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KNESER COLORINGS OF POLYHEDRA

BY

K.S. SARKARIA

1. Introduction

(1.1) It is well known that if a 1-dimensional simplicial complex, i.e., a "graph", K^1, embeds in a 2-dimensional manifold M^2, then its chromatic number is less than a certain constant c, which depends only on the topology of M^2. We have proved elsewhere various generalisations of this result which apply to higher dimensional simplicial complexes K^n: see [18], [19] and [20].

In this paper we turn things around and show that if a simplicial complex K^n can be suitably colored by not too many colors, then it p.l. embeds in a given $\mathbf{R}^m$. As typical specimens of such results we have the following two:

THEOREM 2 (2.5.1). *Let $G(K^n_\square)$ denote the graph whose vertices are pairs (v,θ) where v is a vertex of K^n and θ a maximal simplex of K^n not containing v, with (v_1,θ_1) adjacent to (v_2,θ_2) iff $v_1 \in \theta_2$ and $v_2 \in \theta_1$. If $G(K^n_\square)$ has chromatic number $\leq m+1$ and $2m \geq 3(n+1)$, or else $n=1$ and $m=2$, then K^n p.l. embeds in $\mathbf{R}^m$.*

THEOREM 6 (3.2.1). *Let $G(X^n)$ denote the graph whose vertices X^n_i are closures of the non-singular edge-less components of the underlying polyhedron X^n of K^n, with X^n_i adjacent to X^n_j iff X^n_i is disjoint from X^n_j. If $G(X^n)$ is bichromatic and $n \neq 2$ then K^n p.l. embeds in $\mathbf{R}^{2n}$.*

Note that Theorem 6 above includes the well known fact that an n-pseudo-manifold p.l. embeds in $\mathbf{R}^{2n}$. The hypotheses of this theorem are relaxed considerably in Theorem 8 (3.4.2) whose statement involves some equivariant cohomology.

In Theorem 2 above, $K^n_\square$ denotes a self-dual poset, *the dual deleted product*, which we associate canonically to each simplicial complex K^n. Theorems 3 and 4 of (2.5) are analogues of Theorem 2 for graphs $G(K^n_\Sigma)$ arising out of some sub self-dual posets K^n_Σ of $K^n_\square$. Theorem 3 is in fact a common generalization of Theorem 2 above and the Lovász-Kneser Theorem [12] which appears in this setting only as a very special colorability implies embeddability

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theorem. We thus obtain, besides the classical Kneser graphs, a host of new examples of highly chromatic graphs which are triangle-free (and much more).

(1.2) *Method of proof*.

There are two main ingredients involved in our proofs:

The first of these is the idea, going back at least to van Kampen [25], 1932, of using the *deleted product* K_*^n of a simplicial complex K^n. Recall that K_*^n is the cell complex consisting of all cells of the type $\sigma \times \theta$ where σ and θ are disjoint simplices of K^n. One equips K_*^n with the involution $(x, y) \mapsto (y, x)$. The deleted product is important in Embedding Theory because embeddings $f\colon K^n \to \mathbb{R}^m$ yield $\mathbb{Z}_2$-maps $f_*\colon K_*^n \to S^{m-1}$ by

$$f_*(x, y) = \frac{f(x) - f(y)}{|f(x) - f(y)|}.$$

Conversely, Weber's Theorem [28], 1967, tells us that for $2m \geq 3(n+1)$ the existence of such an f_* implies the existence of an embedding f.

The second idea came to the fore with Lovász's proof [12], 1978, of Kneser's Conjecture [11], 1955, viz. the idea of *using a coloring to construct a suitable $\mathbb{Z}_2$-map*. We do not however use Lovász's "neighborhood complexes" and instead, taking our cue from a subsequent paper [13] of Lovász on "strongly self-dual polytopes", find it convenient to work with *self-dual posets*, i.e., finite partially ordered sets equipped with non-degenerate order reversing involutions. This idea ties up with deleted products because the latter are self-dual posets provided one uses the opposite order on the second factor.

(1.3) *Summary*.

In (2.1) to (2.3) we review the basic definitions. Then we prove Theorem 1 (2.4.2): *The graph of a self-dual poset has chromatic number $\leq k+2$ only if there is a $\mathbb{Z}_2$-map from its space to S^k.* In (2.5) we prove the aforementioned Theorems 2, 3 and 4 pertaining to the self-dual poset $K_\square^n$ and its subposets K_Σ^n. In particular we obtain *a new proof for the Lovász-Kneser Theorem*. Then we show in (3.1) that an "obvious" generalization of the classical Kneser graphs is not of much interest. In (3.2) we give a direct proof of Theorem 6 by using the van Kampen cone construction.[1] In (3.3) we review the equivariant cohomology of fixed point free $\mathbb{Z}_2$-spaces. This is used in (3.4.1) to obtain some corollaries of Weber's Theorem, e.g., that *for $n \neq 2$ a polyhedron X^n unknots in $\mathbb{R}^{2n+1}$ only if it embeds in $\mathbb{R}^{2n}$*. Next we state and prove Theorem 8,

[1]This is one of the two main constructions needed in Embedding Theory. The other is the Whitney Trick [29].

the aforementioned stronger form of Theorem 6, using Weber's Theorem. To do this we introduce a certain interesting sub $\mathbb{Z}_2$-polyhedron of the deleted product which we call the *Kneser deleted product* (3.4.2). For some combinatorial purposes *deleted joins* $K^n_{\#}$—which go back at least to Flores [4], 1933—are more convenient than deleted products K^n_*. We use this idea in (3.5) to give a polynomial bound for the codimension one least valence of K^n in terms of the "extent" to which K^n can p.l. knot in $\mathbb{R}^{2n+1}$. We conclude by pointing out some evidence which tends to support the conjectures made in [19].

Since we are going to stick to the piecewise linear category *the adjective p.l. is quite often omitted*. Unless otherwise stated all spaces, maps, embeddings, homeomorphisms, etc., should be assumed p.l. (The main exception will be the use of some projection maps. These too can be replaced by p.l. maps by means of the standard way of getting around "the standard mistake"; e.g., see [10], pp. 20–21.)

2. Self-dual posets

(2.1) *Posets, complexes, spaces.*

(2.1.1) We will deal mainly with the following categories.

(A) Finite partially ordered sets or POSETS and functions between them which are MONOTONE, i.e., order preserving or order reversing.

(B) Finite abstract simplicial COMPLEXES[2] and functions between them which are SIMPLICIAL.

(C) Piecewise linear SPACES and MAPS between them which are piecewise linear.

The isomorphisms of category (C) are called piecewise linear *homeomorphisms*. We say that X_1 *embeds* piecewise linearly in an X_2 if X_1 is homeomorphic to a closed subspace $\tilde{X}_1$ of X_2.

(2.1.2) Also we will need the corresponding equivariant categories for the group $\mathbb{Z}_2$:

A $\mathbb{Z}_2$*-object* is an object equipped with an involutive morphism and between $\mathbb{Z}_2$-objects one considers $\mathbb{Z}_2$*-morphisms*, i.e., those which commute with the involutions.

(2.2) *Space of a poset.*

(2.2.1) We have the following functors between the above categories.

(B) $\overset{\subseteq}{\to}$ (A). Each simplicial complex is considered as a poset under the partial order $\subseteq$. This makes (B) into a subcategory of (A).

(A) $\overset{\prime}{\to}$ (B). The *derived* functor associates to each poset P the simplicial complex P' of *chains* (i.e., totally ordered subsets) of P.

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[2] Our simplices will be *non-empty* finite sets.

(B) $\xrightarrow{|\cdot|}$ (C). This functor associates to each abstract simplicial complex K the piecewise linear space $|K|$ underlying the corresponding geometrical simplicial complex obtained by thinking of each vertex v as the v-*th* unit vector of the vector space $\mathbb{R}^{\mathrm{vert}\, K}$.

A simplicial complex K is called a piecewise linear *triangulation* of the compact piecewise linear space X if $|K|$ is p.l. homeomorhpic to X.

(2.2.2) A simplicial complex K *embeds rectilinearly* in a real vector space V if one has a linear map

$$\mathbb{R}^{\mathrm{vert}\, K} \xrightarrow{\eta} V$$

which is one-one on $|K|$.

It is known that each compact piecewise linear space X occurs as the image $\eta(|K|)$ of some such rectilinear embedding. (Such a K may be called a *rectilinear triangulation* of X.) Likewise each piecewise linear map $f\colon X_1 \to X_2$ between compact spaces can be written as $\eta_2|\varphi|\eta_1^{-1}$ for suitable choices of K_1, K_2, η_1, η_2, and simplicial map $\varphi\colon K_1 \to K_2$.

Examples of Cairns [3] and van Kampen [26] show that the p.l. embeddability of a $|K|$ in V does not guarantee the rectilinear embeddability of K in V.

(2.2.3) Any homeomorph of $|P'|$ will be denoted by X_P and said to be the *space of the poset* P. Note that for any simplicial complex K, K' can be identified with the first barycentric derived of K and so $|K'|$ is homeomorphic to $|K|$. Hence the space of any poset is homeomorphic to that of its derived.

By a *cell* we understand the convex hull of a non-empty finite subset of a vector space V. A finite set P of cells of V constitutes a *cell complex* if (1) the relative interiors of any two distinct cells are disjoint and (2) the relative boundary of any i-dimensional cell σ, $i > 0$, is a union of cells of dimensions $< i$: these, and σ itself, are called the *faces* of σ. We make P into a poset by letting $\theta \leq \sigma$ iff θ is a face of σ. If P is a cell complex P' can once again be identified with the simplicial complex arising as the first barycentric derived of P and so the space of the poset P is homeomorphic to the subspace $|P|$ of V covered by the cell complex P.

Note that a cell complex is a geometrical simplicial complex iff it is isomorphic as a poset to an abstract simplicial complex.

(2.3) *Deleted products, Weber's Theorem.*

(2.3.1) The cartesian product $X \times X$ of a space with itself shall be equipped with the *involution* s which switches the coordinates: $s(x_1, x_2) = (x_2, x_1)$. The *deleted product* X_* of X is the $\mathbb{Z}_2$-subspace of $X \times X$ consisting of all pairs (x_1, x_2), $x_1 \neq x_2$. Each embedding $\varphi\colon X \to Y$ induces a $\mathbb{Z}_2$-map $\varphi_*\colon X_* \to Y_*$ by

$$\varphi_*(x_1, x_2) = (\varphi(x_1), \varphi(x_2)).$$

Note that if an embedding φ_0 is *isotopic* to φ_1 via the embeddings $\varphi_t, 0 \le t \le 1$, then φ_{0*} is $\mathbb{Z}_2$-homotopic to φ_{1*} via the $\mathbb{Z}_2$-maps $\varphi_{t*}, 0 \le t \le 1$.

Any space p.l. homeomorphic to an m-dimensional cell (resp. its boundary) is called an *m-ball* B^m (resp. $(m-1)$*-sphere* S^{m-1}). The 0-sphere S^0 consists of 2 points and has the involution which interchanges the 2 points. We will equip S^k, the $(k+1)$-fold join of S^0, with the *antipodal involution*, i.e., the $(k+1)$-fold join of the involution of S^0.

