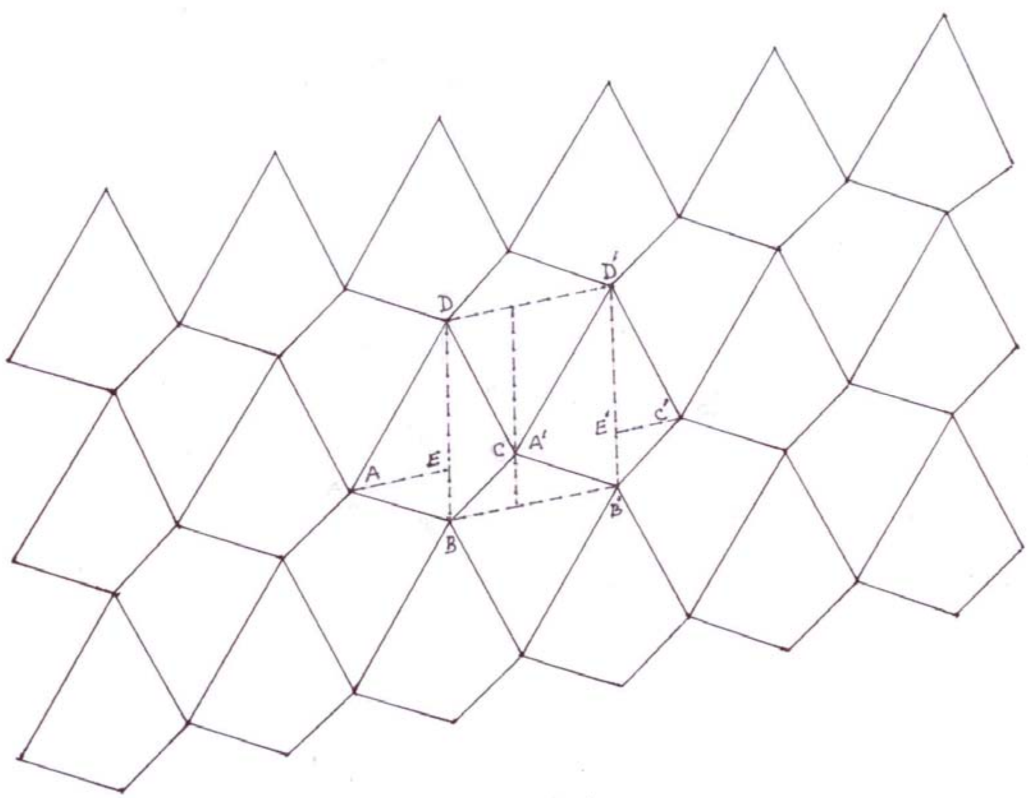



Figure 4

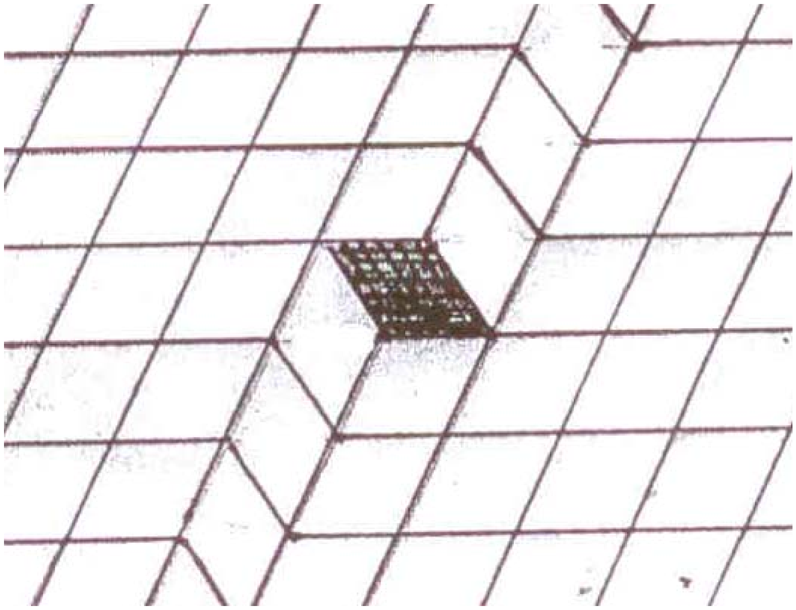


