Linear Embeddability –

① A matroid M (poss. oriented) is realizable over a field K iff
∃ a polynomial in $I^K_{n,d} \cap (S^K_M + I^K_M)$

(see B-St p. 71)

Similar theorem ??

② Decidability thm. Does over $\mathbb{R}^n$ imply over A^n?

③ Realization space $\mathcal{R}(K) \subset G^{\mathbb{R}}_{N,m}$ or of $G^{\mathbb{R}}_{N,N-m}$.
(N vertices, m ambient dim n)

Like spaces for matrices etc.

As union of oriented matd realizations

Realization of linear embeddability of simpl. complexes ⟺ one of the matrices (involves $K \neq$!)

④ Equiv. cohomology classes represented by any oriented matroid for each class of Univ. bdl. sphere $\mathbb{R}_2$?

⑤ Simplicity & linear embeddability / Vanishing of Pont. classes.

⑥ Hypersimplices (convex hulls of barycenters of p-simplices? in the σ^N) bdry formula

$(\Delta \cdot)_{\#} = 2$-sphere

⑦ $K^n \subset \mathbb{R}^{n+1}$ / wild
cone-rigidity thm. / spheres.

⑧ Dancis thm.

⑨ Fary thm.

⑩ p.q knots on torus.
If linear ⇒ torus is a knot

⑪ Least no. of vertices on a linear knot

⑫ Dulce's [?] paper & refs. there.

14 → 19

(13) Simple mflds (≠ spheres)
don't embed linearly.
(convexity enters naturally!)

(14) ~~The Knudsen problem~~.

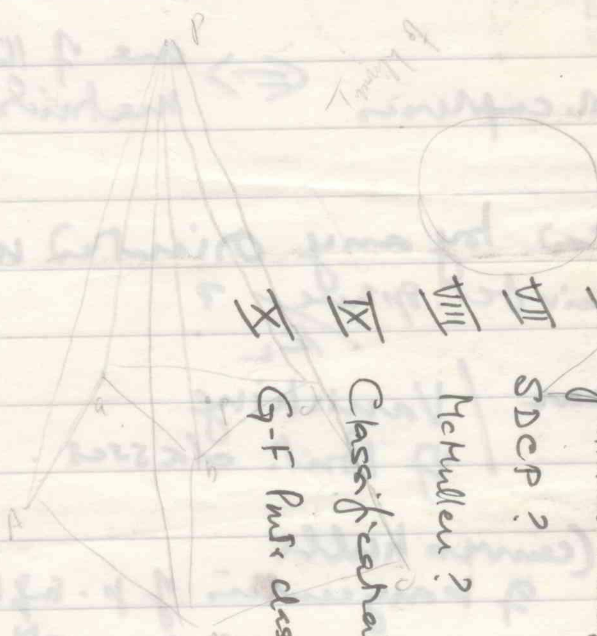
(15) The Möbius strip
which ~~knots~~ doesn't
embed linearly in R^2.

(16) ~~Radon's thm globalized~~.

(17) ~~Tverberg thms globalization~~??

(18) ~~Quantitative questions, SDCP e.g.~~??

4:36



I Rigidity
II Doubling examples
III Tverberg-Vrećica conjecture
IV Oriented matroids/min cl. classes
V Stellar theorem
VI Symmetric stiefling preserves how[?]
VII SDCP?
VIII McMullen?
IX Classification Theorem.
X G-F Pontr classes via *?

Chapter IV. Linear Embeddability

§1. Definition.

A simplicial complex K with vertices $e_1, \ldots, e_N$ has a canonical geometrical realization K in $\mathbb{R}^N$, obtained by thinking of these vertices as the canonical basis vectors of $\mathbb{R}^N$.[1] In this chapter we plan to study geometric or (simplex-wise) **LINEAR EMBEDDINGS** $f : K \to \mathbb{R}^m$, i.e. one-one maps which occur as the restrictions of linear maps $f : \mathbb{R}^N \to \mathbb{R}^m$.

