From Bina's Garden to Khovanov's Homology

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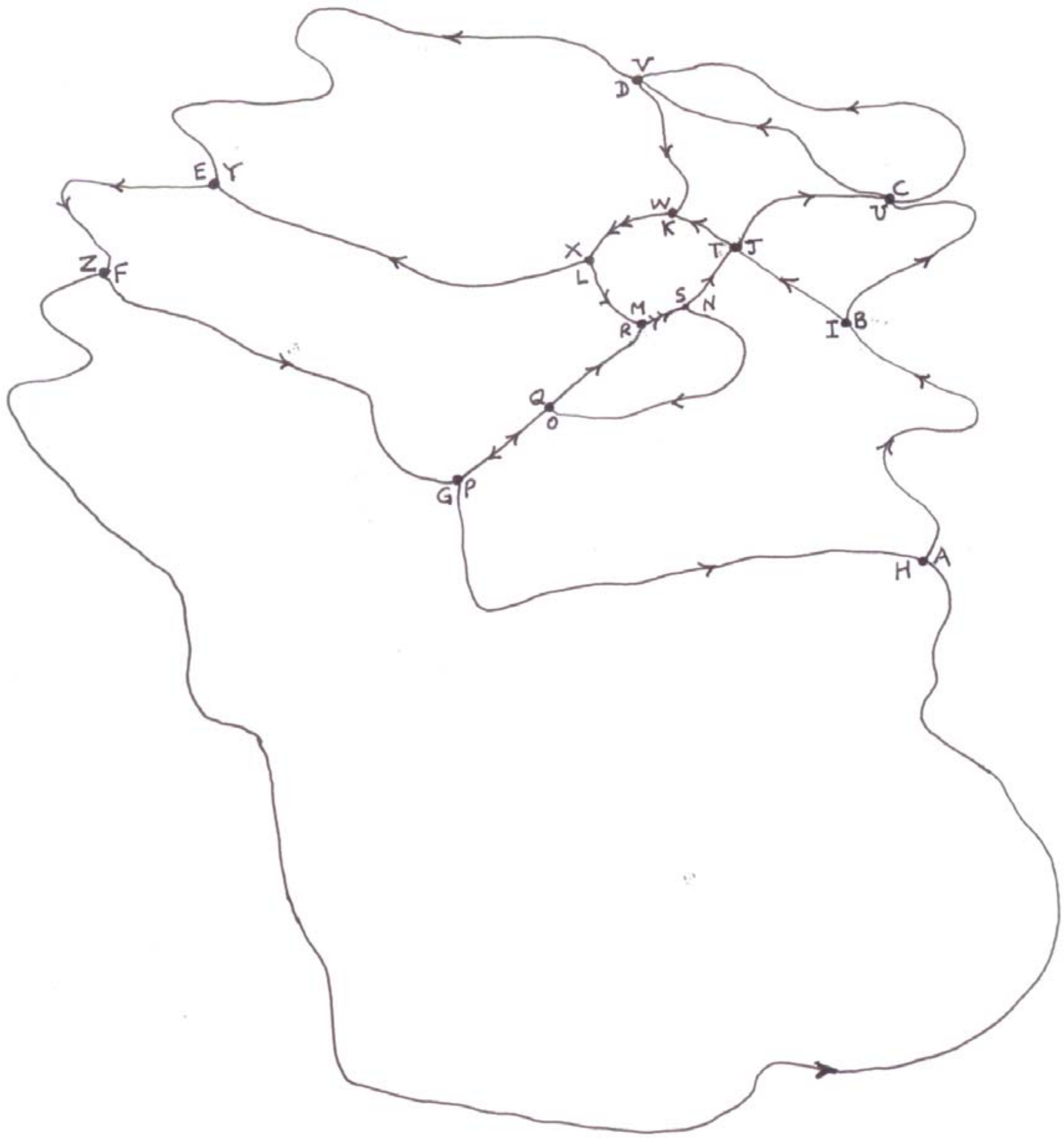
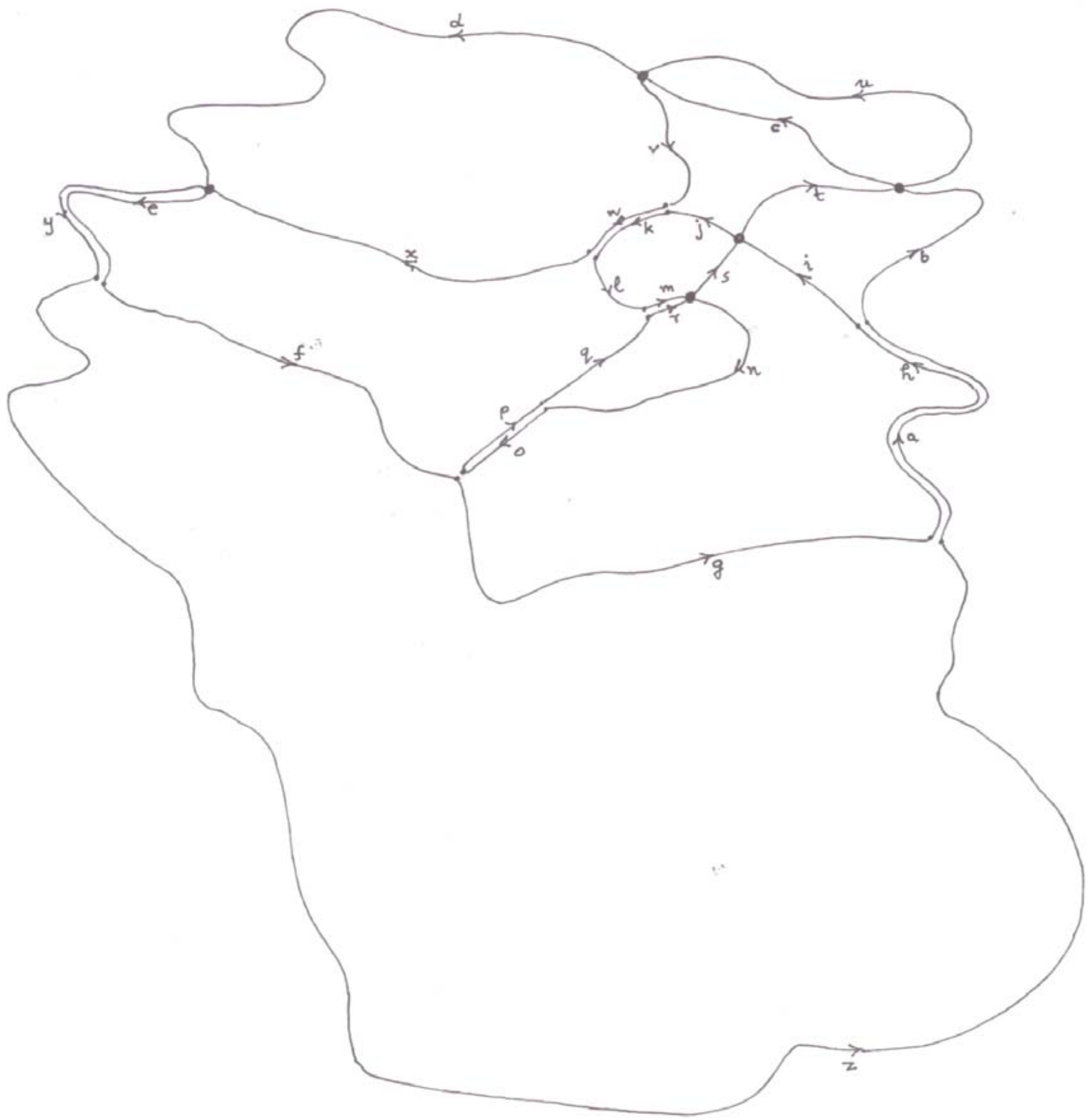
The map above focusses on the pretty pattern of paths in a *petit bois*, but it has much more. It is thick with the thorny bushes and evergreen trees—their trunks are shown—native to these hills, and amongst these, *une jardinière* has artfully planted all sorts of flowers like roses, dalhias, chrysanthemums, marigolds, gladioli and irises. Again, to depict the beautiful undulations of this hillside would have entailed many contour lines, which would have cluttered up the map, so only the relative height of a few spots is indicated. The height above sea level is about 5500 feet, which is *haut* alright, but that cottage you see is no *institut*. There has never been more than one *chercheur* residing in it, and he too spends most of the day outside it, taking in the vista—a broad valley with villages and a winding railway line, and with hills on three sides—that opens up to his eyes through that arc of pine trees. And when dusk falls, he is reluctant to go in before he has spotted at least one flying squirrel—there is one in the map!—gliding from tree to tree.



