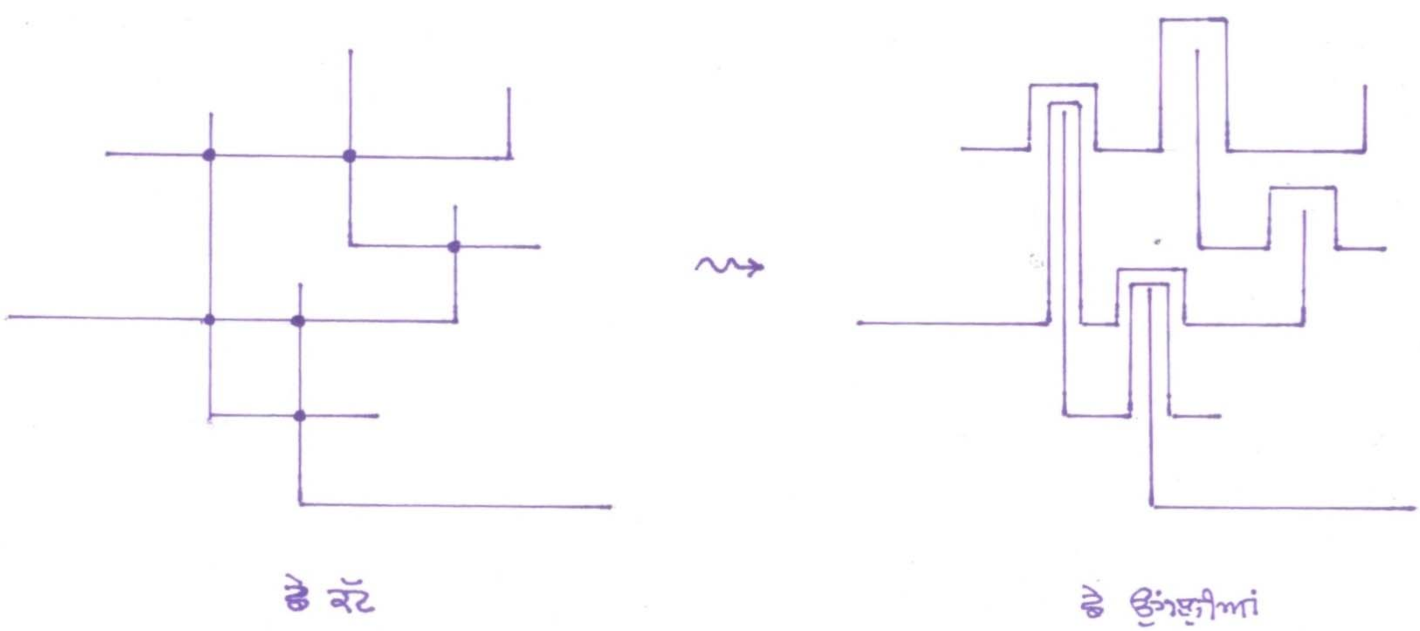
**Fingers and Tiles**

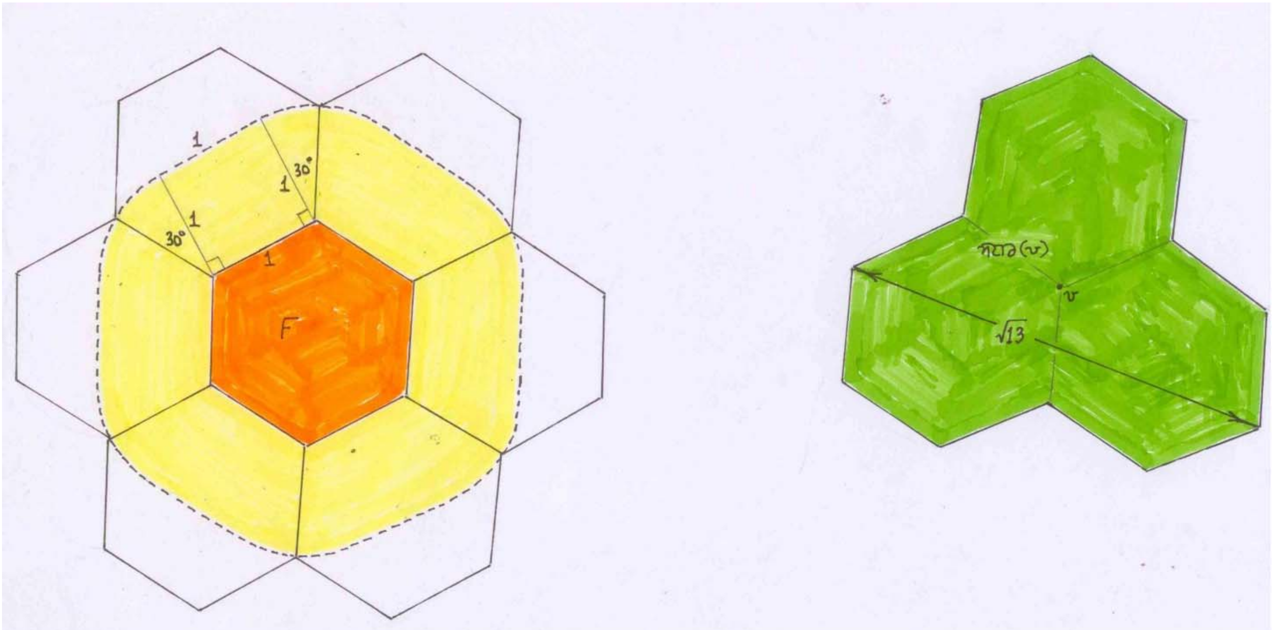
(translation of ਉਂਗਲ਼ੀਆਂ ਅਤੇ ਟਾਇਲਾਂ )

**Theorem** Given in the plane any finite set of disjoint pairs of points distance 1 or less apart we can join them by disjoint arcs whose diameters are less than an absolute constant $\alpha$.

**Proof 1** Here is a construction which works with $\alpha = \sqrt{2}$ and uses rectangular axes such that there is at most one of the given points on any horizontal or vertical line. We first join each pair of points by an angle : we go from the lower point horizontally in that direction such that a ninety degree turn then takes us to the higher point. The cuts between these angles shall now be eliminated in decreasing order of their height. Note that all cuts at the same height are situated on the horizontal arm of just one angle, which we now replace by a new angle with fingers : the vertical arm of the old angle stays put, but a small interval around each cut is deleted from its horizontal arm, and joined instead by an arc that first goes up, then around the higher vertex of the intersecting angle, and then down again. If these fingers are thin enough – diagram below has six cuts so six fingers – they'll have height less than 1 and by using thinner fingers at lower levels we can ensure also that they go comfortably inside the shorter higher level fingers if any which might already have been made around the same vertical arm. This new angle-with-fingers is contained in the rectangle of height 1 on the horizontal arm of the old angle, so its diameter is less than $\sqrt{2}$. q.e.d.