This geometric notion is related to the previous, more flexible, notion of piecewise linear embeddability, as follows.

The p. l. embeddability of K in $\mathbb{R}^m$ amounts to requiring only that some *linear subdivision* K' of K – i.e. a simplicial complex which embeds linearly in $\mathbb{R}^N$ with image K – be linearly embeddable in $\mathbb{R}^m$.

Further, one can here even assume (see Ch.II.2, ...) that K' is a *stellar subdivision* of K, i.e. that K' can be obtained from K as the end result of a sequence of "starring" operations: one stars a simplex σ by replacing $\overline{St\sigma}$ by the cone of its boundary over a new vertex $\hat{\sigma}$.

It is natural thus to consider, for the set of all simplicial complexes, the following two binary relations:

$$\cong_{\mathbf{St}} \;=\; \{(K, L) : K \text{ and } L \text{ have a common stellar subdivision}\}.$$

$$\cong_{\mathbf{Pl}} \;=\; \{(K, L) : K \text{ and } L \text{ have a common linear subdivision}\}.$$

The second relation, i.e. that of being *p. l. homeomorphic* , is obviously an equivalence relation. On the other hand the answer to the following purely combinatorial problem is still unknown.

Q. *Is $\cong_{\mathbf{St}}$ an equivalence relation?*[2]

Despite this ignorance, the following fundamental result shows that *stellar theory* provides a purely combinatorial development of piecewise linear topology.

---

[1] N will always denote the number of vertices of K.

[2] Answer is 'yes': the proof involves the old Newman-Alexander stellar theory.

NEWMAN'S THEOREM. *The equivalence relation generated by* $\cong_{\mathbf{St}}$ *coincides with* $\cong_{\mathbf{Pl}}$.

Proof. By placing the new vertex $\hat{\sigma}$ at, say, the barycenter of σ, one sees that starring σ leads to a p.l. homeomorphic complex. So $[\cong_{\mathbf{St}}]$, the equivalence relation generated by $\cong_{\mathbf{St}}$, is no bigger than $\cong_{\mathbf{Pl}}$.

Before taking up the converse we first list the results from stellar theory which we will use. Their purely combinatorial proofs are given in full in Alexander[...].

(a) A stellar $(n-1)$-sphere, coned over a new vertex, yields a stellar n-ball.

(b) The boundary of any stellar n-ball is a stellar $(n-1)$-sphere.

(c) The union of two stellar n-balls, intersecting in a stellar $(n-1)$-ball common to their boundaries, is a stellar n-ball.

Here, a simplicial complex B^n, resp. S^{n-1}, is called a *stellar* n*-ball* , resp. *stellar* $(n-1)$*-sphere* , iff $B^n[\cong_{St}]\overline{\sigma^n}$, resp. $S^{n-1}[\cong_{St}]\partial\sigma^n$.

The required implication $K^n \cong_{Pl} L^n \implies K^n[\cong_{St}]L^n$ will now be established by induction on n.

It obviously suffices to show that *any linear subdivision* L *of the closed* n*-simplex* $\overline{\sigma^n}$ *is a stellar* n*-ball* . A little more thought shows that in fact it would suffice to prove this assertion only for the case when the subdivision L is related to a finite set Ω of $(n-1)$-dimensional affine hyperplanes as follows:

Ω determines a subdivision of $\overline{\sigma^n}$ into convex cells $\{\omega^i\}$. We want that L should be a subdivision of this cell complex, with each $L|\omega^n = \overline{St_L w^n}$ for some vertex $w^n \in L$.

Assume inductively that the assertion has been established when the number of hyperplanes is lesser.