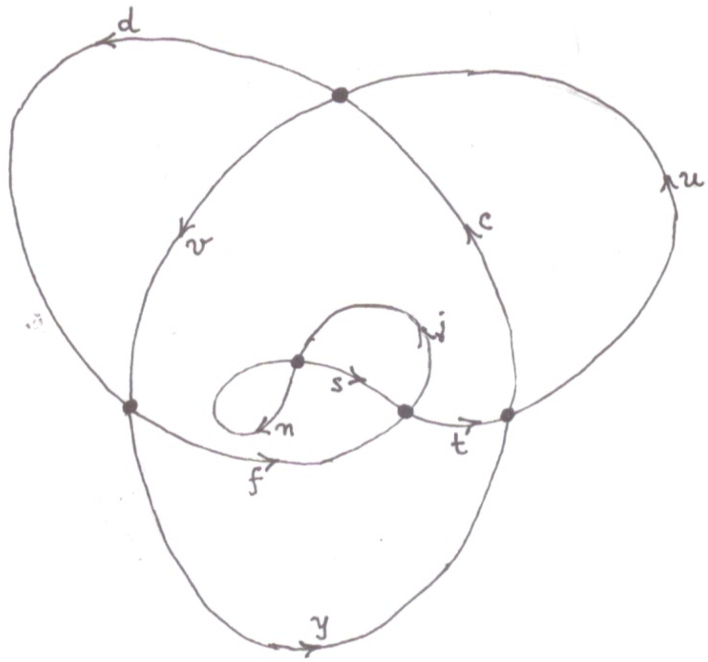
Also, he is given to touring all the paths at least once each day, and this he is wont to do in a rather peculiar way. For he imagines the pattern of paths—in his mind's eye—as the graph that is drawn above, and wants to repeat the least number of edges. His route is ***A to Z*** in alphabetical order and then ***to A***. Which is optimal because only five edges get repeated—the ones which have two arrows marked on them—and the ten vertices incident to them account for all the vertices of valence three, the remaining three vertices of the graph being of valence four.

More generally, the great Euler, after touring those notorious seven bridges of Kőnigsberg, had discovered long ago that, *for any connected graph, the least number of edges that one must repeat is that which converts it into a mod 2 cycle*, i.e., a graph with all vertices of even valence. Since $\partial\partial = 0$, the number of vertices of odd valence is always even, and obviously we must repeat at least half this number of edges. This sufficed for the graph above, but there are graphs—consider any subdivided arc—for which all the edges have to be repeated.