LEMMA 1. *There is a strong $\mathbb{Z}_2$-deformation of the deleted product of an m-dimensional Euclidean space, or of an m-ball, onto an $(m-1)$-sphere.*

Proof. Consider the orthogonal projection of $\mathbb{R}^m \times \mathbb{R}^m$ onto the orthogonal complement

$$\Delta^\perp = \{(v, -v) : v \in \mathbb{R}^m\}$$

of the diagonal subspace

$$\Delta = \{(v, v) : v \in \mathbb{R}^m\}.$$

Thus each isotopy

$$X \xrightarrow{\varphi_t} \mathbb{R}^m$$

determines a $\mathbb{Z}_2$-homotopy

$$X \xrightarrow{\varphi_{t*}} S^{m-1}.$$

(2.3.2) The following result illustrates the importance of the deleted product functor.

WEBER'S CLASSIFICATION THEOREM. *If $2m > 3(n+1)$ (resp. $2m = 3(n+1)$), and X^n is compact, then $\varphi \mapsto \varphi_*$ sets up a bijective (resp. surjective) correspondence between isotopy classes of embeddings of X^n in $\mathbb{R}^m$ and $\mathbb{Z}_2$-homotopy classes of $\mathbb{Z}_2$-maps $X^n_* \to S^{m-1}$.*

This theorem is due to Weber [28], 1967; the surjectivity of $[\varphi] \mapsto [\varphi_*]$ in the special case $m = 2n, n \ge 3$, was conjectured (and partly proved) by van Kampen [25], 1932, and proved, independently, by Wu [31], 1956, and Shapiro [22], 1957. A theorem exactly analogous to Weber's is valid for smooth embeddings of smooth n-manifolds in $\mathbb{R}^m$ and was proved by Haefliger [8], 1962.

(2.3.2a) The results of van Kampen-Wu-Shapiro were stated differently in terms of the vanishing of some cohomology classes (see (3.4.1) below). It was Haefliger who reformulated this vanishing condition into an equivalent one involving the existence of a $\mathbb{Z}_2$-map: this quickly led to the aforementioned stronger results of Haefliger and Weber. We note the following *addendum* to Weber's Theorem for the case $n = 1$, $m = 2$.

A GRAPH PLANARITY CRITERION. *A graph X^1 is planar iff there is a $\mathbb{Z}_2$-map from X^1_* to S^1.*

This surprisingly little known result must have been known to van Kampen by 1932, but the only place where I could find it explicitly stated (in its cohomological formulation) is p. 210 of Wu's book [32]. One can prove it directly without defining the van Kampen Obstruction as follows.

It is easily seen that there is a $\mathbb{Z}_2$-map from X^1_* to S^1 iff there is a $\mathbb{Z}_2$-map from the deleted join (see (3.5.1) below) $X^1_{\ast}$ to S^2. But the deleted joins of the Kuratowski Graphs are $\mathbb{Z}_2$-homeomorphic to S^3 (see (3.5.2)). Hence by the Borsuk-Ulam Theorem, X^1 cannot have a subspace homeomorphic to a Kuratowski Graph, and so must be a planar graph.

We will show elsewhere that one can turn things around and prove this criterion without using Kuratowski's Theorem, and then deduce the latter from it. By using Weber's Theorem such techniques also yield some higher dimensional generalizations of Kuratowski's Theorem: see [34].

(2.3.3) For any poset P (and in particular for simplicial complexes) we equip $P \times P$ with the product partial order, $(a_1, a_2) \leq (b_1, b_2)$ iff $a_1 \leq b_1$, $a_2 \leq b_2$, and the involution s which switches the coordinates, $s(a_1, a_2) = (a_2, a_1)$. If K is a simplicial complex its *deleted product* K_* will be the $\mathbb{Z}_2$-subposet of $K \times K$ consisting of all pairs (σ_1, σ_2), $\sigma_1 \cap \sigma_2 = \varnothing$. Note that K_* can be considered as a $\mathbb{Z}_2$-cell complex covering a compact portion of the deleted product of the space X_K.[3] In fact one can say more.

PROPOSITION 1. *For any simplicial complex K, X_{K_*} is a strong $\mathbb{Z}_2$-deformation retract of $(X_K)_*$.*

Thus the $\mathbb{Z}_2$-homotopy type of the space of K_* depends only on the homeomorphism type of the space of K. This result is due to Wu [30]. (Or else see [32] or [9]. Note that the argument on p. 257 of [22] is flawed because $\beta(p, q)$ does not vary continuously with p and q.)

(2.4) *Self-dual posets.*

(2.4.1) A $\mathbb{Z}_2$-poset (P, ν) is called a *self-dual poset* if its involution ν is order reversing and non-degenerate, i.e., if $a \nleq \nu(a)$ for $a \in P$.

The *ν-product* P_ν of a self-dual poset (P, ν) is the $\mathbb{Z}_2$-subposet of $P \times P$ consisting of all pairs (a, b), $a \leq \nu(b)$.

[3]In this context we will denote the cells of K^n_* by $\sigma_1 \times \sigma_2$ rather than by (σ_1, σ_2).

LEMMA 2. *For any self-dual poset (P, ν), X_P and X_{P_ν} are $\mathbb{Z}_2$-homotopy equivalent.*

Proof. Define θ: $P_\nu' \to P'$ by associating to each chain $(a_1, b_1) < \cdots < (a_n, b_n)$ of P_ν the chain $a_1 \leq \cdots \leq a_n \leq \nu(b_n) \leq \cdots \leq \nu(b_1)$ of P. Though it is not simplicial θ is surjective, inclusion preserving and commutes with the simplicial involutions s': $P_\nu' \to P_\nu'$ and ν': $P' \to P'$. For each simplex

$$\sigma^m = \{c_0 < c_1 < \cdots < c_m\}$$

of P', the subcomplex σ_m^m of all faces of σ^m has as its pullback the subcomplex $\theta^{-1}(\sigma_m^m)$ of P_ν' consisting of all chains of the form

$$(c_{i_0}, \nu(c_{j_0})) < \cdots < (c_{i_r}, \nu(c_{j_r}))$$

where $0 \leq i_0 \leq \cdots \leq i_r \leq j_r \leq \cdots \leq j_0 \leq m$. Since $\theta^{-1}(\sigma_m^m)$ is a cone over the vertex $(c_0, \nu(c_m))$ its space is contractible. Thus a $\mathbb{Z}_2$-section Φ: $X_{P'} \to X_{P_\nu'}$ of the $\mathbb{Z}_2$-map $|\theta'|$: $X_{P_\nu'} \to X_{P'}$ can be constructed by an upward induction on the m-skeletons $(X_{P'})^m = \bigcup_m X_{\sigma_m^m}$. This Φ is the requisite homotopy inverse of $|\theta'|$.

(2.4.2) The *graph of a self-dual poset* $G(P, \nu)$ is the graph (i.e., 1-dimensional simplicial complex) whose vertices are the minimal elements of P with 2 vertices a and b joined iff $a \leq \nu(b)$. The chromatic number of this graph will also be referred to as the chromatic number of the self dual poset.

THEOREM 1. *A self-dual poset P has chromatic number $\leq k + 2$ only if there is a $\mathbb{Z}_2$-map from X_P to S^k. Thus if the space X_P of a self-dual poset P is k-connected then its chromatic number is at least $k + 3$.*

Proof. Let P_0 denote the subset of P consisting of the minimal elements and let φ: $P_0 \to \{1, 2, \ldots, k+2\}$ be a function such that $a \leq \nu(b)$ implies $\varphi(a) \neq \varphi(b)$. Let σ_{k+1}^{k+1} denote the simplicial complex whose simplices are all the non-empty subsets of $\{1, 2, \ldots, k+2\}$. We can define a $\mathbb{Z}_2$-monotone function

$$f: P_\nu \to (\sigma_{k+1}^{k+1})_*$$

by

$$(a, b) \mapsto (\varphi(A), \varphi(B))$$

where A (resp. B) denotes the subset of all minimal elements $c \leq a$ (resp. $c \leq b$). This follows because if $a_0 \in A$, $b_0 \in B$, then $a_0 \leq a \leq \nu(b) \leq \nu(b_0)$ and so $\varphi(a_0) \neq \varphi(b_0)$: thus $\varphi(A) \cap \varphi(B) = \varnothing$. Since σ_{k+1}^{k+1} triangulates a $(k+1)$-disk its deleted product has, by Proposition 1, the $\mathbb{Z}_2$-homotopy type of the deleted product of a $(k+1)$-disk, and so, by Lemma 1, that of the k-sphere S^k. (In fact it is easy to see directly that $|(\sigma_{k+1}^{k+1})_*|$ is $\mathbb{Z}_2$-*homeomorphic* to S^k.) On the other hand Lemma 2 gives us a $\mathbb{Z}_2$-map from X_P to X_{P_*}. Thus we get a $\mathbb{Z}_2$-map from X_P to S^k.

To see the second part note that under the given connectedness hypothesis one has a $\mathbb{Z}_2$-map $S^{k+1} \to X_P$: for each $0 \leq r \leq k$ suitably extend the $\mathbb{Z}_2$-map $S^r \to X_P$ from the equator S^r of S^{r+1} to the northern and southern hemispheres. If chromatic number were less than $k+3$ we would have a $\mathbb{Z}_2$-map $S^{k+1} \to S^k$. This contradicts Borsuk's Theorem [2].

The following cases of Theorem 1 are due to Lovász [12], [13], and Walker [27]: (1) P is a "strongly self-dual polytope"; (2) P is "the proper part $\bar{\mathscr{L}}(G)$ of the ortholattice $\mathscr{L}(G)$ of a graph G": such an $\bar{\mathscr{L}}(G)$ has the same homotopy type as $\mathscr{N}(G)$, "the neighborhood complex of the graph G".

(2.5) *Dual deleted product.*

(2.5.1) If we equip the set

$$\{(\sigma_1, \sigma_2) : \sigma_1 \in K, \sigma_2 \in K, \sigma_1 \cap \sigma_2 = \varnothing\}$$

of all ordered pairs of disjoint simplices of K with the involution s and the partial order $(\sigma_1, \sigma_2) \preccurlyeq (\theta_1, \theta_2)$ iff $\sigma_1 \subseteq \theta_1, \theta_2 \subseteq \sigma_2$ (instead of the product partial order of (2.3.3)) then we get a self-dual poset $K_\square$ which will be called the *dual deleted product* of K.

THEOREM 2. *If $G(K_\square^n)$ has chromatic number $\leq m+1$, (here $m > 0$), then there is a $\mathbb{Z}_2$-map $(X_{K^n})_* \to S^{m-1}$. Thus if further $2m \geq 3(n+1)$, or else $n = 1$ and $m = 2$, then X_{K^n} embeds in $\mathbb{R}^m$.*

Proof. The second part will follow from Weber's Theorem. To prove the first part we note that by Proposition 1 it suffices to find a $\mathbb{Z}_2$-map

$$X_{K_*^n} \to S^{m-1}.$$

Under the given chromatic hypothesis Theorem 1 supplies us with a $\mathbb{Z}_2$-map

$$X_{K_\square^n} \to S^{m-1}.$$

Thus the result follows from the following lemma.