Figure 5

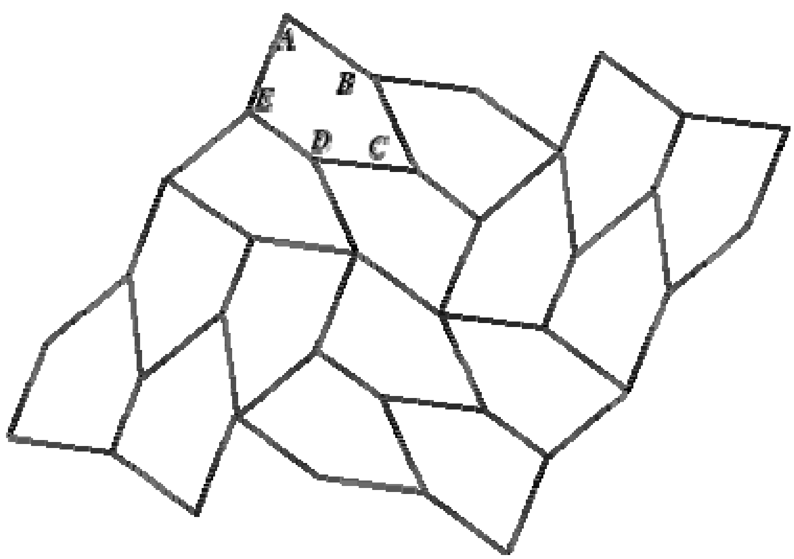



Figure 6

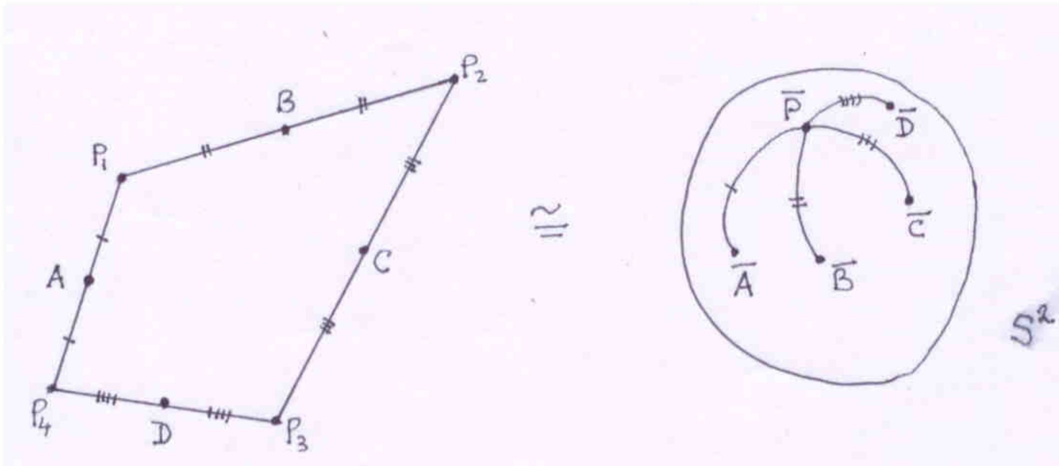



