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A Minimal Triangulation of the Hopf Map and its Application

K. V. MADAHAR* and K. S. SARKARIA
Department of Mathematics, Panjab University, Chandigarh 160014, India.
e-mail: kvm%maths@puniv.chd.nic.in

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Abstract. We give a minimal triangulation $\eta: S^3_{12} \to S^2_4$ of the Hopf map $h: S^3 \to S^2$ and use it to obtain a new construction of the 9-vertex complex projective plane.

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Key words: Hopf map, Villarceau circles, complex projective plane, triangulations of manifolds.

1. Introduction

As is well known, any smooth map of constant rank between two closed smooth manifolds is *triangulable*, i.e., it is, up to homeomorphisms, the geometrical realization of some simplicial map between two finite simplicial complexes. However, it is usually a hard problem to determine *the least number of vertices* required for such a triangulation. In this note we shall describe (Section 2) a simplicial map $\eta: S^3_{12} \to S^2_4$ from a 12-vertex 3-sphere S^3_{12}, onto the 4-vertex 2-sphere S^2_4, which triangulates the Hopf map $h: S^3 \to S^2$ (Section 3) with the least number of vertices. Another description of η is given in Section 4. As an application of this minimal simplicial Hopf map, the first author found a new construction (Section 5) of the 9-vertex complex projective plane. This 9-vertex triangulation of $\mathbb{C}P^2$ was originally discovered via a computer search by Kühnel in 1980 and an interesting account of it is given in Kühnel–Banchoff [1] and Bagchi–Datta [2].

2. Definition of $\eta: S^3_{12} \to S^2_4$

Let S^2_4 denote all proper faces of the tetrahedron ABCD. The simplicial complex S^3_{12} is the union of two solid tori having a common boundary. One of these – the pre-image under η of the triangle ABC of S^2_4 – is the solid 9-vertex torus M^3_9 shown in Figure 1(b) below. Its boundary is the 9-vertex 2-torus T^2_9 shown in Figure 1(a).

The solid torus N^3_{12} – the pre-image under η of $S^2_4\backslash$int. (ABC) – has 12-vertices. Its boundary is the 9-vertex 2-torus of Figure 2(a) which is isomorphic and will be

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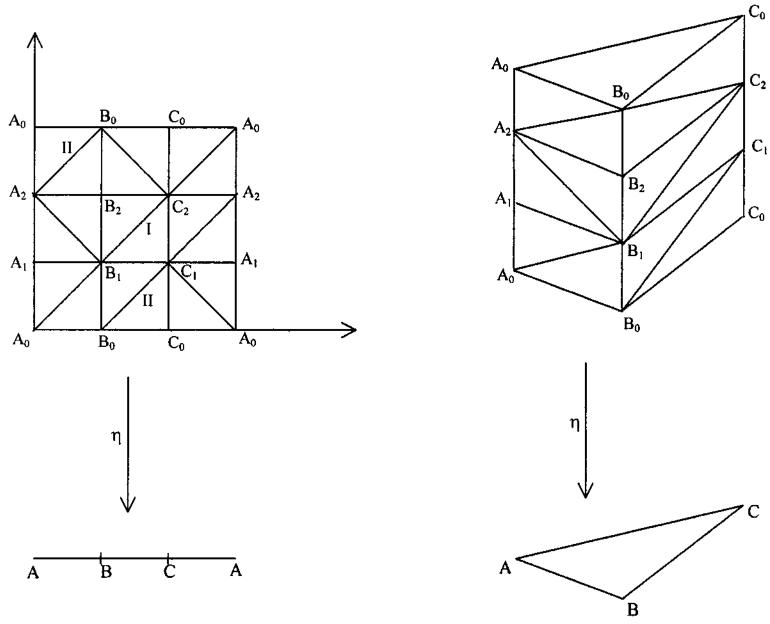


{Fig. 1a -- $T_9^2 = \partial M_9^3$} {Fig. 1b -- M_9^3}

Figure 1.