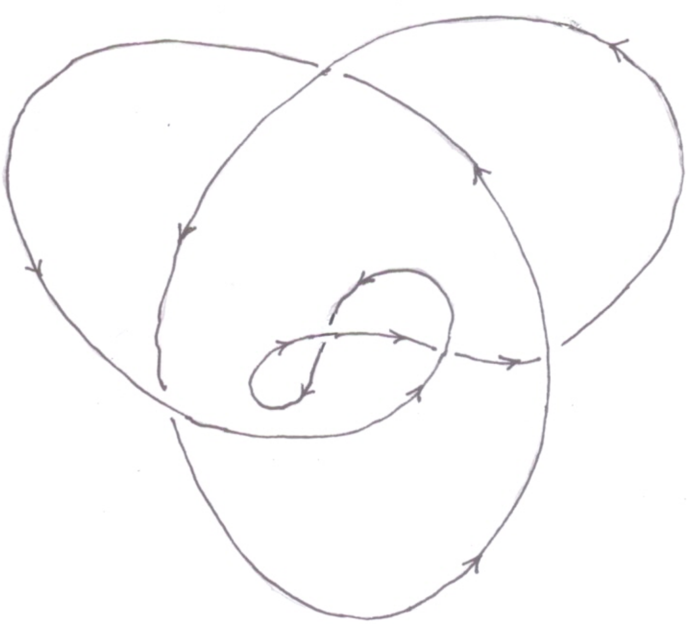
The nine bounded components of the complement of the above planar graph are 2-cells, and so is the tenth if we add a *point at infinity*, obtaining thus a cell subdivision of the 2-dimensional sphere having 13, 21 and 10 cells of dimensions 0, 1 and 2 respectively. A sweeping theorem of Poincaré, who came long after Euler, tells us that, *if we count each cell as* +1 *or* – 1 *depending on whether its dimension is even or odd, then the number of cells in any cell complex depends only on the underlying space*. This invariant was dubbed the *euler characteristic* because it was known to Euler that the *2-dimensional sphere has characteristic two*—e.g. 13 – 21 + 10 = 2—but this fact was known also to Descartes long before Euler was born!



The remembrance of this far-sighted thinker, more particularly his doctrine that, *matter is but extension, and is differentiated only by its various motions, which must perforce be in simple closed loops,* inspires our peripatetic to tread those garden paths more carefully. If you were to watch him closely now, you'll observe that he almost tip-toes on those errant paths! His route is more accurately the one depicted above, and involves *no repetition of edges at all*—these 26 edges a, b, …z correspond to the previous AB, BC, …, ZA—*but, apparently, there remain some pesky vertices*, namely those five bigger black dots. Keeping just these vertices—and denoting the paths running between them by the first letters of their cyclic words, for example, yzab by y and jklm by j—and making some, from the cartesian point of view unimportant, irrotational motions of the underlying plane, we obtain the following directed graph, in which *all vertices are of valence four and all crossings are transverse* (the associated cell subdivision of the 2-sphere has now 5, 10 and 7 cells of dimension 0, 1 and 2 respectively, but of course 5 – 10 + 7 is still two).



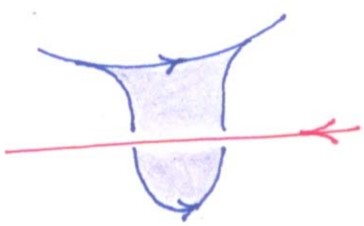
Which brings us to the five funny hops that are now being made by our promenader in the garden! He is mulling *a cartesian string of matter whirring on a plane surface with in-coming traffic from the right making an infinitesimal jump to avoid each apparent collision.*

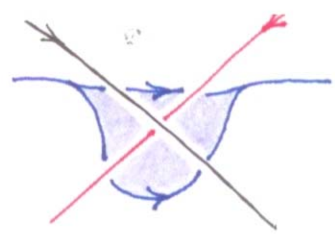


Thus the jumps shown above are by amounts *bigger than zero but less than all positive real numbers!* We recall that the *calculus of infinitesimals* was unveiled by Leibniz a scant 25 years after Descartes (also that, it was a fortuitous discovery in the 19th century of some notes that Leibniz had made, from a posthumous and now lost manuscript of Descartes, which shows us that the latter had a beautiful proof of V – E + F = 2 for all convex 3-polytopes)!

As is usual, we'll assume everything *smooth,* i.e. that Leibniz's calculus applies without any let or hindrance. So the instantaneous motion at each point x(t) of any string proceeds with a well-defined velocity v = dx/dt. Moreover this tangent vector v *is necessarily nonzero*: for, if it were zero, then the string decomposes into the orbits of a gradient vector field on its ambient space, and per the cartesian doctrine, such irrotational motions are to be discounted.

