JOURNAL OF COMBINATORIAL THEORY, Series A **46**, 50–78 (1987)

Heawood Inequalities

K. S. SARKARIA

Department of Mathematics, George Mason University, Fairfax, Virginia 22030

Communicated by the Managing Editors

Received November 4, 1985

We study the effect of ambient topology on least valences, and so also on the chromatic numbers of embedded simplicial complexes. For example, we prove that a simplicial complex K can be embedded in an n-dimensional pseudomanifold X^n, $n \geqslant 2$, only if one of its $n-2$-simplices is incident to δ $n-1$-simplices, where $\delta < n(n+1)/(n-1)$ or $\binom{\delta+n-2}{n} - 2/(n+1)\binom{\delta+n-1}{n} \leqslant \dim H_{n-1}(X^n; \mathbb{Z}_2) - 1$. For the case of surfaces these are precisely the classical Heawood bounds. We show also that the close connection between Heawood bounds and $f_0(X)$, the least number of vertices required to triangulate X, generalises to dimensions $\geqslant 3$. Amongst the results needed to establish these inequalities is the fact that the Kruskal–Katona simplicial complexes maximise homology.

1. INTRODUCTION

In this paper we prove a number of results showing how ambient topology can effect the least valences, and so also the chromatic numbers, of an embedded simplicial complex. These results are discussed in Subsections (A) and (C). In (B) we discuss results used in their proofs; included here is the fact that the Kruskal–Katona simplicial complexes maximise homology.

(A) *Effect of Ambient Topology on Least Valences*

It is well known that a *graph* (undirected, without loops, and without multiple edges, i.e., a one-dimensional *simplicial complex*) can be embedded in the plane only if one of its vertices is incident to less than six edges. Our object is to prove some higher-dimensional analogues of this result.

THEOREM 1 (4.2.1). *A simplicial complex can be embedded*[1] *in an n-*

[1] See Section 2 for the topological definitions and assumptions adopted throughout this paper.

0097-3165/87 $3.00

dimensional space X^n, $n \geqslant 1$, *only if one of its* $(n-1)$*-simplices is incident to* δ n*-simplices, where*

$$\delta < n+1 \qquad or \qquad \binom{\delta+n-1}{n+1} \leqslant \dim H_n(X^n; \mathbb{Z}_2). \tag{1}$$

THEOREM 2 (4.3.3). *A simplicial complex can be embedded in an* n*-dimensional pseudomanifold* X^n, $n \geqslant 2$, *only if one of its* $(n-2)$*-simplices is incident to* δ $(n-1)$*-simplices, where*

$$\delta < \frac{n(n+1)}{n-1} \qquad or \qquad \binom{\delta+n-2}{n} - \frac{2}{n+1}\binom{\delta+n-1}{n}$$
$$\leqslant \dim H_{n-1}(X^n; \mathbb{Z}_2) - 1. \tag{2}$$

Specializing to graphs one gets the following known results.

COROLLARY 1 (Ershov–Kozhukin [6]). *If* G *is a graph, one of the vertices of* G *must be incident to* δ *edges, where*

$$\delta \leqslant \frac{1+\sqrt{1+8 \dim H_1(G; \mathbb{Z}_2)}}{2}. \tag{3}$$

COROLLARY 2 (Heawood [9] for (one-sided or two-sided) surfaces; Sarkaria [15] for all 2-pseudomanifolds). *A graph can be embedded in a 2-dimensional pseudomanifold* X^2 *only if one of its vertices is incident to* δ *edges, where*

$$\delta < 6 \qquad or \qquad \delta \leqslant \frac{5+\sqrt{1+24 \dim H_1(X^2; \mathbb{Z}_2)}}{2}. \tag{4}$$

These corollaries follow because the square root inequalities (3) and (4) are, for $\delta \geqslant 0$ and $\delta \geqslant 2$, the same as $\binom{\delta}{2} \leqslant \dim H_1(G; \mathbb{Z}_2)$ and $\binom{\delta}{2} - \frac{2}{3}\binom{\delta+1}{2} \leqslant \dim H_1(X^2; \mathbb{Z}_2) - 1$, respectively.

Note that Corollary 2 contains the fact regarding planar graphs mentioned in the beginning.

It is well known that Corollary 2 implies the following, where $f_0(X)$, *a topological invariant of* X, denotes the least number of vertices required to triangulate X.

COROLLARY 3. *A graph can be embedded in a surface* X^2 (X^2 *other than the* 2*-sphere*) *only if one of its vertices is incident to less than* $f_0(X^2)$ *edges.*

Thus, e.g., a 2-torus requires 7 vertices to be triangulated and it is true that *any* graph which can be embedded in the 2-torus has a vertex which is incident to less than 7 edges. For the 2-sphere the analogous statement is false.

Remark 1. It is important not to confuse Corollary 3, and its higher-dimensional generalisation, Theorem 3, with the trivial fact that an i-simplex, of any simplicial complex having f_0 vertices, is always incident to less than $f_0 - i$ $i+1$-simplices.

THEOREM 3 (4.3.7). *A simplicial complex can be embedded in an n-dimensional pseudomanifold X^n, $n \geqslant 2$, $H_n(X^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ (X^n other than those having $f_0(X^n) <]\, n(n+1)/(n-1)\, [+n-2$) only if one of its $(n-2)$-simplices is incident to less than f_0 (X^n)-$n+2$ $(n-1)$-simplices.*

Recall that Corollary 2 implies Corollary 3 because the classical Heawood bound, i.e., the expression in (4) involving the square root, happens to be less than the least number of vertices required to triangulate X^2. Likewise Theorem 3 follows from Theorem 2 because the generalized Heawood bound (2) behaves analogously:

THEOREM 4(4.2.3), (4.3.6). (a) *For any n-dimensional space X^n, $n \geqslant 1$,*

$$\dim H_n(X^n; \mathbb{Z}_2) \leqslant \binom{f_0(X^n)-1}{n+1}. \tag{5}$$

(b) *For any n-dimensional pseudomanifold X^n, $n \geqslant 2$,*

$$\dim H_{n-1}(X^n; \mathbb{Z}_2) - \dim H_n(X^n; \mathbb{Z}_2) \leqslant \binom{f_0(X^n)-1}{n} - \frac{2}{n+1}\binom{f_0(X^n)}{n}. \tag{6}$$

(B) *Method of Proof*

We will denote the number of j-simplices in a simplicial complex K by $f_j(K)$. Also $p_i, p_{i-1},\ldots, p_t$, will denote the numbers entering in the canonical representation of an integer $p \geqslant 0$ with respect to the integer $i+1$

$$p = \binom{p_{i+1}}{i+1} + \binom{p_{i-1}+1}{i} + \cdots + \binom{p_t+1}{t+1}, \qquad p_i > p_{i-1} > \cdots > p_t \geqslant t \geqslant 0. \tag{7}$$

An essential ingredient in our proof of Theorems 1 and 2 is the following well-known result:

KRUSKAL-KATONA THEOREM [12, 11, 3]). *On the set of all simplicial* *complexes having p i-simplices, the functions* $f_j(K)$, $0 \leqslant j \leqslant i$, *have the minimum values*

$$\binom{p_i+1}{j+1}+\binom{p_{i-1}+1}{j}+\cdots+\binom{p_t+1}{t+1-i+j}, \tag{8}$$

and these minimum values are attained at the Kruskal–Katona simplicial complex $\sum_i^p$ (see (3.1.1) *for a definition*).

Besides, we will need the following result, which is obviously interesting for its own sake.

THEOREM 5 (3.1.5). *On the set of all simplicial complexes having p i-simplices, the function* dim $H_i(K; \mathbb{Z}_2)$ *has the maximum value*

$$\binom{p_i}{i+1}+\binom{p_{i-1}}{i}+\cdots+\binom{p_t}{t+1} \tag{9}$$

and this maximum value is attained at the Kruskal–Katona simplicial complex $\sum_i^p$.

Remark 2. After receiving a preliminary version of this paper, Anders Björner was kind enough to inform me that he and Gil Kalai had also proved Theorem 5. An announcement of their work appears in [21].

Finally, our proof of Theorem 2 requires the following result which exploits the fact that X^n is a pseudomanifold, i.e., has no codimension one singularities. We remark that a slightly weaker result appeared in [15] and sufficed—without the use of any of the above extremal set theory—to prove Corollary 2 and weaker versions of Theorem 2.

THEOREM 6 (4.3.2). *A simplicial complex K can triangulate a subspace Y of an n-dimensional pseudomanifold* X^n, $n \geqslant 2$, *only if*

$$\dim H_n(X, Y; \mathbb{Z}_2) \leqslant \dim H_n(X; \mathbb{Z}_2) \qquad \text{or}$$

$$\dim H_n(X, Y; \mathbb{Z}_2) - \dim H_n(X; \mathbb{Z}_2) + 1 \leqslant \frac{2}{n+1} f_{n-1}(K). \tag{10}$$

Remark 3. We can prove that *for any i such that* $2i+2 \geqslant n$, *a simplicial complex* triangulates X^n *only if its least i*th *valence is bounded above by some number* $\Delta_i(X^n)$, *depending only on the topology of* X^n; furthermore, in the absence of singularities in dimensions $\geqslant i+1$, these bounds are very similar to those given in Theorems 1 and 2. However, this much easier result cannot be considered a generalisation of Theorems 1 and 2 since it does not apply to all embedded subcomplexes of X^n.

A set $\mathscr{T}$ of simplicial complexes is called a *topological class* if it satisfies the following property: $K \in \mathscr{T}$, L embeddable in the space of $K \Rightarrow L \in \mathscr{T}$ (cf. Saaty–Kainen [22, p. 193]; also cf. Theorem 3 with their "metaconjecture"). Note that the set of all simplicial complexes embeddable in an X^n is a topological class, while that of all triangulations of X^n is not. It seems to us that an inequality $\delta_i(K) \leqslant C$ (here $\delta_i(K)$ denotes the least ith valence of simplicial complex K) deserves to be called a *Heawood Inequality* only if it is valid for all K belonging to some non-trivial topological class $\mathscr{T}$.