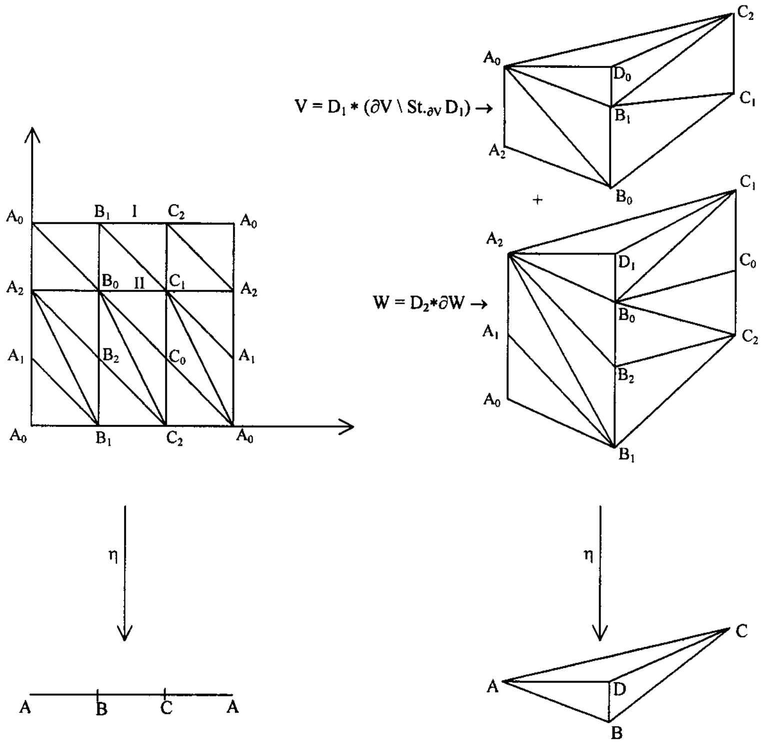
identified with T_9^2 under the isomorphism provided by the indicated vertex labeling (note that the previously slanting circles I and II of Figure 1(a) have become horizontal). This solid torus N_{12}^3 is the union of two 3-balls V and W shown in Figure 2(b).

The boundary 2-spheres of these balls are made up of two cylinders of T_9^2 bounded by I and II, capped by the 2-disks obtained by coning over D_0 and D_1 respectively. Note that one of these balls is triangulated as a cone over a boundary vertex D_1, while the second is a cone over an interior vertex D_2. The simplicial map $\eta\colon S_{12}^3 \to S_4^2$ is well defined because under $A_i \to A$, $B_i \to B$, $C_i \to C$ and $D_i \to D$ $\forall i \in \mathbb{Z}/3$, each simplex of S_{12}^3 gets mapped to a simplex of S_4^2.

3. Hopf Map h: $S^3 \to S^2$

We recall that, if one thinks of S^3 as a unit sphere of $\mathbb{C}^2$, and of S^2 as the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then the Hopf map is defined by $h(z_1, z_2) = z_1/z_2$.

Let $D^2 = \{u \in \mathbb{C}: |u| \leqslant 1\}$ and $S^1 = \partial D^2$. Then $(u, z) \to (1 + |u|^2)^{-1/2} \cdot (uz, z)$ gives a homeomorphism $D^2 \times S^1 \to h^{-1}(D^2)$ under which the fibers of the projection map $D^2 \times S^1 \to D^2$ are mapped onto those of the Hopf map. Likewise $(u, z) \to (1+|u|^2)^{-1/2} \cdot (z, \bar{u}z)$ provide us with a fiber preserving homeomorphism $D^2 \times S^1 \to h^{-1}(\hat{\mathbb{C}} \backslash \text{int. } D^2)$. Hence, up to homeomorphism, the Hopf map is same



{Fig. 2a -- $T_9^2 = \partial N_{12}^3$} {Fig. 2b -- N_{12}^3}

Figure 2.

as the map obtained from two copies of the projection map $D^2 \times S^1 \to D^2$, by identifying the two boundary tori $T^2 = S^1 \times S^1$ by means of fiber preserving homeomorphism $(u, z) \to (u, \bar{u}z)$.