LEMMA 3. *For any simplicial complex K, $X_{K_\square}$ has the same $\mathbb{Z}_2$-homotopy type as X_{K_*}.*

Proof. Let $X = |K|$ and identify K with the geometrical simplicial complex covering X and thus $K_* \subset K \times K$ with the cell complex consisting of all cells $\sigma \times \theta, \sigma \in K, \theta \in K, \sigma \cap \theta = \varnothing$. The space $|K_*|$ covered by K_* is thus a compact $\mathbb{Z}_2$-subspace Y of the deleted product, $X_* \subset X \times X = |K| \times |K|$, of X.

Choose a barycenter $\hat{\sigma}$ in each $\sigma \in K$ and think of K' as the barycentric derived of K, i.e., each $\sigma' \in K'$ which is a chain $\sigma_0 \subsetneq \cdots \subsetneq \sigma_k$ of K, is thought of as the geometrical simplex which is the convex hull $[\hat{\sigma}_0, \dots, \hat{\sigma}_k]$. Let us now cut up the cells $\sigma \times \theta$ of $K \times K$ into smaller cells of the form $\sigma' \times \theta', \sigma' \in K', \theta' \in K', \sigma' \subseteq \sigma, \theta' \subseteq \theta$. This gives us the cell complex $K' \times K'$; we denote by C the subcomplex of $K' \times K'$ covering $|K_*|$.

Consider the cells $\sigma' \times \theta'$ of C,

$$\sigma' = [\hat{\sigma}_0, \dots, \hat{\sigma}_k], \quad \sigma_0 \subsetneq \cdots \subsetneq \sigma_k,$$
$$\theta' = [\hat{\theta}_0, \dots, \hat{\theta}_l], \quad \theta_0 \subsetneq \cdots \subsetneq \theta_l.$$

We can further cut them up into simplices

$$\left[\left(\hat{\sigma}_{i_0}, \hat{\theta}_{j_0}\right), \dots, \left(\hat{\sigma}_{i_r}, \hat{\theta}_{j_r}\right)\right], \quad 0 \le i_0 \le \cdots \le i_r \le k, 0 \le j_0 \le \cdots \le j_r \le l.$$

This gives us a simplicial complex isomorphic to K_*'. Alternatively we can cut them up into the simplices

$$\left[\left(\hat{\sigma}_{i_0}, \hat{\theta}_{j_0}\right), \dots, \left(\hat{\sigma}_{i_r}, \hat{\theta}_{j_r}\right)\right], \quad 0 \le i_0 \le \cdots \le i_r \le k, l \ge j_0 \ge \dots j_r \ge 0.$$

This gives us a simplicial complex isomorphic to a subcomplex of $K_\square'$. Thus $|K_*'| = Y$ and $|K_\square'| \supseteq Y$ (this already suffices to complete the proof of Theorem 2). Figure 1 shows an example where $|K_\square'|$ is not homeomorphic to Y.

To get an isomorph $\tilde{K}_\square'$ of $K_\square'$ we have thus to consider also the simplices

$$\left[\left(\hat{\sigma}_{i_0}, \hat{\theta}_{j_0}\right), \dots, \left(\hat{\sigma}_{i_r}, \theta_{j_r}\right)\right],$$

$0 \le i_0 \le \cdots \le i_r, l \ge j_0 \le \cdots \ge j_r, \sigma_{i_0} \cap \theta_{j_0} = \cdots = \sigma_{i_r} \cap \theta_{j_r} = \varnothing$, arising in the above fashion, within cells $\sigma \times \theta$ of $K \times K$ which are *not* in K_*. Since $\sigma \cap \theta \neq \varnothing$, note that any such simplex of $K_\square'$ cannot have the barycenter $(\hat{\sigma}, \hat{\theta})$ of (σ, θ) as a vertex. For each cell $\sigma \times \theta$ of $(K \times K) - K_*$ let $r_{\sigma \times \theta}$ denote the radial deformation of $(\sigma \times \theta) - (\hat{\sigma}, \hat{\theta})$ from the barycenter $(\hat{\sigma}, \hat{\theta})$ towards the boundary $\partial(\sigma \times \theta)$. *This deformation* $r_{\sigma \times \theta}$ *maps* $|K_\square'| \cap |\sigma \times \theta|$ *into itself*. To see this we note that a simplex of above type lies outside $\partial(\sigma \times \theta)$ iff $\theta_{j_0} = \theta$ and $\sigma_{i_r} = \sigma$ and that under the deformation it moves over the region given as the convex hull of the vertices of the simplex and the points $(\hat{\sigma}_{i_k}, \hat{\theta}_{j_l}), 0 \le k < l \le r, \theta_{j_k} = \theta, \sigma_{i_l} = \sigma$ (cf. shaded areas in Figure 1). Since $\sigma_{i_k} \times \theta_{j_l} \in K_*$, it is easy to check that this region is covered by simplices of $K_\square'$. We can now obtain a $\mathbb{Z}_2$-deformation of $|K_\square'|$ onto $|K_*|$ by a step-by-step

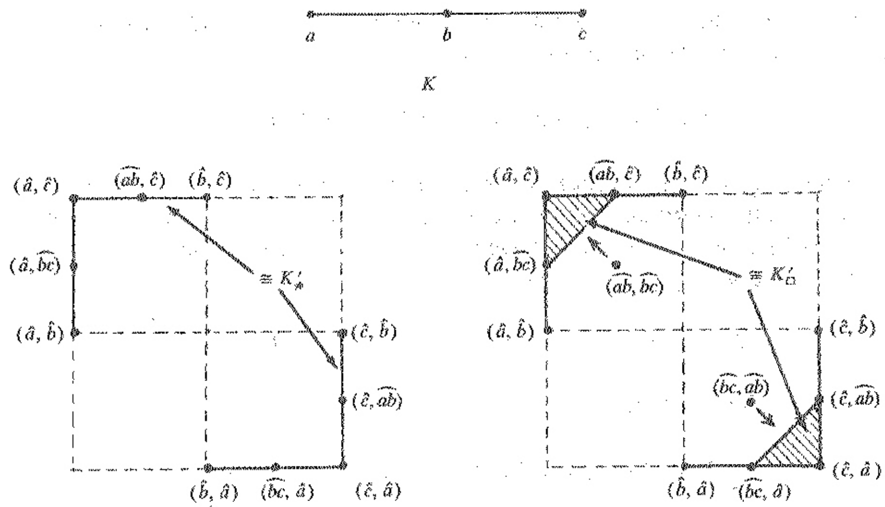

FIG. 1

procedure in which at each step these radial deformations are used in those highest dimensional antipodal pair of cells $\{\sigma \times \theta, \theta \times \sigma\}$ of $K \times K - K_*$ which still contain some part of the deformed $|K'_\square|$.

(2.5.2) *Remarks.*

(a) Twisted triangulations analogous to $K'_\square$ can be defined also on higher "deleted powers" and "configuration complexes" of K. We will show elsewhere that this leads to interesting generalizations of the results of this paper for finite groups G other than $\mathbb{Z}_2$.

(b) Note that the argument used to prove Lemma 3 is applicable to any $\mathbb{Z}_2$-subcomplex of $K \times K$. Thus one has:

LEMMA 3'. *If a $\mathbb{Z}_2$-subcomplex E of the cell complex $K \times K$ is considered as a poset P under $\preccurlyeq$, then X_P has the same $\mathbb{Z}_2$-homotopy type as X_E.*

This observation will be used in the proof of Theorem 3 below.

(c) In the above proof of Theorem 2 *one can avoid using Theorem* 1 *by means of the following direct construction.*

Let $S \subset K_\square$ denote the set of all minimal elements. Let

$$f\colon S \to \{0,1,\dots,m\} = \sigma^m$$

denote the given coloring. So

$$(v_1,\theta_1),(v_2,\theta_2) \in S,\, v_1 \in \theta_2,\, v_2 \in \theta_1 \Rightarrow f(v_1,\theta_1) \neq f(v_2,\theta_2). \tag{1}$$

We now define a (non-simplicial) function $F\colon K_\square \to \sigma_m^m$ (the simplicial complex containing all faces of σ^m) by

$$F(\alpha,\beta) = \{ f(v,\theta) \mid (v,\theta) \in S, v \in \alpha, \theta \supseteq \beta \}. \tag{2}$$

Note that $F(\alpha,\beta)$ is indeed always a non-empty subset of σ^m. (Had we started from a coloring f defined over some other $S \subset K_\square$ we would have got an F with a possibly smaller domain $\hat{S} \subseteq K_\square$.) Next we note that $F(\alpha,\beta)$ is always disjoint from $F(\beta,\alpha)$ because

$$v_1 \in \alpha, \theta_1 \supseteq \beta, v_2 \in \beta, \theta_2 \supseteq \alpha \Rightarrow v_1 \in \theta_2, v_2 \in \theta_1 \Rightarrow f(v_1,\theta_1) \neq f(v_2,\theta_2)$$

by (1). Finally note that

$$\alpha_1 \subseteq \alpha_2, \beta_1 \supseteq \beta_2 \Rightarrow F(\alpha_1,\beta_1) \subseteq F(\alpha_2,\beta_2).$$

Thus we can define a simplicial map $f_\square\colon K'_\square \to (\sigma_m^m)'_\square$ by mapping each simplex

$$\{(\alpha_1,\beta_1),(\alpha_2,\beta_2),\dots\}, \quad \alpha_1 \subseteq \alpha_2 \subseteq \dots, \beta_1 \supseteq \beta_2 \supseteq \dots,$$

of $K'_\square$ to the simplex

$$\{(F(\alpha_1,\beta_1), F(\beta_1,\alpha_1)), (F(\alpha_2,\beta_2), F(\beta_2,\alpha_2)),\dots\}$$

of $(\sigma_m^m)'_\square$. Then $|f_\square|$ is the requisite $\mathbb{Z}_2$-map from $X_{K_\square}$ to S^{m-1}.

(d) Even the graph-theoretical case $n = 1$, $m = 2$, of Theorem 2 seems to be new. It says that for any non-planar graph K^1, the associated graph $G(K^1_\square)$ must have chromatic number bigger than 3. In the opposite direction we have:

A FOUR COLOR THEOREM. *For any planar graph K^1 the associated graph $G(K^1_\square)$ has chromatic number ≤ 4.*

We will show elsewhere that it is enough to consider the case when K^1 is hamiltonian. Let $L^1 \subseteq K^1$ be a circle containing all the vertices of K^1. Choose an embedding of K^1 in $\mathbb{R}^2$ and let the vertices of K^1 be $1, 2, \dots, t$, as one proceeds along L^1 in a clockwise direction. A vertex $(\{a\},\{b,c\})$, $b < c$, of $G(K^1_\square)$ will be given (i) the color α if $a < b < c$, (ii) the color β if $b < a < c$ and $\{b,c\}$ is either $\{1,t\}$ or an edge lying in the bounded component of $\mathbb{R}^2 - |L^1|$, (iii) the color γ if $b < a < c$ and $\{b,c\}$ is in the unbounded component of $\mathbb{R}^2 - |L^1|$, and (iv) the color δ if $b < c < a$. It is easily verified that this is a good 4-coloring of the vertices of $G(K^1_\square)$.

I do not know whether the above bound is the best possible.

Also one has Heawood-type upper bounds for the chromatic number of $G(K^1_{\square})$ as K^1 runs over all graphs embeddable in *any* fixed 2-pseudomanifold X^2.

