Hyperbolic Manifolds*

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1. Towards the end of [1] on pages 553-554 Sullivan sketched why *closed* [and almost-parallelizable][1] *hyperbolic manifolds exist in all dimensions.* We'll aim for this result[1] starting from the much simpler goings-on of [2]. In like manner we'll review in a sequel[1] how this fact was used in this amazing 1977 talk to show that *any non-four dimensional manifold admits a unique lipschitz structure,* surely one of the outstanding results of twentieth century mathematics.

2. In the eighth 'lecture' of [2] we saw that, once one thinks of lines as 'infinite circles' one is led to a notion of reflection in a circle (inversion) under which a finite open disk is preserved by reflections in all circles normal to its boundary. This much generalizes easily to higher dimensions. However, quite unlike the $\{p, q\}$ tilings of the finite open 2-disk constructed in [2], *for $n > 4$ a finite open n-ball does not admit a regular tiling by copies of a spherically curved regular n-polytope.* We'll come to the proof later[2], for the moment we'll merely lower our ambitions from regularity to *crystallographicity* (as in the fifth 'lecture' of [2] for the classical case), i.e., the symmetries preserving the tiling shall be required to be transitive on tiles (but maybe not vertices), and assume first that such a tiling of the finite open n-ball B is given.

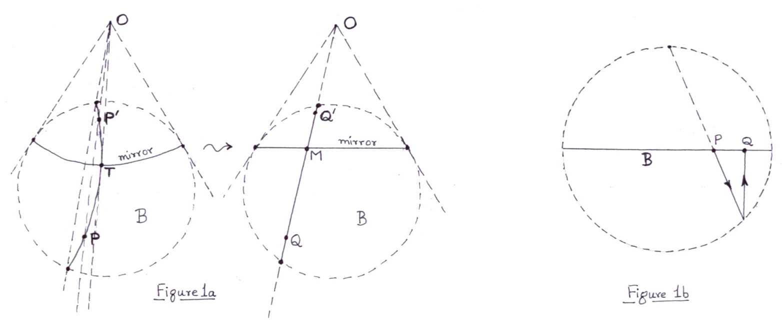
3. The *symmetries of B*—that is, compositions of reflections in $(n-1)$-spheres normal to ∂B—form a continuous group G, and those preserving the tiling a discrete subgroup Γ thereof. The tiling being crystallographic, the orbit space B/Γ is obtainable, from any tile of a simply transitive subdivision, by boundary identifications. So B/Γ is compact, which is the same as saying that the discrete subgroup Γ is *co-compact,* i.e., the coset space G/Γ is compact. This follows because B is the quotient of G by the isotropy subgroup of any point, and these are compact, for example, compositions of reflections in $(n-1)$-planes passing through the centre of B constitute the isotropy subgroup H of the centre. Conversely, given a co-compact discrete subgroup Γ of G, one can construct a crystallographic tiling on B, however to begin with we'll focus only on the compactum B/Γ. This is a manifold at least when Γ is *torsion-free,* for then $\Gamma \cap H = \{1\}$. We'll show that, *co-compact and torsion-free discrete subgroups Γ of G exist in great abundance in all dimensions* [and moreover, one can also ensure that B/Γ minus a point is parallelizable][1].

*August 6, 2012 (footnotes added in March 2016).

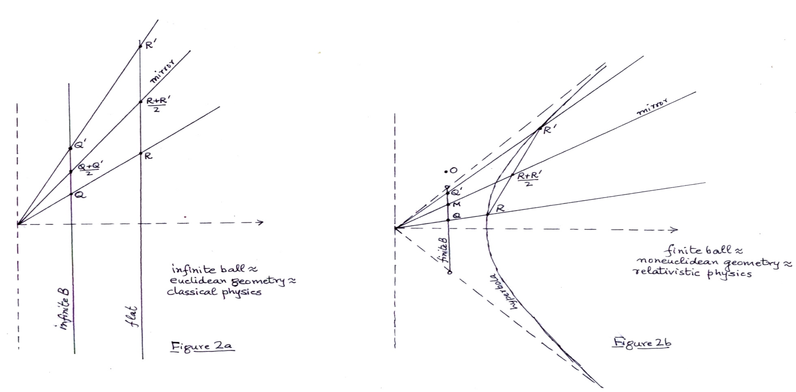
[1] I was unable to learn the hard part of this result to my full satisfaction in 2011-12, so it is not in this unfinished paper ... the sequel also remains on my to-do list.

[2] Only references to Coxeter and Vinberg in an end note are there.

4. However we'll work not with G, but with an isomorph of it which consists of linear transformations. To get to this representation we'll use, *a self-homeomorphism of the closed n-ball, identity on its boundary, which straightens its curved mirrors and transforms reflection into harmonic conjugation,* that is – Figure 1a – each circular arc normal to ∂B is mapped onto the segment with the same end-points, and inversion $P \leftrightarrow P', OP.OP' = OT^2$ in the former becomes harmonic conjugation $Q \leftrightarrow Q', QO/QM = Q'O/Q'M$ in the latter. This homeomorphism simply pushes each P out radially to Q as in Figure 1b: project B stereographically, from the north pole of an n-sphere having ∂B as its equator onto the southern hemisphere, and then project this hemisphere vertically back onto B. The stereographic map preserves angles, so the circular arcs normal to ∂B first become vertical circular arcs on the hemisphere with the same end-points, which project vertically back to the segments. The verification of the new recipe for reflection is not so hard either, for we need to check it only on its *axis,* i.e., the diameter incident to O, but we'll omit the details.



Straightening



Linearizing

5. The symmetries of euclidean n-space are commonly linearized by identifying it projectively – Figure 2a – with a constant-time flat of a vector space having one more dimension called 'time'. Likewise, *the reflections* $Q \leftrightarrow Q', QO/QM = Q'O/Q'M$ *of a finite n-ball* B *are linearized if we identify* B *projectively with the surface obtained by revolving, around the time axis, a hyperbola asymptotic to rays passing through the ends of any diameter.* For, a computation which we omit shows that, if Q and Q' are on the axis of this reflection, and if the rays through them meet such a hyperbola in R and R' – Figure 2b – then the ray through M shall pass through the mid-point of the chord RR'. So our identification transforms the reflection of B into the *linear* transformation of space-time which switches R and R', and keeps the other $n-1$ components same; therefore G identifies with the group generated by these linear reflections.

6. In the language of physics, Figure 2 identifies each point of a 3-ball with an *inertial frame,* i.e., a parallel pencil of free particles in 4-dimensional space-time. Since Galileo, the laws of classical physics – see Figure 2a – had been required to be independent of the observer's inertial frame, so invariant with respect to the group of euclidean symmetries of an infinite 3-ball. The finiteness of B allowed Poincaré to show that the equations proposed by Maxwell to explain the propagation of light were, analogously, invariant with respect to the group G of its noneuclidean symmetries. *The finiteness of* B *encodes the law that all particles must have speeds less than an absolute constant* c : indeed, if in Figure 2b, the ball B is situated at time $t = 1$, then its radius is precisely c.

7. In cartesian coordinates $(t, x_1, \ldots x_n)$ the group G consists of *real matrices preserving the quadratic form* $-c^2t^2 + x_1^2 + \cdots + x_n^2$ for the hyperbolic surface of Figure 2b is given by setting it equal to a constant; indeed, G *contains all such matrices* $g = [g_{i,j}]$ *with* $g_{1,1}$ *positive.* This extra condition ensures that the g-image of the unit time vector is also in the forward cone; if it is distinct we multiply g with the linear reflection which switches $g(1, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)$ to obtain $\begin{bmatrix} 1 & O \\ O & h \end{bmatrix}$ – these matrices form the subgroup H of G – and use the fact that h, an isometry of the euclidean vector subspace $t = 0$, is a product of at most n reflections of the same, which can be proved thus: if h does not fix any unit vector v, multiply it with the reflection in the right bisector of a v and $h(v)$, now consider the euclidean subspace orthogonal to v, etc.

8. For example, the ray through $(1, c \tanh\theta, 0, \ldots, 0) \in B$ meets the tangent hyperbola in $(\cosh\theta, c\sinh\theta, 0, \ldots, 0)$ and $\begin{bmatrix} \cosh\theta & -c^{-1}\sinh\theta & O \\ c\sinh\theta & -\cosh\theta & O \\ O & O & I \end{bmatrix} \in G$ is the linear reflection that switches this point with the unit time vector. Multiplying this reflection with the reflection $x_1 \leftrightarrow -x_1$ one gets a one-parameter subgroup $\begin{bmatrix} \cosh\theta & c^{-1}\sinh\theta & O \\ c\sinh\theta & \cosh\theta & O \\ O & O & I \end{bmatrix}$ of G consisting of *translations* – i.e., products of two reflections having the same axis – of B. Multiplying any orientation preserving $g \in G$ with a suitable translation one can fix the unit time vector, i.e., obtain an orientation preserving euclidean isometry of $t = 0$, which can be decomposed into mutually orthogonal two-dimensional rotations.

9. For the infinite n-ball n independent translations generate a co-compact subgroup of motions $\Gamma \cong \mathbb{Z}^n$, and dividing out its action gives the n-torus, a closed parallelizable manifold. Likewise, for a finite 1-ball B, the powers of a non-identity translation form a discrete subgroup isomorphic to $\mathbb{Z}$ and dividing by its action gives the circle. For $n = 2$ just two translations won't do because there is no rectangle with circular sides normal to ∂B, and for higher n also n translations seem too few, however one should be able to do the job with more: *it seems to us that G should have discrete subgroups Γ of—or at least generated solely by—translations which are co-compact, and that the closed n-manifold obtained by dividing the finite ball B out by any such Γ should be almost parallelizable?* For $n = 2$ this is true and more: using the regular tilings of [2] one gets lots of discrete co-compact groups consisting solely of translations and any closed orientable 2-manifold is almost parallelizable. This last because any closed manifold admits a nonzero tangent vector field in the complement of a point—since the finitely many singularities of a generic vector field can be enclosed in a ball—and when the dimension is two and one has orientability there is also a second normal tangent vector field. However the torus does not occur, more generally, *for any even n, B/Γ can never be parallelizable,* as follows by using Gauss-Bonnet, see e.g., Ratcliffe [3], p. 528. On the other hand, when n is odd, B/Γ may well be parallelizable, for example, all closed orientable 3-manifolds are parallelizable, so this shall always be the case for $n = 3$.

10. The group of motions of an infinite n-ball, assumed identified with the flat $t = 1$, consists of all matrices which preserve this flat and its quadratic form $x_1^2 + \cdots + x_n^2$. Its discrete subgroup, of all matrices with integer entries, is co-compact because it contains the n translations $x_i \mapsto x_i + 1$. The group G of motions of an n-ball B of radius c on $t = 1$, assumed identified with its tangent hyperboloid, consists of all matrices with first entry positive which preserve $-c^2t^2 + x_1^2 + \cdots + x_n^2$. *Is the discrete subgroup Γ of G, of all matrices with integer entries, co-compact*? A couple of examples for *the case $n = 1$* will show that the answer depends on the nature of c:-

(a) *If $c = 2$*—more generally if c is any rational number—*then Γ is of order* 2*,* i.e., it coincides with the isotropy subgroup $\Gamma \cap H$ of the unit time vector (in general the n reflections $x_i \leftrightarrow -x_i$ generate this subgroup of order 2^n), because *the hyperbola $-2^2t^2 + x^2 = -2^2$ has no integral point with $t > 1$,* this because $t^2 - 1$ is not the square of any integer for $t > 1$. (On the other hand the 'discrete pythagorean theorem' of [2] shows that for c rational the hyperbola $-c^2t^2 + x^2 = -c^2$ has infinitely many rational points.)

(b) *If $c^2 = 2$ then Γ is co-compact and the hyperbola $-2t^2 + x^2 = -2$ has infinitely many integral points with $t > 0$ all obtainable from $(1, 0)$ by the action of an index 2 subgroup $\mathbb{Z}$ of Γ.* We note that there is no integral point with $t = 2$ but for $t = 3$ we have $(3, \pm 4)$; the linear reflection $(1, 0) \leftrightarrow (3, 4)$ is given by $\begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$ and multiplying it with the reflection $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ one gets the translation $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$ whose powers applied to $(1, 0)$ must generate all the integral points of our hyperbola, for, if there was any other, then, by applying a suitable power of this translation to it, we would get an integral point on the hyperbola nearer to $(1, 0)$ than $(3, \pm 4)$.

The above method works even if 2 is replaced by any whole number d which is not a perfect square, and is essentially Brahmagupta's, however the existence of an integral point with $t > 1$—the least such t can be very big indeed!—on the hyperbola $-dt^2 + x^2 = -d$ was proved much later by Lagrange, who also showed that these integral points were intimately related to the continued fraction of $\sqrt{d}$. For d square-free, these integral points correspond (cf. §18) to the first columns of the matrices of Γ, which are also determined by their orthogonal second columns, which correspond to the integral points on the hyperbola $-dt^2 + x^2 = 1$, and its factorization $(x + t\sqrt{d})(x - t\sqrt{d}) = 1$ shows that this method gives also the group of units of the ring $\mathbb{Z}[\sqrt{d}\,]$ of algebraic integers. Lagrange's argument was simplified—using his pigeon hole principle—by Dirichlet, who also worked out the group of units of any algebraic number field. *Setting* $n = 1$ *one can read* §§ 11, 13 *below as yet another proof of Lagrange's theorem,* and the fact that this argument works for all n gives similar examples—see §12—of discrete co-compact subgroups Γ of G in dimensions 2 and 3.

11. A *lattice of* $\mathbb{R}^{n+1}$ consists of all integral combinations of any basis – so two bases give the same lattice iff they are related by an integral matrix whose inverse is also integral – and lattices are deemed close to each other iff they can be generated by bases close to each other. The absolute value of the determinant is the same for all generating bases and is called the *volume* of a lattice, and another important invariant of a lattice is the minimum *separation* between its elements. *A set of lattices has compact closure if and only if separation is bounded from below and volume from above over it;* we'll prove the non-trivial 'if' part of this criterion in §13, after giving some corollaries in §12.