This self-homeomorphism of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is induced by the linear automorphism of $\mathbb{R}^2$ provided by the unimodular matrix $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. So it is the geometrical realization of the simplicial isomorphism, which we had used to identify the toral triangulation of Figure 1(a) with that of Figure 2(a). On the other hand Figures 1(b) and 2(b) gave us triangulations of two copies of the projection map $D^2 \times S^1 \to D^2(\approx \mathrm{ABC})$. So $\eta\colon S^3_{12} \to S^2_4$ triangulates $h\colon S^3 \to S^2$.

4. Hopf Map and Villarceau Circles [4]

Let d be the metric on S^3 defined as $d(u, v) = \cos^{-1}\langle u, v\rangle \forall u, v \in S^3$ (here by the symbol $\langle u, v\rangle$ we mean inner product or scalar product of the position vectors of u and v). In other words $d(u, v)$ is the length of the shorter arc (between u and v) of the great circle of S^3 through u and v. For any great circle C of S^3 define $d(u, C) = \inf\{d(u, v)$ for all $v \in C\}$ and for any two great circles C_1 and C_2

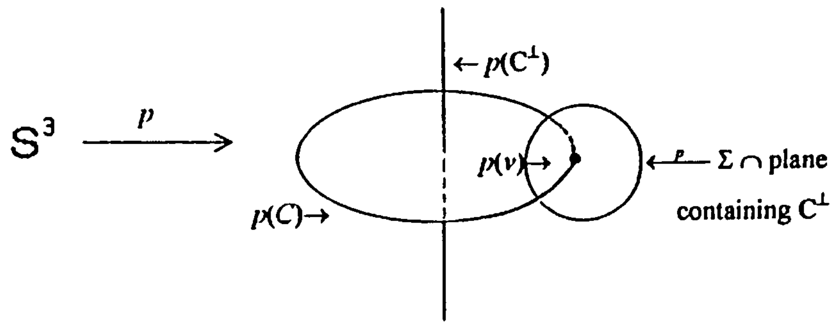


Figure 3.

define $d(C_1, C_2) = \inf\{d(u, C_2)$ for all $u \in C_1\}$. Two great circles C_1 and C_2 of S^3 are called Clifford parallel if $d(u, C_2) = d(v, C_2)$ for all $u, v \in C_1$. It can be easily seen that any two fibers of the Hopf map $h: S^3 \to S^2 = \hat{\mathbb{C}}$ are Clifford parallel to each other. This follows from the fact that the action of S^1 on S^3 i.e. $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2)\, \forall \lambda \in S^1$ is isometric and transitive on each fiber. For any given fiber $C = h^{-1}(0)$ (say) and for each $\alpha \in [0, \pi/2]$ define

$$C_\alpha = \{u \in S^3 | d(u, C) = \alpha\}.$$

Clearly $C_0 = C$ and $C_{\pi/2} = \{u \in S^3 | \cos^{-1}\langle u, v\rangle = \pi/2$, where $v \in C$ is such that $d(u, v) = d(u, C)\}$ i.e. $C_{\pi/2}$ is the set of all points of S^3 whose position vectors are orthogonal to the plane containing the circle C. So we can write $C_{\pi/2} = C^{\perp}$.

To show C_α is topologically $S^1 \times S^1$ for each $\alpha \in (0, \pi/2)$.

For any $v \in C$ the set $\Sigma = \{u \in S^3 | d(u, v) = \alpha\}$ is a small 2-sphere in S^3 with centre v and radius α. Note that only those points of Σ which are also the points of the plane containing the circle $C^{\perp}$, lie in C_α. But the points that are common to both Σ and the plane containing the circle $C^{\perp}$ form a small circle in S^3 (a stereographic projection p of this small circle, in $\mathbb{R}^3$, is shown in Figure 3).