(e) It is easy to check that for any graph K^1 the associated graph $G(K^1_{\square})$ does not contain a complete graph on four vertices. However *there are graphs K^1 for which the chromatic number of $G(K^1_{\square})$ can be arbitrarily big*. For example let $\sigma^N = \{0, 1, \ldots, N\}$ and let $K^1 = \sigma_1^N$, the complete graph on $N + 1$ vertices. Suppose there is a good vertex coloring of $G((\sigma_1^N)_{\square})$ with m colors. To any 2-simplex $\{a, b, c\} \subset \sigma^N$, $a < b < c$, assign the color of the vertex $(b, \{a, c\})$ of $G((\sigma_1^N)_{\square})$. We note that any 3-simplex $\{e, f, g, h\} \subset \sigma^N$, $e < f < g < h$, has two incident 2-simplices, namely $\{e, f, g\}$ and $\{f, g, h\}$ which have different colors. By Ramsey's Theorem no such m-coloring of the 2-simplices of σ^N is possible if N is sufficiently big.

(2.5.3) Let $G_s(K_{\square})$, $s \geq 0$, denote the graph whose vertices are pairs (σ^s, θ) where σ^s is an s-simplex of K and θ a maximal simplex of K disjoint from σ^s, with (σ_1^s, θ_1) adjacent to (σ_2^s, θ_2) iff $\sigma_1^s \subseteq \theta_2$ and $\sigma_2^s \subseteq \theta_1$. So $G_0(K_{\square}) = G(K_{\square})$. Note further that if a vertex (σ^s, θ) of $G_s(K_{\square})$ has $\dim \theta < s$, then it is an isolated vertex. Modulo such vertices $G_s(K_{\square})$ coincides with $G(K_{\Sigma})$, the graph of minimal elements of the sub self-dual poset K_{Σ} of $K_{\square}$ consisting of all pairs (α, β) with $\dim \alpha \geq s, \dim \beta \geq s$. The following result generalizes Theorem 2.

THEOREM 3. *If $G_s(K^n_{\square})$ has chromatic number $\leq m + 1 - 2s$ (here $m > 2s$), then there is a $\mathbb{Z}_2$-map $(X_{K^n_*}) \to S^{m-1}$. Thus if further $2m \geq 3(n + 1)$ then X_{K^n} embeds in $\mathbb{R}^m$.*

Proof. Because of Theorem 2 we can assume that K_{Σ} is a proper subset of $K_{\square}$. Since $m - 1 - 2s \geq 0$, the given chromatic hypothesis and Theorem 1 (or else the construction of (2.5.2)(c)) supply us with a $\mathbb{Z}_2$-map

$$|(K^n_{\Sigma})'| \xrightarrow{f} S^{m-1-2s}.$$

Any chain of $K^n_{\square}$ is built up in a unique way from a chain of K^n_{Σ} and a chain of the complementary self-dual poset $P = K^n_{\square} - K_{\Sigma}$. We note that $|(K^n_{\Sigma})'|$ and $|P'|$ are disjoint subspaces of $|(K^n_{\square})'|$ and that any point x of $|(K^n_{\square})'|$ which is in neither of these subspaces is an interior point of a unique geometrical simplex of $(K^n_{\square})'$ having some vertices in $(K^n_{\Sigma})'$ and some in P'. In other words $|(K^n_{\square})'|$ is a $\mathbb{Z}_2$-subspace of the join $|(K^n_{\Sigma})'| * |P'|$. Hence if we could produce a $\mathbb{Z}_2$-map $g: |P'| \to S^{2s-1}$, then we would have the desired $\mathbb{Z}_2$-map

$$X_{K^n_{\square}} \cong |(K^n_{\square})'| \xrightarrow{f * g} S^{m-1-2s} * S^{2s-1} \cong S^{m-1}.$$

Note that the cells $\sigma \times \theta \in P$ are all those cells of K_*^n for which either $\dim \sigma$ or $\dim \theta$ is less than s. Thus they constitute a subcomplex E of the cell complex K. By Lemma 3′, $|P'|$ has the same $\mathbb{Z}_2$-homotopy type as its subspace $|E'|$. Thus there will be a $\mathbb{Z}_2$-map $g\colon |P'| \to S^{2s-1}$ iff such a $\mathbb{Z}_2$-map can be defined on the subspace $|E'|$. Let T denote the full subcomplex of E' spanned by vertices of the type $(\hat{\sigma}, \hat{\theta})$ where either $\dim \sigma \geq s, \dim \theta < s$ (the full subcomplex of T determined by all such vertices will be called T_1) or $\dim \sigma < s, \dim \theta \geq s$ (these determine the subcomplex T_2 of T). We note that an edge of T cannot have one vertex in T_1 and the other in T_2. So T is the disjoint union of the antipodal subcomplexes T_1 and T_2. Thus we can find a $\mathbb{Z}_2$-map

$$|T| \xrightarrow{g_1} S^0.$$

The remaining vertices of E' are of the type $(\hat{\sigma}, \hat{\theta}), \dim \sigma < s, \dim \theta < s$, and thus determine a full subcomplex U of E' of dimension $\leq 2(s-1)$. By working up inductively on the skeletons of U we see that there is no obstruction to finding a $\mathbb{Z}_2$-map

$$|U| \xrightarrow{g_2} S^{2(s-1)}.$$

This gives us the required $\mathbb{Z}_2$-map

$$|E'| \xrightarrow{g = g_1 * g_2} S^0 * S^{2(s-1)} \cong S^{2s-1}.$$

(2.5.4) *Remarks.* (a) *The graphs $G_s(K_\square^n)$ cannot contain a complete graph on more than*

$$\left[1 + \frac{n+1}{s+1}\right]$$

vertices: If $(\sigma_1^s, \theta_1), \ldots, (\sigma_t^s, \theta_t)$ are all mutually adjacent, then $\sigma_1^s, \sigma_2^s, \ldots, \sigma_{t-1}^s$ would be mutually disjoint subsets of the simplex θ_t. Since $\dim \theta_t \leq n$ we thus get $(t-1)(s+1) \leq n+1$ and so

$$t \leq \frac{n+1}{s+1} + 1.$$

(b) *Theorem 3 gives a host of new examples of triangle-free graphs $G_s(K_\square^n)$, $n \leq 2s$, $n - s$ large, having large chromatic numbers*:

For instance if we make sure that the n-dimensional simplicial complex K^n does not p.l. embed in $\mathbb{R}^{2n}$, then, by Theorem 3, the chromatic number of $G_s(K_\square^n)$ will be at least $2(n-s)+2$. Or, again, if n is a power of 2, and K^n

is a triangulation of $\mathbb{RP}^n$, the real n-dimensional projective space, then the chromatic number of $G_s(K_\square^n)$ is at least $2(n-s)+1$. This follows because such a projective space does not p.l. embed in $\mathbb{R}^{2n-1}$ (e.g., see Steenrod [24], p. 34, or Milnor [15], p. 120).

Since $n \le 2s$ the high chromaticity of these graphs is obviously not due to any contained isomorphs of the "classical Kneser graphs"; see proof of Theorem 4 below.

(c) *Conjecture.* The chromatic number of the graph $G_{n-1}(K_\square^n)$ is bounded as K^n runs over all n-dimensional simplicial complexes embeddable in $\mathbb{R}^{2n}$, and, more generally, as K^n runs over all n-dimensional simplicial complexes embeddable in a fixed X^{2n} (cf. the analogous Conjecture 2 of [19] which deals with "Ramsey Colorings", and (2.5.2)(d) above).

(2.5.5) We now take a closer look at the chromatic lower bounds which must be satisfied by the graphs $G_s(K_\square^n)$ merely by virtue of the local topological fact that K^n is n-dimensional. Since there is no $\mathbb{Z}_2$-map from $|K_*^n|$ to S^{n-2}, the following result is included in Theorem 3. We give below another argument which further clarifies the constructions introduced above.

THEOREM 4. *For any n-dimensional simplicial complex K^n the chromatic number of $G_s(K_\square^n)$ (here $n \ge 2s$) is greater than $n-2s$.*

Proof. For any subcomplex L of K one can find an isomorph of $G_s(L_\square)$ in the graph $G_s(K_\square)$: assign to each vertex (σ^s, θ) of $G_s(L_\square)$ a vertex (σ^s, θ') of $G_s(K_\square)$ where θ' is a maximal simplex of K disjoint from σ^s and containing θ. Thus it is enough to prove the result when $K^n = \sigma_n^n$, the simplicial complex consisting of all the faces of an n-simplex $\sigma^n = \{0, 1, \ldots, n\}$. But $G_s((\sigma_n^n)_\square)$ is obviously isomorphic to the *s-th classical Kneser graph* $G_s(\sigma_n^n)$ of σ_n^n, i.e., the graph whose vertices are the s-faces of σ^n with 2 vertices adjacent iff they are disjoint. The theorem follows because the well-known Kneser Conjecture [11], 1955, proved by Lovász [12], 1978, tells us that the chromatic number of this graph is exactly $n-2s+1$.

The Lovász-Kneser Theorem can also be proved as follows. That $n-2s+1$ colors suffice is very easy to see: assign to each s-face having a vertex in

$$\{0, 1, \ldots, n-2s-1\}$$

its first vertex and to all other s-faces the vertex $n-2s$. Clearly $n-2s=0$ or 1 colors won't do if $n=2s$ or $n=2s+1$. So assume $n-1>2s$. If $n-2s$ colors would do, then Theorem 1 (or else the construction of (2.5.2)(c)) supplies us with a $\mathbb{Z}_2$-map

$$|(\sigma_n^n)_\Sigma{}'| \to S^{n-2s-2},$$

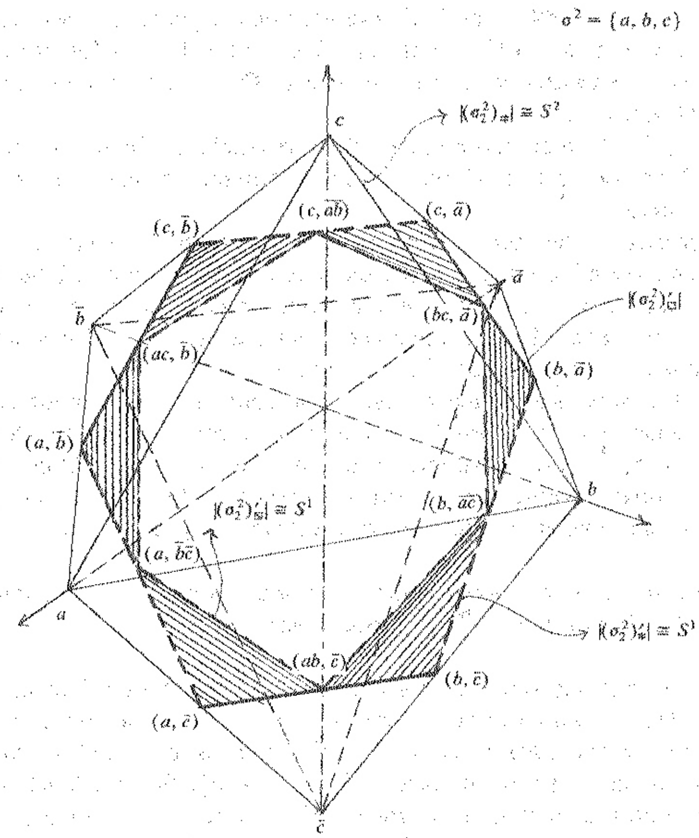

FIG. 2

where $(\sigma_n^n)_\Sigma$ is the subposet of $(\sigma_n^n)_\square$ consisting of all disjoint pairs of faces (α, β) of σ_n^n with $\dim \alpha \geq s, \dim \beta \geq s$, i.e., all pairs (α, β) with

$$s \leq \dim \alpha, \dim \beta \leq n - s - 1.$$

Consider also the subposet $(\sigma_n^n)_{\boxslash}$ of $(\sigma_n^n)_\square$ consisting of all (α, β) with $\dim \alpha + \dim \beta = n - 1$, i.e., all (α, β) for which $\beta = \sigma^n - \alpha$. Any chain of $(\sigma_n^n)_{\boxslash}$ has at most n members (corresponding to $0 \leq \dim \alpha \leq n - 1$), out of which at most $n - (n - 2s) = 2s$ (i.e., those with $\dim \alpha$ not in $[s, n - s - 1]$) are outside $(\sigma_n^n)_\Sigma$. So (as in the proof of Theorem 3) we can construct a $\mathbb{Z}_2$-map from $|(\sigma_n^n)'_{\boxslash}|$ to $S^{n-2s-2} * S^{2s-1} \cong S^{n-2}$. This is not possible because the poset $(\sigma_n^n)_{\boxslash}$ is isomorphic to the poset underlying the simplicial complex,