Our inability to formulate and prove suitable analogues of Theorem 6 prevents us from using the methods of this paper to prove analogues of Theorems 1 and 2 in codimensions $\geqslant 3$. Still, there are reasons to believe that the following is true:

CONJECTURE 1. *For each X^n there exist integers $\delta_i(X^n)$, $2i+2 \geqslant n$, depending solely on the topology of X^n, such that a simplicial complex can be embedded in X^n only if one of its i-simplices is incident to $\leqslant \delta_i(X)$ $(i+1)$-simplices.*

Note that this conjecture is false without the condition $2i+2 \geqslant n$: if $2i+2 < n$, any $(i+1)$-dimensional simplicial complex can be embedded in an n-dimensional X^n. Likewise, by using (5.1.2) of [18], one can see that even the result mentioned in Remark 3 is false without the condition $2i+2 \geqslant n$.

Remark 4. In particular, Conjecture 1 implies that there is an integer C_n such that an n-dimensional simplicial complex can be embedded in $\mathbb{R}^{2n}$ only if one of its $(n-1)$-simplices is incident to $\leqslant C_n$ n-simplices. We have proved the existence of C_n for somewhat smaller topological classes (see [23] and [24]). Of these results the simplest to state is the following: *if $n \neq 2$, an n-dimensional simplicial complex unknots in $\mathbb{R}^{2n+1}$ only if it embeds in $\mathbb{R}^{2n}$; and, for any such simplicial complex, one of the $(n-1)$-simplices is incident to less than $3(n+1)$ n-simplices.* The methods used to obtain these results are very different and make essential use of the embedding and unknotting theorems of van Kampen–Wu–Shapiro and Weber.

(C) *Effect of Ambient Topology on Chromatic Numbers*

A well-known argument enables one to deduce from Corollary 2 the following:

COROLLARY 4 (5.2.3). *A graph can be embedded in a 2-dimensional pseudomanifold X^2 with $H_2(X^2; \mathbb{Z}_2) \cong \mathbb{Z}_2$ (X^2 other than the 2-sphere) only if its chromatic number is less than or equal to*

$$\frac{7+\sqrt{1+24 \dim H_1(X^2; \mathbb{Z}_2)}}{2}. \tag{11}$$

Let us recall this "well-known argument." For this purpose it is convenient to denote by $\delta_i(X^n)$ the best possible of the numbers mentioned in Conjecture 1. Then the essential point used in the proof is that a graph G embeds in X^2 only if its chromatic number is less than or equal to $\delta_0(X^2)+1$ (cf. Szekeres–Wilf [19]).

In order that Theorems 1 and 2 have similar chromatic implications, one must thus define the chromatic numbers of simplicial complexes in such a way that a "generalized Szekeres-Wilf bound" is available. Such a definition was introduced in [15] (and, in a different context, before by Erdös–Hajnal [5]): The *weak ith chromatic number*, $c_i(K)$, of a simplicial complex K is the smallest number of colors that can be assigned to the i-simplices of K in such a way that *not all* of the i-dimensional faces of any $(i+1)$-simplex have the same color. With this one does have the required generalisation: A simplicial complex K embeds in the space X only if $c_i(K) \leqslant \delta_i(X)+1$ (5.1.1). Using this lemma we deduce *some higher-dimensional analogues of Corollary* 4 from Theorems 1 and 2 (see (5.1.2), (5.1.3)).

It is convenient to denote by $c_i\,(X^n)$ the supremum of the numbers $c_i(K)$ as K runs over all simplicial complexes which can be embedded in X^n. Note that Conjecture 1 implies the weaker

Conjecture 2. *$c_i(X^n)$ is finite if $2i+2 \geqslant n$.*

Even in this weaker conjecture one cannot drop the condition $2i+2 \geqslant n$: By Ramsey's Theorem the weak ith chromatic number of the $(i+1)$-skeleton of an N-simplex is arbitrarily large for N sufficiently big, and—if $2i+2<n$—this $(i+1)$-skeleton can be embedded in X^n.

Methods establishing their finiteness seem also to put the exact calculation of the numbers $\delta_i(X)$, $i>0$, within reach. But, because of their Ramsey nature, there is not much hope for an exact calculation of the finite numbers $c_i(X^n)$, $i>0$, in very many cases. However, the zeroth chromatic numbers, $c_0(X^2)$, of surfaces have now been calculated: $c_0(X^2)$ equals (11), except for the Klein Bottle, when it is one less (Appel–Haken [1], Ringel [13], etc.). This raises the question whether a similar improvement of Corollary 4 can be made for all 2-pseudomanifolds. In this context we have

Theorem 7 (5.2.5). (a) *Let S^2_k, $k \geqslant 2$, denote the 2-pseudomanifold obtained by identifying k distinct points of the 2-sphere to a single point. Then $c_0\,(S^2_k)=5$ for all $k \geqslant 2$.*

(b) *Let $S^2_{(k)}$, $k \geqslant 1$, denote the 2-pseudomanifold obtained by identifying k distinct pairs of points of the 2-sphere. Then $c_0(S^2_{(k)})=4+k$ for $1 \leqslant k \leqslant 4$ while $c_0\,(S^2_{(5)})=8$.*

This improves on Dewdney [4] where Theorem 7(a) is proved for $k=2$, and Theorem 7(b) for $k=1, 2$. Dewdney's conjecture that $c_0(S^2_{(k)}) = 4+k$, $\forall k$, turns out to be false for all $k>4$.

Last, we have *some results regarding the strong chromatic numbers*, $C_i(K)$ (see (5.3.2)). $C_i(K)$ is defined like $c_i(K)$ except that, this time, we require that *no two* of the i-dimensional faces of any $(i+1)$-simplex have the same color.

2. Topological Definitions and Assumptions

(2.1) *All Our SIMPLICIAL COMPLEXES Are Finite*

From the combinatorial viewpoint, a simplicial complex K is thus a finite set of finite sets, obeying the condition $\sigma \in K$, $\theta \subseteq \sigma \Rightarrow \theta \in K$. A geometric *realisation* of K (in some Euclidean space, by means of rectilinear simplices) will also be denoted by K; the underlying space of such a geometric realisation will be denoted by $|K|$ or, if there is no danger of confusion, by K.

(2.2) *All Our SPACES and MAPS Are Piecewise Linear*

Realisations of simplicial complexes and simplicial maps: in particular this implies that all spaces are compact, and that one speaks of a *subspace* only if the inclusion map is piecewise linear. A one-to-one (resp. one-to-one onto) map is called an *embedding* (resp. *homeomorphism*). The phrases "$|K|$ embeds in X," "K embeds in X," and "K is a *subcomplex* of (the space) X" are to be considered synonymous; likewise "$|K|$ homeomorphic to X," "K homeomorphic to X," and "K is a *triangulation* of X" are equivalent.

Remark 5. Thus "K embeds in X" is the same as saying that "K triangulates a subspace Y of X," but it is not the same as saying that "K is a subcomplex of a simplicial complex L which triangulates X." (Easy examples show that such an L may not exist.) However, one can show easily that "K embeds in X" is the same as saying that "one can *subdivide* K to obtain a K' for which there exists an $L \supseteq K'$ such that L triangulates X."

Remark 6. With some extra, but routine, effort, Theorems 1 and 2 can be extended, with essentially the same proofs, to more general spaces. By restricting to the piecewise linear category we avoid the bother of using singular homology, dimension theory, etc., and still retain all the characteristic features of an embeddability problem. Note that compactness is not much of a restriction; e.g., embeddability in Euclidean space $\mathbb{R}^m$ is to be understood as embeddability in the the piecewise linear sphere S^m. Note also that "K embeds in $\mathbb{R}^m$" is not the same as saying that "K can be realised in $\mathbb{R}^m$;" rather it is the same as saying that "some subdivision K' of K can be realised in $\mathbb{R}^m$."

See [10] or [14] for more regarding the concepts of piecewise linear topology.

(2.3) *Topological Invariants Defined as Supremums or Infimums*

We will take supremums as K runs over the set of all embedded *subcomplexes of X*. It is important to remember (see Remark 5) that this set is generally larger than the set of all subcomplexes of triangulations of X.

$\delta_i(X)$. For each simplex σ of K, the number of incident $(\dim \sigma + 1)$-simplices of K is called the *valence* of σ in K and denoted by $\delta_K(\sigma)$. The least of the numbers $\delta_K(\sigma^i)$, $\sigma^i \in K$, is called the ith-*least valence* of K and denoted by $\delta_i(K)$. We put $\delta_i(X) = \sup_K \delta_i(K)$. So $\delta_i(X)$ is an integer or ∞.

$d_i(X)$. The number of i-simplices in K is denoted $f_i(K)$. If $f_i(K) > 0$, the average of the valences of the i-simplices of K, i.e., $(i+2) f_{i+1}(K)/f_i(K)$, will be denoted by $d_i(K)$. We put $d_i(X) = \sup_K d_i(K)$. So $d_i(X)$ is a real number or ∞.

$c_i(X)$. The ith *weak chromatic number* $c_i(K)$ of simplicial complex K is the least number of colors that can be assigned to the i-simplices of K so that *not all* the i-faces of any $(i+1)$-simplex have the same color. We put $c_i(X) = \sup_K c_i(K)$. So $c_i(X)$ is an integer or ∞.

$C_i(X)$. The ith *strong chromatic number* $C_i(K)$ of simplicial complex K is the least number of colors that can be assigned to the i-simplices of K so that *no two* of the i-faces of any $(i+1)$-simplex have the same color. We put $C_i(X) = \sup_K C_i(K)$. So $C_i(X)$ is an integer or ∞.

$f_0(X)$. In contrast to the above this time we take the infimum of the numbers $f_0(K)$ as K runs over all *triangulations* of X. (Thus Theorem 3 appears to be a curious "minimax" theorem.) Unlike the above numbers the integral function $f_0(X)$ is not monotone: one can have $Y \subseteq X$ and $f_0(Y) > f_0(X)$.

(2.4) *Notation*

We conform to standard usage and notation regarding simplicial complexes. "Dimensions"—the dimension of a simplex σ is one less than its cardinality $\#(\sigma)$—are occasionally denoted by superscripts; the union of two disjoint simplices σ and θ is denoted by $\sigma \cdot \theta$ (their "join"); "stars," and "links" of simplices are denoted as usual by $\mathrm{St}_K \sigma$ and $\mathrm{Lk}_K \sigma$, etc.

The only homology that will be used in this paper is *reduced homology* *with* $\mathbb{Z}_2$ *coefficients.* The dimensions of the vector spaces $H_i(K;\mathbb{Z}_2)$, $H_i(X;\mathbb{Z}_2)$ are sometimes denoted by $b_i(K)$, $b_i(X)$.