Using (a) and the inductive hypothesis on n, each $L|\omega^n$ is a stellar n-ball. Choose any hyperplane $\omega \in \Omega$. For each cell $\omega^{n-1} \subset \omega$ which is incident to two n-cells ω^n (c) shows that the subcomplex of L covering these two cells is a stellar n-ball. We replace each of these balls by the cone of its boundary over a new vertex. By (b) and (a) this too is a stellar n-ball.

The stellary equivalent simplicial complex L' thus obtained from L is related to the smaller set $\Omega' = \Omega \setminus \{\omega\}$ of hyperplanes in the required way. So L', and thus also L, is stellarly equivalent to $\overline{\sigma^n}$. **q.e.d.**

An affirmative answer to the above question would improve the above theorem to $\cong_{\mathbf{St}} \;=\; \cong_{\mathbf{Pl}}$.

However, as Milnor [...] first showed, the binary relation $\cong_{\mathbf{Top}}$, i.e. that of being *homeomorphic* , is strictly bigger. In fact Edwards [...] has even given an example of a simplicial 5-sphere which is not p. l. homeomorphic to the boundary of a 6-simplex!!

§2. Radon's Theorem.

For linear embeddability the following useful weakening of the definition is available.

THEOREM. *A linear map* $f : \mathbb{R}^N \to \mathbb{R}^m$ *is one-one on* K *if and only if* $f(\sigma) \cap f(\theta) = \emptyset$ *for all* $\sigma \in K$, $\theta \in K$, *such that* $\sigma \cap \theta = \emptyset$.

Proof. Restricted to each simplex ξ such an f must be one-one. Otherwise by Radon's Theorem (see below) the f-images $\overline{\sigma}$ and $\overline{\tau}$, of some 2 disjoint faces $\sigma, \tau \subset \xi$, would intersect.

The possibility remains that there exist non-disjoint simplices ξ and η of K with $\overline{\xi} \cap \overline{\eta}$ bigger than $\overline{\zeta}$ where $\zeta = \xi \cap \eta$. If so, consider such a pair (ξ, η) with $dim\xi + dim\eta$ least. Any $q \in (\overline{\xi} \cap \overline{\eta}) \setminus \overline{\zeta}$ can not belong to a proper face $\overline{\xi'}$ of $\overline{\xi}$, for then (ξ', η) would be another such non-disjoint pair with lesser dimension sum. But then, the segment going from q to $\overline{w}$, $w \in (\eta \setminus \zeta)$ takes us from $int\overline{\xi}$ to a point outside $\overline{\xi}$, and so, by linearity of f, yields a $q' \in (\overline{\xi} \cap \overline{\eta}) \setminus \overline{\zeta}$, with q' belonging to a proper face $\overline{\xi'}$ of $\overline{\xi}$. **q.e.d.**

The above theorem can be reformulated using deleted joins:

COROLLARY. *A linear map* $f : \mathbb{R}^N \to \mathbb{R}^m$ *is one-one on* K *if and only if the induced linear* $\mathbb{Z}_2$*-map* $f \cdot f : \mathbb{R}^{2N+1} \to \mathbb{R}^{2m+1}$ *images the deleted join* K_* *into the complement of the fixed subspace* $\Delta^m = \{(u, u, 0); u \in \mathbb{R}^m\}$.

Here of course we think of $\mathbb{R}^{2m+1}$ (and likewise $\mathbb{R}^{2N+1}$) as the direct sum $\mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}$ with $\mathbb{Z}_2$-action given by $(u, v, t) \leftrightarrow (v, u, -t)$, and the map $f \cdot f$ is defined by $(x, y, t) \mapsto (f(x), f(y), t)$.

Note that we have thus globalized

RADON'S THEOREM. *Any* $m+2$ *points of* $\mathbb{R}^m$ *admit a partition into* 2 *disjoint subsets* S *and* T *such that* $conv(S) \cap conv(T) \neq \emptyset$.

This is indeed a special case of the previous theorem because, otherwise, the $(m+1)$-dimensional σ^{m+1}_{m+1} would embed linearly in $\mathbb{R}^m$.