σ^n_{n-1}, of all proper faces of σ^n (under $(\alpha, \beta) \mapsto \alpha$), and so its space $|(\sigma^n_n)'_{\boxtimes}|$ is homeomorphic to the $(n-1)$-sphere $|\sigma^{n\prime}_{n-1}|$.

One can check either directly, or else using Lemmas 2 and 3, that the $2(n-1)$-dimensional $\mathbb{Z}_2$-space $|(\sigma^n_n)'_{\square}|$ has the same $\mathbb{Z}_2$-homotopy type as the $\mathbb{Z}_2$-subspaces $|(\sigma^n_n)'_{\boxtimes}|$ and $|(\sigma^n_n)'_{*}|$, these being $\mathbb{Z}_2$-homeomorphs of S^{n-1}. Figure 2 shows all these spaces and the deleted join $|(\sigma^n_n)_{\#}|$ (see (3.5.1)) for $n = 2$.

3. Kneser graphs of polyhedra

(3.1) *Interior of K_*.*

(3.1.1) Besides the generalisation $G_s(K^n_{\square})$—or even $G(S)$, $S \subset K^n_{\square}$—of the classical Kneser graphs, it is of interest also to examine some others. For example one can define the *i-th Kneser graph* $G_i(K)$ of a simplicial complex K to have as vertices the i-simplices α^i of K with α^i_1 joined to α^i_2 iff $\alpha^i_1 \cap \alpha^i_2 = \varnothing$.

THEOREM 5. *Let K^n be a homogeneously n-dimensional simplicial complex whose deleted product has the same $\mathbb{Z}_2$-homotopy type as its interior in $(X_{K^n})_*$. If $G_n(K)$ has chromatic number $\leq m+1$, then there is a $\mathbb{Z}_2$-map $(X_{K^n})_* \to S^{m-1}$. Thus if further $2m \geq 3(n+1)$ then X_{K^n} embeds in $\mathbb{R}^m$.*

Proof[4]. Let $S_n(K)$ be the set of all n-simplices of K and let

$$\varphi\colon S_n(K) \to \sigma^m = \{1, 2, \ldots, m+1\}$$

be the postulated coloring: so $\alpha^n_1 \cap \alpha^n_2 = \varnothing$ implies $\varphi(\alpha^n_1) \neq \varphi(\alpha^n_2)$. Let $P(K)$ denote the sub $\mathbb{Z}_2$-poset of $K \times K$ consisting of all cells $\sigma_1 \times \sigma_2 \subseteq \operatorname{int}|K_*|$. (Since we are considering elements of $K \times K$ as cells we write them as $\sigma_1 \times \sigma_2$ instead of as pairs (σ_1, σ_2). From the abstract viewpoint we have $P(K) = \{(\sigma_1, \sigma_2)\colon \sigma_1 \in K, \sigma_2 \in K, \overline{St}_K\sigma_1 \cap \overline{St}_K\sigma_2 = \varnothing\}$.) For each $\sigma_1 \times \sigma_2 \in P(K)$, the sets Σ_1 and Σ_2 of all n-simplices incident to σ_1 and σ_2 respectively, are non-empty and each α^n_1 in Σ_1 is disjoint from each α^n_2 in Σ_2. Thus we can define an order reversing $\mathbb{Z}_2$-morphism

$$f\colon P(K) \to (\sigma^m_m)_*$$

by

$$(\sigma_1 \times \sigma_2) \mapsto (\varphi(\Sigma_1) \times \varphi(\Sigma_2)).$$

[4] See also [35] where a variant of this simple argument is used to establish a generalized Erdős-Kneser Conjecture.

In the barycentric subdivision $(K \times K)'$, the derived complex $(P(K))'$ occurs as the subcomplex made up of all simplices having vertices in $\operatorname{int}|K_*|$. Each point of $|K \times K|$ lying in neither $|(P(K))'|$ nor in the complement of $\operatorname{int}|K_*^n|$ lies in a unique open simplex of $(K \times K)'$ having some vertices in $(P(K))'$ and some outside $\operatorname{int}|K_*^n|$; thus it is an interior point of a unique line segment having one end in $|(P(K))'|$ and one outside $\operatorname{int}|K_*^n|$. By pushing along these line segments towards $|(P(K))'|$ we see that $|(P(K))'|$ is a $\mathbb{Z}_2$-deformation retract of $\operatorname{int}|K_*^n|$. Also, by hypothesis, $|K_*^n|$—which by Proposition 1 has the $\mathbb{Z}_2$-homotopy type of $(X_{K^n})_*$—has the same $\mathbb{Z}_2$-homotopy type as $\operatorname{int}|K_*^n|$, and we know that $|(\sigma_m^m)_*|$ has the $\mathbb{Z}_2$-homotopy type of S^{m-1}. Thus f furnishes us with a $\mathbb{Z}_2$-map $(X_{K^n})_* \to S^{m-1}$.

(3.1.2) *Remarks.* (1) The condition $|K_*^n| \simeq \operatorname{int}|K_*^n|$ used in the above theorem holds whenever K^n is an n-manifold, and its deleted product K_*^n a $2n$-manifold-with-boundary. However very few such K^ns will satisfy the chromatic hypothesis of Theorem 5.

(2) *A colorability implies embeddability theorem of the above type cannot hold unconditionally.* To see this consider the n-skeleton of a $(2n+2)$-simplex σ_n^{2n+2}. It was proved by van Kampen [25] and Flores [4] that σ_n^{2n+2} does not embed in $\mathbb{R}^{2n}$. On the other hand the Lovász-Kneser Theorem tells us that the chromatic number of $G_n(\sigma_n^{2n+2})$ is only 3.

(3) However that is about as bad as things can be. *Any K^n with $G_n(K^n)$ bichromatic embeds in $\mathbb{R}^{2n}$.* Indeed we will proceed now to show that there are graphs much smaller than $G_n(K^n)$, and depending only on the homeomorphism type of the underlying polyhedron $X^n = |K^n|$, whose bichromaticity still forces the same conclusion.

(3.2) *The van Kampen construction.*

(3.2.1) A point x of a compact n-dimensional space $X^n, n \geq 1$, is called a *singular point* (or "of intrinsic dimension $\leq n-1$" in the terminology of Akin [1]) if no triangulation of X^n contains x in the interior of an n-simplex. A singular point will be said to lie on the *edge* of X^n if, in some triangulation of X^n, it is incident to exactly one n-simplex. Let $\operatorname{sing}(X^n)$ denote the subpolyhedron of all singular points, and X_i^n the closures, in X^n, of the components of $X^n - \operatorname{sing}(X^n)$. We define $G(X^n)$, the *Kneser graph of polyhedron* X^n, to have as vertices all those X_i^n which have no points on the edge of X^n, with vertices X_i^n and X_j^n joined iff $X_i^n \cap X_j^n = \varnothing$.

A polyhedron X^n is said to *unknot* in $\mathbb{R}^m$ if X^n embeds in $\mathbb{R}^m$, and any two embeddings of X^n in $\mathbb{R}^m$ are ambient isotropic to each other. Hudson's Isotopy Extension Theorem—see Hudson [10] for background and Akin [1], Corollary 17, p. 465—assures us that, for $m - n \geq 3$, this is equivalent to just demanding that any two embeddings of X^n in $\mathbb{R}^m$ be isotopic to each other.

We now strengthen the theorem of Sarkaria [21] to the following.

THEOREM 6. *The graph $G(X^n)$, $n \neq 2$, can be 2-colored only if X^n embeds in $\mathbb{R}^{2n}$. And $G(X^n)$, $n \neq 1$, can be 1-colored only if X^n unknots in $\mathbb{R}^{2n+1}$.*

Proof. We'll color the vertices of $G(X^n)$ by w and b. Let W be the union of all $X_i^n \notin G$ (i.e., those having a point on the edge of X^n), and all those $X_i^n \in G$ which have color w; and let B be the union of all $X_i^n \in G$ having color b. Since $W \cap B$ has dimension $\leq n-1$ it embeds in the $2n-1$ dimensional vector subspace

$$\mathbb{R}^{2n-1} = \{(x_1, x_2, \ldots, x_{2n}) : x_{2n} = 0\}$$

of $\mathbb{R}^{2n}$, and we can extend this embedding to a general position map $f: X^n \to \mathbb{R}^{2n}$ which images $W - W \cap B$ (resp. $B - W \cap B$) into the half space $x_{2n} < 0$ (resp. $x_{2n} > 0$). The map f is one-one except for a finite number of pairs of double points (p_1, p_2) which are non-singular.

If one of these double points lies in an $X_i^n \notin G$ we join it, via non-singular non-double points, to a point on the edge of X^n and delete from X^n an open regular neighborhood of this arc. The resulting polyhedron being homeomorphic to X^n, this pair of double points gets eliminated.

Otherwise $p_1 \in X_i^n \in G$, $p_2 \in X_j^n \in G$ with X_i^n and X_j^n having the same color, and so $X_i^n \cap X_j^n \neq \varnothing$. Such a pair (p_1, p_2) is eliminated, when $n \geq 3$, by means of the van Kampen-Penrose-Whitehead-Zeeman cone construction. (See [25], p. 152, and [16]; also [33], p. 66.) Briefly, as in [21], join p_1 to p_2 via an arc A, all of whose (other) points are non-singular non-double points, with at most one exception, which is a singular point. Since $n \geq 3$, the circle $C = f(A)$ bounds a 2-disk $D \subseteq \mathbb{R}^{2n}$ meeting $f(X^n)$ only in C. Its regular neighborhood $N(D)$—a $2n$-disk—meets $f(X^n)$ in $f(N(A))$ with $\partial N(D) \cap f(X^n) = f(\partial N(A))$. Here $N(A)$ denotes regular neighborhood of A. From hypotheses on A, $N(A)$ is a cone over its boundary $\partial N(A)$. So we can alter f on $N(A)$ by coning $f(\partial N(A))$ over an interior point of $N(D)$.