Figure 7

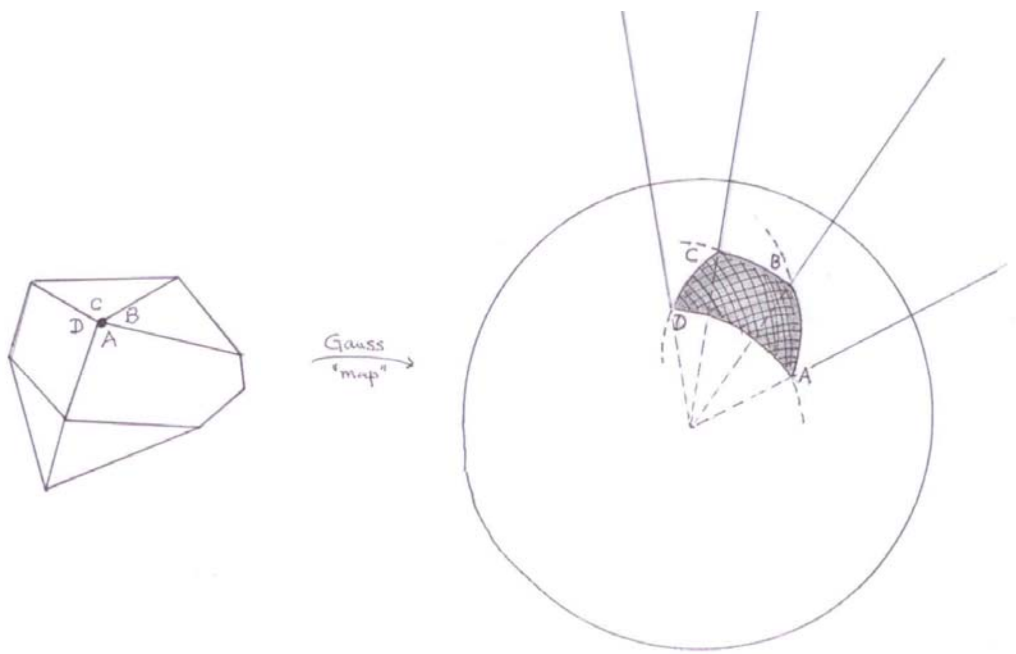
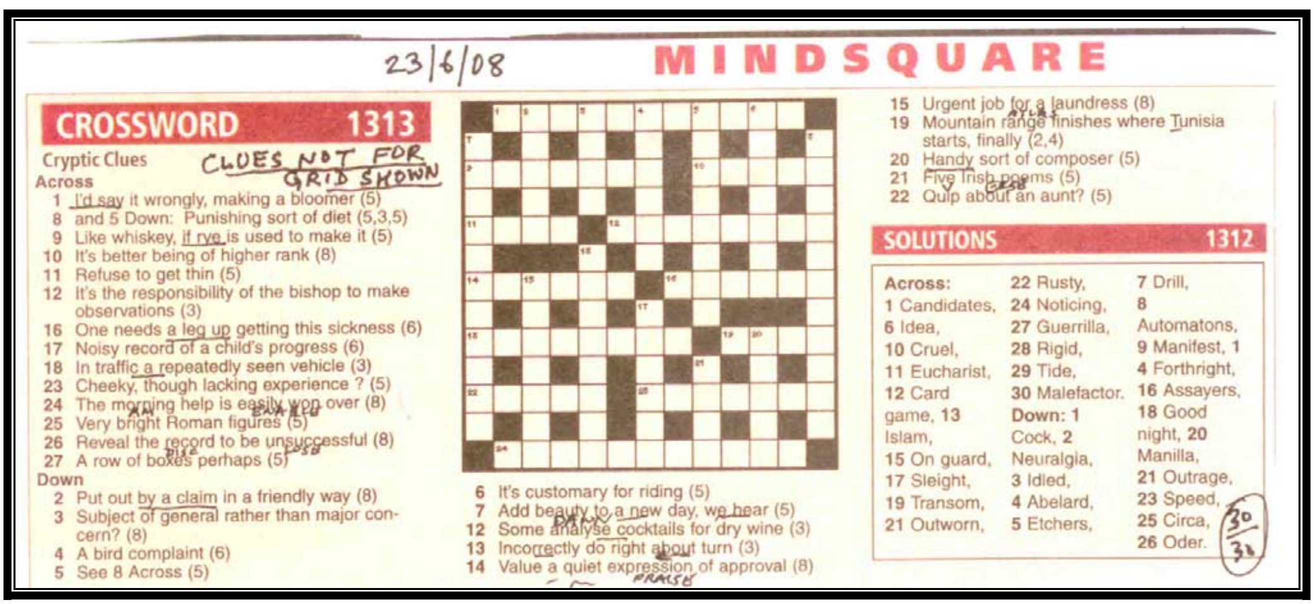