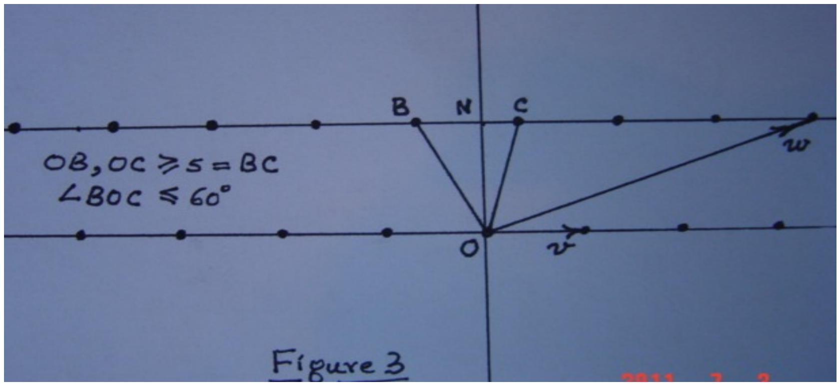
These all pertain to the group G of linear isomorphisms of space-time preserving the quadratic form $Q := -c^2t^2 + x_1^2 + \cdots + x_n^2$ *with* c^2 *integral* and its discrete subgroup Γ of integral matrices. The matrices of G are those whose columns have 'squared lengths' $-c^2, 1, \ldots, 1$ and are pairwise orthogonal with respect to the quadratic form, and all such matrices have determinants ± 1; so, $g \mapsto g(\mathbb{Z}^{n+1})$ *identifies* G/Γ *with a closed subset of lattices of volume one;* furthermore, *separation is bounded from below if and only if* $-c^2t^2 + x_1^2 + \cdots + x_n^2 = 0$ *has no nontrivial integral solution.* Indeed, if $Q(v) = 0$ for a nonzero $v \in \mathbb{Z}^{n+1}$, we can find $g_m \in G$ such that $g_m(v)$ is on the same ray as v and arbitrarily close to the origin, for example, in any 2-plane through this ray—cf. Figure 2b—we can reflect first in the time axis, and then use the linear reflection which interchanges the resulting point with the required $g_m(v)$; so separation cannot be bounded below. Conversely, if v_m is a sequence of nonzero vectors of $\mathbb{Z}^{n+1}$ such that $g_m(v_m)$ approaches the zero vector, then $Q(g_m(v_m)) = Q(v_m)$ approaches zero, i.e., these integers are eventually equal to zero.

12. So integral matrices preserving $-t^2 + x_1^2 + \cdots + x_n^2$ give a non co-

compact Γ for all $n \geq 1$, while $-2t^2 + x_1^2 + \cdots + x_n^2$ gives a co-compact Γ for $n = 1$ only. Again, $-3t^2 + x_1^2 + \cdots + x_n^2$, *more generally* $c^2 \equiv 3$ *mod* 4, *gives a co-compact* Γ *if and only if* $n = 1$ *or* 2, because it is easily checked that even the congruence $-3t^2 + x_1^2 + x_2^2 \equiv 0 \bmod 4$ has no nontrivial integral solution. Likewise, $-7t^2+x_1^2+\cdots+x_n^2$, *more generally* $c^2 \equiv 7$ *mod* 8, *gives us a co-compact* Γ *if and only if* $n = 1, 2$ *or* 3, because the congruence $-7t^2 + x_1^2 + x_2^2 + x_3^2 \equiv 0$ mod 8 has no nontrivial integral solution: for then it would have a solution with one of the squares odd, that term would be 1 mod 8, but the remaining three terms – each 0, 1 or 4 mod 8 – can't add up to the requisite 7 mod 8.

However, *for* $n \geq 4$, *and any positive integer* m, *integer matrices preserving* $-mt^2 + x_1^2 + \cdots x_n^2$ *always give us a non co-compact discrete subgroup* Γ *of* G. Indeed, $-mt^2+x_1^2+\cdots x_n^2 = 0$ has a solution with $t = 1$, because another famous theorem of Lagrange assures us that any whole number m can be written as a sum of four squares. (More generally, for $n \geq 4$, and m_i any positive integers, it is known that the equation $-mt^2 + m_1x_1^2 + \cdots + m_nx_n^2 = 0$ has a nontrivial integral solution. This means that we still won't get this way, for $n \geq 4$, a compact G/Γ, even if we were to modify Figure 2b to an ellipsoidal hyperboloid, with the n-disk B now an oblique section of its asymptotic cone.)

13. Since the set of lattices which admit bases whose members have lengths in a closed subinterval of the positive reals is compact, the criterion of §11 follows from: *if a lattice* L *of* $\mathbb{R}^{n+1}$ *has separation* s *and volume* V *then it has a basis whose members have lengths* $\leq (\frac{2}{\sqrt{3}})^{\frac{n(n+1)}{2}} \frac{V}{s^n}$. To see this choose a nonzero $v \in L$ of minimum length s. Since v extends to a basis of L, orthogonal projection of L gives us a lattice $\overline{L}$ in the n-dimensional orthogonal complement of v in $\mathbb{R}^{n+1}$. Clearly $\overline{L}$ has volume V/s. Further, if $w \in L$ has a nonzero projection $ON \in \overline{L}$, then ON has length at least $\frac{\sqrt{3}}{2}s$, because it is the altitude of an acute angled lattice triangle OBC – see Figure 3 – on its smallest side BC of length s. We can assume inductively that $\overline{L}$ has a basis of vectors $u_2, \ldots, u_{n+1}$ of lengths $\leq (\frac{2}{\sqrt{3}})^{\frac{(n-1)n}{2}} \frac{V/s}{(\frac{\sqrt{3}}{2}s)^{n-1}}$. For each of these vectors $u = ON \in \overline{L}$ we define $\tilde{u} \in L$ to be either OB or OC depending on which makes the smaller angle with ON. This gives us a basis $v, \tilde{u_2}, \ldots, \tilde{u_{n+1}}$ of L with lengths at most $\frac{2}{\sqrt{3}}$ times bigger, so with lengths obeying the required bound.



14. The natural idea of restricting c^2 to $\mathbb{Z}$ gave us lots of discrete subgroups Γ of G, and quite a few of these were co-compact in dimensions ≤ 3, but in higher dimensions they were all much too small. This suggests using a bigger subring of the reals than $\mathbb{Z}$, but since such a subring is dense in $\mathbb{R}$, the fear now is that we'll lose the discreteness of Γ in G. However, this is not necessarily so, to show which we'll now work out a third one-dimensional example:-

The two previous examples in §10 had dealt with $c = 2$ and $c^2 = 2$, now we'll do $c^4 = 2$. In particular we'll find *all points on the hyperbola* $-\sqrt{2}t^2 + x^2 = 1$ *with coordinates of the type* $p + q\sqrt{2}$ *where* p *and* q *are integers.* These numbers $p + q\sqrt{2}$ are closed under addition and multiplication and replacing each $t = p+q\sqrt{2}$ by $\bar{t} = p - q\sqrt{2}$ preserves these operations. Further, *all pairs* $(t, \bar{t})$ *form a lattice of* $\mathbb{R}^2$ – see Figure 4 – because any $(p + q\sqrt{2}, p - q\sqrt{2})$ is an integral combination of the orthogonal vectors $(1, 1)$ and $(\sqrt{2}, -\sqrt{2})$. Moreover, *for the sought-for points* (t, x) *on the hyperbola, the lattice points* $(t, \bar{t})$ *must be in the strip* $|\bar{t}| < 2^{-\frac{1}{4}}$, this because $-\sqrt{2}t^2 + x^2 = 1$ implies $x \neq 0$ and $\sqrt{2}\,\bar{t}^2 + \bar{x}^2 = 1$. So, hoping that there exists a solution with a positive t, we'll merely totally order these lattice points in the strip by increasing first coordinate, and just start checking one by one. The first such lattice point $T_1 = (1 + \sqrt{2}, 1 - \sqrt{2})$ does not work because $-\sqrt{2}(1 + \sqrt{2})^2 + x^2 = 1$ gives us $x^2 = 5 + 5\sqrt{2}$ which is not equal to $(p + q\sqrt{2})^2 = (p^2 + 2q^2) + 2pq\sqrt{2}$ for any integers p and q; likewise, $T_2 = (2 + \sqrt{2}, 2 - \sqrt{2})$ leads to $x^2 = 9 + 6\sqrt{2}$ which is not of the desired type either; but quite fortunately for us, the very next lattice point $T_3 = (2 + 2\sqrt{2}, 2 - 2\sqrt{2})$ gives $x^2 = 17 + 12\sqrt{2} = (3 + 2\sqrt{2})^2$.

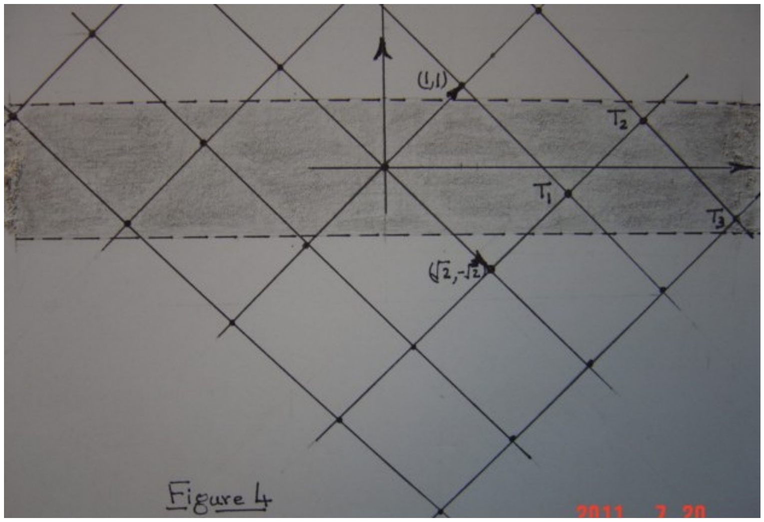


Figure 4

The solution $(2+2\sqrt{2}, 3+2\sqrt{2})$ nearest to $(0,1)$ on $-\sqrt{2}t^2+x^2=1$ determines in turn the translation $\begin{bmatrix} 3+2\sqrt{2} & 2+2\sqrt{2} \\ 4+2\sqrt{2} & 3+2\sqrt{2} \end{bmatrix} \in G$ whose first column is nearest to $(1,0)$ of the required kind on $-\sqrt{2}t^2+x^2=-\sqrt{2}$. It follows – cf. the second example – that, *the subgroup* Γ *of matrices with entries* $p+q\sqrt{2}$ *is discrete and co-compact,* is generated by this matrix and $x \leftrightarrow -x$, and the Γ-orbits of $(1,0)$ and $(0,1)$ give us all points with such coordinates on both hyperbolas.

Indeed 'Brahmagupta's method' extends not just to $\sqrt{2}$ but to *any positive real algebraic integer whose square root is of a higher degree and whose other conjugates are all real and negative*: one strip-searches similarly for a non-trivial solution of the required kind using a lattice whose dimension equals the degree of the algebraic integer, with the success of the search now guaranteed – cf. Fricke [4] – by Dirichlet's generalization of Lagrange's theorem mentioned in §10. However, postponing all this, we'll now show directly that our third example–unlike our second–works in all dimensions.

15. *Matrices with entries* $p+q\sqrt{2}$ *which preserve* $-\sqrt{2}t^2+x_1^2+\cdots+x_n^2$ *form a discrete and co-compact subgroup* Γ *of* G *for all* $n \geq 1$. For the discreteness of Γ in G we note from Figure 4 that, though any bounded open interval of the reals has infinitely many $p+q\sqrt{2}$'s, only *finitely* many of them have $|p-q\sqrt{2}|$ less than a given bound; and that this remark is applicable to our matrix entries because their $p-q\sqrt{2}$'s are the corresponding entries of a matrix whose columns are orthogonal and have squared lengths $\sqrt{2}, 1, \ldots, 1$ with respect to the positive definite quadratic form $\sqrt{2}t^2+x_1^2+\cdots+x_n^2$, so $|p-q\sqrt{2}| \leq 2^{\frac{1}{4}}$.

The planar lattice of Figure 4 generalizes to *the lattice* $L \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ *of all pairs of vectors* $(v;\overline{v})$ with corresponding entries $p+q\sqrt{2}$ and $p-q\sqrt{2}$, it has as integral basis the $2(n+1)$ orthogonal vectors $(\ldots 0,1,0\ldots;\ldots 0,1,0\ldots)$ and $(\ldots 0,\sqrt{2},0\ldots;\ldots 0,-\sqrt{2},0\ldots)$. Given any $g \in G$, a real matrix (with first entry positive) which preserves $Q := -\sqrt{2}t^2+x_1^2+\cdots+x_n^2$, and another real matrix h of the same size (no condition on the first entry now) which preserves the positive quadratic form $\overline{Q} := \sqrt{2}t^2+x_1^2+\cdots+x_n^2$, we'll denote by $(g;h)(L) \subset \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ the lattice consisting of all pairs $(g(v);h(\overline{v}))$. We have $(g;h)(L) = L$ *if and only if* $g \in \Gamma$ *and* $h = \overline{g}$, the matrix obtained from g by changing each entry $p+q\sqrt{2}$ to $p-q\sqrt{2}$: to see this note that $(g(\ldots 0,1,0\ldots);h(\ldots 0,1,0\ldots))$ is the ith column of g followed by that of h. So $(g;h)(L) \mapsto g$ induces a surjection from the set of all these lattices to G/Γ, and since it is obviously continuous, it suffices to show that, *this set* $\{(g;h)(L)\}$ *of lattices is compact.*

This we'll check much as before in §11. That, *this set is closed,* follows from the nature of the defining prescriptions on the columns of the real matrices g and h, viz., that they be orthogonal and have squared lengths $\mp\sqrt{2},1,\ldots,1$ with respect to the quadratic forms Q and $\overline{Q}$. Further, since g and h have determinants ± 1, *all these lattices have the same volume as* L, that is $(2\sqrt{2})^{n+1}$, the $(n+1)$th power of that of the planar lattice of Figure 4. Finally, *separation is bounded from below on this set of lattices.* Otherwise, we can find a sequence of matrices g_m and h_m together with a sequence of nonzero vectors $(v_m;\overline{v_m})$ of L such that the $2(m+1)$-vector $(g(v_m);h(\overline{v_m}))$ approaches the zero vector. Evaluating the pair of quadratic forms on this sequence, the pairs of numbers $(Q(v_m),\overline{Q}(\overline{v_m}))$ of type $(p_m+q_m\sqrt{2},p_m-q_m\sqrt{2})$ approach zero. Applying again the remark used before for discreteness, this means that for all m big these numbers are zero, which means there exists a nonzero vector $(v;\overline{v})$ of L such that $\overline{Q}(\overline{v}) = 0$, an impossibility because $\overline{Q}$ is positive definite.