Though Radon's Theorem has been much generalized (see e.g. Tverberg's Theorem below) the simple proof of the original result still retains its interest:

Proof. Any cardinality $m+2$ subset $\{e_1, \ldots, e_{m+2}\}$ of $\mathbb{R}^m$ is affinely dependent, i.e. $\sum_{i=1}^{m+2} x_i e_i = 0$, for some real numbers x_i, not all zero, such that $x_1 + \cdots + x_{m+2} = 0$. So the subsets $S = \{e_j : x_j > 0\}$ and

$T = \{e_j : x_j < 0\}$, are nonempty and disjoint, and contain $\frac{\sum_{j\in S} x_j e_j}{\sum_{j\in S} x_j} = \frac{\sum_{j\in T}(-x_j)e_j}{\sum_{j\in T} -x_j}$ in the intersection of their convex hulls. **q.e.d.**

A slight elaboration of the above proof shows that if the $m+2$ points are in general position, then there is only *one* such partition.

In general it is of interest to find the least number of disjoint simplices of K which must intersect under any linear map $f : \mathbb{R}^N \to \mathbb{R}^m$. This number is an obvious measure of the non linear embeddability, of the given simplicial complex K, in $\mathbb{R}^m$.

We will see in §8 that a suitable analogue of the notion "one-one" (for an integer $q > 2$) enables one to globalize also the following more general local result and conjecture.

TVERBERG'S THEOREM. *Any $(m+1)(q-1)+1$ points in $\mathbb{R}^m$ can be partitioned into q pairwise disjoint subsets whose convex hulls have a common point.*

SIERKSMA'S DUTCH CHEESE PROBLEM. *In fact under above hypotheses one must have at least $((q-1)!)^m$ such partitions.*

Proof (of Tverberg's theorem). We use induction on q, starting from the trivial case $q = 1$.

Without loss of generality we can assume that the given set $\Omega = \{e_0, e_1, \ldots, e_{(m+1)(q-1)}\}$ is in a *generic*[3] position in $\mathbb{R}^m$. This ensures that whenever $\Omega_1, \Omega_2, \ldots, \Omega_t$ are disjoint subsets of Ω one has

$$codim\ (\cap_{i=1}^t\ aff\ (\Omega_i)) \ = \ \sum_{i=1}^{t} codim\,(aff\,(\Omega_i)).$$

The inductive hypothesis ensures that there exist q-tuples $\Omega_0, \Omega_1, \ldots,$ Ω_{q-1} of disjoint nonempty subsets of E, such that $\cap_{i=1}^{q-1} conv\,(\Omega_i)$ is nonempty. Out of these select one for which the distance from $conv\,(\Omega_0)$ to $\cap_{i=1}^{q-1} conv\,(\Omega_i)$ is minimal.

If possible suppose this minimal distance is nonzero and let $x \in conv\,(\Omega_0) \subset aff\,(\Omega_0) \subseteq H_x$ and $y \in \cap_{i=1}^{q-1} conv\,(\Omega_i) \subset \cap_{i=1}^{q-1} aff\,(\Omega_i) \subseteq H_y$ be two points between which this minimal distance is attained. Here H_x and H_y are the hyperplanes of $\mathbb{R}^m$, through x and y, perpendicular to the segment $[xy]$. So $aff\,(\Omega_0)$ and $\cap_{i=1}^{q-1} aff\,(\Omega_i)$ do not intersect, which by genericity is possible only if $\sum_{i=0}^{q-1} |E_i| \le (m+1)(q-1)$.

[3] to ...

So one of the points of Ω, say e_0, is not contained in any Ω_i. This point e_0 can not be on the same side of H_x as y, for then the distance from $conv\,(\Omega_0 \cup \{e_0\})$ to $\cap_{i=1}^{q-1} conv\,(\Omega_i)$ would be even smaller than $|xy|$.