3. Kruskal-Katona Complexes

(3.1.1) Definition. The Kruskal–Katona simplicial complex $\sum_i^p$ (mentioned in subsect. 1(B)) can be defined as follows. If $p=\binom{p_i+1}{i+1}$, then $\sum_i^p=\sigma_i^{p_i}$, the simplicial complex consisting of all simplices of dimensions $\leqslant i$ having vertices in $\{0,1,2,..., p_i\}$. (So in particular $\sum_0^p=\sigma_0^{p-1}=\{0,1,2,..., p-1\}$.) If $\binom{p_i+2}{i+1}>p>\binom{p_i+1}{i+1}$, $\sum_i^p$ is a subcomplex of $\sigma_i^{p_i+1}$ defined by $\sum_i^p=\sigma_i^{p_i}\cup(p_i+1)\cdot\sum_{i-1}^{m_i}$, where $m_i=p-\binom{p_i+1}{i+1}$.

(3.1.2) Homology of $\sum_i^p$. *One has* $H_j(\sum_i^p)=0$ *for all* $j\neq i$; *further, if* p *has the canonical representation*

$$p=\binom{p_i+1}{i+1}+\binom{p_{i-1}+1}{i}+\cdots+\binom{p_t+1}{t+1},\qquad p_i>p_{i+1}>\cdots>p_t\geqslant t\geqslant 0,\tag{1}$$

then

$$\dim H_i\left(\sum_i^p\right)=\binom{p_i}{i+1}+\binom{p_{i-1}}{i}+\cdots+\binom{p_t}{t+1}.\tag{2}$$

We prove this by induction on i. The result is obvious when $\sum_i^p=\sigma_i^{p_i}$: If $j<i$, $H_j(\sigma_i^{p_i})=H_j(\overline{\sigma^{p_i}})=0$; so, by Euler's formula, $\dim H_i(\sigma_i^{p_i})=f_i(\sigma_i^{p_i})-f_{i-1}(\sigma_{i-1}^{p_i})+\cdots\pm 1=\binom{p_i+1}{i+1}-\binom{p_i+1}{i}+\cdots\pm 1=\binom{p_i}{i+1}$. Otherwise $\sum_i^p$ is the union of $\sigma_i^{p_i}$ and a cone over a subcomplex of $\sigma_i^{p_i}$. Since this cone has trivial homology, the Mayer–Vietoris exact homology sequence of $(\sigma_i^{p_i}, (p_i+1)\cdot\sum_{i-1}^{m_i})$ reduces to

$$0\to H_i(\sigma_i^{p_i})\to H_i\left(\sum_i^p\right)\to H_{i-1}\left(\sum_{i-1}^{m_i}\right)\to 0.\tag{3}$$

So $\dim H_i(\sum_i^p)=\dim H_i(\sigma_i^{p_i})+\dim H_{i-1}(\sum_{i-1}^{m_i})$ which implies (2).

(3.1.3) Number of Simplices. *Again one can use a similar easy induction on* i. *Thus* $f_j(\sum_i^p)=f_j(\sigma_i^{p_i})+f_{j-1}(\sum_{i-1}^{m_i})$, $j\leqslant i$, *leads to*

$$f_j\left(\sum_i^p\right)=\binom{p_i+1}{j+1}+\binom{p_{i-1}+1}{j}+\cdots+\binom{p_t+1}{t+1-i+j}.\tag{4}$$

This shows that (2) coincides with Euler's formula for $\sum_i^p$.

(3.1.4) KRUSKAL–KATONA THEOREM [12, 11]. *$\sum_i^p$ minimises the functions $f_j(\,.\,)$ on the set of all simplicial complexes having p i-simplices. So*

$$f_j(K) \geqslant f_j\left(\sum_i^{f_i(K)}\right) \qquad \textit{for all } K. \tag{5}$$

(3.1.5) THEOREM 5. *$\sum_i^p$ maximises the function* $\dim H_i(\,.\,;\mathbb{Z}_2)$ *on the set of all simplicial complexes having p i-simplices. So*

$$\dim H_i(K;\mathbb{Z}_2) \leqslant \dim H_i\left(\sum_i^{f_i(K)};\mathbb{Z}_2\right) \qquad \textit{for all } K.$$

(3.2) *Proof of Theorem* 5.

We will essentially mimic Katona's proof [11] of (3.1.4), and, therefore, it will be convenient to adopt, for the course of this proof *only*, a notation parallel to that used in [11].

If natural number n has, with respect to natural number l, the canonical representation

$$n = \binom{a_l}{l} + \binom{a_{l-1}}{l-1} + \cdots + \binom{a_t}{t}, \tag{7}$$

then we define

$$F_l(n) = \binom{a_l}{l-1} + \binom{a_{l-1}}{l-2} + \cdots + \binom{a_t}{t-1}, \tag{8}$$

$$E_l(n) = \binom{a_l - 1}{l-1} + \binom{a_{l-1} - 1}{l-2} + \cdots + \binom{a_t - 1}{t-1}, \tag{9}$$

and

$$\Phi_l(n) = \binom{a_l - 1}{l} + \binom{a_{l-1} - 1}{l-1} + \cdots + \binom{a_t - 1}{t}. \tag{10}$$

Note that (2) and (4) tell us that

$$\dim H_{l-1}\left(\sum_{l-1}^{n}\right) = \Phi_l(n) \tag{11}$$

and

$$f_{l-2}\left(\sum_{l-1}^{n}\right) = F_l(n). \tag{12}$$

Furthermore, by (26) of [11, p. 194], one has

$$\frac{n \cdot l}{a_l^*} \leqslant E_l(n), \tag{13}$$

where a_l^* is the least integer such that $\binom{a_l^*}{l} \geqslant n$.

(3.2.1) We will prove the following three statements by induction on l. In [11], Katona proves three analogous statements regarding $F_l(n)$ in the same way.

(A_l) *If $h \geqslant 1$, $1 \leqslant l \leqslant h$, $1 \leqslant n \leqslant \binom{h}{l}$ and K is any simplicial complex with $f_{l-1}(K) = n$ and $f_0(K) \leqslant h$, then*

$$\dim H_{l-1}(K) \leqslant \Phi_l(n). \tag*{$(\alpha)_l$}$$

(B_l) *If $h \geqslant 1$, $1 \leqslant l \leqslant h$, $\binom{h}{l} \leqslant n \leqslant 2\binom{h}{l}$ and $\{K_1, K_2\}$ is any pair of disjoint simplicial complexes with $f_{l-1}(K_1) + f_{l-1}(K_2) = n$ and $f_0(K_1)$, $f_0(K_2) \leqslant h$, then*

$$\dim H_{l-1}(K_1) + \dim H_{l-1}(K_2) \leqslant \binom{h-1}{l} + \Phi_l\left(n - \binom{h}{l}\right). \tag*{$(\beta)_l$}$$

(C_l) *If $l \geqslant 2$, $n_1 \geqslant 1$, $n_2 \geqslant 1$, $n = n_1 + n_2$, and $n_2 \leqslant E_l(n)$, then*

$$\Phi_l(n_1) + \Phi_{l-1}(n_2) \leqslant \Phi_l(n). \tag*{$(\gamma)_l$}$$

From (11) it follows that (A) is equivalent to (6). By taking $K_1 \cong \sigma_{l-1}^{h-1}$ and $K_2 \cong \sum_{l-1}^{n-\binom{h}{l}}$ one sees that $(\beta)_l$ is also best possible. Finally the identity $\binom{a-2}{l} + \binom{a-2}{l-1} = \binom{a-1}{l}$ shows that $(\gamma)_l$ too is best possible.

(3.2.2) One has $\Phi_1(n) = n - 1$ and $\dim H_0(K) = \{$number of components of $K\} - 1$. This verifies (A) and (B) for $l = 1$.

For the canonical representations of n_1, n_2, with respect to l, $l-1$, let us write

$$n_1 = \binom{b_l}{l} + \binom{b_{l-1}}{l-1} + \cdots + \binom{b_\gamma}{\gamma},$$

$$n_2 = \binom{c_{l-1}}{l-1} + \binom{c_{l-2}}{l-2} + \cdots + \binom{c_s}{s}.$$

Assuming that (A), (B), and (C) have been proved for values less than l, we prove (C_l) as follows:

The hypothesis $n_2 \leqslant E_l(n)$ implies (see [11, p. 191]) that one of the following cases must arise. If $l = 2$, this verification already completes the proof because $(\gamma)_2$ is equivalent to $b_2 \geqslant a_2 - 1$. So in the arguments given below one can take $l \geqslant 3$.

Case a. $b_l = a_l$. Subtracting $\binom{a_l - 1}{l}$ from both sides of $(\gamma)_l$ we see that it is same as

$$\Phi_{l-1}\left(n_1 - \binom{a_l}{l}\right) + \Phi_{l-1}(n_2) \leqslant \Phi_{l-1}\left(n - \binom{a_l}{l}\right),$$

which follows from A_{l-1} because, for $l \geqslant 3$, the left side is precisely the dimension of the $(l-2)$th homology of the disjoint union $\sum_{l-2}^{n_1 - \binom{a_l}{l}} \amalg \sum_{l-2}^{n_2}$.

Case ba. $b_l = a_l - 1$, $n_2 \geqslant \binom{a_l - 1}{l-1}$. In this case one also has [11, p. 193] $c_{l-1} = a_l - 1$ and $n_2 - \binom{a_l - 1}{l-1} \leqslant E_{l-1}(n - \binom{a_l}{l})$.

Subtracting $\binom{a_l - 2}{l} + \binom{a_l - 2}{l-1}$ from the left, and the equal amount $\binom{a_l - 1}{l}$ from the right side of $(\gamma)_l$, we can rewrite $(\gamma)_l$ as

$$\Phi_{l-1}\left(n_1 - \binom{a_l - 1}{l}\right) + \Phi_{l-2}\left(n_2 - \binom{a_l - 1}{l-1}\right) \leqslant \Phi_{l-1}\left(n - \binom{a_l}{l}\right)$$

which follows from C_{l-1}.

Case bb. $b_l = a_l - 1$, $n_2 < \binom{a_l - 1}{l-1}$. In this case both $\sum_{l-2}^{n_1 - \binom{a_l - 1}{l}}$ and $\sum_{l-2}^{n_2}$ have $\leqslant a_l - 1$ vertices [11, p. 194].