16. Matrices of the above Γ, which map to the identity matrix when p and q are considered mod ℓ, form a normal subgroup Γ_ℓ, and we'll now show that, *if* ℓ *is an odd prime, then* B/Γ_ℓ *is a closed hyperbolic n-manifold,* i.e., it is compact without singularities and has B as an unbranched covering space.

The group Γ/Γ_ℓ is finite, so the compactness of B/Γ implies that of B/Γ_ℓ. Moreover, Γ_ℓ *is torsion-free for* $\ell \neq 2$. Indeed, in §15, we saw that Γ was isomorphic to a subgroup of isomorphisms of the lattice $L \subset \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$, so, using that integral basis of L, to a group of integral matrices of size $2n+2$. So it suffices to show: *an integral matrix of type* $I+\ell^d A$*, where* $d \geq 1$ *and the nonzero matrix* A *is not divisible by* ℓ*, has no prime power* k *equal to the identity matrix* I. This is so because $(I+\ell^d A)^k = I + k\ell^d A + \binom{k}{2}\ell^{2d}A^2+\cdots$, and, for $k \neq \ell$, resp. $k = \ell$, the second term is the *only* one on the right which is divisible *exactly* by the dth, resp. the $(d+1)$th, power of ℓ. On the other hand, for $\ell = 2 = k$ and $d = 1$, the second and third terms are both divisible exactly by 2^2; and indeed, Γ_2 *has some* 2*-torsion,* e.g., the reflections $x_i \leftrightarrow -x_i$; the remaining torsion of Γ lies outside these congruence subgroups Γ_ℓ.

The action of a discrete subgroup of symmetries is *discontinuous* on B, so, each $x \in B$ has a neighbourhood V such that $g(V) = h(V)$ iff $g(x) = h(x)$ and $g(V)\cap h(V) = \emptyset$ otherwise. If the discrete subgroup happens to be torsion-free, the compactness of the isotropy subgroup of x implies $g(x) = h(x)$ iff $g = h$, so then B is an unbranched covering of the orbit space.

17. All this and more, for example, fundamental domains and tilings, about the action of discrete subgroups of symmetries, especially those of a disk—the case considered in the 'eighth lecture' of [2]—has been known for very long, however we'll approach these matters differently here.

We recall that it has often been argued—see for example [5]—that 'physical space-time' is discrete. If so, a natural candidate for this appellation is the subset of geometric space-time consisting of points whose coordinates are integers, or perhaps, some algebraic integers. So, the associated *cartesian subdivision* $\mathfrak{S}$ of the hyperboloid $-c^2t^2 + x_1^2 + \cdots + x_n^2 = -c^2$ into closed *cells,* each made up of points around an integral point which are separated from it by no more than from any other integral point, deserves our close attention. Indeed, this *separation* between two points P and Q of the hyperboloid, that is, the positive length of the 'space-like vector' $\overrightarrow{PQ}$ with respect to our quadratic form, is a physically measurable quantity for $n = 3$: see Synge [6], pp. 24-26.

The "cell" (the inverted commas here, and at some places below, are reminders that it may not be homeomorphic to a closed n-ball) around each integral point consists, alternatively, of points of the hyperboloid contained in all the closed half-spaces containing our point and bounded by hyperplanes which are *right-bisectors,* with respect to the quadratic form, of the segments joining it to other integral points. So if, as before, we projectively identify the hyperboloid with the ball B of radius c which is tangent to it at its centre, then each cell of B is *convex* in the usual sense.

We'll see from the examples below that these convex "cells" are often compact, even polytopes, however there is, as such, no reason to believe that $\mathfrak{S}$ is always crystallographic, or even that its polytopes are congruent to each other. The point being that, though our hyperplane arrangement, so also our subdivision, is preserved by the action of Γ, *it is seldom the case that* Γ *acts transitively on the integral points of the hyperboloid.* However, it may be transitive in some examples in all dimensions, which shall suffice to give us, *a crystallographic tiling* $\mathfrak{S}$ *of hyperbolic n-space by convex polytopes?*

18. Indeed, Γ *can act non-transitively even in dimension one.* For example, if (t, x) is an integral point of $-2m^2t^2 + x^2 = -2m^2$, then x is divisible by m, and $(t, x/m)$ is an integral point of $-2t^2 + x^2 = -2$, which we considered before in §10. So now $(3, \pm 4m)$ are the integral points nearest to $(1, 0)$. The translation which moves $(1, 0)$ to $(3, 4m)$ is given by $\begin{bmatrix} 3 & 2/m \\ 4m & 3 \end{bmatrix}$, and more generally its jth power is given by likewise dividing and multiplying the off-diagonal entries of $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}^j$ by m. So, if m does not divide the second entry of the first row of $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}^j$ for $1 \leq j \leq k$, these powers of the translation are given by non-integral matrices. So, the translation generates all the integral points and only these, but the subgroup of integral translations is smaller with an arbitrarily large index. This doesn't happen if c^2 is a square-free integer.

19. Turning now to 2-*dimensional examples,* we'll first work out the central cell of the subdivision $\mathfrak{S}$ of the disk B of radius c, tangent to the hyperboloid

$-c^2t^2+x^2+y^2=-c^2$ at $(1,0,0)$, for $c^2=1,2,3,7,11$ and $\sqrt{2}$.

The points (t,x,y) equidistant—with respect to the quadratic form—to two distinct points $P=(t_P,x_P,y_P)$ and $Q=(t_Q,x_Q,y_Q)$, i.e., the right-bisecting plane of PQ, has equation $-c^2(t-\frac{t_P+t_Q}{2})(t_P-t_Q)+(x-\frac{x_P+x_Q}{2})(x_P-x_Q)+(y-\frac{y_P+y_Q}{2})(y_P-y_Q)=0$. So, if P and Q are on the hyperboloid, it is the plane through the equidistant origin, which contains the line given by $t=\frac{t_P+t_Q}{2}$ and $(x-\frac{x_P+x_Q}{2})(x_P-x_Q)+(y-\frac{y_P+y_Q}{2})(y_P-y_Q)=0$. The rays through the points $(\frac{t_P+t_Q}{2},x,y)$ of this line meet B in points $(1,X,Y)$ satisfying $(X-\frac{x_P+x_Q}{t_P+t_Q})(x_P-x_Q)+(Y-\frac{y_P+y_Q}{t_P+t_Q})(y_P-y_Q)=0$, so this is the equation of the *right-bisector* of the points $P=(1,\frac{x_P}{t_P},\frac{y_P}{t_P})$ and $Q=(1,\frac{x_Q}{t_Q},\frac{y_Q}{t_Q})$ of B under our identification of the hyperboloid with this disk. Using $Q=(1,0,0)$ we see that, *the central cell of* $\mathfrak{S}$ *is defined by the inequalities* $(X-\frac{x_P}{t_P+1})x_P+(Y-\frac{y_P}{t_P+1})y_P\le 0$ *as* P *runs over* all *the integral points of the hyperboloid,* however only a handful of P's are needed to determine it in the examples considered:-

For $c^2=1$ *it is a* 4*-gon minus its vertices,* which are on the boundary of B, its sides being the bisectors of the segments from $(1,0,0)$ to the four integral points $(3,\pm 2,\pm 2)$. Note that—in sharp contrast to §10—*our hyperboloid has an infinity of integral points* (this is true for any rational c^2), more precisely, a theorem of Fermat tells us that, t_P^2-1 is a sum $x_P^2+y_P^2$ of two squares iff no prime equal to 3 mod 4 occurs an odd number of times in either t_P-1 or t_P+1. Also note that, *if* (t,x,y) *is on the hyperboloid, then so is the point obtained by interchanging, or by changing the sign of one or both of, the last two coordinates.* So, to confirm that Figure 5(i) is the central "cell" it suffices to check that, if (t_P,x_P,y_P) is any integral point with $t_P>1$ and $0\le x_P\le y_P$, then the vertex $(0,1)$ satisfies $(X-\frac{x_P}{t_P+1})x_P+(Y-\frac{y_P}{t_P+1})y_P\le 0$, i.e., $-\frac{x_P}{t_P+1}x_P+(1-\frac{y_P}{t_P+1})y_P\le 0$, i.e., $(t_P+1)y_P\le x_P^2+y_P^2=t_P^2-1$, i.e., $y_P\le t_P-1$, which is true because t_P has a bigger square than y_P.

For $c^2=2$ *too* – Figure 5(ii) – *it is a* 4*-gon minus its vertices,* its sides the bisectors of the segments which join $(1,0,0)$ to the integral points $(3,\pm 4,0)$ and $(3,0,\pm 4)$. To confirm this it suffices to check that, if (t_P,x_P,y_P) is any integral point with $t_P>1$ and $0\le x_P\le y_P$, then the vertex $(1,1)$ satisfies $(X-\frac{x_P}{t_P+1})x_P+(Y-\frac{y_P}{t_P+1})y_P\le 0$, i.e., $(1-\frac{x_P}{t_P+1})x_P+(1-\frac{y_P}{t_P+1})y_P\le 0$, i.e., $(x_P+y_P)(t_P+1)\le x_P^2+y_P^2=2(t_P^2-1)$, i.e., $x_P+y_P\le 2(t_P-1)$. Which is true because $y_P^2=2t_P^2-x_P^2-2=(2t_P-x_P)^2-2(t_P-x_P)^2-2$ shows that the square $(2t_P-x_P)^2$ exceeds the square y_p^2 but has the same parity, so it is not the next square, so $y_p\le 2t_P-x_P-2$, i.e., $x_P+y_P\le 2(t_P-1)$.

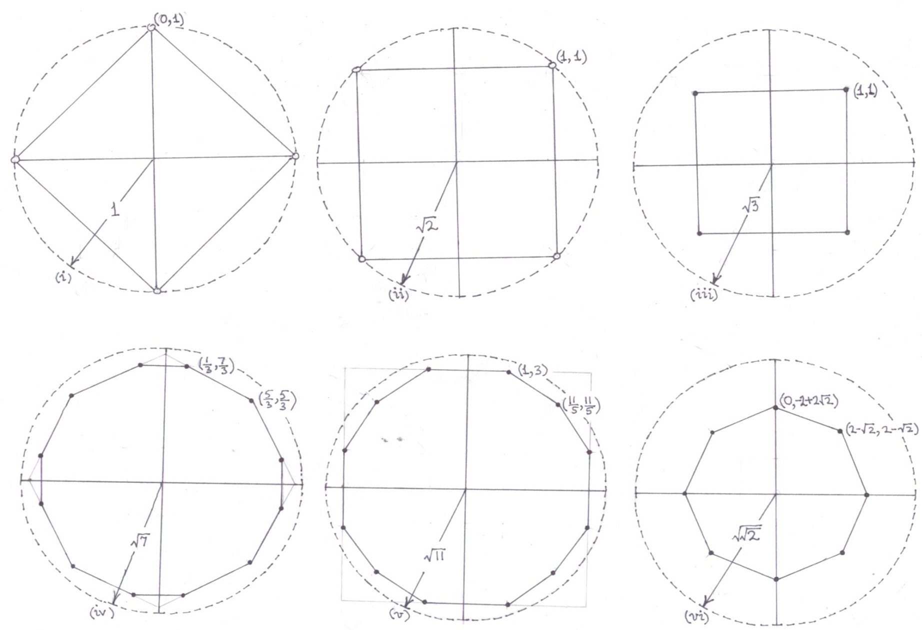


Figure 5

For $c^2 = 3$ *it is a* 4*-gon* – Figure 5(iii) – its sides bisectors of segments from $(1, 0, 0)$ to the integral points $(2, \pm 3, 0)$ and $(2, 0, \pm 3)$ of $-3t^2 + x^2 + y^2 = -3$. Simply by noting whether or not $3(t^2 - 1)$ is a sum of two squares, and if it is, writing it as such in all possible ways, it is easy to find integral points close to $(1, 0, 0)$, given below are all with $1 \le t_P \le 10$ and $0 \le x_P \le y_P$.

t_P	1	2	4	5	7
x_P	0	0	3	6	0
y_P	0	3	6	6	12

The circumcircle of our 4-gon has radius $r = \sqrt{2}$. Also, *the right-bisector of the segment from* $(1, 0, 0)$ *to a point* (t_P, x_p, y_P) *on* $-c^2t^2 + x^2 + y^2 = -c^2$ *is tangent to the central circle of* B *of radius* r *iff* $t_P = \frac{c^2+r^2}{c^2-r^2}$. For, this can happen iff the mid-point $(\alpha, \beta) := (\frac{x_P}{t_P+1}, \frac{y_P}{t_P+1})$ is its point of tangency, when $-c^2t_P^2 + \alpha^2(t_P+1)^2 + \beta^2(t_P+1)^2 = -c^2$ gives us $-c^2t_P^2 + r^2(t_P+1)^2 = -c^2$, solving which quadratic in t_P we get the stated value. For example, $\frac{3+2}{3-2} = 5$, and the bisector of the segment from $(1, 0, 0)$ to the point $(5, 6, 6)$ of $-3t^2+x^2+y^2 = -3$ is tangent to the circumcircle at $(\frac{6}{5+1}, \frac{6}{5+1}) = (1, 1)$, a vertex of our 4-gon. Since bisectors of segments to integral points with bigger t_P's won't intersect this circle, it only remains to check that $(1, 1)$ satisfies $(X - \frac{x_P}{t_P+1})x_P + (Y - \frac{y_P}{t_P+1})y_P \le 0$ for the integral point $(t_P, x_P, y_Q) = (4, 3, 6)$. Substitution shows that it is so, in fact one has equality, this bisector too passes through the vertex $(1, 1)$.