But the possibility that e_0 is on the same side of H_y as x can also be ruled out as follows:

With y as origin we can – since $(\cap_{i=1}^{q-1} aff(\Omega_i))^{\perp} = \oplus_{i=1}^{q-1}(aff(\Omega_i))^{\perp}$ – choose new affine coordinates $t_1, \ldots, t_m$, such that $aff(\Omega_1)$ is described by the vanishing of the first C_1 coordinates, $aff(\Omega_2)$ by the vanishing of the next C_2, and so on … . Also we can adjust these coordinates so that e_0 is given by $t_1 = t_2 = \cdots = t_m = 1$. Thus H_y, which contains $\cap_{i=1}^{q-1} aff(\Omega_i)$, is given by a linear equation $a_1 t_1 + \cdots + a_m t_m = 0$, with $a_i = 0\ \forall i > |C_1| + \cdots + |C_{q-1}|$. Choose a j, $1 \leq j \leq q-1$, such that $a_{|C_1|+\cdots+|C_{j-1}|+1} + \cdots + a_{|C_1|+\cdots+|C_j|}$ has the same sign, say positive, as the nonzero number $a_1 + \cdots + a_m$. For such a j we assert that *the distance from* $conv(\Omega_0)$ *to* $conv(\Omega_1) \cap \cdots \cap conv(\Omega_j \cup \{e_0\}) \cap \cdots \cap conv(\Omega_{q-1})$ *is smaller than* $|xy|$. For this verification we will, by a change of notation, assume $j = 1$.

The point $e_0' = (\underbrace{1, \ldots, 1}_{C_1 times}, 0, \ldots, 0)$ is in $aff(\Omega_i)$ for $2 \leq i \leq q-1$, and is also in $aff(\Omega_1 \cup \{e_0\})$ because it is the sum of e_0 and the point $e_0'' = (\underbrace{0, \ldots, 0}_{C_1 times}, -1, \ldots, -1)$ of $aff(\Omega_1)$. So all points on the open segment $(0, e_1')$, which are sufficiently close to the origin, are contained in $conv(\Omega_1 \cup \{e_0\}) \cap conv(\Omega_2) \cap \cdots \cap conv(\Omega_{q-1})$. But, since $a_1 + \cdots + a_{C_1} > 0$, this segment is contained in the half space bounded by H_y which contains x. Thus such points have distances from x less than $|xy|$. **q.e.d.**

In §8 we will use complex roots of unity to give another, more conceptual, proof of Tverberg's Theorem, which generalizes the proof of Radon's Theorem given before.

§3. Oriented matroids.

A free $\mathbb{Z}_2$-simplicial set $\mathcal{O}$, consisting of *circuits* $\Sigma = (\sigma, \theta)$with σ and θ disjoint subsets of $\{e_1, \ldots, e_N\}$, and having no proper inclusion relations, will be called an **oriented matroid** if the following axiom holds.

[o.m.] For any 2 distinct circuits Σ and Θ of $\mathcal{O}$, having a common vertex p, there exists a circuit Φ of $\mathcal{O}$ contained in $(\Sigma \cup \nu(\Theta)) \setminus \{p, \nu(p)\}$.

THEOREM 1. *A simplicial complex K embeds linearly in R^m if and only if its deleted join K_* is disjoint from some* linear *$(m+1)$-dimensional oriented matroid $\mathcal{O}$.*

Proof. Without loss of generality we can assume that the given linear embedding $f : K \to \mathbb{R}^m$ is in general position. Now consider minimal simplices $\Sigma = (\sigma, \theta)$, with σ and θ disjoint subsets of $\{e_1, e_2, \ldots, e_N\}$, such that $f(\sigma) \cap f(\theta) \neq \emptyset$. These have cardinality $m+2$ and obey the above axiom **[o.m]** They furnish us with the required oriented matroid $\mathcal{O}$ disjoint from K_*.