The graph-theoretical case $n = 1$ is trivial. In fact, there is no $X^1 = |K^1|$ which satisfies the hypothesis *and* for which all vertices of K^1 have valence ≥ 3: This follows by noting that the subcomplexes W^1 and B^1 of K^1 determined by the white and black edges of K^1 must in fact be in the closed stars of two vertices w and b, but one cannot have $K^1 = (\overline{St}_{K^1} w) \cup (\overline{St}_{K^1} b)$.

The second part follows by noting that if $n \geq 2$ and $G(X^n)$ has no edge, then an analogous elimination of double points converts any general position map

$$X^n \times [0,1] \to \mathbb{R}^{2n+1} \times [0,1]$$

into a concordance between the pair of embeddings

$$X^n \times \{0,1\} \to \mathbb{R}^{2n+1} \times \{0,1\}.$$

(3.2.2) *Remarks.* (1) The first part of Theorem 6 is probably true also for $n = 2$ but we have not written a complete proof for this so far.[5] The second part of Theorem 6 is *not* true for $n = 1$: $G(S^1) = pt.$ is 1-colorable but S^1 knots in $\mathbb{R}^3$. However note that any two embeddings of S^1 in $\mathbb{R}^3$ are still isotopic though of course not necessarily ambient isotopic.

(2) *If* $X^n = |K^n|$, *then* $G(X^n)$ *can be identified to a subgraph of* $G_n(K^n)$ by choosing an n-simplex in each vertex X_i^n of $G(X^n)$. Obviously $G(X^n)$ is usually much smaller than $G_n(K^n)$.

(3) And thus Theorem 6 easily checks the embeddability in $\mathbb{R}^{2n}$ and the unknotting in $\mathbb{R}^{2n+1}$ of many interesting n-dimensional spaces. However *the sufficient conditions for embeddability and unknotting given in Theorem 6 are far from being necessary.* There exist polyhedra X^n, with chromatic number of $G(X^n)$ arbitrarily big, which embed in $\mathbb{R}^{2n}$ (or unknot in $\mathbb{R}^{2n+1}$). If X^n is the disjoint union of N n-spheres then X^n embeds in $\mathbb{R}^{n+1}$ and $G(X^n)$ has chromatic number N. Again take the Zeeman Dunce Hat Z^2. It is a contractible 2-dimensional polyhedron without an edge. If X^n is the disjoint union of N copies of $Z^2 \times S^{n-2}$, $G(X^n)$ is once again the complete graph on N vertices. That X^n unknots in $\mathbb{R}^{2n+1}$ follows from *Price's Theorem* [17]: If $n \geq 2$ and $H^n(X^n; \mathbb{Z}) = 0$, then X^n unknots in $\mathbb{R}^{2n+1}$.

(3.3) *Equivariant cohomology*

We recall some simple facts regarding the cohomology of a polyhedron E equipped with a *fixed point free involution* ν.

(3.3.1) The (singular, integral) cochain complex $C(E)$ of E, the total space of the 2-fold cover $\pi: E \to E/\mathbb{Z}_2$, has two important subcomplexes. The first, $C_s(E)$, consisting of *symmetric cochains* c invariant under the involution of E, $\nu^*c = c$, can be identified with the pull back of cochains of $E/\mathbb{Z}_2$. The second, $C_a(E)$, consisting of *antisymmetric cochains* c which change sign under the involution of E, $\nu^*c = -c$, can be identified with the pull back of cochains of $E/\mathbb{Z}_2$ with twisted integer coefficients $\hat{\mathbb{Z}} = E \times_{\mathbb{Z}_2} \mathbb{Z}$. The cohomologies of these two subcomplexes can thus be denoted by

$$H^i(E/\mathbb{Z}_2; \mathbb{Z}) \quad \text{and} \quad H^i\big(E/\mathbb{Z}_2; \hat{\mathbb{Z}}\big).$$

LEMMA 4. *In* $H^m(E^m/\mathbb{Z}_2; \mathbb{Z})$ *or* $H^m(E^m/\mathbb{Z}_2; \hat{\mathbb{Z}})$, *multiplication by 2 is surjective iff it is bijective iff this group is finite and has no elements of order two. Further, this happens iff* $H^m(E^m/\mathbb{Z}_2; \mathbb{Z}_2) = 0$.

Proof. The first part follows because $H^m(E^m/\mathbb{Z}_2; \mathbb{Z})$ and $H^m(E^m/\mathbb{Z}_2; \hat{\mathbb{Z}})$ are finitely generated Abelian groups. To see the second part one recalls that

[5] A proof of this delicate case is proposed in [34].

the short exact coefficient sequences

$$0 \to \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \to \mathbf{Z}_2 \to 0 \quad \text{and} \quad 0 \to \hat{\mathbf{Z}} \xrightarrow{\times 2} \hat{\mathbf{Z}} \to \mathbf{Z}_2 \to 0$$

give rise to long exact Bockstein sequences

$$\ldots \to H^m(E^m/\mathbf{Z}_2; \mathbf{Z}) \xrightarrow{\times 2} H^m(E^m/\mathbf{Z}_2; \mathbf{Z}) \to H^m(E^m/\mathbf{Z}_2; \mathbf{Z}_2) \to 0$$

and

$$\ldots \to H^m(E^m/\mathbf{Z}_2; \hat{\mathbf{Z}}) \xrightarrow{\times 2} H^m(E^m/\mathbf{Z}_2; \hat{\mathbf{Z}}) \to H^m(E^m/\mathbf{Z}_2; \mathbf{Z}_2) \to 0.$$

(Note that for $H^m(E^m/\mathbf{Z}_2; \mathbf{Z})$ one has an analogue for primes $p \neq 2$ also.)

(3.3.2) There is a natural bijective correspondence between equivariant maps $E \to S^i$ and sections of the i-sphere bundle $E \times_{\mathbf{Z}_2} S^i \to E/\mathbf{Z}_2$ associated to the 2-fold cover $\pi: E \to E/\mathbf{Z}_2$. For each $i \geq 0$, i even (resp. i odd), obstruction theory (Steenrod [23], pp. 177–198, or Milnor [15], pp. 139–148) provides us with an *obstruction class*

$$\mathfrak{o}_{i+1}(E) \in H^{i+1}(E/\mathbf{Z}_2; \hat{\mathbf{Z}}) \quad (\text{resp. } \mathfrak{o}_{i+1}(E) \in H^{i+1}(E/\mathbf{Z}_2; \mathbf{Z}))$$

which is zero iff this i-sphere bundle has a cross-section over the $i+1$-skeleton of any triangulation of $E/\mathbf{Z}_2$. In particular if $m = \dim E$ there is an equivariant map $E^m \to S^m$. Further, there is a *Hopf classification* of such maps: the equivariant homotopy classes of equivariant maps $E^m \to S^m$ can be put in bijective correspondence with the elements of the cohomology group $H^m(E^m/\mathbf{Z}_2; \hat{\mathbf{Z}})$ or $H^m(E^m/\mathbf{Z}_2; \mathbf{Z})$ depending on the parity of m.

LEMMA 5. *A non-zero obstruction class has order* 2.

Proof. Under the map $\pi: E \to E/\mathbf{Z}_2$, $\mathfrak{o}_{i+1}(E)$ pulls back to the $i+1$-*th* obstruction class of the pulled back i-sphere bundle $\pi^*(E \times_{\mathbf{Z}_2} S^i)$. But this is just the trivial i-sphere bundle $E \times S^i \to E$. Thus if i is even (resp. odd), and antisymmetric (resp. symmetric) cocycle z represents $\mathfrak{o}_{i+1}(E)$, then we must have a cochain c of E such that $\delta c = z$; therefore $\delta(c - \nu^* c) = 2z$ (resp. $\delta(c + \nu^* c) = 2z$) and so $2\mathfrak{o}_{i+1}(E) = 0$.

(3.3.3) One checks easily that cochain complex sequences

$$0 \to C_s(E) \subseteq C(E) \xrightarrow{\mathrm{Id} - \nu^*} C_a(E) \to 0$$

and

$$0 \to C_a(E) \subseteq C(E) \xrightarrow{\mathrm{Id}+\nu^*} C_s(E) \to 0$$

are exact. Their long exact sequences, which run

$$\text{(1)} \qquad \dots \to H^i(E/\mathbb{Z}_2;\mathbb{Z}) \to H^i(E;\mathbb{Z}) \to H^i(E/\mathbb{Z}_2;\hat{\mathbb{Z}}) \xrightarrow{\delta_s^i} H^{i+1}(E/\mathbb{Z}_2;\mathbb{Z}) \to \dots,$$

$$\text{(2)} \qquad \dots \to H^i(E/\mathbb{Z}_2;\hat{\mathbb{Z}}) \to H^i(E;\mathbb{Z}) \to H^i(E/\mathbb{Z}_2;\mathbb{Z}) \xrightarrow{\delta_a^i} H^{i+1}(E/\mathbb{Z}_2;\hat{\mathbb{Z}}) \to \dots,$$

are called the *Smith-Richardson sequences* (e.g., see Wu [32], Ch. II) of E. One has the following characterisation of the obstruction classes, in terms of the connecting homomorphisms of these sequences:

$$\text{(3)} \qquad o_1(E) = \delta_a^0(1),\ o_2(E) = \delta_s^1(o_1(E)),\ o_3(E) = \delta_a^2(o_2(E)),\dots.$$

LEMMA 6.

$$H^m(E^m/\mathbb{Z}_2;\mathbb{Z}) \xrightarrow{\pi^*} H^m(E^m;\mathbb{Z}) \quad \left(\text{resp. } H^m(E^m/\mathbb{Z}_2;\hat{\mathbb{Z}}) \xrightarrow{\pi^*} H^m(E^m;\mathbb{Z})\right)$$

is surjective iff it is bijective iff

$$H^m(E^m/\mathbb{Z}_2;\hat{\mathbb{Z}}) = 0 \left(\text{resp. } H^m(E^m/\mathbb{Z}_2;\mathbb{Z}) = 0\right).$$

Proof. That surjectivity is equivalent to the vanishing of the stated group is clear from the exact sequence (1) (resp. (2)). But this in turn ensures, by Lemma 4, that in $H^m(E^m/\mathbb{Z}_2;\mathbb{Z})$ (resp. $H^m(E^m/\mathbb{Z}_2;\hat{\mathbb{Z}})$) multiplication by 2 is bijective. At the cochain level multiplication by 2 can be written as the composite

$$C_s(E) \subseteq C(E) \xrightarrow{\mathrm{Id}+\nu^*} C_s(E) \quad \left(\text{resp. } C_a(E) \subseteq C(E) \xrightarrow{\mathrm{Id}-\nu^*} C_a(E)\right).$$

So π^*, which is induced by the first factor $C_s(E) \subseteq C(E)$ (resp. $C_a(E) \subseteq C(E)$) must be injective.

(3.4) *Kneser deleted product.*

(3.4.1) Before returning to the Kneser graph $G(X^n)$ let us record some consequences of Weber's Theorem.