For $c^2 = 7$ it is a 12-gon – Figure 5(iv) – its sides bisectors of segments from $(1, 0, 0)$ to the eight integral points with $t_p = 6$ and the four with $t_P = 8$. These are the first two values of $t > 1$ for which $7(t^2 - 1) = 7(t - 1)(t + 1)$ *contains any prime equal to* 3 *mod* 4—so in particular seven—*an even number of times,* a necessary and sufficient condition for it to be a sum of two squares, and given below are all the integral points with $1 \leq t_P \leq 26$ and $0 \leq x_P \leq y_P$.

t_P	1	6	8	15
x_P	0	7	0	28
y_P	0	14	21	28

We'll first work out the 8-gon determined by the points with $t_P = 6$. The bisector of the segment to $(6, 7, 14)$ is $(X - \frac{7}{7})7 + (Y - \frac{14}{7})14 = 0$, i.e., $X + 2Y = 5$. Its intersections with the bisectors of the segments to $(6, -7, 14)$ and $(6, 14, 7)$, i.e., $-X + 2Y = 5$ and $Y + 2X = 5$, give us the vertices $(0, \frac{5}{2})$ and $(\frac{5}{3}, \frac{5}{3})$, the other six vertices are symmetrical. Since $\frac{5}{3}\sqrt{2} < \frac{5}{2} < \sqrt{7}$, this 8-gon is contained in B and has circumradius $r = \frac{5}{2}$. Also, since $\frac{c^2+r^2}{c^2-r^2} = \frac{53}{3} < 18$, only the bisectors contributed by integral points with $t_P \leq 17$ can cut its circumcircle. The midpoint of the segment to $(8, 0, 21)$ is $(0, \frac{21}{9})$ and $\frac{21}{9} < \frac{5}{2}$, so the bisectors of the segments to the four points with $t_P = 8$ do chop off four corners of the 8-gon leaving us with, *the 12-gon whose vertices in the first quadrant are* $(\frac{1}{3}, \frac{7}{3})$, $(\frac{5}{3}, \frac{5}{3})$ *and* $(\frac{7}{3}, \frac{1}{3})$. However, the points with $t_P = 15$ don't matter, because the midpoint of the segment to $(15, 28, 28)$ is $(\frac{28}{16}, \frac{28}{16})$ and $\frac{28}{16} > \frac{5}{3}$.

For $c^2 = 11$ too it is a 12-gon – Figure 5(v) – its sides bisectors of segments from $(1, 0, 0)$ to the four integral points with $t_P = 10$ and the eight with $t_P = 12$. All the integral points with $1 \leq t_P \leq 44$ and $0 \leq x_P \leq y_P$ are given below.

t_P	1	10	12	21
x_P	0	0	22	22
y_P	0	33	33	66

The bisector of the segment to $(10, 0, 33)$ is $Y = 3$, and meets $X = 3$, the bisector of the segment to $(10, 33, 0)$, in $(3, 3)$; so, since $3\sqrt{2} > \sqrt{11}$, the 4-gon determined by the points with $t_P = 10$ is not contained in B. The bisector of the segment to $(12, 22, 33)$ is $(X - \frac{22}{13})22 + (Y - \frac{33}{13})33 = 0$, i.e., $2X + 3Y = 11$. It meets $Y = 3$ at $(1, 3)$, and $2Y + 3X = 11$, the bisector of the segment to $(12, 33, 22)$, at $(\frac{11}{5}, \frac{11}{5})$. So, *the vertices of our 12-gon in the first quadrant are* $(1, 3)$, $(\frac{11}{5}, \frac{11}{5})$ *and* $(3, 1)$. Since $\frac{11}{5}\sqrt{2} < \sqrt{10} < \sqrt{11}$, this 12-gon is contained in B and has circumradius $r = \sqrt{10}$. Also $\frac{c^2+r^2}{c^2-r^2} = 21$, so the bisectors of the segments to the eight points with $t_P = 21$ are tangent to the circumcircle at their midpoints. These points of tangency coincide with eight of the vertices, for example, the bisector of the segment to $(21, 22, 66)$ is tangent at $(1, 3)$. The bisectors of segments to points with $t_P > 21$ don't cut the circumcircle.

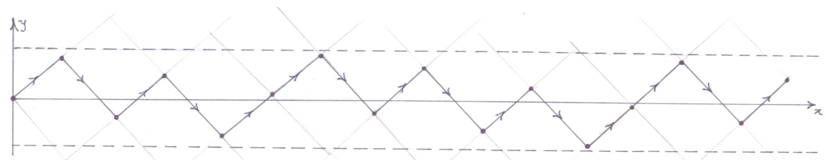
If c^2 is irrational, $(1, 0, 0)$ *is the only point on* $-c^2t^2 + x^2 + y^2 = -c^2$, $t > 0$ *with coordinates in* $\mathbb{Z}$*,* so our "cell" is all of B! However, *the* **integers** *over which* *the hyperboloid is defined is the subring of* $\mathbb{R}$ *generated by* 1 *and* c^2*,* accordingly, in the next example, it is this bigger ring $\mathbb{Z}[\sqrt{2}]$*, not* $\mathbb{Z}$*,* that concerns us.

For $c^2 = \sqrt{2}$ *the central cell of* $\mathfrak{S}$ *is an* 8*-gon* — Figure 5(vi) – its sides bisectors of segments from $(1,0,0)$ to the 8 integral points with $t = 1+\sqrt{2}$ on $-\sqrt{2}t^2 + x^2 + y^2 = -\sqrt{2}$. The bisector of the segment to $(1+\sqrt{2}, 1, 1+\sqrt{2})$ is $(X - \frac{1}{2+\sqrt{2}}) + (Y - \frac{1+\sqrt{2}}{2+\sqrt{2}})(1+\sqrt{2}) = 0$, i.e., $X + (1+\sqrt{2})Y = 2$, because $\frac{1}{2+\sqrt{2}} + \frac{(1+\sqrt{2})^2}{2+\sqrt{2}} = \frac{4+2\sqrt{2}}{2+\sqrt{2}} = 2$. It meets $-X + (1+\sqrt{2})Y = 2$ — the bisector of the segment to $(1+\sqrt{2}, -1, 1+\sqrt{2})$ — in $(0, \frac{2}{1+\sqrt{2}}) = (0, -2+2\sqrt{2})$; and, it meets $Y + (1+\sqrt{2})X = 2$ — the bisector of the segment to $(1+\sqrt{2}, 1+\sqrt{2}, 1)$ — in $(\frac{2}{2+\sqrt{2}}, \frac{2}{2+\sqrt{2}}) = (2-\sqrt{2}, 2-\sqrt{2})$. These then are two of the vertices of our 8-gon, and the remaining six are clear by symmetry. The 8-gon is contained in B, indeed all its vertices are on the circle of radius r where $r^2 = (-2+2\sqrt{2})^2 = 4(3-2\sqrt{2}) = 2(2-\sqrt{2})^2$ is less than $c^2 = \sqrt{2}$. So $\frac{c^2+r^2}{c^2-r^2} = \frac{\sqrt{2}+4(3-2\sqrt{2})}{\sqrt{2}-4(3-2\sqrt{2})} = \frac{12-7\sqrt{2}}{-12+9\sqrt{2}} = \frac{(12-7\sqrt{2})(-12-9\sqrt{2})}{144-162} = \frac{(-144+126)+(84-108)\sqrt{2}}{-18} = \frac{-18-24\sqrt{2}}{-18} = 1 + \frac{4}{3}\sqrt{2} < 2 + \sqrt{2}$. That is, less than all positive t other than $t = 1+\sqrt{2}$ such that $t \in \mathbb{Z}[\sqrt{2}]$ and $|\bar{t}| < 1$ – cf. §14 and Figure 4 – a necessary condition for (t,x,y) to be an integer point, for $-\sqrt{2}t^2 + x^2 + y^2 = -\sqrt{2}$ iff $\sqrt{2}\,\bar{t}^2 + \bar{x}^2 + \bar{y}^2 = \sqrt{2}$. So the bisectors of segments to integral points with $t_P > 1+\sqrt{2}$ don't cut the 8-gon.

That there are 8 and only 8 integral points with $t = 1+\sqrt{2}$ can be seen thus. We must have $|\bar{x}|, |\bar{y}| < 2^{\frac{1}{4}}$, but 0, 1 and $1+\sqrt{2}$ are the only non-negative integers $\leq \sqrt{2}((1+\sqrt{2})^2 - 1)$ which satisfy this condition, from which it follows easily that $\sqrt{2}((1+\sqrt{2})^2 - 1) = 1^2 + (1+\sqrt{2})^2$ is the unique way of writing the left hand side as a sum of two squares in $\mathbb{Z}[\sqrt{2}]$.

t_P	1	$1+\sqrt{2}$	$3+2\sqrt{2}$	$5+3\sqrt{2}$		$5+4\sqrt{2}$
x_P	0	1	0	$1+\sqrt{2}$	$3+2\sqrt{2}$	$2+2\sqrt{2}$
y_P	0	$1+\sqrt{2}$	$4+2\sqrt{2}$	$5+4\sqrt{2}$	$5+3\sqrt{2}$	$6+4\sqrt{2}$

We in fact computed, *all integral points* P *with* $1 \leq t_P \leq 7+5\sqrt{2}$ *and* $0 \leq x_P \leq y_P$*,* these are shown above. Starting with 0, and adding 1 or $\sqrt{2}$ depending on which gives a number whose conjugate is less than $2^{\frac{1}{4}}$ in absolute value, we got a long list without gaps of such integers. Then, we squared them. Then, subtracted 1 from each and multiplied by $\sqrt{2}$ to get the possible values of $\sqrt{2}(t^2-1)$. Finally, we scanned the previous line to obtain all the ways, if any, in which such a possibility is a sum of two squares.



List: $0, 1, 1+\sqrt{2}, 2+\sqrt{2}, 2+2\sqrt{2}, 3+2\sqrt{2}, 4+2\sqrt{2}, 4+3\sqrt{2}, 5+3\sqrt{2}, 5+4\sqrt{2}, 6+4\sqrt{2}, 6+5\sqrt{2}, 7+5\sqrt{2}, 8+5\sqrt{2}, 8+6\sqrt{2}, 9+6\sqrt{2}$

Squares: $0, 1, 3+2\sqrt{2}, 6+4\sqrt{2}, 12+8\sqrt{2}, 17+12\sqrt{2}, 24+16\sqrt{2}, 34+24\sqrt{2}, 43+30\sqrt{2}; 57+40\sqrt{2}, 68+48\sqrt{2}, 86+60\sqrt{2}, 99+70\sqrt{2}, 114+80\sqrt{2}, 136+96\sqrt{2}, 153+108\sqrt{2}$

$\sqrt{2}(t^2-1)$: $-\sqrt{2}, 0, 4+2\sqrt{2}, 8+5\sqrt{2}, 16+11\sqrt{2}, 24+16\sqrt{2}, 32+23\sqrt{2}, 48+33\sqrt{2}, 60+42\sqrt{2}, 80+56\sqrt{2}, 96+67\sqrt{2}, 120+85\sqrt{2}, 140+98\sqrt{2}, 160+113\sqrt{2}, 192+135\sqrt{2}, 216+152\sqrt{2}$

Figure 6

This computation is depicted in Figure 6, which uses again the lattice of §14, but now that strip has width $2.2^{\frac{1}{4}}$. The list gives in order the x-coordinate of its lattice points. That both operations—adding 1 and adding $\sqrt{2}$—can't keep us in the strip is clear because the vertical height $1+\sqrt{2}$ of the lattice rectangles is more than the width of the strip. However, *a machine performing this computation shall eventually halt.* This because $\mathbb{Z}[\sqrt{2}]$ is dense in the reals, so we shall reach a lattice point whose distance from the upper boundary of the strip is at most 1, and, from the lower boundary at most $\sqrt{2}$. Then neither operation keeps us in the strip, and the next lattice point in the strip is the diagonally opposite point of the lattice rectangle, so we must now add $1+\sqrt{2}$ if we want to continue the list. Alternatively, *we can avoid this halting problem by searching the slightly wider strip of width* $1+\sqrt{2}$*,* when the machine won't stop because there is no lattice point with $y = \frac{1+\sqrt{2}}{2} - 1$.

20. To propagate the properties of the central cells of these examples to all the cells of $\mathfrak{S}$ we'll use a method which is quite general.

The subdivision $\mathfrak{S}$ of the open n-ball B of radius c is preserved by any symmetry of B which restricts to a permutation of the integral points. This is immediate from the definition of the subdivision. From now on by *a symmetry of* $\mathfrak{S}$ we'll always understand any such symmetry of B, and $\mathfrak{S}$ shall be called *crystallographic* if its symmetries act transitively on its cells.

We can join the centre to any integral point by a path which avoids points on the boundaries of cells lying on more than one bisectors. Clearly the given integral point can be enclosed in a concentric open n-ball B_r of radius $r < c$. Also, *given any B_r we can find another $B_{r'}$ which contains it and is such that the bisector of any segment PQ with $P \in B_r$ and $Q \in B \setminus B_{r'}$ does not meet B_r.* Indeed, this bisector is the member, of the pencil of hyperplanes orthogonal to the line PQ, which passes through the mid-point of the segment PQ. As Q recedes away from P along this line towards the horizon, so does this mid-point, so there comes a stage after which this bisector does not cut the closed ball $\overline{B}_r$. The assertion follows by using the compactness of the space of all directions at all the points of $\overline{B}_r$. So, *B_r is covered by the cells σ_P of $\mathfrak{S}$ around the finitely many integral points P in $B_{r'}$ and the boundaries in B_r of these $B_r \cap \sigma_P$'s are on the bisectors of the finitely many pairs of integral points in $B_{r''}$.* The points of these boundaries which are contained in more than one of these finitely many hyperplanes form a closed subset of B_r of codimension ≥ 2, so its complement in B_r is path-connected. In particular, there is a path in this complement from the centre to the given integral point.