On the other hand an oriented matroid $\mathcal{O}$ of dimension $m+1$, and having N pairs of vertices, is called *linear* if and only if it arises from N points in R^m in the manner just described. So the converse follows by using §2. **q.e.d.**

Identifying each vertex with its antipode one gets from $\mathcal{O}$ the **matroid** $\overline{\mathcal{O}}$ i.e. a simplicial set whose member circuits $\overline{\Sigma}$ have no proper inclusion relations, and which obeys the following axiom.

[m.] For any 2 distinct circuits $\overline{\Sigma}$ and $\overline{\Theta}$ of $\overline{\mathcal{O}}$ having a common vertex $\overline{p}$, there exists a circuit $\overline{\Phi}$ of $\overline{\mathcal{O}}$ contained in $(\overline{\Sigma} \cup \overline{\Theta}) \setminus \{\overline{p}\}$.

Note that $\mathcal{O}$ does indeed "orient" $\overline{\mathcal{O}}$ in the sense that each circuit $\overline{\Sigma}$ of $\overline{\mathcal{O}}$ is covered by exactly 2 circuits of $\mathcal{O}$. Generalizing this one can analogously define G-fold coverings of the circuits of a matroid for groups G other than $\mathbb{Z}_2$.

The oriented matroid $\mathcal{O}$ of above theorem can clearly be assumed to be **simple,** i.e. all circuits are of cardinality $m+2$, and any set consisting of $m+2$ pairs of antipodal vertices contains (exactly) two antipodal circuits.

Convex, projective, and combinatorial embeddability.

At this point it seems worthwhile to examine the definition of §1 more closely, and discuss similar notions for other m-spaces.

Our canonical embedding of K was in fact in $\mathbb{R}A^{N-1} \subset \mathbb{R}^N$, the *affine space* given by $x_1 + \cdots + x_N = 1$, and the linear embeddability[4] of K in m-space amounts to the existence of an *affine map* $\mathbb{R}A^{N-1} \to \mathbb{R}A^m$ – i.e. the restriction of some linear map $\mathbb{R}^N \to \mathbb{R}^{m+1}$ – which injects K into $\mathbb{R}A^m$. One can assume this map to be in general position.

We will say that K is *convexly embeddable* in an m-sphere if there exists a general position linear map $f : \mathbb{R}^N \to \mathbb{R}^{m+1}$ which injects K in the boundary of the convex hull of the points $f(e_1), \ldots, f(e_N)$.

The canonical embedding of K in R^N also induces one in the *projective space* $\mathbb{R}P^{N-1}$ i.e. the space obtained from $\mathbb{R}^N$ by identifying lines through the origin to points. We will say that K is *projectively embeddable* in projective m-space if there exists a *projective map* $\mathbb{R}P^{N-1} \to$

[4]or, perhaps more suitably, the *affine* linear embeddability

$\mathbb{R}P^m$ – i.e. quotient of some linear map $\mathbb{R}^N \to \mathbb{R}^{m+1}$ – which injects K in $\mathbb{R}P^m$.

THEOREM 2. *Convex embeddability into*[5] *an m-sphere $\Longrightarrow$ linear embeddability in m-space $\Longrightarrow$ projective embeddability*[6] *in projective m-space, with both implications strict. Furthermore, if a projective embedding $K \to \mathbb{R}P^m$ lifts to the 2-fold cover S^m, then K embeds linearly in $\mathbb{R}^m$.*

Proof. The convex embedding misses the interior of at least one m-dimensional facet of the simplicial polytope of $\mathbb{R}^{m+1}$ occuring as the convex hull of the images of the vertices. So the first implication follows by projecting into this facet, i.e. by considering the *Schlegel diagram* of the polytope determined by this facet.

The second implication is trivial since the linear embedding in fact induces a projective embedding into the open m-ball of $\mathbb{R}P^m$ determined by the lines of $\mathbb{R}^{m+1}$ passing through its origin and the affine subspace $\mathbb{R}A^m$.