Figure 8


Figure 9



23/6/08

MINDSQUARE

**CROSSWORD 1313**

Cryptic Clues — CLUES NOT FOR GRID SHOWN

**Across**

1 I'd say it wrongly, making a bloomer (5)
8 and 5 Down: Punishing sort of diet (5,3,5)
9 Like whiskey, if rye is used to make it (5)
10 It's better being of higher rank (8)
11 Refuse to get thin (5)
12 It's the responsibility of the bishop to make observations (3)
16 One needs a leg up getting this sickness (6)
17 Noisy record of a child's progress (6)
18 In traffic a repeatedly seen vehicle (3)
23 Cheeky, though lacking experience ? (5)
24 The morning help is easily won over (8)
25 Very bright Roman figures (5)
26 Reveal the record to be unsuccessful (8) LOSE
27 A row of boxes perhaps (5) PILE

**Down**

2 Put out by a claim in a friendly way (8)
3 Subject of general rather than major concern? (8)
4 A bird complaint (6)
5 See 8 Across (5)
6 It's customary for riding (5)
7 Add beauty to a new day, we hear (5)
12 Some analyse cocktails for dry wine (3)
13 Incorrectly do right about turn (3)
14 Value a quiet expression of approval (8) PRAISE
15 Urgent job for a laundress (8)
19 Mountain range finishes where Tunisia starts, finally (2,4) ATLAS
20 Handy sort of composer (5)
21 Five Irish poems (5)
22 Quip about an aunt? (5)

**SOLUTIONS 1312**

**Across:** 1 Candidates, 6 Idea, 10 Cruel, 11 Eucharist, 12 Card game, 13 Islam, 15 On guard, 17 Sleight, 19 Transom, 21 Outworn, 22 Rusty, 24 Noticing, 27 Guerrilla, 28 Rigid, 29 Tide, 30 Malefactor. **Down:** 1 Cock, 2 Neuralgia, 3 Idled, 4 Abelard, 5 Etchers, 7 Drill, 8 Automatons, 9 Manifest, 14 Forthright, 16 Assayers, 18 Good night, 20 Manilla, 21 Outrage, 23 Speed, 25 Circa, 26 Oder.