Now suppose that, *the central cell* σ_0 *of* $\mathfrak{S}$ *is a polytope,* and its facial structure is known to us (some or all of its vertices may be on the horizon). Then, *for* $\mathfrak{S}$ *to be a* (face-to-face) *crystallographic tiling by congruent polytopes, it is necessary that each facet* s *of the central polytope is shared by just one other cell of* $\mathfrak{S}$ *which is its image under a symmetry* f_s *of* $\mathfrak{S}$ (note that f_s must image some facet of the central cell, possibly s itself, onto s).

This obviously necessary condition is also sufficient. Consider any path of the above kind from the centre to any integral point P, and write in order the cells of $\mathfrak{S}$ which this path meets: $\sigma_0, \sigma_1, ..., \sigma_k$. Since the path does not go through any face of σ_0 of dimension less than $n-1$, we must have $\sigma_1 = f_{s_1}(\sigma_0)$ for some facet s_1 of σ_0. So σ_1 is a congruent polytope, and, for each of its facets $f_{s_1}(s)$, the symmetry $f_{s_1} f_s f_{s_1}^{-1}$ of $\mathfrak{S}$ throws this polytope on a unique cell sharing this facet with it. Once again, because our path avoided the lower dimensional faces of σ_1, the next cell σ_2 must be one of these, say $f_{s_1} f_{s_2}(\sigma_0)$. Continuing in this manner we obtain after k steps a symmetry $f_{s_1} f_{s_2} \dots f_{s_k}$ of $\mathfrak{S}$ that throws the central cell σ_0 on the cell σ_k around P.

The examples of §19 *give crystallographic tilings* $\mathfrak{S}$ *of the open disk* B *of radius* c. We'll show this by displaying, for each of these examples, an f_s as above for a maximal set—it has cardinality at most two—of sides s of the central polygon, which are inequivalent under the obvious symmetries, (t, x, y) maps to $(t, -x, y), (t, x, -y)$ or (t, y, x), of $\mathfrak{S}$ around the centre. Therefore, *the matrices displayed below, together with these central symmetries, generate from* $(1, 0, 0)$ *all the integral points of the hyperboloid,* for each of these examples.

For $c^2 = 1$*,* let $f_{s_1} \in G$ be the *reflection* in the side s_1 of the central "4-gon" which bisects (the segment joining) the integral points $(1, 0, 0)$ and $(3, 2, 2)$ of $-t^2 + x^2 + y^2 = -1$. From the fact that f_{s_1} switches $(1, 0, 0)$ and $(3, 2, 2)$ and keeps all orthogonal vectors, e.g. $(0, 1, -1)$, fixed, an easy calculation shows that its matrix is $\begin{bmatrix} 3 & -2 & -2 \\ 2 & -1 & -2 \\ 2 & -2 & -1 \end{bmatrix}$. Since this matrix is integral, this linear map of determinant -1 restricts to a permutation of the integral points of space-time, so also to a permutation of the integral points on the hyperboloid.

For $c^2 = 2$*,* the reflection in the side s_1 of the "4-gon" bisecting the points $(1, 0, 0)$ and $(3, 4, 0)$ of $-2t^2+x^2+y^2 = -2$ has an integral matrix $\begin{bmatrix} 3 & -2 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so again this reflection can serve as our f_{s_1}.

The above two examples are in fact isomorphic in a natural sense, but we'll defer this to the next section. The other examples are not isomorphic to these or to each other. However, it turns out that, with just one exception, reflections in the sides of the central polygon continue to do the needful.

For $c^2 = 3$*,* let $f_{s_1} = \begin{bmatrix} 3 & -2 & -2 \\ 2 & -1 & -2 \\ 2 & -2 & -1 \end{bmatrix}$, the reflection in the side s_1 of the 4-gon bisecting the points $(1,0,0)$ and $(2,0,3)$ of $-3t^2 + x^2 + y^2 = -3$.

For $c^2 = 7$*,* the reflection in the side s_1 of the central 12-gon bisecting the integral points $(1,0,0)$ and $(8,0,21)$ of $-7t^2 + x^2 + y^2 = -7$ has the integral matrix $\begin{bmatrix} 8 & 0 & -3 \\ 0 & 1 & 0 \\ 21 & 0 & -8 \end{bmatrix}$ and it shall be our f_{s_1}. However the reflection in the side s_2 bisecting $(1,0,0)$ and $(6,7,14)$ does not have an integral matrix, and it maps the integral point $(6,14,7)$ of the hyperboloid to its non-integral point $(6, 84/5, 63/5)$. So we turn to the *half-turn* $f_{s_2} \in G$ which switches $(1,0,0)$ and $(6,7,14)$, and reverses all orthogonal vectors, e.g. $(0,2,-1)$. An easy calculation shows that its matrix is $\begin{bmatrix} 6 & -1 & -2 \\ 7 & -2 & -2 \\ 14 & -2 & -5 \end{bmatrix}$, which is integral, so f_{s_2} is a symmetry of $\mathfrak{S}$. Moreover $f_{s_2}(s_2) = s_2$ because the mid-point $(7/2, 7/2, 14/2)$ of the segment joining $(1,0,0)$ and $(6,7,14)$ identifies with $(1,2) \in B$ which is also the mid-point of the side s_2 – see Figure 5(iv) – so, as required, the image of the central 12-gon under f_{s_2} is another cell of $\mathfrak{S}$ sharing this side with it.

For $c^2 = 11$*,* let $f_{s_1} = \begin{bmatrix} 10 & 0 & -3 \\ 0 & 1 & 0 \\ 33 & 0 & -10 \end{bmatrix}$, the reflection in the side s_1 of the 12-gon bisecting the points $(1,0,0)$ and $(10,0,33)$ of $-11t^2 + x^2 + y^2 = -11$, and let $f_{s_2} = \begin{bmatrix} 11 & -2 & -3 \\ 22 & -3 & -6 \\ 33 & -6 & -8 \end{bmatrix}$, the reflection in the side s_2 of the 12-gon bisecting the points $(1,0,0)$ and $(11,22,33)$.

For $c^2 = \sqrt{2}$*,* the reflection f_{s_1} in the side s_1 of the 8-gon bisecting the points $(1,0,0)$ and $(1+\sqrt{2}, 1, 1+\sqrt{2})$ of the hyperboloid $-\sqrt{2}t^2 + x^2 + y^2 = -\sqrt{2}$ has the matrix $\begin{bmatrix} 1+\sqrt{2} & -\sqrt{2}/2 & -(2+\sqrt{2})/2 \\ 1 & 1/2 & -(1+\sqrt{2})/2 \\ 1+\sqrt{2} & -(1+\sqrt{2})/2 & -(1+2\sqrt{2})/2 \end{bmatrix}$. The entries of this matrix are not all in $\mathbb{Z}[\sqrt{2}]$, so f_{s_1} does not induce a bijection of $(\mathbb{Z}[\sqrt{2}])^3$. Nevertheless, f_{s_1} *restricts to a permutation of the points* (t,x,y) *on the hyperboloid with coordinates in* $\mathbb{Z}[\sqrt{2}]$. We note that $(t,x,y) = (t_1 + \sqrt{2}t_2, x_1 + x_2\sqrt{2}, y_1 + y_2\sqrt{2})$ *where* x_1, y_1 *and* $x_2 + y_2$ *are either all odd or all even.* For, equating the rational parts on both sides of $\sqrt{2}(t^2 - 1) = x^2 + y^2$, we get $4t_1t_2 = x_1^2 + y_1^2 + 2(x_2^2 + y_2^2)$ which shows that $x_1^2 + y_1^2$, so $x_1 + y_1$, is even, so x_1 and y_1 have the same parity. When they are both odd, $x_1^2 + y_1^2 = 2 \bmod 4$ which is possible only if $x_2^2 + y_2^2$, so $x_2 + y_2$, is odd. When they are both even, $x_1^2 + y_1^2 = 0 \bmod 4$ which is possible only if $x_2^2 + y_2^2$, so $x_2 + y_2$, is even. These parity conditions and matrix multiplication show that all coordinates of $f_{s_1}(t,x,y)$ are in $\mathbb{Z}[\sqrt{2}]$.

§21. Brahmagupta's $(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2$ showed that sums of squares are closed under products; this identity involves *complex multiplication* $(x_1, y_1)(x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$; when using this operation we'll put $(t, x, y) = (t, z)$, where $z = x + \imath y$ and $\imath^2 = -1$.

For example, $(t, x, y) \mapsto (t, x - y, x + y)$ is the same as multiplying z by $1 + \imath$, but $x^2 + y^2$, that is $|z|^2$, is equal to $\frac{1}{2}|z(1 + \imath)|^2$, so this map throws the hyperboloid $-t^2 + x^2 + y^2 = -1$ onto the hyperboloid $-2t^2 + x^2 + y^2 = -2$. Further, *this linear isomorphism restricts to a bijection of the integral points of these hyperboloids.* Indeed, $t^2 - 1$ and $2(t^2 - 1)$ are sums of two squares for the same integral values of t, and since for any such t the two hyperboloids have the same number of integral points—because 2 is a sum of a unique ordered pair of squares $(1, 1)$—the induced injection between these equicardinal finite sets is a bijection. So, as was stated already in §20, the subdivisions $\mathfrak{S}_1$ and $\mathfrak{S}_2$ of the examples $c^2 = 1$ and $c^2 = 2$ are isomorphic to each other. Likewise, $\mathfrak{S}_m$ *is isomorphic to* $\mathfrak{S}_{2m}$ for any whole number m, and the same is true if m is multiplied by a square which is not a sum of two nonzero squares.

However, the subdivision can be quite different if we multiply $c^2 = m$ by a prime p equal to 1 mod 4, despite the fact that, by a theorem of Fermat, there is one and only one pair $\{a, b\}$ of whole numbers such that $p = a^2 + b^2$. The point being simply that $a \neq b$, so we have now not one, but two ordered pairs (a, b) and (b, a) to contend with, which makes all the difference! As *the example* $c^2 = 5$, which we'll now work out in full, shall clearly show us.

The second table below lists, *all the integral points* (t_P, x_P, y_P) *of* $-5t^2 + x^2 + y^2 = -5$ *with* $0 \leq x_P \leq y_P$ *and* $1 \leq t_P \leq 32$. A comparison with the first table, which lists the same points for the example $c^2 = 1$, shows that for some values of t_P, we have now twice as many—or even more—integral points.

$c^2 = 1$:-

t_P	1	3	9	17	19
x_P	0	2	4	12	6
y_P	0	2	8	12	18

$c^2 = 5$:-

t_P	1	3	9		17	19	
x_P	0	2	0	12	12	6	30
y_P	0	6	20	16	36	42	30

Our previous examples may suggest that the central cell of $\mathfrak{S}_m$ is compact if and only if m is not a sum of two squares; more so because this condition on the whole number m characterizes the co-compactness of Γ_m; and indeed co-compactness, i.e., the compactness of the cells of the Voronoi subdivision determined by the Γ_m-orbit of $(1, 0, 0)$ does imply the compactness of the central cell of $\mathfrak{S}_m$; but the converse is false: *the central cell of* $\mathfrak{S}_5$ *is compact!*

The central cell of $\mathfrak{S}_5$ *is an* 8*-gon* – see Figure 7 – its sides the bisectors of (the segments joining $(1, 0, 0)$ to) the eight integral points on the hyperboloid $\mathcal{H}_5$ with $t_P = 3$. The bisector of $(3, 2, 6)$ is $(X - \frac{2}{4})2 + (Y - \frac{6}{4})6 = 0$, i.e., $X + 3Y = 5$, which meets $-X + 3Y = 5$, i.e. the bisector of $(3, -2, 6)$, in $(0, \frac{5}{3})$, and $Y + 3X = 5$, i.e. the bisector of $(3, 6, 2)$, in $(\frac{5}{4}, \frac{5}{4})$. These then are two of the vertices of the 8-gon, the other six are clear by symmetry. Further, $\frac{5}{3} < \frac{5}{4}\sqrt{2} < \sqrt{5}$, so our 8-gon is in the open disk $\mathcal{B}_5$, and since $\frac{c^2+r^2}{c^2-r^2} = \frac{5+25/8}{5-25/8} = \frac{65}{15} < 5$, only bisectors of integral points with $t_P \leq 4$ can meet its circumcircle, and since we have taken account of all such, this is indeed the central cell.

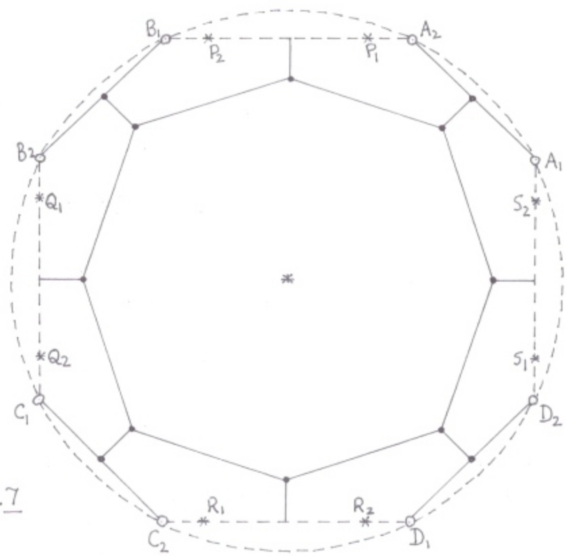
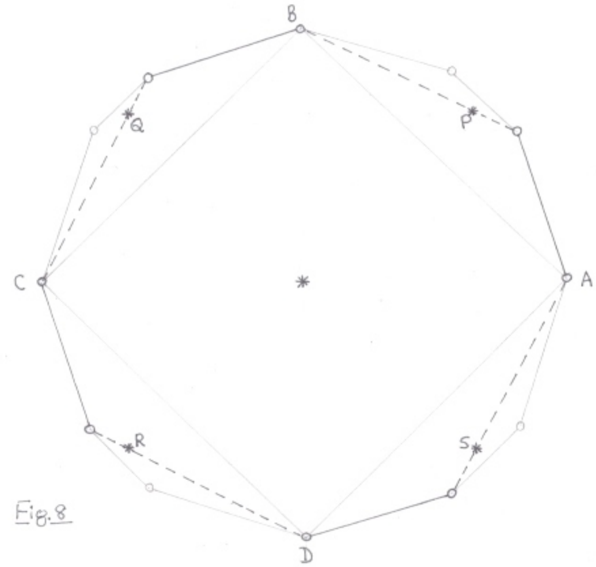


Fig.7

In fact, *the central cell of* $\mathfrak{S}_p$ *is an* 8*-gon for any prime* $p = 1$ *mod* 4*!* For, as noted before, the integral points for the example $c^2 = pm$ are obtained from those for the example $c^2 = m$ by complex multiplication by $a+\imath b$ or $b+\imath a$, where $\{a,b\}$ is the unique pair of whole numbers such that $p = a^2 + b^2$. This implies that the central cell of $\mathfrak{S}_{pm}$ is the intersection of the images of the central cell of $\mathfrak{S}_m$ under these two linear isomorphisms. In particular, for $c^2 = 1$ the central cell is the "4-gon" $ABCD$ with vertices $(1,0)$, etc. – Figure 5(i) – so the central cell for $c^2 = p$ must be, *the* 8*-gon which is the intersection of the "4-gons"* $A_1B_1C_1D_1$ *and* $A_2B_2C_2D_2$ *formed by the alternate vertices of the "*8*-gon" with vertices* (a,b)*, etc.,* on the boundary of the open disk $\mathcal{B}_p$ of radius $\sqrt{p}$.