Unusually so, we'll assume as ambient space for the last picture, *all points above the plane surface and at an infinitesimal distance from it!* That is, the disjoint union of planes, each consisting of all points at a fixed infinitesimal height above the surface. Our string is a union of strands lying on different layers of this space, but its infinitesimal breaks are invisible! We'll now describe how, over an interval of time, the last picture can change because of some, from the cartesian point of view unimportant, irrotational motions of this ambient space.



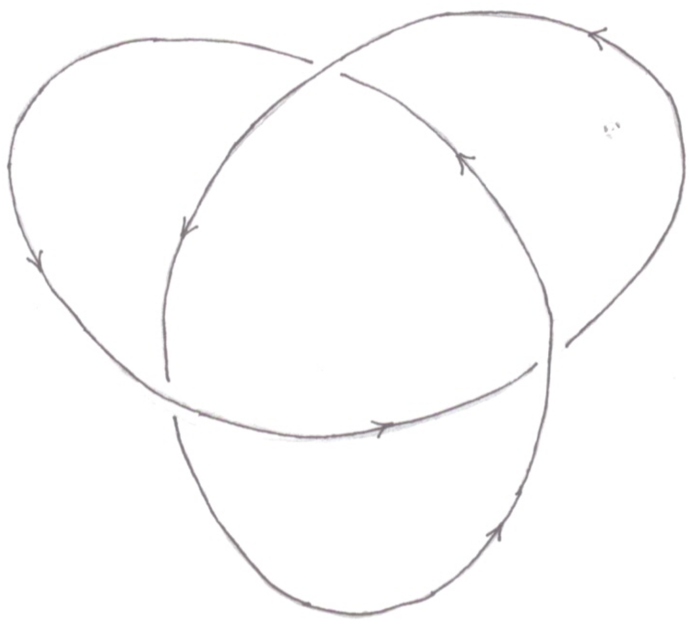


For example, as shown above, there might appear or disappear two apparent crossings of a strand in a sliding layer with another strand; or else, a strand in this sliding layer might slide over to the other side of the apparent crossing of two strands in two other layers. Moreover, it can be shown that, *the picture can change only up to a finite sequence of such elementary moves.* This implies in particular that, *there is no picture of our string with less than five hops!*

Indeed, for one of the two hops in the first picture above, it is the in-coming traffic from the right that jumps up, while for the other hop, it is that coming in from the left. On the other hand, the second move does not change either the number or the nature of the hops. So, quite generally, given any finite set of strings whirring on the plane surface, if we take any snapshot of these strings, and then *count each hop in the picture as* +1 *or* – 1 *depending on whether it is the in-traffic from the right or that from the left that jumps up,* we obtain an invariant of the strings, which we'll call their *writhing number.* This invariant being +5 for the case of our string, there is a picture of this string with n hops iff n is odd and bigger than or equal to five.

Is it possible, our walker muses, that for the strings making our world, we can assume that it is *always* the in-traffic from the right (or always left) that jumps? Such a bias in its very monads could perhaps explain why *all living things of our world are made from DNA that always spirals in the right-handed way* ... Of course quite a few of us are left-handed, and about 1 in 10,000 even have *situs inversus*—heart on the right side, etc—but these variations perhaps occur, he muses further, because mitosis and morphogenesis are not continuous, they somehow involve, *a sequence* *of reflections into and back from a mirror world* ... Coming out of this reverie, he consoles himself by recalling a beautiful fact from elementary geometry, viz. that, *all distance-preserving motions are compositions of reflections,* which serves, once again, to reinforce his deeply-held conviction that, *the discrete and the continuous are the two sides of the same coin!*

[initial pages only from an unfinished paper, of which the next figure was the following]



[see also *The Joys of Forgetting* and *Adieu 2013!*]