VAN KAMPEN-WU-SHAPIRO THEOREM. *If $n \neq 2$, X^n embeds in $\mathbb{R}^{2n}$ iff the obstruction class $\mathfrak{o}_{2n}(X^n_*) \in H^{2n}(X^n_*/\mathbb{Z}_2; \mathbb{Z})$ vanishes.*

Since there is a $\mathbb{Z}_2$-map $X^n_* \to S^{2n-1}$ iff $\mathfrak{o}_{2n}(X^n_*) = 0$ this older result is, for $n \geq 3$, merely a special case of Weber's Theorem. If $n = 1$ the above characterization of planar graphs follows easily from the well-known Kuratowski Theorem; see Wu [32], p. 210. The 4-dimensional methods of Freedman [5] may have a bearing on the unknown case $n = 2$.[6]

THEOREM 7. (a) *If $n \geq 2$, isotopy classes of embeddings $X^n \to \mathbb{R}^{2n+1}$ are in bijective correspondence with the elements of $H^{2n}(X^n_*/\mathbb{Z}_2; \hat{\mathbb{Z}})$.*

(b) *For $n \geq 2$, $\pi^*: H^{2n}(X^n_*/\mathbb{Z}_2; \mathbb{Z}) \to H^{2n}(X^n_*; \mathbb{Z})$ is surjective iff it is bijective iff $H^{2n}(X^n_*/\mathbb{Z}_2; \hat{\mathbb{Z}}) = 0$ iff X^n unknots in $\mathbb{R}^{2n+1}$.*

(c) *For $n \neq 2$, X^n unknots in $\mathbb{R}^{2n+1}$ only if it embeds in $\mathbb{R}^{2n}$.*

Proof. To see (a) and (b) use Weber's Theorem in conjunction with (3.3.2) and Lemma 6. (Note that (a) and (b) are not true for $n = 1$: a circle S^1 has $H^2(S^1_*/\mathbb{Z}_2; \hat{\mathbb{Z}}) = 0$ but does knot in $\mathbb{R}^3$.) To see (c) for $n \geq 3$ note that if

$$H^{2n}\left(X^n_*/\mathbb{Z}_2; \hat{\mathbb{Z}}\right) = 0$$

then, by Lemma 4, $H^{2n}(X^n_*/\mathbb{Z}_2; \mathbb{Z})$ is finite with no elements of order two, and so, by Lemma 5, $\mathfrak{o}_{2n}(X^n_*) = 0$. (It would be interesting to have a purely geometric proof of this.) For $n = 1$, (c) follows because an X^1 unknots in $\mathbb{R}^3$ iff it has no loops, in which case it does embed in $\mathbb{R}^2$.

PRICE'S THEOREM. *If $H^n(X^n; \mathbb{Z}) = 0$, then $H^{2n}(X^n_*/\mathbb{Z}_2; \hat{\mathbb{Z}}) = 0$.*

Proof. In fact, $H^n(X^n; \mathbb{Z}) = 0$ implies $H^{2n}(X^n \times X^n; \mathbb{Z}) = 0$ (because if n-cochain b^n is coboundary of c^{n-1} in K^n, then $\delta(c^{n-1} \times e^n) = b^n \times e^n$ in $K^n \times K^n$) which in turn implies $H^{2n}(X^n_*; \mathbb{Z}) = 0$ (by exact cohomology sequence of pair $(X^n \times X^n, X^n_*)$) and this implies $H^{2n}(X^n_*/\mathbb{Z}_2; \hat{\mathbb{Z}}) = 0 = H^{2n}(X^n_*/\mathbb{Z}_2; \mathbb{Z})$ by using exact sequences (1) and (2) of (3.3.3).

(3.4.2) We define X^n_{**}, the *Kneser deleted product* of the polyhedron X^n, to be the $\mathbb{Z}_2$-subpolyhedron of X^n_* given by

$$\bigcup\left\{ X^n_i \times X^n_j : \left\{ X^n_i, X^n_j \right\} \quad \text{an edge of } G\left(X^n\right)\right\}.$$

[6] A direct proof is proposed in [34].

It is important to observe that *in general X^n_{**} does not have the same $\mathbf{Z}_2$-homotopy type as X^n_**; e.g., in the case of a circle S^1 the deleted product S^1_* has the homotopy type of S^1 while the Kneser deleted product S^1_{**} is empty. Again, note that the 2-fold cover $X^n_* \to X^n_*/\mathbf{Z}_2$ is hardly ever trivial; the only connected X^n's for which it is trivial are the point and the closed interval. On the other hand the bichromaticity of $G(X^n)$ ensures that the 2-fold cover $X^n_{**} \to X^n_{**}/\mathbf{Z}_2$ is trivial.

However we will now check that X^n_{**} does retain *some* of the information contained in X^n_*:

PROPOSITION 2. *The inclusions $X^n_{**} \subseteq X^n_*$ and $X^n_{**}/\mathbf{Z}_2 \subseteq X^n_*/\mathbf{Z}_2$ induce isomorphisms for the $2n$-th homologies and cohomologies.*

Proof. Choose a triangulation K^n of X^n. Since each X^n_i is covered by a subcomplex K^n_i of K^n, it follows that X^n_{**} is covered by a subcomplex K^n_{**} of K^n_*. Proposition 1 shows that our result will follow if we can show that $K^n_{**} \subseteq K^n_*$ induces isomorphisms for the $2n$-th cohomologies and homologies. Any $2n$-cell $\Psi^{2n} = \sigma^n \times \theta^n$ of $K^n_* - K^n_{**}$ either (1) lies in an $X^n_i \times X^n_j$ with $X^n_i \cap X^n_j \neq \varnothing$ or (2) lies in an $X^n_i \times X^n_j$ with $X^n_i \cap X^n_j = \varnothing$ with at least one of the X^n_i, X^n_j (say X^n_i) having some points on the edge of X^n. Note that int X^n_i (resp. int $X^n_i \times X^n_j$) is a connected open n-dimensional (resp. $2n$-dimensional) manifold. In case (1) choose a sequence of (open) n-simplices $\sigma^n = \sigma^n_0, \sigma^n_1, \ldots, \sigma^n_r$ of int X^n_i (resp. $\theta^n = \theta^n_0, \theta^n_1, \ldots, \theta^n_s$ of int X^n_j), each sharing a common $(n-1)$-simplex of int X^n_i (resp. int X^n_j) with the next one, such that $\sigma^n_r \cap \theta^n_s \neq \varnothing$ but $\sigma^n_a \cap \theta^n_b = \varnothing$ if $a < r$ or $b < s$. Then

$$\Psi^{2n} = \Psi^{2n}_0, \psi^{2n}_1 = \sigma^n_1 \times \theta^n, \psi^{2n}_2 = \sigma^n_1 \times \theta^n_1, \Psi^{2n}_3 = \sigma^n_2 \times \theta^n_1, \ldots, \Psi^{2n}_l = \sigma^n_r \times \theta^n_{s-1}$$

is a sequence of open $2n$-cells of $K^n_* \cap \operatorname{int}(X^n_i \times X^n_j)$, each sharing a common $(2n-1)$-face, Φ^{2n-1}_t, $1 \leq t \leq l-1$, of $\operatorname{int}(X^n_i \times X^n_j)$ with the next one, with the very last one Ψ^{2n}_l having a $(2n-1)$-face $\Phi^{2n-1}_l = \sigma^n_r \times (\theta^n_{s-1} \cap \theta^n_s)$ which is incident to no other $2n$-cell of K. In case (2) choose a sequence of n-simplices $\sigma^n = \sigma^n_0, \sigma^n_1, \ldots, \sigma^n_l$ of int X^n_i, each sharing a common $(n-1)$-simplex of int X^n_i with the next one, and with the last, σ^n_l, having an $(n-1)$-face ξ^{n-1}_l incident to no other n-simplex of K^n. Now

$$\Psi^{2n} = \Psi^{2n}_0, \Psi^{2n}_1 = \sigma^n_1 \times \theta^n, \ldots, \Psi^{2n}_l = \sigma^n_l \times \theta^n, \Phi^{2n-1}_l = \xi^{n-1}_l \times \theta^n,$$

has exactly the same properties as before. With appropriate orientations of the cells one has the coboundary formula $\delta(\sum^l_{a=1} \Phi^{2n-1}_a) = \Psi^{2n}$; this shows that any $2n$-cochain of K^n_* is cohomologous to one which is supported on K^n_{**}.

Likewise the boundary formula,

$$\partial\left(\sum_{a=0}^{l} c_a\Psi_a^{2n} + \text{terms involving other } 2n\text{-cells of } K_*^n\right)$$
$$= \sum_{a=1}^{l-1}(c_a - c_{a-1})\Phi_a^{2n-1} + c_l\Phi_l^{2n-1} + \text{terms involving other}$$
$$(2n-1)\text{-cells of } K_*^n,$$

shows that Ψ^{2n} cannot occur (with a non-zero coefficient) in any $2n$-cycle of K_*^n, and thus any $2n$-cycle of K_*^n is supported on K_{**}^n.

With reference to the above proof let us observe that with the appropriate orientations all $2n$-cells of each $X_i^n \times X_j^n \subseteq X_{**}^n$ occur with the same coefficient in a $2n$-cycle of K_{**}^n. Thus each $2n$-cycle $e \in H_{2n}(X_*^n/\mathbb{Z}_2; R)$ is *supported* (i.e., has non-zero coefficients) on a $\mathbb{Z}_2$-subspace E of X_{**}^n which is a union of some $X_i^n \times X_j^n, \{X_i^n, X_j^n\}$ an edge of $G(X^n)$. For any such E the corresponding subgraph of $G(X^n)$ determined by these edges and their vertices will be denoted by $G(E)$.

LEMMA 7. *$G(E)$ is bichromatic only if there is a $\mathbb{Z}_2$-map $f\colon E \to S^0$; also conversely provided $G(E)$ is known to be a full triangle-free subgraph of $G(X^n)$.*

Proof. Let us denote S^0 by $\{1,2\}$ and let 1 and 2 also denote the two colors. If $G(E)$ is 2-colored then the $\mathbb{Z}_2$-function $f\colon E \to S^0$ imaging each $X_i^n \times X_j^n$ to the color of X_i^n is easily checked to be continuous. Conversely, $G(E)$ is triangle-free, so $X_i^n \cap X_j^n = \varnothing = X_i^n \cap X_k^n$ only if $X_j^n \cap X_k^n \neq \varnothing$, and thus for any $\mathbb{Z}_2$-map $f\colon E \to S^0$ one has $f(X_i^n \times X_j^n) = f(X_i^n \times X_k^n)$. So we can 2-color $G(E)$ by assigning to X_i^n the color $f(X_i^n \times X_j^n)$.

If each element of $H_{2n}(X_*^n/\mathbb{Z}_2; R)$ can be written as a sum of elements e with chromatic number of $G(E) \leq k$, then we say that $H_{2n}(X_*^n/\mathbb{Z}_2; R)$ *has chromatic number* $\leq k$.

For all k sufficiently large, $H_{2n}(X_*^n/\mathbb{Z}_2; \mathbb{Z}_{2^k})$ is independent of k. This homology group will be denoted by $H_{2n}(X_*^n/\mathbb{Z}_2; \mathbb{Z}_{2^\infty})$.

We now strengthen the first part of Theorem 6 to the following.