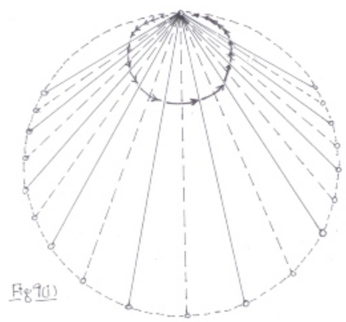
Let us double-check this general fact for $p = 5$: now $\{a,b\} = \{2,1\}$, and sure enough, the alternate sides $X + 3Y = 5$ and $-Y + 3X = 5$ of the central cell meet at $A_1 = (2,1)$ on the horizon of $\mathcal{B}_5$. Quite special however to the prime 5 is the fact that, *the only integral points other than the centre in this "*8*-gon" are the eight points determining the central cell, and these are on its boundary,* viz., $(3,2,6)$, i.e., $P_1 = (\frac{2}{3},2)$, etc., shown starred in Figure 7. To prove this assertion we note that any such point is obtained, from an integral point for $c^2 = 1$ lying in the images of this "8-gon" under *complex division* by $2+\imath$ or $1+2\imath$, by complex multiplying the same by $2+\imath$ or $1+2\imath$, respectively.

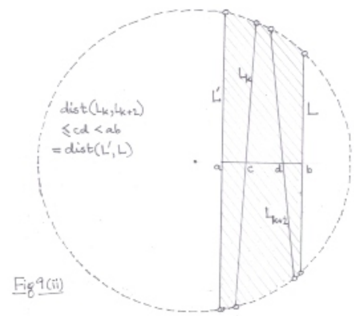


These two image "8-gons" in $\mathcal{B}_1$ are bounded by four sides and four diagonals, one each from the primary reflections of the central "4-gon" of $\mathfrak{S}_1$ *in its edges,* the image under complex division by $2+\imath$ being as in Figure 8, because $B_1 \mapsto B$, $P_1 \mapsto P$ and linearity imply A_2 must go to the opposite vertex of the "4-gon" of $\mathfrak{S}_1$ around P, etc. Since the centre and P, Q, R, S are the only integral points in these five "4-gons" of $\mathfrak{S}_1$, the assertion follows.

From the description of $\mathfrak{S}_1$ worked out in §20, we know that if we now add to Figure 8, $ABCD$'s secondary reflections, then tertiary, etc., we shall exhaust the open unit disk. In fact, *repeatedly reflecting* any *cyclic polygon in its edges exhausts the enclosed open disk!* In other words, no sequence of distinct polygons, with the kth obtained by reflecting the previous in L_k, is such that these non-intersecting chords or *lines* converge to a line L.

We equip the open disk with a *distance* preserved by all its reflections. The minimum distance between *parallel lines,* i.e., those meeting on the horizon, is zero, only that between *ultraparallel lines,* which don't meet even on the boundary, is positive. However, applying a *parallel displacement,* i.e., a product of reflections in two parallel lines, repeatedly to any one of them, gives a non-convergent pencil of parallel lines, see Figure 9(i). So we can assume that L_k and L_{k+2} are ultraparallel infinitely often. But, the distance between any such pair of lines is at least ϵ, the minimum positive distance between the finitely many edges of the given cyclic polygon, and those of its primary reflections. So, not a single such pair can exist in the region – see Figure 9(ii) – bounded by any line L and a line L' at a positive distance less than ϵ from it. *q.e.d.*





In particular, we can tile the open disk of Figure 7 this way using that "8-gon," which is pertinent for $c^2 = 5$ because, *the reflections of the "8-gon" have integral matrices,* so they are symmetries of $\mathfrak{S}_5$. For example, the edge $Y = 2$ is the bisector of the segment joining the centre to the integral point $(9, 0, 20)$ and reflection in it is given by $\begin{bmatrix} 9 & 0 & -4 \\ 0 & 1 & 0 \\ 20 & 0 & -9 \end{bmatrix}$, while $X + Y = 3$ is the bisector of the segment to $(19, 30, 30)$ and reflection in it is $\begin{bmatrix} 19 & -6 & -6 \\ 30 & -9 & -10 \\ 30 & -10 & -9 \end{bmatrix}$.

So it suffices to know $\mathfrak{S}_5$ *in the "8-gon," which is as shown in* Figure 7, the solid lines being (the restrictions of) the edges of the cells of $\mathfrak{S}_5$. For example, $X+Y = 3$ is the bisector of the segment joining $(3, 2, 6)$ and $(9, 12, 16)$, because the second matrix above interchanges these two points, alternatively, a formula given in §19 shows that the said bisector is $(X - \frac{2+12}{3+9})(2-12) + (Y - \frac{6+16}{3+9})(6-16) = 0$ which simplifies to $X + Y = 3$.

Repeated reflections in the edges of the "8-gon" propagate this local picture to the entire open disk. Thus, *the cells of* $\mathfrak{S}_5$ *are* 8*-gons or* 7*-gons,* the former compact and congruent to the central 8-gon, the latter non-compact and congruent to any of the eight 7-gons encircling it, of which one-half of each is drawn in Figure 7, and which have one vertex at infinity. So the group of symmetries of $\mathfrak{S}_5$ is no bigger than Γ_5 and is generated by the two matrices above and the obvious symmetries $x \leftrightarrow -x$, $y \leftrightarrow -y$ and $x \leftrightarrow y$. However the symmetry group of $\mathfrak{S}_5$ does *not* act transitively on its cells, there are two orbits, one with all cells 8-gons, the other with all cells non-compact 7-gons; note also that the orientation preserving symmetries act faithfully on the second orbit.

In fact, $\mathfrak{S}_p$ *is non-crystallographic for any prime* $p = 1$ *mod* 4*,* because we saw above that the central cell is compact, while on the other hand, any fundamental domain of Γ_p, though non-compact, can be shown to have finitely many integral points. However, working out the exact nature of these subdivisions for bigger primes p seems to entail progressively more and more work.

§22. Multiplying the integral points of $c^2 = 3$ by $2+\imath$ and $1+2\imath$ gives those of *the example* $c^2 = 15$; we list below all with $0 \le x_P \le y_p$ and $1 \le t_P < 14$.

$c^2 = 3$:-

t_P	1	2	4	5	7	11
x_P	0	0	3	6	0	6
y_P	0	3	6	6	12	18

$c^2 = 15$:-

t_P	1	2	4		5	7	11	
x_P	0	3	0	9	6	12	6	30
y_P	0	6	15	12	18	24	42	30

Multiplying the central 4-gon $ABCD$—here $A = (1, -1)$, etc., see Figure 5(iii)—of the subdivision $\mathfrak{S}_3$ of $\mathcal{B}_3$ by $2 + \imath$ and $1 + 2\imath$ gives us two 4-gons $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ in $\mathcal{B}_{15}$ whose intersection—see Figure 10—is the central 8-gon of $\mathfrak{S}_{15}$. But, as we'll see, these images of vertices, $A_1 = (3, -1), A_2 = (3, 1)$, etc., are *not* vertices of $\mathfrak{S}_{15}$, also, there are *no* integral points other than the centre, either within, or on the boundary of their convex hull.

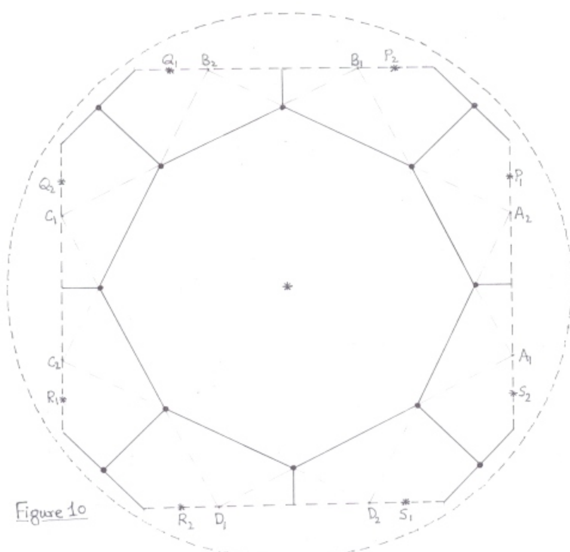


Figure 10

However, the eight integral points determining the central cell, viz., $(2, 6, 3)$, i.e., $P_1 = (3, \frac{3}{2})$, etc., are on A_1A_2 produced, etc., and a bigger 8-gon $\mathcal{F}$, with vertices $(3, 2)$, etc., serves us well. For starters, *the reflections of $\mathcal{F}$ have integral matrices,* so they are symmetries of $\mathfrak{S}_{15}$, for example, the edge $Y = 3$ is the bisector of the segment joining the centre to the integral point $(4, 0, 15)$ and reflection in it is given by $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 1 & 0 \\ 15 & 0 & -4 \end{bmatrix}$, while $X + Y = 5$ is the bisector of the segment to $(11, 30, 30)$ and reflection in it is $\begin{bmatrix} 11 & -2 & -2 \\ 30 & -5 & -6 \\ 30 & -6 & -5 \end{bmatrix}$.

Also, *iterated reflections of* $\mathcal{F}$ *tile the open disk* $\mathcal{B}_{15}$. That, by reflecting any polygon, in an edge incident to the exit point of a curve $\mathcal{C}_P$ going to a point P, we'll reach P in finitely many steps, is even easier when the vertices of the polygon are not at infinity. However, in general, the choice of the curve $\mathcal{C}_P$ to P matters, this tiling process is not well-defined. It is so in our case because, *all angles of* $\mathcal{F}$ *are right angles,* so four tiles around any vertex always bring us back to the original tile. For this note that, the integral points $(2,3,6)$ and $(4,9,12)$ are both on $Y=3$, and the right-bisector of the segment joining them is $(X-\frac{12}{6})6+(Y-\frac{18}{6})6=0$, i.e., $X+Y=5$. So the adjacent edges of $\mathcal{F}$ lying on these two lines intersect at right angles at the vertex $(2,3)$ of $\mathcal{F}$, which corresponds to the mid-point $(3,6,9)$ of this segment.

So it suffices to know $\mathfrak{S}_{15}$ *in* $\mathcal{F}$ *which is as shown in* Figure 10, the solid lines being (the restrictions of) the edges, and the black dots the vertices, of this subdivision of the open disk $\mathcal{B}_{15}$. Essentially, it only remains to confirm that there are no integral points on $\mathcal{F}$ other than the nine starred points, which can now be done more easily and without using $\mathfrak{S}_3$. The points $(X=x/t, Y=y/t)$ of the open disk with $X^2+Y^2\le r^2$ correspond to points of $-c^2t^2+x^2+y^2=-c^2$ with $1\le t\le\sqrt{\frac{c^2}{c^2-r^2}}$. So integral points in the circumcircle of $\mathcal{F}$ must satisfy $1\le t_P\le\sqrt{\frac{15}{15-13}}<3$, but these nine are the only such points.

Repeated reflections in the edges of $\mathcal{F}$ propagate this local picture to the entire open disk. Thus, *the cells of* $\mathfrak{S}_{15}$ *are* 8*-gons or* 6*-gons,* the former congruent to the central 8-gon, the latter to any of the eight 6-gons encircling it, of which one-half of each is drawn in Figure 10. The group of symmetries of $\mathfrak{S}_{15}$ is no bigger than Γ_{15} and is generated by the two matrices above and the obvious symmetries $x\leftrightarrow -x$, $y\leftrightarrow -y$ and $x\leftrightarrow y$. However the symmetry group of $\mathfrak{S}_{15}$ does not act transitively on its cells, there are *two orbits,* one with all cells 8-gons, the other with all cells 6-gons; the index 2 subgroup of orientation preserving symmetries acts faithfully on the second orbit.

§23. These examples have, as such, not much to do with the aim – §1 – with which we had set out on this journey. Yes, in §16 we showed that there exist, in all dimensions n, hyperbolic manifolds which are closed, but we have all but forgotten that extra requirement of almost-parallelizability, i.e. that, their tangent bundles be trivial outside a point-puncture, or equivalently, that they admit, *punctured immersions into* n*-space.*[(a)]

Before we turn to these, there are some bits of wisdom, picked up in the course of this journey so far, that can bear repetition at this point. For example: limiting the physical space of all inertial frames to those with speeds less than c replaces classical by relativistic physics. Likewise: limiting geometrical space to an open ball of radius c, and thinking of its chords as lines, liberates us at once from the strait-jacket of the parallel postulate. In stark contrast, from the point of view of topology, that is, if we liberate ourselves from all the postulates of geometry excepting those of continuity, the n-space and an open n-ball are equivalent, and as Sullivan stresses in [1], some remarkable constructions of topology are at heart based only on this *obvious fact.*

But, even these postulates are moot: the continuum of physical space may be something imagined by our mind around the matter of a *discontinuum.* And, cartesian simplicity suggests this discontinuum: go up one dimension to linearize the group of motions and consider all points of this space-time that are integral in our coordinates. This discrete subset makes the continuum inhomogenous, but in a controlled way: the points closest to any integral point are bounded by only finitely many hyperplanes. However, some vertices of these polytopes may be on the horizon in the relativistic case: but not, if we postulate c^2 to be a suitable integer. Strikingly, for dimensions bigger than one in the relativistic case, the discontinuum also tends to become inhomogenous: two integral points may not be related by a permutation which extends to a motion of the ambient space. But once again, this inhomogeneity is controlled: there are only finitely many such orbits or *particles.* Our closed manifolds are tied to these particles as follows: we cut down the discontinuum to one orbit, consider the associated bigger polytopal subdivision of the continuum, and roll up the same under a subgroup of extendable permutations acting faithfully on the orbit.