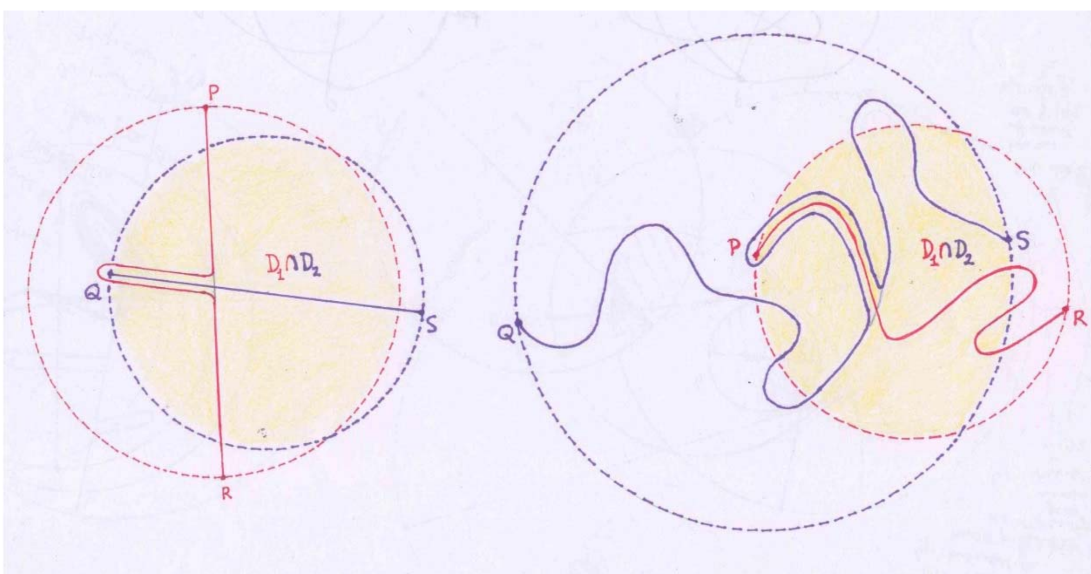



**Proof 2** This method works only with the much bigger constant $\alpha = \sqrt{13}$ but has its own charm. It uses a regular unit hexagonal tiling chosen such that the given points are in the interiors of the tiles. We'll inductively join the pairs of points by disjoint arcs such that any arc which meets a tile is either in its interior or cuts its boundary just once in one of its six edges. Therefore the other points of any tile which are not in any of these arcs form a path connected set. So, if the next pair of points is contained in a tile F, we can join them in the interior of this tile by a new disjoint arc. If only one point of this pair is in F, the other must be in one of the two other tiles of the star of some vertex v of F, with the line segment joining them contained in this star(v). This because any line segment of length at most one and starting from F cannot protrude beyond the yellow ribbon depicted in the next figure. So the two points of our pair are in tiles F and G sharing a common edge e. Using again the aforementioned path connectivity we can join each of them in its own tile by a new disjoint arc to the same point in e. Lastly note that $\sqrt{13}$ is the diameter of any star(v). q.e.d.




**Proof 3** In fact any $\alpha > 1$ works ! This is a corollary of the following result which applies to an open and dense subset of the space of all finite subsets of disjoint pairs of points of the plane.

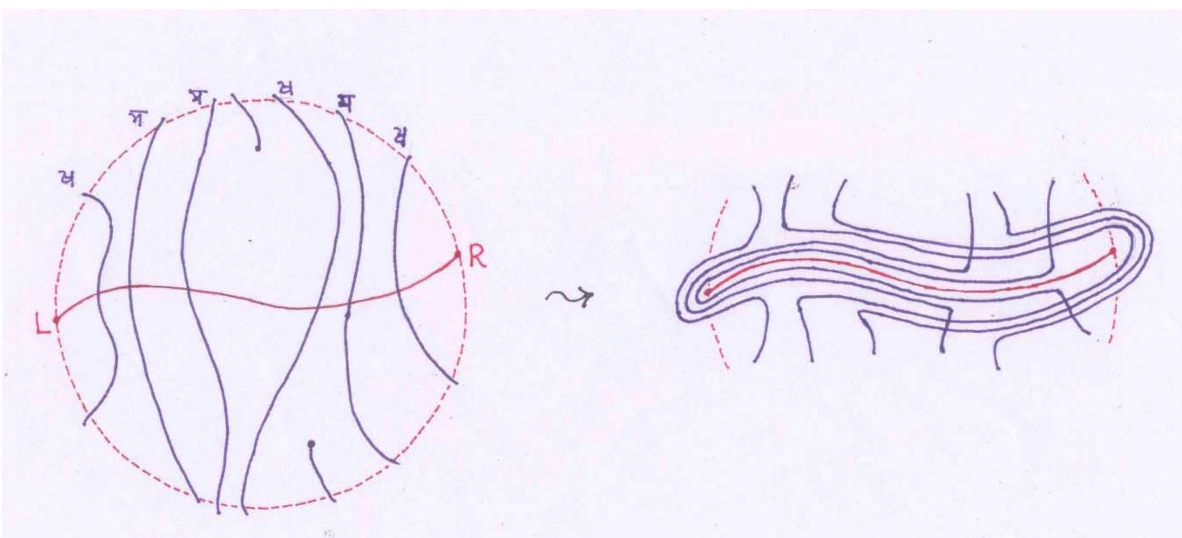
(ਓ) <u>If the circles having the pairs as antipodal points, and the segments joining the pairs, cut only two at a time and transversely, then the segments can be replaced by disjoint arcs which too are, but for their ends, in the open disks enclosed by these circles, as well as in an arbitrarily small neighbourhood of the union of the segments.</u>