Subtracting $\binom{a_l - 2}{l}$ from $(\gamma)_l$ we can rewrite it as

$$\Phi_{l-1}\left(n_1 - \binom{a_l - 1}{l}\right) + \Phi_{l-1}(n_2) \leqslant \binom{a_l - 2}{l-1} + \Phi_{l-1}\left(n - \binom{a_l}{l}\right),$$

which follows from B_{l-1} applied to $K_1 \cong \sum_{l-2}^{n_1 - \binom{a_l - 1}{l}}$, $K_2 \cong \sum_{l-2}^{n_2}$, because

$$n_1 - \binom{a_l - 1}{l} + n_2 - \binom{a_l - 1}{l-1} = n - \binom{a_l}{l}.$$

(3.2.3) Fix an $l \geqslant 2$ and assume that A and B have both been proved for all values less than l, and that C has been proved for all values up to and including l. We will now prove (A_l) and (B_l) by induction on h.

For $h = l$ the left sides of $(\alpha)_l$ and $(\beta)_l$ are zero; so the induction on h can start.

Choose a vertex v of K which is incident to $n_2 \leqslant n.l/f_0(K)$, $(l-1)$-simplices of K. So we have $n_2 \leqslant n.l/a_l^*$ and so $\leqslant E_l(n)$ by (13). The Mayer–Vietoris sequence of the pair $\{K - \mathrm{St}_K v, \overline{\mathrm{St}_K v}\}$ runs

$$\cdots \to H_{l-1}(K - \mathrm{St}_K v) \oplus H_{l-1}(\overline{\mathrm{St}_K v}) \to H_{l-1}(K) \to H_{l-2}(\mathrm{Lk}_K v) \to \cdots. \tag{14}$$

Since $H_{l-1}(\overline{\mathrm{St}_K v})=0$, the exactness of (14) shows that

$$\begin{aligned}\dim H_{l-1}(K) &\leqslant \dim H_{l-1}(K-\mathrm{St}_K v)+\dim H_{l-2}(Lk_K v)\\ &\leqslant \Phi_l(n-n_2)+\Phi_{l-1}(n_2)\\ &\qquad \text{(since } A_l \text{ can be applied to}\\ &\qquad \text{the complex } K-\mathrm{St}_K v \text{ which has less}\\ &\qquad \text{vertices and } A_{l-1} \text{ can be applied}\\ &\qquad \text{to } \mathrm{Lk}_K v)\\ &\leqslant \Phi_l(n) \text{ (by } C_l).\end{aligned}$$

To prove (B_l) we choose vertices v_1 and v_2 which are incident to $r_2 \leqslant r \cdot l/h$, resp. $s_2 \leqslant s \cdot l/h$, $(l-1)$-simplices of K_1, K_2; here $f_{l-1}(K_1)=r$ and $f_{l-1}(K_2)=s$. A Mayer–Vietoris sequence shows that

$$\begin{aligned}\dim H_{l-1}(K) &\leqslant \dim H_{l-1}(K-\mathrm{St}_{K_1}v_1-\mathrm{St}_{K_2}v_2)\\ &\quad +\dim H_{l-2}(\mathrm{Lk}_{K_1}v_1 \cup \mathrm{Lk}_{K_2}v_2)\\ &\leqslant \binom{h-2}{l}+\Phi_l\left(n-r_2-s_2-\binom{h-1}{l}\right)\\ &\quad +\binom{h-2}{l-1}+\Phi_{l-1}\left(r_2+s_2-\binom{h-1}{l-1}\right)\\ &\qquad \text{(by applying } B_l \text{ to the lesser-vertex-pair}\\ &\qquad \{K_1-\mathrm{St}_{K_1}v_1, K_2-\mathrm{St}_{K_2}v_2\};\\ &\qquad \text{and } B_{l-1} \text{ to the pair } \{Lk_{K_1}v_1, Lk_{K_2}v_2\})\\ &=\binom{h-1}{l}+\Phi_l\left(n-r_2-s_2-\binom{h-1}{l}\right)\\ &\quad +\Phi_{l-1}\left(r_2+s_2-\binom{h-1}{l-1}\right)\\ &\leqslant \binom{h-1}{l}+\Phi_l\left(n-\binom{h}{l}\right)\\ &\qquad \text{(by } C_l\text{: the requisite condition } r_2+s_2-\binom{h-1}{l-1}\\ &\leqslant E_l\left(n-\binom{h}{l}\right) \text{ is verified on p. 197 of [11]).}\end{aligned}$$

If one of the complexes K_1 and K_2 already has less than h vertices then we only have to extract a suitable St v from the other complex, and repeat the same argument.

(Observe that formulae (2) and (6), and all of the above argument, are valid even if $\mathbb{Z}_2$ is replaced by some other field of coefficients.)

4. Least Valences

(4.1.1) Lemma 1. *The Kruskal–Katona simplicial complexes maximise average valences, i.e.,*

$$d_i(K) \leqslant d_i\left(\sum_i^{f_{i+1}(K)}\right) \qquad \textit{for all } K. \tag{1}$$

This follows immediately from the Kruskal–Katona Theorem (3.1.4).

(4.1.2) Lemma 2. *For any simplicial complex* K,

$$f_i(K) \geqslant \binom{\delta_i(K)+i+1}{i+1} \tag{2}$$

In fact, if N denotes the largest integer such that $f_i(K) \geqslant \binom{N+1}{i+1}$, *then*

$$N \geqslant \delta_i(K)+i. \tag{3}$$

Proof. By (4) and (5) of §3, $f_{i+1}(K) < \binom{N+2}{i+2}$. Thus, by definition (3.1.1), $\sum_{i+1}^{f_{i+1}(K)}$ is a proper subcomplex of σ_{i+1}^{N+1}. Since σ_{i+1}^{N+1} has $N+2$ vertices, each i-simplex of $\sum_{i+1}^{f_{i+1}(K)}$, which has $i+1$ vertices of its own, must have valence $\leqslant N+2-(i+1)$. Further, an i-simplex of $\sum_{i+1}^{f_{i+1}(K)}$ incident to an $i+1$-simplex of σ_{i+1}^{N+1} not lying in $\sum_{i+1}^{f_{i+1}(K)}$, has valence strictly less. Hence the average ith valence of $\sum_{i+1}^{f_{i+1}(K)}$, and so by (1) that of K, is less than $N-i+1$. So K must have an i-simplex of valence $\leqslant N-i$.

(4.1.3) Lemma 3. *Let K be any $(i+1)$-dimensional simplicial complex and let* $\binom{N_i+1}{i+1}+\binom{N_{i-1}+1}{i}+\cdots$ *denote the canonical representation of* $f_i(K)$ *with respect to* $i+1$. (So N_i is same as N of Lemma 2.) *Then*

$$f_{i+1}(K)-f_i(K) \leqslant \dim H_{i+1}(K;\mathbb{Z}_2)-\dim H_i(K;\mathbb{Z}_2) \\ -\binom{N_i}{i}-\binom{N_{i-1}}{i-1}-\cdots \tag{4}$$

Proof. Let $\partial_j(K)$ denote the mod 2 boundary map $C_j(K)\to C_{j-1}(K)$. Applying the alternating sum formula to the chain complex $C_{i+1}(K)$

$\to^{\partial_{i+1}} C_i(K) \quad \to^{\partial_i} C_{i-1}(K)$ we get $f_{i+1}(K) - f_i(K) + \dim \operatorname{Im} \partial_i(K) = b_{i+1}(K) - b_i(K)$. To obtain (4) we note that

$$\begin{aligned}
\dim \operatorname{Im} \partial_i(K) &= f_i(K) - \dim \ker \partial_i(K) \\
&= f_i(K) - b_i(K^i), \\
&\qquad \text{where } K^i \text{ is the } i\text{-skeleton of } K \\
&\geqslant f_i(K) - b_i\left(\sum_i^{f_i(K)}\right), \\
&\qquad \text{by Theorem 5 (3.1.5)} \\
&= \left[\binom{N_i+1}{i+1} + \binom{N_{i-1}+1}{i} + \cdots\right] - \left[\binom{N_i}{i+1} + \binom{N_{i-1}}{i} + \cdots\right], \\
&\qquad \text{by (2) of Sect. 3} \\
&= \binom{N_i}{i} + \binom{N_{i-1}}{i-1} + \cdots \\
&\qquad \text{because } \binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}.
\end{aligned}$$

(4.2) *Least Valences in Codimension One*

(4.2.1) THEOREM 1. *A simplicial complex K can embed in an n-dimensional space X^n, $n \geqslant 1$, only if*

$$\delta_{n-1}(K) < n+1 \qquad \textit{or} \qquad \binom{\delta_{n-1}(K)+n-1}{n+1} \leqslant \dim H_n(X^n; \mathbb{Z}_2). \tag{5}$$

Proof. K embeds in X^n iff some subdivision K' of K is contained in a triangulation L of X (Remark 5, Sect. 2). Since simplicial complex L is n-dimensional, $\dim H_n(K; \mathbb{Z}_2) = \dim H_n(K'; \mathbb{Z}_2) \leqslant \dim H_n(L; \mathbb{Z}_2) = \dim H_n(X; \mathbb{Z}_2)$. Now we note that

$$\begin{aligned}
&\binom{\delta_{n-1}(K)+n-1}{n+1} \\
&\quad = \binom{\delta_{n-1}(K)+n}{n+1} - \binom{\delta_{n-1}(K)+n}{n} + \binom{\delta_{n-1}(K)+n-1}{n-1} \\
&\quad = \frac{\delta_{n-1}(K)}{n+1}\binom{\delta_{n-1}(K)+n}{n} - \binom{\delta_{n-1}(K)+n}{n} + \binom{\delta_{n-1}(K)+n-1}{n-1}
\end{aligned}$$

$$\leqslant \frac{\delta_{n-1}(K)}{n+1} f_{n-1}(K) - f_{n-1}(K) + \binom{\delta_{n-1}(K)+n-1}{n-1}$$

by (2) of Lemma 2, *provided* $\delta_{n-1}(K) \geqslant n+1$

$$\leqslant \frac{d_{n-1}(K)}{n+1} f_{n-1}(K) - f_{n-1}(K) + \binom{\delta_{n-1}(K)+n-1}{n-1}$$

$$= f_n(K) - f_{n-1}(K) + \binom{\delta_{n-1}(K)+n-1}{n-1}$$

$$\leqslant \dim H_n(K; \mathbb{Z}_2) - \binom{N}{n-1} + \binom{\delta_{n-1}(K)+n-1}{n-1}$$

by Lemma 3

$$\leqslant \dim H_n(X^n; \mathbb{Z}_2) \qquad \text{by (3) of Lemma 2.}$$

(4.2.2) *Remark* 8. It is well known that the inequality $\dim H_n(K; \mathbb{Z}_2) \leqslant \dim H_n(X; \mathbb{Z}_2)$ is true in much more generality, e.g., whenever K can be topologically embedded in a separable metric space X having dimension n. One can extend Theorem 1 to any such situation. On the other hand note that *for* $n \geqslant 2$, *the alternative* $\delta_{n-1}(K) \leqslant n$ *is necessary in Theorem* 1. To see this consider the *Zeeman dunce that* Z^2, i.e., the 2-dimensional space obtained by making identifications $\mathbf{AB} = \mathbf{AC} = \mathbf{CB}$ on the boundary of a triangle **ABC**. It is well known that Z^2 is contractible. So $H_n(Z^2 \times S^{n-2}; \mathbb{Z}_2) = 0$ for all $n \geqslant 2$. For any triangulation K of $Z^2 \times S^{n-2}$, $\delta_{n-1}(K) = 2$ and only the first of the inequalities (5) is valid.