If a projective embedding lifts to S^m then by suitably projecting from a point of S^m, missing this lifting, one obtains a linear embedding in $\mathbb{R}^m$.

The complete graph on 5 vertices is non-planar but embeds projectively in the projective plane (see *fig. 1*), so the second implication is strict.

Linear embeddability in $\mathbb{R}^3$ is apparently much *weaker than convex embeddability into a 3-sphere.* (And likewise for all $m > 3$. For $m = 2$ a theorem of Steinitz – see xxxxx– says that the two notions are equivalent.) For example

(a)Brückner's 3-sphere minus a suitable 3-simplex ,

(b)Rudin's *non-shellable* linear subdivision of the tetrahedron (see xxx), with boundary, excepting one 2-simplex, coned over a 15th vertex, *or*

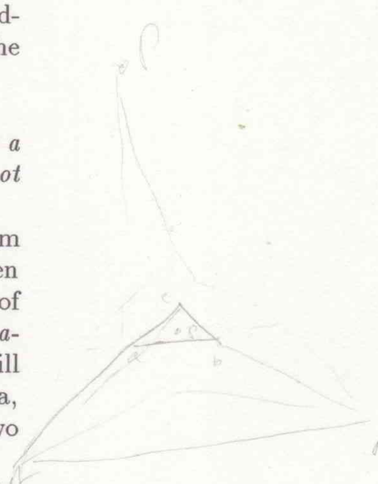
[5] i.e. as a *proper* subset of

[6] cf. Kempf et al

(c) Connelly-Henderson's *non-flexible* linear subdivision of the tetrahedron (see xxxx), with boundary, excepting one 2-simplex, coned over a new vertex,

are all linear subdivisions of the tetrahedron (with only 4 boundary vertices) which fail to embed convexly in a 3-sphere.

For (b) and (c) this follows because convex simplicial polytopal boundaries are shellable and flexible – see xxxx and xxxx – while for (a) the argument is given below. **q.e.d.**



BRÜCKNER-GRÜNBAUM-SREEDHARAN SPHERE. $\exists$ *a 8-vertex neighbourly simplicial 3-sphere which, minus a suitable, but not any, 3- simplex, yields a linear subdivision of the tetrahedron.*

Proof. Consider the triangular prism $abcABC$ of *fig.2,* whose bottom is much larger than its top. Its faces, other than the bottom, have been coned over the point P. Now, first *derive* this polyhedral subdivision of $PABC$ over the barycentre p of abc, and then make a small *perturbation* to a general position subdivision. Note that this last process will change the 3 quadrilaterals $AabB$, $BbcC$ and $CcaA$ into 3 tetrahedra, and thus each of their incident quadrilateral pyramids will split into two 3-simplices.

The neighbourly simplicial 3-sphere $\mathcal{B}$ obtained by adding $PABC$ to this simplicial complex has thus the property that $\mathcal{B} \setminus \{PABC\}$ *embeds linearly in* $\mathbb{R}^3$(: so, unlike the examples to be considered in §5, the 2-skeleton of $\mathcal{B}$ also embeds linearly in $\mathbb{R}^3$).

On the other hand $\mathcal{B} \setminus \{AabB\}$ *does not embed linearly in* $\mathbb{R}^3$, i.e. a tetrahedron $AabB \subset \mathbb{R}^3$ cannot be linearly subdivided by the simplicial complex $\mathcal{B} \setminus \{AabB\}$. To see this – cf. [xxx], p.447 – first put in the 4 tetrahedra incident to the bounding triangles. In this solid, the vertices P and p occur as "saddle points", and thus linearity forces the triangle $\partial(abc)$ to be such that abc (which is not in $\mathcal{B}$) cuts pP somewhere in the middle. So by adding $\overline{St_{\mathcal{B}}\{pP\}} = pP \cdot \partial(abc)$, which has to be convex, we will be left with a 3-ball which is not star shaped with respect to