It is clear that C_α is the surface obtained by revolving this small circle (by keeping its center at C) about $C^{\perp}$. So it is topologically $S^1 \times S^1$.

Villarceau circles

Through each $m = (z_1, z_2) \in C_\alpha$ there are two great circles (of S^3) (i) $= \{(\lambda z_1, \lambda z_2)\, \forall \lambda \in S^1\}$ and (ii) $= \{(\lambda z_1, \bar{\lambda} z_2)\, \forall \lambda \in S^1\}$, which are Clifford parallel to C. This shows that the relation of Clifford parallelism is not a transitive relation. Now we can draw four families of circles on $p(C_\alpha)$, (1) usual meridians and parallels, (2) the image under p of two kinds (i.e. (i) and (ii)) of Clifford parallels to C contained in C_α. The last two families are called *Villarceau circles*. In Figure 4, Villarceau circles of only one kind are drawn over $p(C_\alpha)$ for some α.

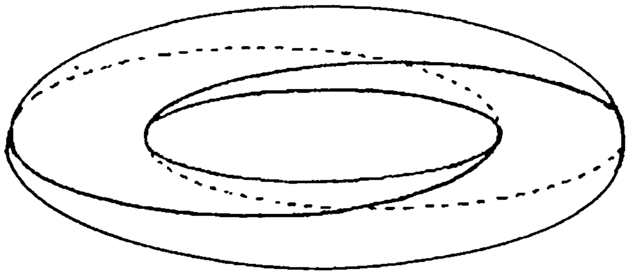


Figure 4.

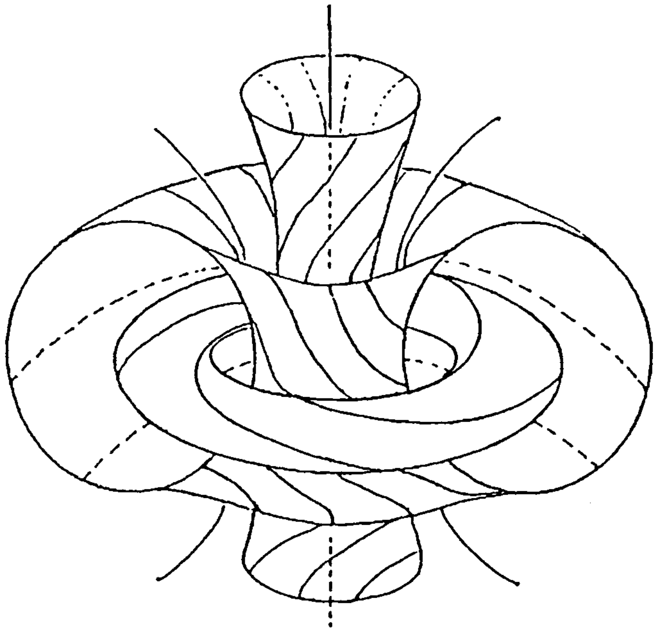


Figure 5. (From [4], Vol. II, p. 305.)

Decomposition of S^3 into two solid tori

By fixing some α say $\pi/4$ we can express our S^3 as union of two solid tori $\Xi_1 = \cup_{\alpha\in[0,\pi/4]} C_\alpha$ and $\Xi_2 = \cup_{\alpha\in[\pi/4,\pi/2]} C_\alpha$. The two tori are glued together at their common boundary along the Villarceau circles. As α varies from 0 to $\pi/2$ we get the following picture of the stereographic projection of S^3 in $\mathbb{R}^3$. (see Figure 5)

Villarceau triangles

The following picture shows the triangles η^{-1}(A), η^{-1}(B) and η^{-1}(C) of $M_9^3 \subset S_{12}^3$ linking each other in $\mathbb{R}^3$. They triangulate three Villarceau circles on $p(C_{\pi/4})$, and the fourth triangle η^{-1}(D), which links them all, triangulates $C^\perp$. We remark that this is a nonlinear imbeding of M_9^3 in $\mathbb{R}^3$: the edges A_0C_1, A_2B_1 and B_0C_2 (not drawn in Figure 6) bend when they cross the line segments A_1B_2, B_2C_0 and C_0A_1 respectively (these line segments are not edges of M_9^3).