(4.2.3) Theorem 4. (a) *For any n-dimensional space* X^n, $n \geqslant 1$,

$$\dim H_n(X^n; \mathbb{Z}_2) \leqslant \binom{f_0(X)-1}{n+1}. \tag{6}$$

Proof. Let L denote a *minimal triangulation* of X^n, i.e., one with $f_0(X)$ vertices. Then

$$\dim H_n(X^n; \mathbb{Z}_2) = \dim H_n(L; \mathbb{Z}_2)$$

$$\leqslant \dim H_n\left(\sum_n^{f_n(L)}; \mathbb{Z}_2\right) \qquad \text{by Theorem 5 (3.1.5),}$$

$$\leqslant \dim H_n(\sigma_n^{f_0(X)-1}; \mathbb{Z}_2) \qquad \text{because } f_n(L) \leqslant \binom{f_0(X)}{n+1}$$

$$\text{and thus } \sum_n^{f_n(L)} \subseteq \sigma_n^{f_0(X)-1}$$

$$= \binom{f_0(X)-1}{n+1}.$$

(4.2.4) PROPOSITION 1. *A simplicial complex K can embed in an n-dimensional space X^n, $n \geqslant 1$, only if one of its $(n-1)$-simplices is incident to less than f_0 (X)-$n+1$ n-simplices.*

Note that this follows from (5) and (6) provided $f_0(X) \geqslant 2n$. However, the following simple argument proves it unconditionally.

Proof. The essential point used is that the process of subdivision does not increase the valence of a codimension one simplex; this is obviously false in codimensions $\geqslant 2$.

We choose a subdivision K' of K which is contained in a subdivision L' of a minimal triangulation L of X. Then

$$\begin{aligned}
\delta_{n-1}(K) &= \delta_K(\sigma^{n-1}) && \text{for some } \sigma^{n-1} \in K \\
&= \delta_{K'}(\tau^{n-1}) && \text{for all } \tau^{n-1} \in K' \text{ such that } \tau^{n-1} \subseteq \sigma^{n-1} \\
&\leqslant \delta_{L'}(\tau^{n-1}) && \text{because } K' \subseteq L' \\
&= \delta_L(\theta^{n-1}) && \textit{provided} \text{ one has a } \theta^{n-1} \in L \\
& && \text{such that } \theta^{n-1} \supseteq \tau^{n-1} \\
&\leqslant f_0(L) - n \\
&= f_0(X) - n.
\end{aligned}$$

If no such $\theta^{n-1} \in L$ exists, then the open $(n-1)$-simplex τ^{n-1} of L' is contained in an open n-simplex of L, and we must have $\delta_{n-1}(K) \leqslant \delta_{L'}(\tau^{n-1}) = 2$. But $\delta_{n-1}(K) \leqslant 2$ ensures $\delta_{n-1}(K) \leqslant f_0(X) - n$ unless $f_0(X) = n+1$, i.e., unless X is a closed n-simplex. In this cases $\delta_{n-1}(K) = 1$ and the inequality is still true.

(4.3) *Least Valences in Codimension Two*

If one allows X^n, $n \geqslant 2$, to run over all n-dimensional spaces, then $\delta_{n-2}(X^n)$ cannot be bounded by a function of n and $H_*(X^n)$ (see Remark 9). It seems that the most natural subclass of spaces for which such a bound can be found is the one of all n-dimensional pseudomanifolds.

(4.3.1) DEFINITION. If a point x, of an n-dimensional space X^n, does not lie in the interior of any n-simplex of any triangulation of X^n, then it is called a *singular point* of X^n. It is easy to check that if $X = |L|$, then there is a subcomplex $L_0 \subseteq L$ such that the set of singularities, $\operatorname{sing}(X^n)$, equals $|L_0|$. Thus $\operatorname{sing}(X^n)$ is a subspace of dimension $\leqslant n-1$. An n-dimensional space X^n, $n \geqslant 2$, is called a *pseudomanifold* if $\dim \operatorname{sing}(X^n) \leqslant n-2$. This is equivalent to saying that, for any triangulation of X^n, each $(n-1)$-simplex is incident to exactly two n-simplices.

(4.3.2) THEOREM 6. *A simplicial complex K can triangulate a subspace Y of an n-dimensional pseudomanifold X^n, $n \geqslant 2$, only if*

$$b_n(X, Y; \mathbb{Z}_2) \leqslant b_n(X; \mathbb{Z}_2) \qquad \textit{or}$$

$$b_n(X, Y; \mathbb{Z}_2) - b_n(X; \mathbb{Z}_2) + 1 \leqslant \frac{2}{n+1} f_{n-1}(K). \tag{7}$$

Proof. Without loss of generality we can assume $\dim K \leqslant n-1$, for, if K is replaced by its $(n-1)$-skeleton K^{n-1}, then the only change in (7) is that $b_n(X, Y)$ is replaced by $b_n(X, K^{n-1}) = b_n(X, Y) + b_n(Y, K^{n-1}) = b_n(X, Y) + f_n(K)$.

Let K' be a subdivision of K which extends to a triangulation L of X. Elements of $H_n(L, K'; \mathbb{Z}_2)$ are n-chains c with $\partial c \subseteq K'$. Out of all such non-zero chains c, those which are minimal—with respect to set inclusion—constitute the *minimal basis* $\mathscr{B}$ of $H_n(L, K'; \mathbb{Z}_2)$. Likewise let $\mathscr{L}$ denote the minimal basis of $H_n(L)$. Note that if $\sigma^n \in b \in \mathscr{B}$, then b consists of all n-simplices of L which can be "joined" to σ^n via $n-1$-simplices of $L - K'$. Enlarging b to all n-simplices of L which can be "joined" to σ^n via $(n-1)$-simplices of L one gets a $c \in \mathscr{L}$. Thus each $b \in \mathscr{B}$ is contained in a unique $c \in \mathscr{L}$. Hence either $\#(\mathscr{B}) \leqslant \#(\mathscr{L})$ or else at least one c contains two or more b's and so $\#(\mathscr{B}) - \#(\mathscr{L}) + 1 \leqslant \#(\mathscr{B} - \mathscr{L})$.

The chain subdivision map θ induces an isomorphism in $(n-1)$-cycles, $H_{n-1}(K) \to H_{n-1}(K')$. Thus an $n-1$-simplex ξ of K' occurring inside an $(n-1)$-simplex σ of K, i.e., an ξ such that $\xi \in \theta\sigma$, can belong to some $(n-1)$-cycle z of K' iff $\theta\sigma \in z$. Let $\mathscr{E}$ denote the set of all $(n-1)$-simplices σ of K such that $\sigma \in \theta^{-1}\partial b$ for some $b \in \mathscr{B} - \mathscr{L}$, and consider the subset $\mathscr{R} \subseteq (\mathscr{B} - \mathscr{L}) \times \mathscr{E}$ consisting of all pairs (b, σ) such that $\sigma \in \theta^{-1}\partial b$. For each $b \in \mathscr{B} - \mathscr{L}$, $\{\sigma : (b, \sigma) \in \mathscr{R}\}$ is non-empty and constitutes a mod 2 $(n-1)$-cycle of K. So its cardinality must be at least $n+1$. On the other hand for each $\sigma \in \mathscr{E}$ one has $\#\{b : (b, \sigma) \in \mathscr{R}\} \leqslant 2$: this follows because $\sigma \in \theta^{-1}\partial b$ iff $\theta\sigma \subseteq \partial b$ iff $\xi \in \partial b$ for some $\xi \in \theta\sigma$ and this happens for at most two $b \in \mathscr{B} - \mathscr{L}$. Hence $(n+1)\ \#(\mathscr{B} - \mathscr{L}) \leqslant \#(\mathscr{R}) \leqslant 2\#(\mathscr{E})$.

These inequalities imply (7).