The need to proceed with care is clear already from the case of two disjoint pairs {P,R} and {Q,S}. If the segments PR and QS cut, then this cut is in the intersection $D_1 \cap D_2$ of the open disks over them as diameters, and at least one -- possibly two or three, but never all four -- of the ends P, Q, R and S is on the boundary of $D_1 \cap D_2$. This because the angles of the quadrilateral PQRS add to four right angles, so at least one is not acute. We'll remove the cut by putting a thin finger on the other segment past such an end. Indeed, *even if two* ***arcs*** $\widetilde{PR}$ *and* $\widetilde{QS}$ *are, but for their ends, in the open disks* $D_1$ *and* $D_2$*, and have a single transverse cut, then one of them, from this cut to one of its ends, is entirely in* $D_1 \cap D_2$. For, if say the second arc runs out into $D_2 \backslash D_1$ on both sides of the cut, then it separates one end of the first arc from $D_1\setminus D_2$. So again, putting a thin curved finger past such an end shall remove the cut in such a way that the new arc is in the same open disk. *What if $\widetilde{PR}$ and $\widetilde{QS}$ have more transverse cuts?* Then the conclusion is not true for all cuts because an arc can double back in $D_1 \cap D_2$ and cross the other again, that is, because of *some loops in* $D_1 \cap D_2$ *formed by the two arcs between consecutive cuts*. Such pairs of cuts and loops shall be removed by changing the second arc, from just before the first to just after the second cut, so that it is parallel to the first arc [alternatively we can double switch train tracks to remove this pair of cuts and loop]. With all the loops in $D_1 \cap D_2$ now gone the remaining cuts can then be removed one by one by making as before the appropriate finger moves. This means that we have established the following stronger claim but only for the case of two disjoint pairs.

(ਅ) If the circles having the pairs as antipodal points, and some arcs joining them in the open disks enclosed by these circles, cut each other two at a time and transversely, then the pairs can be joined by disjoint arcs, also but for their ends in the same open disks, as well as in an arbitrarily small neighbourhood of the union of the original arcs.

We assume inductively that n-1 of the arcs are disjoint, and will first remove those cuts they make on the nth *red arc* which can be removed by moving this red arc. That is we first remove the loops if any in the intersections $D_i \cap D_n$ by changing its course in the manner indicated above. Then we make all possible finger moves on the red arc within the nth disk from cuts past ends of the cutting arcs lying in this disk. The situation now – in figure ਖ for ਖੱਬੇ = left, and ਸ for ਸੱਜੇ = right -- is that at each cut the cutting ith arc runs out of $D_n$ into $D_i\setminus D_n$ on both sides, so from above the appropriate finger move to make is now on this cutting arc past at least one of the two ends L and R of the red arc.




It may be that the first some cuts can be removed by fingering all these arcs past the left end L, and the remaining by fingers past the right end R. Then of course we are done, but more often, as shown above, this fingering shall introduce new cuts between them. The union of these new n-1 arcs is however disjoint from the red arc, so we can choose a smaller neighbourhood of this union which avoids it. Applying the inductive hypothesis again we now remove these cuts appropriately taking care to stay in this neighbourhood. *q.e.d.*