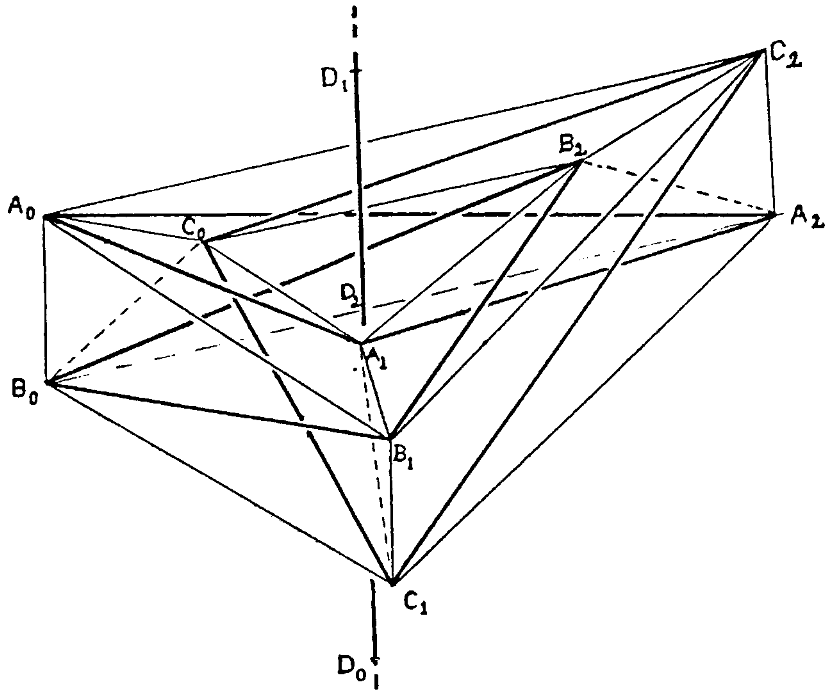


Figure 6.

In the next section we shall construct the triangulation of the complex projective plane. The construction is the simplicial analogue of the well known fact that $\mathbb{C}P^2$ can be obtained by identifying the boundary of a 4-ball with S^2 under the Hopf map $h: S^3 \to S^2$.

5. Construction of the 4-Ball D^4_{17}

We divide our S^3_{12} into five 3-cells Λ_1, Λ_2, Λ_3, Λ_4 and Λ_5 as shown in Figure 7.

The cell Λ_1 is generated by the 3-simplices $\{A_0B_0B_1C_1, A_0B_0C_0C_1, A_0A_1B_1C_1, B_0B_1C_1D_1, B_0C_0C_1D_2, A_0A_1C_1D_2, A_0C_0C_1D_2, B_0C_1D_1D_2\}$ and the simplices of Λ_2 and Λ_3 can be obtained from these by using the permutations $(A_i\ B_{i+1}\ C_{i+2})$ $\forall i \in \mathbb{Z}/3$. The 3-simplices of Λ_4 and Λ_5 are $\{A_0B_0B_1D_1, A_0B_1D_0D_1, B_1C_1C_2D_1, B_1C_2D_0D_1, A_0A_2C_2D_1, A_0C_2D_0D_1\}$ and $\{A_0B_1D_0D_2, A_0A_1B_1D_2, B_1C_2D_0D_2, B_1B_2C_2D_2, A_0C_2D_0D_2, A_0C_0C_2D_2\}$ respectively.