(4.3.3) THEOREM 2. *A simplicial complex K can be embedded in an n-dimensional pseudomanifold X^n, $n \geqslant 2$, only if*

$$\delta_{n-2}(K) < \frac{n(n+1)}{n-1} \quad \textit{or} \quad \binom{\delta_{n-2}(K)+n-2}{n} - \frac{2}{n+1}\binom{\delta_{n-2}(K)+n-1}{n}$$
$$\leqslant \dim H_{n-1}(X^n; \mathbb{Z}_2) - 1. \tag{8}$$

Proof. For any such simplicial complex K, triangulating subspace Y of X, we have

$$\binom{\delta_{n-2}+n-2}{n} - \frac{2}{n+1}\binom{\delta_{n-2}+n-1}{n}$$
$$= \frac{n-1}{n+1}\binom{\delta_{n-2}+n-1}{n} - \binom{\delta_{n-2}+n-1}{n-1} + \binom{\delta_{n-2}+n-2}{n-2}$$
$$= \frac{n-1}{n+1}\frac{\delta_{n-2}}{n}\binom{\delta_{n-2}+n-1}{n-1} - \binom{\delta_{n-2}+n-1}{n-1} + \binom{\delta_{n-2}+n-2}{n-2}$$
$$\leqslant \frac{n-1}{n+1}\frac{\delta_{n-2}}{n} f_{n-2} - f_{n-2} + \binom{\delta_{n-2}+n-2}{n-2}$$

by Lemma 2 (4.1.2)

provided $\delta_{n-2} \geqslant n(n+1)/(n-1)$

$$\leqslant \frac{n-1}{n+1}\frac{d_{n-2}}{n} f_{n-2} - f_{n-2} + \binom{\delta_{n-2}+n-2}{n-2}$$
$$= \frac{n-1}{n+1} f_{n-1} - f_{n-2} + \binom{\delta_{n-2}+n-2}{n-2}$$
$$= f_{n-1} - f_{n-2} - \frac{2}{n+1} f_{n-1} + \binom{\delta_{n-2}+n-2}{n-2}$$
$$\leqslant f_{n-1} - f_{n-2} - b_n(X, Y) + b_n(X) - 1 + \binom{\delta_{n-2}+n-2}{n-2}$$

by Theorem 6 (4.3.2), the first alternative can be ignored because $\delta_{n-2} \geqslant n(n+1)/(n-1)$ implies $f_{n-1} \geqslant (n+1)/2$

$$\leqslant b_{n-1}(K) - \binom{N}{n-2} - b_n(X, Y) + b_n(X) - 1 + \binom{\delta_{n-2}+n-2}{n-2}$$

by Lemma 3 (4.1.3)

$$\begin{aligned}&\leqslant b_{n-1}(K)-b_n(X, Y)+b_n(X)-1\\ &\qquad \text{by (3) of Lemma 2 (4.1.2)}\\ &=b_{n-1}(Y)-b_n(X, Y)+b_n(X)-1\\ &\leqslant b_{n-1}(X)-1,\end{aligned}$$

because the alternating sum formula applied to the exact sequence $0 \to H_n(X) \to H_n(X, Y) \to H_{n-1}(Y) \to H_{n-1}(X) \to \cdots$ yields $b_n(X) - b_n(X, Y) + b_{n-1}(Y) - b_{n-1}(X) \leqslant 0$.

(4.3.4) *Remark* 9. The proof of Theorem 6 makes essential use of the fact that we are using mod 2 coefficients. In fact Corollary 2 (Sect. 1(A)), a special case of Theorem 6's consequence Theorem 2, is false (e.g., for the projective plane) if rationals $\mathbb{Q}$ are used instead of $\mathbb{Z}_2$. Also note that *for each δ there exists a contractible 2-space X^2 with $\delta_0(X^2) \geqslant \delta$*: To construct such a space take a general position map f from the complete graph σ_1^δ into the 2-simplex θ^2. Then $X^2 = (\operatorname{Im}(f) \times [0, 1]) \cup \theta^2 \times \{0\}$ is contractible and a slight "perturbation" of f embeds σ_1^δ in X^2; so $\delta_0(X^2) \geqslant \delta$. This shows that the above bound—or in fact any bound given by a function of the dimension and the homotopy type—cannot apply to *all* n-dimensional spaces X^n, $n \geqslant 2$. Finally note that the second of the inequalities (8) is false if $K^n = \sigma_{n-1}^{n-1}$, $X^n = S^n$, $n \geqslant 2$.

(4.3.5) LEMMA 4. (a) *If $n+1 \leqslant l$, then*

$$\binom{l-1}{n} - \frac{2}{n+1}\binom{l}{n} \leqslant \binom{l}{n} - \frac{2}{n+1}\binom{l+1}{n} \tag{9}$$

with strict inequality if $n+1<l$.

(b) *If $\binom{n+1}{n} \leqslant p \leqslant \binom{k}{n}$, $n \geqslant 1$, then*

$$\dim H_{n-1}\Bigl(\sum_{n-1}^{p}\Bigr) - \frac{2}{n+1}p \leqslant \binom{k-1}{n} - \frac{2}{n+1}\binom{k}{n}. \tag{10}$$

Proof. (a) A simple calculation shows that (9) is same as $n+1 \leqslant l$.

(b) Turning to (10) we see that it is true for $n=1$ because both sides equal -1. So assume $n \geqslant 2$. Also, by (9), we can assume $\binom{k-1}{n} \leqslant p < \binom{k}{n}$, $k > n+1$. So $p = \binom{k-1}{n} + m$ where $m < \binom{k-1}{n-1}$. *If* $\binom{n}{n-1} \leqslant m$ then

$$\begin{aligned}&\dim H_{n-1}\Bigl(\sum_{n-1}^{p}\Bigr) - \frac{2}{n+1}p\\ &\quad = \binom{k-2}{n} + \dim H_{n-2}\Bigl(\sum_{n-2}^{m}\Bigr) - \frac{2}{n+1}p \qquad \text{by (3.1.2),}\end{aligned}$$

$$
\begin{aligned}
&= \left[\binom{k-2}{n} - \frac{2}{n+1} \binom{k-1}{n} \right] \\
&\quad + \left[\dim H_{n-2}\left(\sum_{n-2}^{m} \right) - \frac{2}{n} m \right] + \left(\frac{2}{n} - \frac{2}{n+1} \right)(m) \\
&\leqslant \left[\binom{k-2}{n} - \frac{2}{n+1} \binom{k-1}{n} \right] \\
&\quad + \left[\binom{k-2}{n-1} - \frac{2}{n} \binom{k-1}{n-1} \right] + \left(\frac{2}{n} - \frac{2}{n+1} \right) \binom{k-1}{n-1} \\
&\qquad \text{by inductive hypothesis on } n \\
&= \binom{k-1}{n} - \frac{2}{n+1} \binom{k}{n},
\end{aligned}
$$

and, *if* $0 \leqslant m < n$, then one has

$$
\begin{aligned}
&\dim H_{n-1}\left(\sum_{n-1}^{p} \right) - \frac{2}{n+1} p \\
&\quad = \binom{k-2}{n} - \frac{2}{n+1} p, \qquad \text{because } \sum_{n-1}^{m} \text{ has less than } n \\
&\qquad\qquad (n-2)\text{-simplices and thus no } n-2\text{-cycle} \\
&\quad \leqslant \binom{k-2}{n} - \frac{2}{n+1} \binom{k-1}{n} \\
&\quad \leqslant \binom{k-1}{n} - \frac{2}{n+1} \binom{k}{n} \text{ by (9).}
\end{aligned}
$$

(4.3.6) THEOREM 4. (b) *For any n-dimensional pseudomanifold* X^n, $n \geqslant 1$,

$$
b_{n-1}(X^n) - b_n(X^n) \leqslant \binom{f_0(X^n) - 1}{n} - \frac{2}{n+1} \binom{f_0(X^n)}{n}. \tag{11}
$$

Proof. Let L denote a minimal triangulation of X^n and let L^{n-1} be the $(n-1)$-skeleton of L. Applying the alternating sum formula to the exact sequence $0 \to H_n(X) \to H_n(X, L^{n-1}) \to H_{n-1}(L^{n-1}) \to H_{n-1}(X) \to 0$ we get

$$
\begin{aligned}
&b_{n-1}(X) - b_n(X) \\
&\quad = b_{n-1}(L^{n-1}) - f_n(L) \\
&\quad = b_{n-1}(L^{n-1}) - \frac{2}{n+1} p, \qquad \text{where} \quad p = f_{n-1}(L)
\end{aligned}
$$

$$\leqslant b_{n-1}\left(\sum_{n-1}^{p}\right)-\frac{2}{n+1}p \qquad \text{by Theorem 5 (3.1.5)}$$

$$\leqslant \binom{f_0(X)-1}{n}-\frac{2}{n+1}\binom{f_0(X)}{n} \text{ by (10),}$$

because $\binom{n+1}{n} \leqslant p = f_{n-1}(L) \leqslant \binom{f_0(L)}{n} = \binom{f_0(X)}{n}$.

(4.3.7) THEOREM 3. *A simplicial complex K can embed in an n-dimensional pseudomanifold X^n, $n \geqslant 2$, $H_n(X^n;\ \mathbb{Z}_2) \cong \mathbb{Z}_2$ (X^n other than those having $f_0(X^n) <]n(n+1)/(n-1)[+n-2)$ only if one of its $(n-2)$-simplices is incident to less than $f_0\ (X^n)-n+2$ $(n-1)$-simplices.*

Proof. This follows from (8) and (11) by using Lemma 4(a) because, when only the first of the inequalities (8) is valid, one still has $f_0(X^n) - n+1 \geqslant]n(n+1)/(n-1)[-1 \geqslant \delta_{n-2}(X^n)$.

(4.3.8) *Remark* 10. Theorem 3 is the codimension two analogue of Proposition 1 (4.2.4), but this time we can offer no direct geometric proof. Such a proof would be quite interesting because it may (a) extend Theorem 3 to all n-dimensional spaces (in this context note that $f_0(X)$ is a topological invariant but not a homotopy type invariant: see Remark 9), and (b) show that, for $2i+2=n$, Conjecture 1 is equivalent to the existence of the integers C_n mentioned in Remark 4 of Section 1(A). However, note that (unlike in Proposition 1) *the inequality $\delta_{n-2}(X_n) \leqslant f_0(X^n)-n+1$ of Theorem 3 does not hold for all n-pseudomanifolds X^n, $n \geqslant 2$.* For example, $f_0(S^2)=4$ and $\delta_0(S^2)=5$. Also note that Theorem 3 shows that *Corollary 3 of Section 1 is valid for all 2-pseudomanifolds X^2 with $b_2(X^2)=1$:* this follows because it is easy to check that a 2-pseudomanifold X^2 has $f_0(X^2)<6$ only if $X^2=S^2$.

(4.3.9) *Remark* 11. We obtain *bounds for codimension one and two average valences* as follows: The proof of Lemma 2 (4.1.2) shows that, for $i=0$, $N \geqslant d_0(K)$, and for $i>0$, $N \geqslant d_i(K)+i-1$. Thus, exactly the same proofs as before show, that for $n=1$ and $n=2$ Theorems 1 (4.2.1) and 2 (4.3.3) are valid if $\delta_0(K)$ is replaced by $d_0(K)$, and for higher values of n, that a simplicial complex K embeds in an n-dimensional space X^n resp. pseudo-manifold X^n only if one has the polynomial inequalities

$$d_{n-1}(K) < n+1 \qquad \text{or} \qquad \binom{d_{n-1}(K)+n-1}{n+1} - \binom{d_{n-1}(K)+n-2}{n-2}$$
$$\leqslant b_n(X^n;\mathbb{Z}_2) \tag{12}$$

resp.

$$d_{n-2}(K) < \frac{n(n+1)}{n-1} \quad \text{or} \quad \binom{d_{n-2}(K)+n-2}{n} - \frac{2}{n+1}\binom{d_{n-2}(K)+n-1}{n} - \binom{d_{n-2}(K)+n-3}{n-3} \leqslant b_{n-1}(X^n; \mathbb{Z}_2) - 1. \tag{13}$$

We note also that the apparently stronger conjecture, $d_i(X^n) < \infty$ if $2i+2 \geqslant n$, can be shown to be equivalent to Conjecture 1.