In the classical case, this is the familiar picture of n-space rolling up under integral translations to give the n-torus. Which is parallelizable, also some direct recipes are known for its punctured immersions in n-space, but they all use the special fact that it is the n-fold product of the circle.[(e)] We'll keep things more general by identifying the puncture with the orbit of integral points, and imagine this discontinuum as having an equivariant *force field* which is smooth on its complement, but blows up in the usual way on its matter.[(f)] This n-vector field shall give the required punctured immersion if the zeros of its jacobian are isolated, for then, there will be only finitely many of these zeros in each polytopal cell, and so these can be swallowed with the central puncture in an open n-ball which we can safely delete from our closed n-manifold.[(g,h)]

§24. The 'obvious fact' of §23 has this popular proof: contracting nonzero n-vectors at a point by the tan of their lengths gives a homeomorphism from n-space onto an open n-ball of radius $\pi/2$. The function tan of period π also pops up if we assume, for the 'force field' of §23 for the one-dimensional classical case, that the action of each point of the discontinuum $\mathbb{Z}$ is repulsive and varies inversely with the distance, and that, at any $x \in \mathbb{R} \setminus \mathbb{Z}$, these infinitely many contributions are to be added in order of proximity. This vector field gives us a function $\mathbb{R} \setminus \mathbb{Z} \to \mathbb{R}$ that is continuous, surjective, of period 1, and strictly decreasing on each component interval. In fact, for $0 < x < 1/2$, it is the sum of the convergent alternating series $\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x+2} + \cdots$, which is nothing but *Euler's partial fraction expansion of* $\pi/\tan(\pi x)$. So this periodic function is an instance of the following general definition.

Central force fields. For any n-dimensional classical or relativistic case, let each point a of the discontinuum A act repulsively with magnitude decreasing smoothly from infinity to zero as a given function μ of the distance—a distance ρ preserved by motions of n-space or open n-ball of radius c—with the force field at any x the limit, of the sum of the actions of the finitely many points of A that are within a distance d from x, as d approaches infinity; this gives us an equivariant vector field $\mu[A]$, defined at all x for which the limit exists, in particular, note that it blows up on the discontinuum.

So this vector field $\mu[A](x) = \lim_d \Sigma_a\{\mu(\rho(x,a))u(x,a) : \rho(x,a) < d\}$—where $u(x,a)$ denotes the unit vector at x along the ray from a—*is well-defined on the complement of A for any eventually zero μ.* Besides, there are also many real-analytic μ's for which the same conclusion is true, but before reviewing some of these examples, let us make another general definition.

Central potential fields. The positive function $\mu(A)(x) = \lim_d \Sigma_a\{\mu(\rho(x,a)) : \rho(x,a) < d\}$, defined at all x for which the limit exists, is likewise preserved by all motions which restrict to permutations of A, i.e., by all the symmetries of the cartesian polytopal tiling of A. Per §23, we'll assume these tiles compact, and that these symmetries act transitively on them; so there is a co-compact and torsion free subgroup Γ of finite index acting faithfully on tiles, and à fortiori our force and potential fields $\mu[A]$ and $\mu(A)$ are preserved by Γ.

The positive terms of the series defining $\mu(A)(x)$ dominate the lengths of the corresponding terms of the series defining $\mu[A](x)$, so if the first series converges the second *converges absolutely,* that is, its terms can be summed in any order we like. The domain of $\mu(A)$ is thus contained in that of $\mu[A]$, but it can be much smaller, for example, for $A = \mathbb{Z} \subset \mathbb{R}$ and $\mu(\rho) = \frac{1}{\rho}$, $\mu[A](x) = \frac{\pi}{\tan(\pi x)}\mathbf{i}$ but the domain of $\mu(A)$ is empty: its defining series $\frac{1}{x} + \frac{1}{1-x} + \frac{1}{x+1} + \frac{1}{2-x} + \frac{1}{x+2} + \cdots$ for $0 < x \leq \frac{1}{2}$ diverges because $\frac{1}{x+r-1} + \frac{1}{r-x} \geq \frac{2}{r-x} > \frac{2}{r}$, and we know that the harmonic series is divergent.

The same argument applies to any μ with $\mu(1) + \mu(2) + \cdots = \infty$, which is equivalent – using Figure 11 following Cauchy – to $\int_1^\infty \mu(x)dx = \infty$; on the other hand, *if $\int_1^\infty \mu(x)dx$ is finite* – e.g., if $\mu(\rho) = \frac{1}{\rho^k}$ with $k > 1$ – *then both $\mu(A)$ and $\mu[A]$ are defined and continuous in the complement of $A = \mathbb{Z} \subset \mathbb{R}$.*

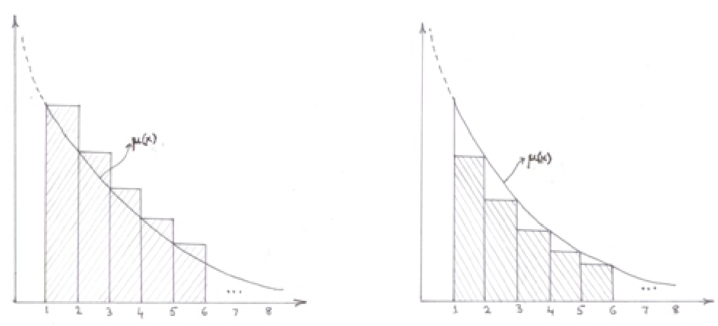


Figure 11

For, if $0 < x \leq \frac{1}{2}$, then $\mu(A)(x) = \mu(x)+\mu(1-x)+\mu(x+1)+\mu(2-x)+\cdots \leq 2\mu(x) + 2\mu(x+1) + \cdots \leq 2\mu(x) + 2\int_x^\infty \mu(x)dx < \infty$. Likewise, if we omit terms after $\mu(x+r)$, the error is less than $2\int_{x+r}^\infty \mu(x)dx < 2\int_r^\infty \mu(x)dx$ for all x, so the series *converges uniformly,* and $\mu(A)(x)$ is continuous. Regarding differentiability, if the negative derivative $-\mu'$ is also decreasing from infinity to zero, then – since its integral from 1 to infinity is $\mu(1)$, hence finite – its force and potential fields are defined on the complement of A by series converging uniformly, so the formulas $-\frac{d}{dx}(\mu(A)(x))\mathbf{i} = (-\mu')[A](x)$ and $-\frac{d}{dx}(\mu[A](x)) = (-\mu')(A)(x)\mathbf{i}$ obtained by term-by-term differentiation are valid.

Indeed, *the integral* $\int_{|x|\geq 1}\mu(|x|)dx$ *makes good sense for any n-dimensional classical or relativistic space,* if we interpret $|x|$ as the distance $\rho(x,0)$ of x from any chosen point 0, and dx as the value at x of the invariant volume form defined by ρ. And, it seems to us that, *the finiteness of this integral always ensures that both the fields $\mu(A)$ and $\mu[A]$ are defined and continuous in the complement of the discontinuum A,* that is, the orbit of the 'centre' 0 under Γ. But, in the proof given below, we'll assume also that μ decays at most exponentially, more precisely that, *the ratio $\mu(\rho-1)/\mu(\rho)$ is eventually bounded.*

Let τ_a be the polytopal tile around $a = g(0), g \in \Gamma$. It suffices to show that the defining series of $\mu(A)(x)$ for $x \in \tau_0 \setminus \{0\}$, viz., the positive series, $\Sigma_a\mu(\rho(x,a)) = \Sigma_g\mu(\rho(x,g(0))) = \Sigma_g\mu(\rho(g^{-1}(x),0)) = \Sigma_g\mu(|g(x)|)$, converges uniformly. Our compact and congruent tiles have the same finite diameter δ and volume ν, also let M_a denote the maximal distance of a point of τ_a from 0. Then ν times the series $\Sigma_{a\neq 0}\,\mu(M_a)$ is less – cf. Fig. 11 – than the integral of $\mu(|x|)$ over all the non-central tiles, which is finite, so this series is convergent. Again, $g(x)$ lies in τ_a, so its distance $|g(x)|$ from 0 is at least $M_a - \delta$, and by the decay condition there is a C such that $\mu(\rho-\delta)/\mu(\rho) < C$ eventually. So C times $\Sigma_a\mu(M_a)$ eventually dominates $\Sigma_g\mu(|g(x)|)$ which proves its convergence. Further, since error can be bounded for all x in terms of the integral of $\mu(|x|)$ over the tiles around the omitted a's, this series converges uniformly.

Not only that, an analogous reasoning using minimal distances m_a of the points of the tiles τ_a from 0, as well as $\mu(\rho+\delta)/\mu(\rho) > 1/C$ eventually, shows that, *if the integral $\int_{|x|\geq 1}\mu(|x|)dx$ is infinite, then the domain of $\mu(A)$ is empty,* but of course, the domain of the force field may be non-empty.

The integral finite condition is more and more restrictive on μ as the dimension n increases, or, for $n \geq 2$, as the radius c decreases. A calculation using polar coordinates shows that $\int_{|x|\geq 1}\mu(|x|)dx = \int_1^\infty \mu(\rho)S^{n-1}(c\sinh\frac{\rho}{c})d\rho$ where $S^{n-1}(r)$ denotes the content of a euclidean $(n-1)$-sphere of radius r : it is 2 if $n=1$, and a constant times r^{n-1} if $n \geq 2$. If $c \to \infty$, $c\sinh\frac{\rho}{c} \to \rho$, so $\mu(\rho) = \rho^{-k}$ *gives us a finite integral over euclidean n-space iff $k > n$,* and no inverse power of ρ works for all n. Or for that matter for any $n \geq 2$ if c is finite, for $c\sinh\frac{\rho}{c}$ grows like $\exp\frac{\rho}{c}$, so the integral shall diverge, but $\mu(\rho) = \rho^{-1-\epsilon}(c\sinh\frac{\rho}{c})^{-n+1}, \epsilon > 0$ *gives a finite integral over the n-ball of radius c.* Also, the decay condition is obeyed by these μ's, so by the theorem proved above, their fields are defined and continuous in the complement of A. The same is, à fortiori, true for any μ which is eventually dominated by these examples, e.g., *the fields of $\mu(\rho) = \rho^{-1}e^{-\rho^2}$ are always defined and continuous in the complement of A,* even though this is a $\mu(\rho)$ that does not obey the decay condition.

Central m-vector fields. Given any m functions $\mu_i(\rho)$ decreasing smoothly from infinity to zero and satisfying above integral convergence criterion for the n-dimensional classical or relativistic space under consideration, and any m orbits A_i of a given torsion free and co-compact discrete group Γ of motions of this n-space, $x \mapsto (\mu_1(A_1)(x), \ldots, \mu_m(A_m)(x))$ defines a Γ-invariant $\mathbb{R}^m$-valued continuous function on the complement of the m orbits. Starting with §25 we'll show how such $\mathbb{R}^n$-valued functions can be used to obtain punctured immersions of the closed smooth n-manifold of Γ-orbits into n-space.

For $n = 2$, using complex multiplication, we also get some nice non-central fields, for example, $\lim_d \Sigma_a\{(z-a)^{-k} : |z-a| < d\}$ converges absolutely and uniformly, for each integer $k > 2$, to a doubly-periodic and meromorphic, i.e., *elliptic function* $\wp_k(z)$ on $\mathbb{C}$, and the formula $\frac{d}{dz}\wp_k(z) = -k\wp_{k+1}(z)$ obtained by term-by-term differentiation is valid. In fact, averaging any rational function $Q(z)$ with $\deg(Q) < -2$ in this manner over A gives an elliptic function, because $|Q(z)|$ is less than a constant times $|z|^{-3}$ for $|z|$ large, and we saw that $\mu(\rho) = \rho^{-3}$ satisfies the integral finite condition for the plane.

On the other hand we saw that $\mu(\rho) = \rho^{-2}$ does not satisfy this condition for the plane, and its potential field $\lim_d \Sigma_a\{|z-a|^{-2} : |z-a| < d\}$ diverges at all $z \in \mathbb{C}$, nevertheless, $\lim_d \Sigma_a\{(z-a)^{-2} : |z-a| < d\}$ *also converges to an elliptic function* $\wp_2(z)$ *and* $\frac{d}{dz}\wp_2(z) = -2\wp_3(z)$. We'll use the fact that at each point of the discontinuum there is a well-defined field due to its remaining points, i.e., $G_2(A) = \lim_d \Sigma\{a^{-2} : 0 < |a| < d\}$ exists, for example, this complex number is obviously zero for lattices A that are generated by periods of equal length making an angle of 60° or 90° with each other. We shall show that $z^{-2} + \lim_d \Sigma_{a\neq 0}\{(z-a)^{-2} : |z-a| < d\} = z^{-2} + \lim_d \Sigma_a\{(z-a)^{-2} : 0 < |a| < d\}$ converges uniformly on any given compact subset of $\tau_0 \setminus \{0\}$. The last equality holds because $a \neq 0, |z-a| < d$ implies $0 < |a| < d+\delta$ and is implied by $0 < |a| < d-\delta$, so $\Sigma_{a\neq 0}\{(z-a)^{-2} : |z-a| < d\}$ differs in absolute value from $\Sigma_a\{(z-a)^{-2} : 0 < |a| < d\}$ by at most $\Sigma_a\{|z-a|^{-2} : d-\delta < |a| < d+\delta\}$, which is bounded by a constant times $d^{-2}(\pi d\delta)$. We now use the fact that $\lim_d \Sigma_a\{(z-a)^{-2} : 0 < |a| < d\} = G_2 + \lim_d \Sigma_a\{(z-a)^{-2} - a^{-2} : 0 < |a| < d\}$, and that—for z in that compact set—$|(z-a)^{-2} - a^{-2}|$ is eventually bounded by a constant times $|a|^{-3}$, so $\wp_2(z)$ has the uniformly and absolutely convergent expansion $G_2 + z^{-2} + \Sigma_{a\neq 0}((z-a)^{-2} - a^{-2})$ whose term-by-term differentiation is valid and gives us $-2z^{-3} - 2\Sigma_{a\neq 0}\{(z-a)^{-3}\} = -2\wp_3(z)$. We note also that, $\lim_d \Sigma_a\{(z-a)^{-1} : |z-a| < d\}$ *does not converge to an elliptic function,* for the integral of any elliptic function around the boundary of τ_0 is zero—because the contributions of sides paired under Γ cancel out—while Cauchy's formula tells us that such a limit function would have integral $\pm 2\pi i$.