THEOREM 8. *If $H_{2n}(X_*^n/\mathbb{Z}_2; \mathbb{Z}_{2^\infty})$ is bichromatic then there is a $\mathbb{Z}_2$-map $X_*^n \to S^{2n-1}$. Thus if we further have $n \neq 2$ then X^n embeds in $\mathbb{R}^{2n}$.*

Proof. Let us choose k so large that $H^{2n}(X_*^n/\mathbb{Z}_2; \mathbb{Z})$ has no elements of order 2^{k+1}. The short exact coefficient sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2^k} \mathbb{Z} \to \mathbb{Z}_{2^k} \to 0$$

induces the Bockstein sequence

$$\cdots \to H^{2n}(X_*^n/\mathbb{Z}_2;\mathbb{Z}) \xrightarrow{\times 2^k} H^{2n}(X_*^n/\mathbb{Z}_2;\mathbb{Z}) \xrightarrow{\kappa} H^{2n}(X_*^n/\mathbb{Z}_2;\mathbb{Z}_{2^k}) \to 0.$$

Observe that $\kappa(\mathfrak{o}_{2n}(X_*^n)) = 0$ implies $\mathfrak{o}_{2n}(X_*^n) = 2^k\psi_{2n}$ for some $\psi_{2n} \in H^{2n}(X_*^n/\mathbb{Z}_2;\mathbb{Z})$; but, by Lemma 5, $2\mathfrak{o}_{2n}(X_*^n) = 2^{k+1}\psi_{2n} = 0$, and so, by our choice of k, $2^k\psi_{2n}$, i.e., $\mathfrak{o}_{2n}(X_*^n)$, is 0. Thus it would suffice to check that the given hypotheses ensure $\kappa(\mathfrak{o}_{2n}(X_*^n)) = 0$. But $H^{2n}(X_*^n/\mathbb{Z}_2;\mathbb{Z}_{2^k})$ is dual to $H_{2n}(X_*^n/\mathbb{Z}_2;\mathbb{Z}_{2^k})$ (e.g., see Maunder [14], p. 166). Hence it would suffice to prove that $(\kappa(\mathfrak{o}_{2n}(X_*^n))(e) = 0$ whenever $e \in H_{2n}(X_*^n/\mathbb{Z}_2;\mathbb{Z}_{2^k})$ has a bichromatic $G(E)$. Observe that e lies in the subgroup $H_{2n}(E/\mathbb{Z}_2;\mathbb{Z}_{2^k})$ and that the restriction of $\mathfrak{o}_{2n}(X_*^n)$ to E is $\mathfrak{o}_{2n}(E)$ (this follows for example by (3) of (3.3.3)). Thus

$$\kappa(\mathfrak{o}_{2n}(X_*^n))(e) = \kappa(\mathfrak{o}_{2n}(E))(e)$$

and it would suffice to prove that $\mathfrak{o}_{2n}(E) = 0$, i.e., that there is a $\mathbb{Z}_2$-map $E \to S^{2n-1}$. But this follows at once from Lemma 7: $G(E)$ is bichromatic so there is in fact a $\mathbb{Z}_2$-map $E \to S^0$.

(3.5) *The deleted join.*

(3.5.1) Since the deleted product K_*^n of a simplicial complex K^n is only a cell complex, for some purposes it is more convenient to use instead the *deleted join* $K_\#^n$ *of* K^n: Take a disjoint copy $\bar{K}$ of K. The join $K \cdot \bar{K}$—i.e., the simplicial complex generated by all simplices of type $\sigma\bar{\theta}(= \sigma \cup \bar{\theta})$, $\sigma \in K$, $\bar{\theta} \in \bar{K}$—is equipped with the involution $\sigma\bar{\theta} \to \theta\bar{\sigma}$. $K_\#$ is the $\mathbb{Z}_2$-subcomplex of $K \cdot \bar{K}$ obtained by omitting those $\sigma\bar{\theta}$ for which $\sigma \cap \theta$ is non-empty. Likewise the deleted join $X_\#^n$ of a space, X^n, is obtained by deleting all points of the type $\frac{1}{2}x + \frac{1}{2}\bar{x}$ from the join of spaces $X * \bar{X}$. Analogously to Proposition 1 one can verify that if K triangulates X, then $|K_\#|$ and $X_\#$ have the same $\mathbb{Z}_2$-homotopy type.

As in [19] we denote by $\delta_i(K)$ the *i-th least valence* (i.e., the least number of $(i+1)$-simplices incident to an i-simplex) of simplicial complex K.

As an illustration of the use of the deleted join functor we prove the following polynomial inequality.

THEOREM 9. *For any n-dimensional simplicial complex K*,

$$(4)\quad \delta_{n-1}(K) < 3(n+1) \quad \text{or} \quad \binom{\delta_{n-1}(K) + n - 1}{2n+2} \le \dim H_{2n}(K_*^n;\mathbb{Z}_2).$$

Proof. The space $|K_\#^n|$ consists of points of the type $tx + (1-t)\bar{y}$, $0 \le t \le 1$, where x and y lie in disjoint closed simplices of $|K^n|$. We note that the subspaces $K^n_{\leq 1/2}$ and $K^n_{\geq 1/2}$ of $|K^n_\#|$, defined by $t \leq 1/2$ and $t \geq 1/2$ respectively, retract to the two ends $t = 0$ and $t = 1$, which are homeomorphic to $|K^n|$; and their intersection, defined by $t = 1/2$, is homeomorphic to the deleted product $|K^n_*|$. Thus the Mayer-Vietoris sequence of the pair $\{K^n_{\geq 1/2}, K^n_{\leq 1/2}\}$ yields the exact sequence

$$(5) \qquad \ldots \to H_i(K^n) \oplus H_i(K^n) \to H_i(K^n_\#) \to H_{i-1}(K^n_*) \to H_{i-1}(K^n) \oplus H_{i-1}(K^n) \to \ldots$$

In particular $H_{2n+1}(K^n_\#; \mathbb{Z}_2) \cong H_{2n}(K^n_*; \mathbb{Z}_2)$; so by applying Theorem 1 of [19] to the simplicial complex $K^n_\#$ we get

$$(6) \quad \delta_{2n}(K^n_\#) < 2n + 2 \quad \text{or} \quad \binom{\delta_{2n}(K^n_*) + 2n}{2n + 2} \leq \dim H_{2n}(K^n_*; \mathbb{Z}_2).$$

The required inequality (4) follows from (6) and the fact (compare (2.2.3) of [20]) that for any simplicial complex K,

$$(7) \quad \delta_{n-1}(K) \leq n \quad \text{or} \quad \delta_{2n}(K^n_\#) \leq \delta_{n-1}(K) \leq \delta_{2n}(K^n_\#) + n + 1.$$

By Theorem 7(a), K^n, $n \geq 2$, unknots in $\mathbb{R}^{2n+1}$ iff $H^{2n}(K^n_*/\mathbb{Z}_2; \hat{\mathbb{Z}}) = 0$, and so only if $H^{2n}(K^n_*/\mathbb{Z}_2; \mathbb{Z}_2) = 0$ and so (using exact sequence (1) of (3.3.3) with $\mathbb{Z}_2$ coefficients) only if $H^{2n}(K^n_*; \mathbb{Z}_2) = 0$. Thus Theorem 9 implies the following result of Sarkaria [20].

Corollary 1. *If K^n unknots in $\mathbb{R}^{2n+1}$, then $\delta_{n-1}(K) < 3(n+1)$.*

As in [19] *the weak i-th chromatic number* of a simplicial complex, $c_i(K)$, is the least number of colors which can be assigned to the i-simplices of K in such a way that no $(i+1)$-simplex has all its i-faces of the same color.

We have conjectured [19] that there exists a constant C_n depending only on n such that $c_{n-1}(K^n) \leq C_n$ for all simplicial complexes K^n embeddable in $\mathbb{R}^{2n}$. In this context we have the following results.

Corollary 2. *For $n \neq 2$, the class of complexes K^n which unknot in $\mathbb{R}^{2n+1}$ is contained in the class of complexes which embed in $\mathbb{R}^{2n}$ and for this smaller class one has $c_{n-1}(K^n) \leq 3(n+1)$.*

This follows from Corollary 1 and Theorem 7(c).

Corollary 3. *For $n \neq 2$, the class of complexes K^n for which the Kneser graph of the underlying polyhedron is bichromatic is contained in the class of* *complexes which embed in* $\mathbb{R}^{2n}$ *and for this smaller class one has* $c_{n-1}(K^n) \leq 6(n+1)$.

In fact if $G(X_{K^n})$ can be well colored by N colors, then for each color t, one gets a subpolyhedron $X_t \subseteq X_{K^n}$ which is the union of all X_i^n's colored t. Since any two X_i^n's in X_t intersect Theorem 6 implies that X_t unknots in $\mathbb{R}^{2n+1}$ and so the subcomplex K_t^n of K^n covering X_t can have its $(n-1)$-simplices well colored by $3(n+1)$ colors. Taking N disjoint sets of $3(n+1)$ colors to color each $K_t^n \subseteq K^n$ we see that the $(n-1)$-simplices of K^n can be well colored by $3N(n+1)$ colors.

Further results regarding the aforementioned conjecture will be given in a sequel to this paper.[7]

(3.5.2) *Remarks*. The following interesting observations regarding the deleted join are due to Flores [4].

(1) *The deleted join of the n-skeleton of a* $(2n+2)$*-simplex is* $\mathbb{Z}_2$*-homeomorphic to* S^{2n+1}.

By exploiting the formula $(K \cdot L)_\# = K_\# \cdot L_\#$ Flores in fact goes on to give some more examples—the join of $n+1$ copies of three points, the join of σ_k^{2k+2} and $\sigma_{n-k-1}^{2(n-k-1)+2}$, etc.—of n-complexes whose deleted join is also $\mathbb{Z}_2$-homeomorphic to S^{2n+1}. (See also Grünbaum [6], exercise 26, p. 67, pp. 210–212, and [7].)

(2) *The deleted join of a simplicial complex is* $\mathbb{Z}_2$*-homeomorphic to the deleted product of its cone.*

To see this homeomorphism

$$|K_\#| \xrightarrow{\cong} |(v \cdot K)_*|,$$

map each line segment $[x, \bar{y}]$ to the broken line $[(x, v), (x, y)] \cup [(x, y), (v, y)]$ with x going to (x, v), $\frac{1}{2}(x + \bar{y})$ to (x, y) and $\bar{y}$ to (v, y). Since K^n embeds in $\mathbb{R}^m$ iff its cone embeds in $\mathbb{R}^{m+1}$ it follows from (2) that a K^n embeds in $\mathbb{R}^m$ only if there is a $\mathbb{Z}_2$-map from the deleted join of K^n to S^m. So Borsuk's Theorem implies that the examples of (1) do not embed in $\mathbb{R}^{2n}$. Grünbaum [7] proves that if one knocks out an n-simplex from any of these complexes then the resulting complex embeds rectilinearly in $\mathbb{R}^{2n}$. Optimal rectilinear immersions (with just one double point) of some of these examples K^n in $\mathbb{R}^{2n}$ had been considered also by van Kampen [25] who used them to show $o_{2n}(K_*^n) \neq 0$ by a direct computation.

[7] By incorporating Kalai's "algebraic shifting" into the above cohomological setup we have now proved the conjectures of [19] as well as (2.5.4)(c); see [34]. However the special cases considered here continue to retain their interest.

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George Mason University
Fairfax, Virginia

Panjab University
Chandigarh, India