Now take five new vertices 1, 2, 3, 4, 5 and consider the simplicial complex D^4_{17} generated by all 4-simplices of the type $\lambda * \theta$, where λ is a simplex of S^3_{12} and $\theta \subseteq \{1, 2, 3, 4, 5\}$ is defined as $\{i : |\lambda| \subset |\Lambda_i|\}$. Here we take the empty set $\emptyset$ as a simplex of S^3_{12} so $\emptyset * 12345 = 12345$ is a simplex of D^4_{17}. Note that D^4_{17} is subdivision of the ten vertex 4-ball obtained by deleting the interior of the 4-simplex $A_0B_1C_2D_1D_2$ from the hyperoctahedral triangulation $\{C_2, 1\} * \{A_0, 2\} * \{B_1, 3\} * \{D_2, 4\} * \{D_1, 5\}$ of S^4.

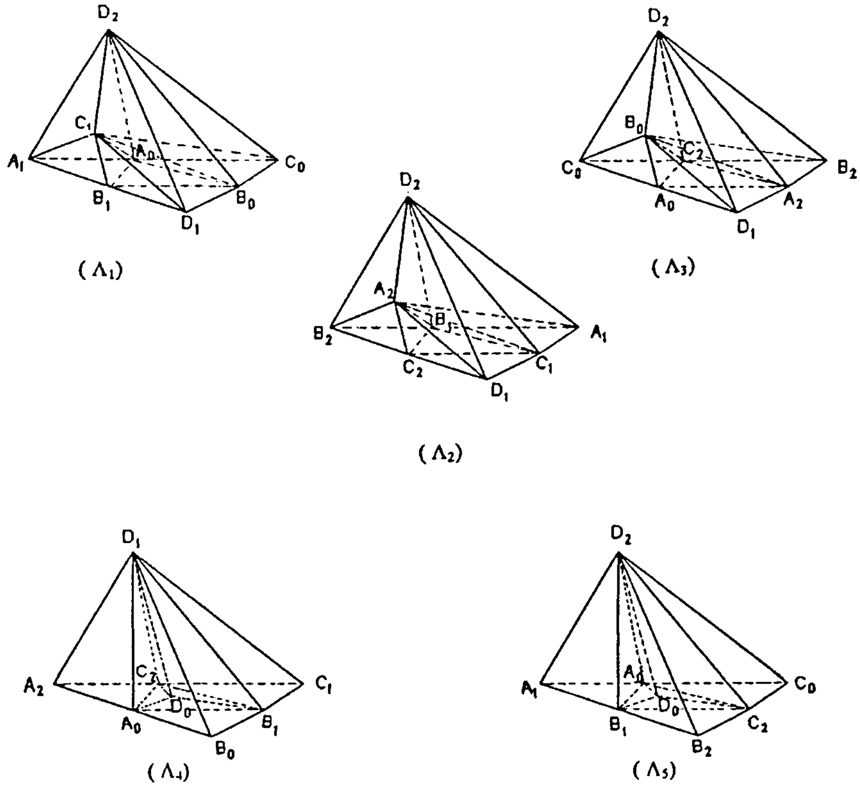


Figure 7.

Our 4-ball D^4_{17} contains eighty-eight 4-simplices; thirty-two of them are listed below and remaining fifty-six simplices can be obtained from these by using the permutations (123)(A_i B_{i+1} C_{i+2}) $\forall i \in \mathbb{Z}/3$.

$1A_0B_0B_1C_1$	$1A_0C_0C_1D_2$	$13A_0B_0C_0$	$14A_0B_0B_1$	$234A_2D_1$	$1345A_0$
$1A_0B_0C_0C_1$	$1B_0C_1D_1D_2$	$13B_0D_1D_2$	$15A_0A_1B_1$	$123D_1D_2$	$1234D_1$
$1A_0A_1B_1C_1$	$4A_0B_0B_1D_1$	$23A_2B_2D_2$	$15A_0A_1D_2$	$125A_1B_1$	$1235D_2$
$1B_0B_1C_1D_1$	$4A_0B_1D_0D_1$	$25A_1B_1D_2$	$34A_0A_2D_1$	$134A_0B_0$	12345
$1B_0C_0C_1D_2$	$5A_0B_1D_0D_2$	$45A_0B_1D_0$	$145A_0B_1$		
$1A_0A_1C_1D_2$	$5A_0A_1B_1D_2$	$34A_0B_0D_1$	$125A_1D_2$		