But, for $n = 1$, $\lim_d \Sigma_a\{(x-a)^{-k} : |x-a| < d\}$ *converges to a singly periodic and meromorphic function* $\mathfrak{e}_k(x)$ *on* $\mathbb{R}$ *for all* $k \geq 1$ *and* $\frac{d}{dz}\mathfrak{e}_k(z) = -k\mathfrak{e}_{k+1}(z)$, because now $\mu(\rho) = \rho^{-k}$ satisfies the integral finite condition for $k \geq 2$, and for $k = 1$ this limit has, as above, on any compact subset of $\tau_0 \setminus \{0\}$, the uniformly and absolutely convergent expansion $\mathfrak{e}_1(x) = x^{-1} + \Sigma_{a\neq 0}((x-a)^{-1} - a^{-1})$, so $\mathfrak{e}_1(x) = \pi/\tan(\pi x)$ for the case $A = \mathbb{Z} \subset \mathbb{R}$.

So the question arises as to whether one can analogously define, on any classical or relativistic n-dimensional space, a Γ-periodic and real-analytic $\mathbb{R}^n$-valued function that has only isolated poles and critical points? However we'll first consider, in the next two sections, the somewhat easier problem in which one demands only smoothness instead of analyticity.

References
(in order of appearance)

[1] D. Sullivan, *Hyperbolic Geometry and Homeomorphisms* (HGH), in, Geometric Topology (Proceedings of the Georgia Topology Conference, Athens, August 1-12, 1977, ed. J. C. Cantrell), Academic Press (1979), pp. 543-555.

[2] K. S. Sarkaria, *"213, 16A" and Mathematics,* kssarkaria.org (2010).

[3] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds,* Springer (2006).

[4] R. Fricke, *Zur gruppentheoretischen Grundlegung der automorphen Functionen,* Math. Ann. 42 (1893), 564-594.

[5] K. S. Sarkaria, *A Topological Paradox of Motion,* Math. Intelligencer 23 (2001), 66-68.

[6] J. L. Synge, *Relativity: The Special Theory,* North-Holland (1972).

[7][3] A. F. Beardon, *The Geometry of Discrete Groups,* Springer (1982).

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[3] There seems to be no appearance of this reference but I did use this book.

Notes, Etc.[4]

(for §1)

The natural vector bundle to consider in the relativistic case is the space-time bundle. To linearize the symmetries of the open n-ball B of finite radius c, we have already gone one dimension up into space-time and identified it with the hyperboloid $-c^2t^2 + x_1^2 + \cdots + x_n^2 = -c^2$. It is the bundle E with each fiber E_x a copy of space-time which is the bundle of choice over B. Within it, is the codimension-one tangent bundle T with fibers T_x all vectors of space-time tangent at x to the hyperboloid, and $E = T \oplus N$, where N is the line bundle orthogonal to T with respect to the quadratic form $-c^2t^2+x_1^2+\cdots+x_n^2$. We've shown there exist free and co-compact discrete subgroups γ of unimodular matrices preserving this form. Dividing out the diagonal action of γ gives the corresponding bundles $E/\gamma = T/\gamma \oplus N/\gamma$ over the closed hyperbolic n-manifold B/γ. The line bundle N/γ is trivial, but the tangent bundle T/γ may not be, for example, by the Gauss-Bonnet theorem, it is non-trivial whenever n is even. *Is the space-time bundle $E/\gamma \to B/\gamma$ always trivial?* The answer was unknown when [1] was written.

However, *its pull-back to some finite cover of B/γ is trivial,* in fact any real vector bundle over a finite polyhedron, having a discrete group of unimodular matrices preserving a non-degenerate quadratic form, pulls back to a trivial bundle over a finite cover. Sullivan says this follows, because all such complex matrices have the homotopy type of all euclidean rotations, by using some étale homotopy theory as in his paper with Deligne [DS]; also that it is not true, see Millson [Mi], for all discrete groups of real unimodular matrices.

So, by replacing B/γ by this finite cover, *let $E/\gamma \to B/\gamma$ be a trivial bundle, then T/γ is almost trivial.* Choose any continuous basis $e_1(x),\ldots,e_{n+1}(x)$ of the fibers of the trivial bundle E/γ and identify sections with their coefficient maps $B/\gamma \to \mathbb{R}^{n+1}$ with respect to it. Let $v(x)$ be an identically nonzero section of the trivial line bundle N/γ. We recall that the first column map $[v, v_1, \ldots, v_n] \mapsto v$ from $GL(n+1,\mathbb{R})$ to $\mathbb{R}^{n+1} \setminus \{0\} \simeq S^n$ has the covering homotopy property, and that, if we puncture each top simplex of a triangulation of B/γ once, the complement retracts to its $(n-1)$-skeleton which maps into S^n inessentially. So, on this complement, E/γ has a continuous basis $v(x), v_1(x), \ldots, v_n(x)$, with the $v_i(x)$'s inducing a continuous basis of $(E/\gamma \bmod N/\gamma) \cong T/\gamma$. The punctures lie in a single n-ball, so T/γ is trivial in the complement of any point.

(for §2)

The result that there are no convex regular tilings of hyperbolic n-space for $n > 4$ is in [Co 1] who seemingly attributes it to Schlegel ? Also compare Vinberg's 'crystallographic', is it just same plus convexity of tiles ? If so his work shows there are no such for $n \geq 30$.

(for §4)

[4] These have been severely pruned and Etc. is now redundant.

Open disk with usual segments simplest way to break fifth postulate. What about other open convex sets, esp., interiors of polytopes, these geometries pop up later in cells of tilings.[5]

(for §6)

Its amazing how quickly above straight lines in open ball geometry ties with relativity under linearization; the first part about Galilean frames and classical relativity was also formulated by Poincaré – see Arnol'd.

(for §7)

A name theorem – Dieudonné ? – is proved jlt : reflections, right bisectors, and not passing to orientation preserving subgroups kept algebra less fierce.

(for §8)

Last line – Bieberbach lemma – is usually via eigenvalues, but it was done differently in *Extracts.*

(for §9)

Question corrected a little may be an open conjecture of Thurston ? The invocation of Gauss Bonnet, a volume invariant, is premature – infinitesimal distance is still way off, but in §17 we got to separation .. how do the two connect? For the complex analogues another volume invariant, 'Dirac one' so signature, proves analogue of Sullivan's result is false here, see HGH.

(for §11)

Venkataramana's notes [Ve] helped a lot in clarifying Sullivan's hints. The assertions made about the columns and the determinant of the matrices of G can be seen thus. Any quadratic form can be written as $X'QX$ where Q is a symmetric matrix and $X' = (\ldots, x_i, \ldots)$. In our case $Q = \text{diag}(-\text{c}^2, 1, \ldots, 1)$. Saying that a linear substitution $X = PY$ of variables preserves the quadratic form is the same as saying that $Q = P'QP$. Now note that the (i,j)th element of $P'QP$ is $C_i'QC_j$ where C_k denotes the kth column of P. So, in our case, $C_i'QC_j = 0$ if $i \neq j$ while $C_i'QC_i = -c^2, 1, \ldots, 1$. That the determinant of P is ± 1 follows by taking the determinants of both sides of $Q = P'QP$.

(for §12)

A whole number is a *square* iff each prime occurs an even number of times. Fermat showed that it is a sum of *two squares* iff each prime equal to 3 mod 4 occurs an even number of times in it; this implies it is of the type $2^a(4b+1)$ but not conversely. However, Gauss showed that a whole number is a sum of *three squares* iff it is not of the type $4^a(8b-1)$, e.g., if m is not divisible by 4 then m or $m-1$ is a sum of three squares, which is stronger than the earlier result of Lagrange that any whole number is the sum of *four squares,* cf. [Se].

[5] In fact I've been subjecting Keerti and others to verbal versions of the opening paragraph of *PG&R* (2013) since at least the mid-1990's !

(for §13)

That factor got pictorially explained thus from jail-book [Co 2], in [Ve] it is called Minkowski reduction, but all this is definitely in Hermite too.

(for §14)

The conditional – strip, a most important picture – discreteness of $\mathbb{Z}[\sqrt{2}]$ in $\mathbb{R}$ became clear from a remark (first sentence in proof on next section in [Fr-K]'s last chapter, paper reference later from Shimura) clarified [Ve].

(for §15)

This section is the high point so far fulfilling Sullivan's assurance only the irrationality of 2 is needed. The one-line proof of Selberg's lemma came from looking at an argument in class notes of someone (?) on the web.

(for §17)

What we called *cartesian subdivision* is often called *Voronoi* subdivision in honour of the substantial use he made of what he called *Dirichlet* subdivision, the discrete subset being now a lattice of euclidean space. But long before them, these subdivisions had figured – the definition is so natural its origin is probably still older – prominently in Descartes' *theory of vortices* [De] a pre-newtonian attempt to give a conceptual description of the cosmos ...[6]

(for §19)

Regarding the computation of page 14, *the machine halts after precisely thirty-four steps at* $20+14\sqrt{2}$*.* This I owe to Keerti, who re-did this computation on *Mathematica.* He found it easier to compute the list of conjugates, in each step adding 1 if the number so obtained is less than $2^{\frac{1}{4}}$, and subtracting $\sqrt{2}$ if the number so obtained is more than $-2^{\frac{1}{4}}$. Likewise, he also found the first 200 outputs of the non-halting computation for the strip of width $1+\sqrt{2}$.

(for §20)

Our definition of crystallographicity of $\mathfrak{S}$ is not stronger than the usual one: any symmetry preserving the tiling automatically has to preserve the integral points. This because centre is by central symmetry of our discrete set, the intersection of any two diagonals of this tile, this geometric property is preserved by the symmetry, thus integral points.

[6] There was much more here that I've omitted, and I've omitted entirely the very long notes to the speculative §23 and §24. As *PG&R* and sequels show cartesian philosophy does give us an intuitive, useful and cohesive picture of a large part of mathematics, but to define the subset which will be seen in an experiment obviously requires more information.

(for §21)

Brahmagupta of Bhillamala (598-668 A.D.) had found integral points on hyperbolas via $(a^2 + Nb^2)(c^2 + Nd^2) = (ac - Nbd)^2 + N(ad + bc)^2$. Complex multiplication and multiplication of matrices are implicit in his approach, and to my mind, a lot of hyperbolic geometry seems to be nothing but Brahmagupta's method generalized to higher dimensions ! Besides this improvement on the work of *Diophantus of Alexandria* (c. 200-284 A.D.) this mathematician was apparently the first to have defined the number zero—thus the witticism: *the contribution of India to mathematics is zero!*—and a beautiful formula for the area of a cyclic 4-gon in terms of the lengths of its sides bears his name. There has been much interest recently in generalizing this formula to cyclic n-gons. See also [Sa 1, 2] and pp. 14-15 of *"213, 16A."*

Brahmagupta's identity $(a^2+b^2)(c^2+d^2) = (ac-bd)^2+(ad+bc)^2$ implies the planar Bunyakowski-Cauchy-Schwarz inequality $(ad+bc)^2 \leq (a^2+b^2)(d^2+c^2)$; more generally, an identity of Lagrange precises the general Schwarz inequality. In [Sa 3], we had shown that this identity also implies *an "odd" Schwarz inequality;* and much before, in [Sa-Z], we had discussed an interesting Cauchy-Schwarz inequality for *vector-valued* bilinear forms.

Vertices of $\mathfrak{S}_1$ = all primitive pythagorean triples = all rational points on unit circle. This plus argument used for 5 might show for any prime 1 mod 4 that fundamental domain of Γ contains finitely many integral points.

Additional References in Notes

[Co 1] H. S. M. Coxeter, *Regular honeycombs in hyperbolic space,* Proceedings ICM Edinburgh (1954).

[Co 2] —do— , *Introduction to Geometry,* Wiley (1969).

[DS] P. Deligne and D. Sullivan, *Fibrés vectoriel complexes à groupes structurel discret,* C. R. Acad. Sci. Paris 281 (1975), 1081-83.

[De] R. Descartes, *Principia Philosophiae,* Amsterdam (1644).

[Fr-K] R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Funktionen,* Leipzig (1897).

[Mi] J. J. Millson, *On the first Betti number of a constant negatively curved manifold,* Ann. of Math. 104 (1976), 235-247.

[Sa 1] K. S. Sarkaria, *Amartya Sen, "The Argumentative Indian" : some impressions,* 3 pp., kssarkaria.org (2005).

[Sa 2] —do— , *Extracts from my notebooks,* 28 pp., kssarkaria.org (2008).

[Sa 3] —do— , *An "odd" Schwarz inequality,* Research Bulletin of the Panjab University, Science 47 (1997) pp. 215-217.

[Sa-Z] K. S. Sarkaria and S. M. Zoltek, *Positively curved bilinear forms,* Proceedings of the American Mathematical Society 97 (1986) pp. 577-584.

[Se] J.-P. Serre, *A Course in Arithmetic,* Springer (1973).

[Ve] T. N. Venkataramana, *Lattices in Lie Groups,* ICM Hyderabad satellite Goa conference notes (2010).