30/31

Figure 10a

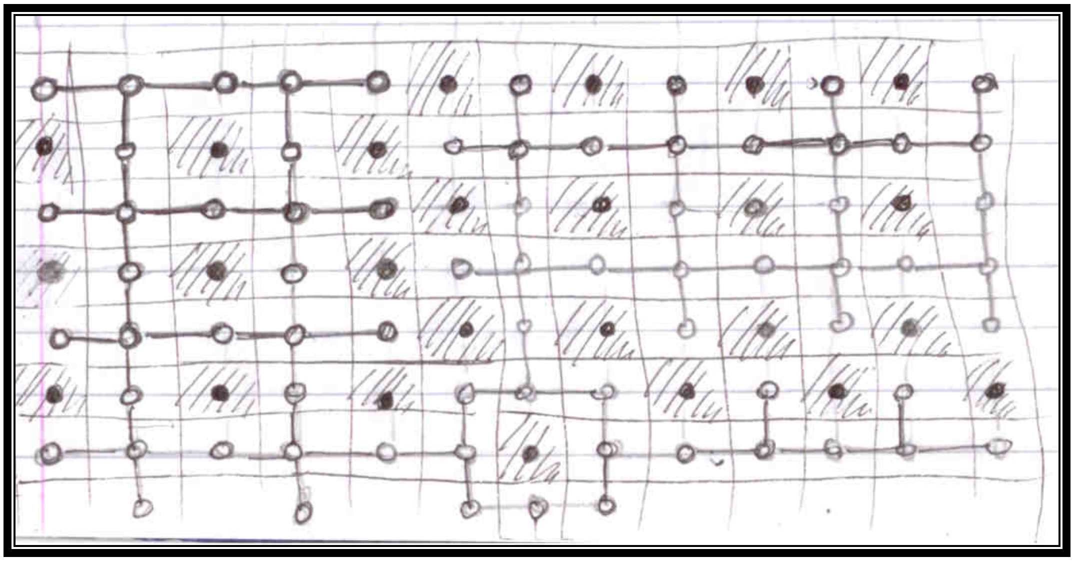


Figure 10b

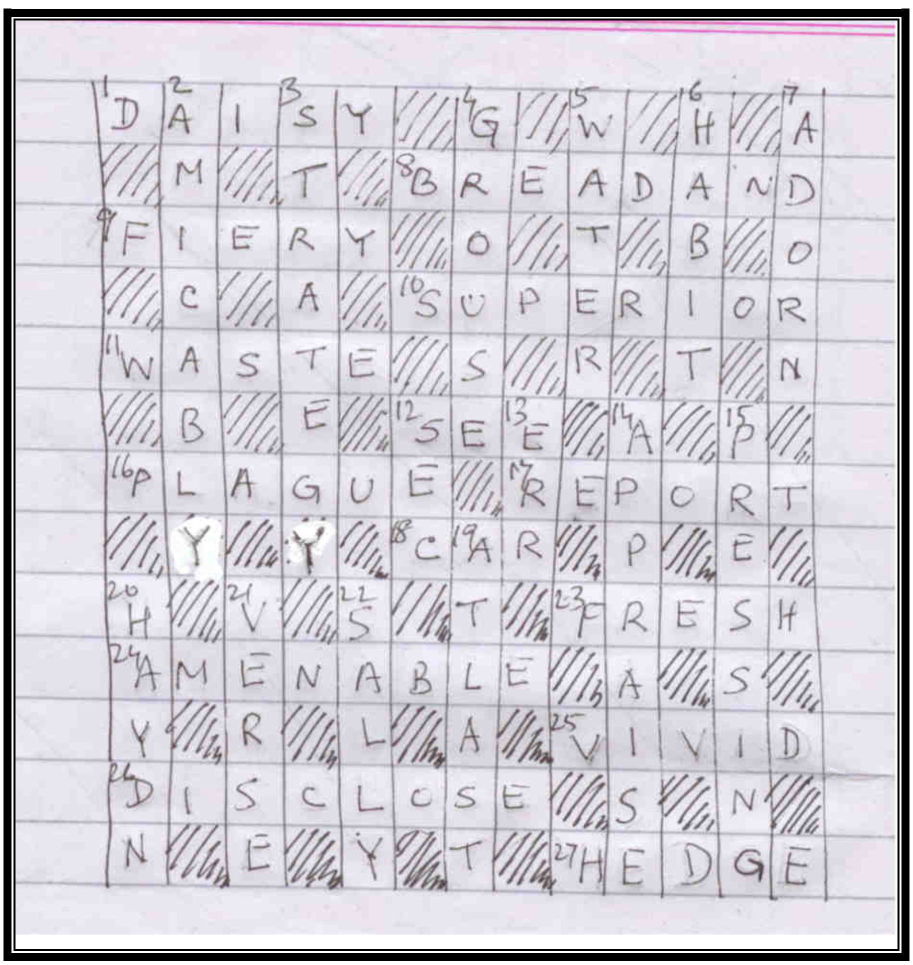



Figure 10c