Construction of $\mathbb{C}P^2_9$. The 9-vertex simplicial complex obtained from D^4_{17} by identifying vertices $A_i \to A$, $B_i \to B$, $C_i \to C$ and $D_i \to D$ $\forall i \in \mathbb{Z}/3$, has following 4-simplices. This simplicial complex is $\mathbb{C}P^2_9$ because by replacing the vertices 1, 2, 3, 4, 5, A, B, C, D by respectively 1, 2, 3, 8, 7, 4, 6, 5, 9 we see that

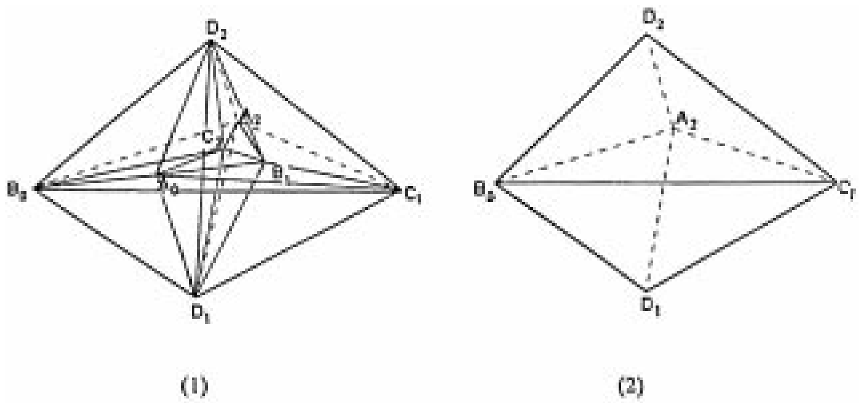


Figure 8.

the following list coincides with the list of top simplices of $\mathbb{C}P_9^2$ given on page 15 of Kühnel–Banchoff [1].

Simplices of $\mathbb{C}P_9^2$

12ABC	45ACD	12345	13BCD	34ABD	235BD
23ABC	45BCD	1234D	35BCD	134BD	23ABD
31ABC	45ABD	1235D	14BCD	234AD	235BC
1345A	125AB	15ACD	135AC	145AB	125AD
1245B	12ACD	345AC	25ABD	135CD	134AB
2345C	124BC	24ACD	245BC	124CD	234AC

6. Remarks

(a) *Minimality of* $\eta\colon S^3_{12} \to S^2_4$. Note that any map in the homotopy class of $h\colon S^3 \to S^2$ is a fibration and we know that in a fibration all fibers are of same homotopy type. As one needs at least 4-vertices to triangulate S^2 and at least 3-vertices to triangulate the circular fiber over each of the four vertices so we need at least 12-vertices in the triangulation of S^3 in order to get a fibration.

(b) We show that S^3_{12} can be obtained by subdividing the *Brückner sphere* (Figure 8) as explained.

Subdivision of Figure 1. Cone the boundary 2-spheres of the three, 3-cells (i) $A_0B_0D_2C_1 \cup A_0B_0D_2C_2$ (ii) $B_1C_1D_2A_0 \cup B_1C_1D_2A_2$ (iii) $A_2C_2D_2B_0 \cup A_2C_2D_2B_1$ over three new vertices C_0, A_1 and B_2, which have been placed in the interior of these 3-cells, respectively. Now insert a vertex D_0 in the 1-simplex D_1D_2 and cone over this vertex by the boundary of the 3-cell generated by $\{A_0B_1D_1D_2 \cup B_1C_2D_1D_2 \cup C_2A_0D_1D_2\}$.