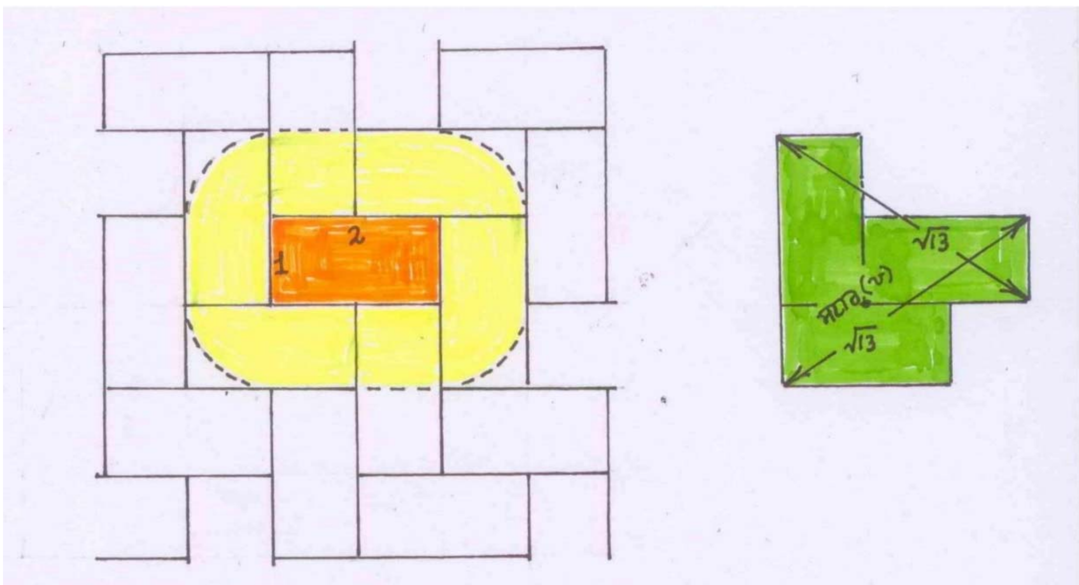
**Notes**

(1) The displayed theorem is tied to a paper of Shub and Smale which appeared in the Annals of Math of 1972 but a proof of the same that was given in a Berkeley seminar in 1971 was wrong. As was pointed out by Thurston by drawing there and then on the board a wonderful picture! Sullivan who was at this seminar made this story popular and in 2014

Calegari made it into a blog post. I read the story but skipped a proof because I wanted to get full enjoyment from this riddle. As a result all the proofs given in this paper are mine only.

(2) It seems uncertain as to who found the first correct proof but Edwards was suspected. Anyway in 1971 itself Bryant had posed the puzzle in print as problem 5787 of the American Math Monthly. The presence of √13 in its statement seems to indicate that he was probably aware of my second proof. Also, reading now the proof in Calegari's blog post, I notice that though it also started with a hexagonal tiling, it is used in a different way, and finally, despite the use of some more weapons, it needs the far bigger constant 42.

(3) However it is possible that Bryant's √13 was hinting only at the square tiling ! Yes, each vertex now has four not three incident tiles, but uniting some pairs of tiles gives us a trivalent crystallographic bricklaying pattern and a fourth proof (the commonest bricklaying pattern also works but needs a bigger constant √17) :-




(4) The Monthly in 1972 printed only the answer of Ungar [and named two others who had also solved the problem] which is essentially the same as my first proof, but in many more words. An editorial note then informs us that by an extensive revision of his procedure Ungar has subsequently shown that any α > 1 works. My third proof -- via the far more general theorem (ℳ) -- seems to be the first time that a proof of this claim has appeared in print.

(5) Theorem (ℳ) remains valid if we replace Euclid's geometry by that of a disk of finite radius c – see my running paper *Plain Geometry and Relativity* – with cayley distance. This because all points at a fixed distance from a given point now form an ellipse but generically if these ellipses cut they cut twice and transversely. This is the axle around which the third proof revolved, so its arguments apply to any distance whose 'circles' have this property.

(6) If we look at the second proof with a discrete or quantum eye then it is 'best possible' too! The superficial continuity or sameness of the plane is but a myth for such an observer and a hexagonal tiling one measure of this uncertainty. Each particle's partner is in the same tile or in a neighbour and we showed the two can indeed interact independently -- those disjoint arcs -- within these two tiles only.

(7) And the same proof works for the relativistic plane and its regular tilings {p.3}, p ≥ 7 …

K S Sarkaria 29 July 2015