5. Chromatic Numbers

(5.1) *Weak Chromatic Numbers*

(5.1.1) Proposition 2. *If $c_i(X) > N$, then X has a subcomplex K in which each i-simplex is incident to at least N $(i+1)$-simplices.* Thus one has the generalized Szekeres–Wilf inequality

$$c_i(X) \leqslant \delta_i(X) + 1. \tag{1}$$

Proof. Let K be any minimal subcomplex of X having $c_i(K) \geqslant N+1$. Each $\sigma^i \in K$ must be incident to at least N $(i+1)$-simplices, for, otherwise, any good coloring of the i-simplices of K-$\mathrm{St}_K \sigma^i$ by N colors would extend to one of K.

Using this, Theorems 1 and 2 yield

(5.1.2) Theorem 8. *A simplicial complex K embeds in an n-dimensional space X^n, $n \geqslant 1$, only if*

$$c_{n-1}(K) \leqslant n+1 \quad \textit{or} \quad \binom{c_{n-1}(K)+n-2}{n+1} \leqslant \dim H_n(X^n; \mathbb{Z}_2). \tag{2}$$

(5.1.3) Theorem 9. *A simplicial complex K embeds in an n-dimensional pseudomanifold X^n, $n \geqslant 2$, only if*

$$c_{n-2}(K) \leqslant \left]\frac{n(n+1)}{n-1}\right[\quad \textit{or} \quad \binom{c_{n-2}(K)+n-3}{n} - \frac{2}{n+1}\binom{c_{n-2}(K)+n-2}{n} \leqslant \dim H_{n-1}(X^n; \mathbb{Z}_2) - 1. \tag{3}$$

(5.2) *Exact Calculations for Some 2-Pseudomanifolds*

(5.2.1) Proposition 3. *For each 2-pseudomanifold X^2 with $H_2(X^2;$

$\mathbb{Z}_2) \cong \mathbb{Z}_2$, one can find a unique connected 2-manifold $\tilde{X}^2$, and a unique monotonic integer sequence $2 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$ such that X^2 is homeomorphic to $\tilde{X}^2_{n_1,n_2,\ldots,n_k}$, the 2-pseudomanifold obtained from $\tilde{X}^2$ by identifying each of k disjoint sets of points, of cardinalities $n_1, n_2,\ldots,$ and n_k, to a single point. Also

$$\dim H_1(\tilde{X}^2_{n_1,n_2,\ldots,n_k}; \mathbb{Z}_2) = \dim H_1(\tilde{X}^2; \mathbb{Z}_2) + \sum_{i=1}^{k} (n_i - 1). \tag{4}$$

Proof. Let $x_1, x_2,\ldots, x_k$ be the singular points of pseudomanifold X^2. Choose a triangulation L of X so fine that no two singular points lie in the same closed simplex. The links of vertices other than the x_i's are circles and each $\mathrm{Lk}_L x_i$ is a disjoint union of n_i circles C_{j_i}, $1 \leqslant j_i \leqslant n_i$, $n_i \geqslant 2$. Without loss of generality we can assume $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$. Now replace each x_i by n_i new vertices $\tilde{x}_{j_i}$ and each $St_L x_i$ by $\bigcup_{i \leqslant j_i \leqslant n_i} \tilde{x}_{j_i} \cdot C_{j_i}$. This gives us a triangulation $\tilde{L}$ of the required 2-manifold $\tilde{X}^2$. The verification that $\tilde{X}^2_{n_1,n_2,\ldots,n_k} \cong \tilde{Y}^2_{m_1,m_2,\ldots,m_t}$ iff $k = t$, $n_1 = m_1$, $n_2 = m_2,\ldots,$ $n_k = m_k$ and $\tilde{X}^2 \cong \tilde{Y}^2$ is routine and is omitted. Formula (4) follows by calculating the Euler characteristics of L and $\tilde{L}$.

(5.2.2) COROLLARY 5. *If a 2-pseudomanifold X^2 with $H_2(X^2; \mathbb{Z}_2) \cong \mathbb{Z}_2$ has $H_1(X^2; \mathbb{Z}_2) = 0$, then X^2 is homeomorphic with S^2.*

Proof. The result is well known for 2-manifolds. But since the left side of (4) is zero we must have $k = 0$ and $\dim H_1(\tilde{X}^2; \mathbb{Z}_2) = 0$; so $\tilde{X}^2 \cong S^2$ and X^2 is obtained from it by no identifications.

(5.2.3) COROLLARY 4. *If a 2-pseudomanifold X^2 with $H_2(X^2; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is other than the 2-sphere, then*

$$c_0(X^2) \leqslant \frac{7 + \sqrt{1 + 24 \dim H_1(X^2; \mathbb{Z}_2)}}{2}. \tag{5}$$

Proof. Corollary 5 shows that this is the case $n = 2$ of Theorem 9 specialised further to 2-pseudomanifolds obeying $b_2(X^2) = 1$.

(5.2.4) *Remark* 12. As is well known, for 2-manifolds, the chromatic estimates (3) admit very little improvement: $c_0(S^2) \leqslant 6$ was improved to $c_0(S^2) \leqslant 5$ by Heawood [9] himself, and to $c_0(S^2) = 4$ by Appel–Haken [1]; c_0 (Klein Bottle) $\leqslant 7$ was improved to c_0 (K.B.) $= 6$ by Franklin [7], and the remaining inequalities were shown to be equalities by Ringel *et al.* [13]. However, for some 2-pseudomanifolds these estimates are quite bad; e.g., the S^2_k, $k \geqslant 2$, of Theorem 7(a) have $c_0 = 5$ even though $b_1 = k - 1$ can be arbitrarily big. For the 2-pseudomanifolds $S^2_{(k)} = S^2_{2,2,\ldots,2(k\text{ times})}$ of Theorem 7(b) one does have $\lim_{k\to\infty} c_0(S^2_{(k)}) = \infty$

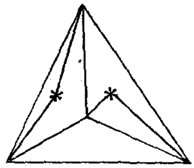

FIGURE 1

(the double points of a complete graph drawn in $\mathbb{R}^2$ can be eliminated after enough pairs of points have been identified); however, there is no reason to believe that (5) is sharp even in this case. Note that $b_1(S^2_{(k)}) = k$ and therefore (5) disproves Dewdney's conjecture [4], $c_0(S^2_{(k)}) = 4 + k$, for all $k > 5$.

(5.2.5) THEOREM 7.(a) $c_0(S^2_k) = 5$ *for all* $k \geqslant 2$.

(b) $c_0(S^2_{(k)}) = 4 + k$ *for* $1 \leqslant k \leqslant 4$ *and* $c_0(S^2_{(5)}) = 8$.

Proof. (a) An embedded graph K of S^2_k, $k \geqslant 2$, either misses the singular point $*$, or else has it in the interior of an edge σ^1, or has it as a vertex. Accordingly the identification map $S^2 \to S^2_k$ allows us to see that either all of K or $K - \sigma^1$ or $K - \mathrm{St}_K *$, can be lifted to S^2. This shows $c_0\ (S^2_k) \leqslant 5$ by the $4CT$. (This use of the $4CT$ is dictated by brevity; we can prove this also by refining Heawood's proof of $c_0(S^2) \leqslant 5$.) But σ^4_1 embeds in S^2_2 (see Fig. 1); thus $c_0(S^2_k) = 5$ for $k \geqslant 2$.

(b) The above lifting argument in fact shows

$$c_0(\tilde{X}^2_{n_1, n_2, \ldots, n_k}) \leqslant c_0(\tilde{X}^2) + k. \tag{6}$$

In particular $c_0(S^2_{(k)}) \leqslant 4 + k$ for all $k \geqslant 1$. Now Fig. 2 gives an embedding of σ^7_1 in $S^2_{(4)}$; hence $c_0(S^2_{(k)}) = 4 + k$ for $1 \leqslant k \leqslant 4$. For $k = 5$ our proof goes as follows. First we check that σ^8_1 cannot embed in $S^2_{(5)}$. If σ^8_1 could be

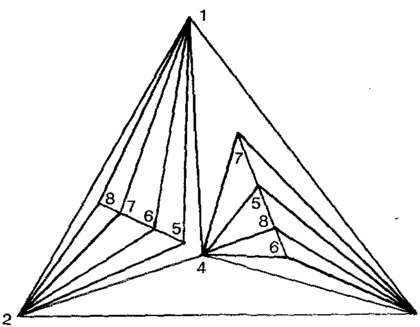

FIGURE 2

embedded in $S^2_{(5)}$, then five of the nine vertices of σ_1^8 must be on the five singular points of $S^2_{(5)}$. The σ_1^4 spanned by these vertices is covered in S^2 by a graph $\tilde{\sigma}_1^4$ having 10 vertices and 10 edges; one can check that this complex must have a circuit of length $\geqslant 5$. This leads to a contradiction of the fact that the embedding of $\tilde{\sigma}_1^8$ in S^2—this complex has 14 vertices and 36 edges—must be triangular. Next, we duplicate the standard Heawood–Franklin argument using $\delta_0(S^2_{(5)}) \leqslant 8$ and $\sigma_1^8 \not\leq S^2_{(5)}$ to conclude the desired $c_0(S^2_{(5)}) = 8$.

(5.3) *Strong Chromatic Numbers*

For antecedents of the "links arguments" of this section see Grünbaum [8], Zykov [20], and Sarkaria [16, 17].

(5.3.1) We denote by $\mathrm{Lk}_i X^n$ the set of (homeomorphism classes of) spaces which occur as the links of i-simplices of triangulations of X^n. By using the well-known fact that the topology of a link is invariant under subdivision (see, e.g., Rourke–Sanderson [14, p. 21]) one can verify that $\mathrm{Lk}_i\ X^n$ *is a finite set.* We define $b_j\ (\mathrm{Lk}_i X^n) = \sup\{\dim H_j(Y; \mathbb{Z}_2)$: $Y \in \mathrm{Lk}_i X^n\}$. (One can check that $b_j(\mathrm{Lk}_i X^n) = \sup\{\dim H_{j+i+1}(X, X - x)$: intrinsic dimension of $x \geqslant i$.)