Subdivision of Figure 2. Join D_1 and D_2 to replace two 3-simplices $A_2B_0C_1D_1$ and $A_2B_0C_1D_2$ by three new 3-simplices $A_2B_0D_1D_2$, $B_0C_1D_1D_2$, $C_1A_2D_1D_2$. Note that simplices in the subdivision of 1 and 2 are same as the simplices in S^3_{12}.

(c) Any S^1 bundle $\delta\colon E^3 \to S^2$ is obtained from the two copies of the projection $D^2 \times S^1 \to D^2$ by identifying the boundary tori $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ under $\begin{bmatrix} \pm 1 & 0 \\ d & \pm 1 \end{bmatrix}$. with -$d$ being the *degree of the classifying map* $S^1 \to S^1$ of this bundle: cf. Steenrod [3], Section 18. Triangulations analogous to one above seem to give some useful upper bounds for $n(\delta)$, the least number of vertices required to triangulate δ. We note that $n(\delta) \to \infty$ as $d \to \pm\infty$.

(d) However there seems to be no obvious ways of triangulating the bundle structure of a nontrivial δ by using only finitely many vertices. For example suppose K triangulates S^2, and that there is given, for each ordered pair (σ, θ) of 2-simplices of K which share an edge, a rotation $g_{\sigma\theta}$ of a fixed polygon P (i.e. a fixed triangulation of S^1) such that (a) $(g_{\sigma\theta})^{-1} = g_{\theta\sigma}$ and (b) $g_{\sigma_1\sigma_2} \circ g_{\sigma_2\sigma_3} \circ \cdots \circ g_{\sigma_{n-1}\sigma_n} = 1$ as one goes cyclically around any vertex of K. We can now patch together the projections $\sigma \times P \to \sigma$ by using these $g_{\sigma\theta}$'s, but this only gives – because S^2 is simply connected while the structure group Rot. (P) is finite – the trivial bundle $S^2 \times S^1 \to S^1$: cf. Steenrod [3] Section 13.

(e) The map η is by no means the only 12-vertex simplicial map in the *homotopy class* of h, e.g. if we alter Figure 1(a) so that all its oblique edges become parallel, then the same method gives another. However the new map won't be in the *homeomorphism class* of h because it transforms a tetrahedron of the new solid 9-vertex torus to an edge of S^2_4. We recall that $[h]$ generates the infinite cyclic group $\pi_3(S^2)$ of homotopy classes $[f]$ of maps $f\colon S^3 \to S^2$, and that if $[f] = t \cdot [h]$, then t is called *Hopf invariant* of f. Analogous triangulations can be used to estimate from above the least number $n[f]$ of vertices required to obtain a simplicial representative of $[f]$.

(f) It seems that up to simplicial homeomorphism η is the *unique* 12-vertex triangulation (of the homeomorphism class) of h. As far as the automorphism group Aut. (S^3_{12}) is concerned, it is cyclic of order three, viz. that generated by $A_i \to B_{i+1}$, $B_i \to C_{i+1}$, $C_i \to A_{i+1}$ and $D_i \to D_i$ $\forall i \in \mathbb{Z}/3$. On M^3_9 this automorphism coincides with the order 3 homeomorphism $(z_1, z_2) \to (e^{i2\pi/3} \cdot z_1, e^{i4\pi/3} \cdot z_2)$. However this free $\mathbb{Z}/3$ - action differs from the simplicial action – which keeps the fiber $h^{-1}(D)$ fixed – on $S^3_{12} \backslash M^3_9$.

(g) We note that the quotient map $D^4_{17} \to \mathbb{C}P^2_9$ not only identifies points of the bounding S^3 under the Hopf map, but also leads to some additional internal identifications. Nevertheless the quotient space is still $\mathbb{C}P^2$! Also it is not known whether there is any other simplicial 4-ball with boundary S^3_{12} which gives $\mathbb{C}P^2_9$ in the same way.

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