(5.3.2) THEOREM 10. (a) *A simplicial complex K embeds in an n-dimensional space X^n, $n \geqslant 2$, only if*

$$C_{n-1}(K) \leqslant 2n \qquad \text{or} \qquad C_{n-1}(K) \leqslant \left[n \cdot \frac{1 + \sqrt{1 + 8b_1(\mathrm{Lk}_{n-2}(X^n))}}{2} \right] + 1. \tag{7}$$

(b) *A simplicial complex K embeds in an n-dimensional pseudomanifold X^n, $n \geqslant 3$, only if*

$$C_{n-2}(K) \leqslant 6(n-1) \qquad \text{or}$$
$$C_{n-2}(K) \leqslant \left[(n-1) \cdot \frac{5 + \sqrt{1 + 24b_1(\mathrm{Lk}_{n-3}(X^n))}}{2} \right] + 1. \tag{8}$$

(The result $C_{n-2}(\mathbb{R}_n) \leqslant 6(n-1)$ is due to Grünbaum [8].)

(c) $C_i(X^n) = \infty$ *if* $n \geqslant i + 3$.

Proof. Let $G_i(K)$ denote the ith *associated graph* of K, i.e., the graph whose vertices are the i-simplices σ^i of K and whose edges $\{\sigma_1^i, \sigma_2^i\}$ are pairs of i-simplices of K incident to the same $(i+1)$-simplex. The definition of the ith strong chromatic number, $C_i(K)$, shows that it is nothing but the chromatic number of this graph:

$$C_i(K) = c_0(G_i(K)). \tag{9}$$

For each subgraph $G \subseteq G_i(K)$, and each $(i-1)$-simplex $\theta^{i-1} \in K$, we define a graph $G[\theta^{i-1}]$ as follows. The vertices, v, of $G[\theta^{i-1}]$ are all those vertices of K for which $v \cdot \theta^{i-1}$ is a vertex of G, and the edges $\{v_1, v_2\}$ of $G[\theta^{i-1}]$ are all those pairs of vertices of K for which $\{v_1 \cdot \theta^{i-1}, v_2 \cdot \theta^{i-2}\}$ is an edge of G. Note that $G[\theta^{i-1}]$ is contained in $G_i(K)[\theta^{i-1}] = [\mathrm{Lk}_K \theta^{i-1}]^1$, the 1-skeleton of the link of θ^{i-1} in K. Also note that

$$\delta_G(\sigma^i) = \sum_{v \in \sigma^i} \delta_{G[\sigma^i \setminus \{v\}]}(v). \tag{10}$$

From this formula it follows that

$$d_0(G) \leqslant (i+1) \sup\{d_0(G[\theta^{i-1}]) : \theta^{i-1} \in K\}. \tag{11}$$

(a) Take $i = n-1$. Let K' be a subdivision of K which is contained in a triangulation L of K; and let θ'^{n-2} be an $(n-2)$-simplex of K' contained in θ^{n-2}. Then the graph $G[\theta^{n-2}]$ embeds in $\mathrm{Lk}_K(\theta^{n-2}) \cong \mathrm{Lk}_{K'}(\theta'^{n-2})$ and so in the 1-dimensional space $\mathrm{Lk}_L(\theta'^{n-2})$.

Thus, by Remark 11, Section 4,

$$d_0(G[\theta^{n-2}]) < 2 \qquad \text{or} \qquad d_0(G[\theta^{n-2}]) \leqslant \frac{1 + \sqrt{1 + 8b_1(\mathrm{Lk}_L(\theta'^{n-2}))}}{2}, \tag{12a}$$

and so, by (11) and (5.3.1),

$$d_0(G) < 2n \qquad \text{or} \qquad d_0(G) \leqslant n \cdot \frac{1 + \sqrt{1 + 8b_1(\mathrm{Lk}_{n-2} X^n)}}{2}. \tag{13}$$

The required inequalities (7) follow by noting that (9) implies $C_{n-1}(K) \leqslant \sup\{\delta_0(G) : G \subseteq G_i(K)\} + 1 \leqslant \sup\{[d_0(G)] : G \subseteq G_i(K)\} + 1$.

(b) Take $i = n$ and proceed exactly as before: this time we note that $\mathrm{Lk}_L(\theta'^{n-3})$ is a 2-pseudomanifold, and so, by Remark 11 of Section 4, one has

$$d_0(G[\theta^{n-3}]) < 6 \qquad \text{or} \qquad d_0(G[\theta^{n-3}]) \leqslant \frac{5 + \sqrt{1 + 24b_1(\mathrm{Lk}_L(\theta'^{n-3})}}{2}. \tag{12b}$$

(c) First, let us note (by assigning to each $\sigma^{i-j-1} \in \mathrm{Lk}_K \theta^j$ the color of the i-simplex $\sigma^{i-j-1} \cdot \theta^j$) that

$$C_i(K) \geqslant C_{i-j-1}(Lk_K \theta^j), \qquad j \leqslant i-1. \tag{14}$$

For each c consider the join K_c^{i+1} of a closed $(i-1)$-simplex θ_{i-1}^{i-1} and a complete graph σ_i^c. Since σ_1^c embeds in a 3-disk, K_c^{i+1} embeds in the join of an $(i-1)$-disk and a 3-disk, i.e., in an $(i+3)$-disk, and so, because $n \geqslant i+3$, also in X^n. So $C_i(K_c^{i+1}) \geqslant C_0(\mathrm{Lk}_{K_c^{i+1}} \theta^{i-1}) = C_0(\sigma_1^c) = c+1$ proves the required result.

Note that for weak chromatic numbers one has a dual of (14),

$$c_i(K) \leqslant \sup\{c_{i-j-1}(Lk_K \theta^j)\}, \qquad j \leqslant i-1. \tag{15}$$

This follows because any weak $(i-j-1)$-coloring of $\mathrm{Lk}_L \theta^j$ induces a weak i-coloring of $\mathrm{St}_L \theta^j$ by assigning to each $\sigma^{i-j-1} \cdot \theta^j$ the color of σ^{i-j-1}. In particular (15) implies $c_{n-2}(X^n) \leqslant \sup\{c_0(Y)\colon Y \in \mathrm{Lk}_{n-3} X^n\}$. Thus $c_{n-2}(X^n) \leqslant 4$ in the absence of codimension 3 singularities, e.g., for all manifolds X^n, $n \geqslant 3$.

Acknowledgments

Thanks are due to the referee for pointing out an elementary binomial identity which shortened many formulae, and also for suggesting that the main results be stated more clearly. I also thank Professors Björner and Stanley for the interest shown towards this paper.

References

1. K. Appel and W. Haken, Every planar map is four colorable, Part I: discharging, *Illinois J. Math.* **21** (1977), 429–490.
2. K. Appel, W. Haken, and J. Koch, Every planar map is four colorable Part II: reducibility, *Illinois J. Math.* **21** (1977), 491–567.
3. D. E. Daykin, A simple proof of the Kruskal–Katona theorem, *J. Combin. Theory Ser. A* **17** (1974), 252–253.
4. A. K. Dewdney, The chromatic number of a class of pseudo-2-manifolds, *Manuscripta Math.* **6** (1972), 311–319.
5. P. Erdős and A. Hajnal, On the chromatic number of graphs and set-systems, *Acta Math. Acad. Sci. Hungar.* **17** (1966), 61–99.
6. A. P. Ershov and G. I. Kozhukhin, Estimates of the chromatic numbers of connected graphs, *Dokl. Akad. Nauk.* **142** (1962), 270–273. [Russian]; *Trans. Soviet Math.* **3** (1962), 50–53.
7. P. Franklin, A six color problem, *J. Math. Phys.* **13** (1934), 363–369.
8. B. Grünbaum, Higher dimensional analogs of the four-color problem and some inequalities for simplicial complexes, *J. Combin. Theory* **8** (1970), 147–153.

9. P. J. HEAWOOD, Map-colour theorems, *Quart. J. Math. Oxford Ser.* **24** (1890), 332–338.
10. J. F. P. HUDSON, "Piecewise Linear Topology," Benjamin, New York, 1969.
11. G. O. H. KATONA, A theorem of finite sets *in* "Theory of Graphs", Proc. Coll. Tihany 1966, pp. 187–207, Akad. Kiado, Budapest, 1968.
12. J. B. KRUSKAL, The number of simplices in a complex *in* "Mathematical Optimization Techniques," pp. 251–278, Univ. Calif. Press, Berkeley, 1963.
13. G. RINGEL, "Map Color Theorem," Springer, Berlin, 1974.
14. C. P. ROURKE AND B. J. SANDERSON, "Introduction to Piecewise Linear Topology," Springer, Berlin, 1972.
15. K. S. SARKARIA, On coloring manifolds, *Illinois J. Math.* **25** (1981), 464–469.
16. K. S. SARKARIA, Addendum to my paper "On coloring manifolds," *Illinois J. Math.* **27** (1983), 612–613.
17. K. S. SARKARIA, A four color theorem for manifolds of dimension $\geqslant 3$, Abstracts Amer. Math. Soc., Providence, R. I., 1982.
18. K. S. SARKARIA, On neighbourly trangulations, *Trans. Amer. Math. Soc.* **277** (1983), 213–239.
19. G. SZEKERES AND H. S. WILF, An inequality for the chromatic number of a graph, *J. Combin. Theory* **4** (1968), 1–3.
20. A. A. ZYKOV, Hypergraphs *Uspekhi Mat. Nauk* **29** (1974), 89–154. [Russian]; *Russian Math. Surveys* **29** (1974), 89–156.
21. A. BJÖRNER AND G. KALAI, A Characterisation of Betti Number and Face Number Sequences, Abstracts Amer. Math. Soc., Providence, RI, 1985.
22. T. L. SAATY AND P. C. KAINEN, "The Four-Color Problem: Assaults and Conquest," McGraw-Hill, New York, 1977.
23. K. S. SARKARIA, Unknotting and coloring of polyhedra, *Bull. Pol. Acad. Sci.*, to appear.
24. K. S. SARKARIA, Kneser colorings of polyhedra, to appear.