Give here A. Marin proof instead (pp. 78-79 of Kühnel's survey paper "Triangulations of manifolds with few vertices") any interior point, and so can not be triangulated linearly by coning its boundary over a vertex C. **q.e.d.**

So it follows that *$\mathcal{B}$ is not the facial boundary of a simplicial 4-polytope.* This conclusion can also be drawn by noting that otherwise the simplicial 2-sphere $Lk_{\mathcal{B}}p$ would extend to a simplicial 3-ball having no vertices or edges in its interior. (This 3-ball is the "back" of the polytope which would be determined by the 7 vertices other than p.) Now an Euler characteristic calculation shows that the interior of this 3-ball must be made up of one 2-simplex and three 3-simplices, which is obviously impossible.

Thus though the notion of "projective subdivision" coincides with that of linear subdivision, that of "convex subdivision" is much more restrictive, and of course of stellar subdivision even more so.

Also note that, analogously to Theorem 1, convex embeddability of K in an m-sphere is equivalent to asking that the free $\mathbb{Z}_2$-simplicial complex formed by all pairs of disjoint simplices, of which at least one is in K, be disjoint from some simple linear $(m+2)$-dimensional oriented matroid on the N vertices.

Since all simple oriented matroids are not linear, Theorem 1 suggests a notion of *combinatorial embeddability* of K in some (as yet undefined![7]) *combinatorial m-space* by requiring that the deleted join K_* be disjoint from some simple $(m+1)$-dimensional oriented matroid.

Of course it would be still nicer if there were a definition of "combinatorial space" sufficient to change the above definition into a pleasant combinatorial theorem, and to yield a useful notion of "combinatorial subdivision".

In any case for $m = 2n$, $n \neq 2$, such a K^n does indeed at least embed piecewise linearly in $\mathbb{R}^{2n}$.

This last follows from the fact proved in §6 that *if $\mathcal{O}$ is a simple oriented $(m+1)$-dimensional matroid on the given N pairs of vertices, then $K_* \cap \mathcal{O}$ is the support of a cocycle of the $(m+1)$th van Kampen obstruction class $\mathfrak{o}^{m+1}(K_*)$.* The notation $\mathcal{O}$ for oriented matroids was chosen to highlight this connection with the characteristic classes $\mathfrak{o}$.

§4. Extending triangulations.

We now discuss some p.l. topological results which will be used in §5 to construct examples of simplicial complexes which embed piecewise

[7] now done: om MacPherson type definition

linearly, but not linearly in some higher dimensional Euclidean space $\mathbb{R}^m$.

THEOREM 1. *A simplicial complex K embeds piecewise linearly in a p.l. manifold M^m if and only if it is a subcomplex of some piecewise linear triangulation L of M^m.*

Proof. K p.l. embeds in M^m if and only if some stellar subdivision K' extends to a p.l. triangulation L' of M^m. Without loss assume that K' is obtained by starring just one simplex σ of K. We first retriangulate the interior of the m-ball $B^m = \overline{St_{\hat{\sigma}} L'}$ by taking the cone of some vertex v of σ over the $(m-1)$-ball , $\partial B^m \setminus St_{\partial B^m} v$, of the boundary. Some simplices of L' not in K or B^m might occur in this new triangulation of the m-ball. We take care of this by starring all of these, and then attaching this newer triangulation of B^m to $L' \setminus B^m$. This gives a triangulation L of M^m containing K. **q.e.d.**

Addendum. On the other hand note that if $X \subset M^m$ is the p.l. subspace occuring as the image of some given p.l. embedding $f : K \to M^m$, then it may NOT be possible to find an extension $L \supset K$ such that (L, K) p.l. triangulates the polyhedral *pair* (M^m, X).

A straightforward elaboration of the above argument – cf. Armstrong [...] – shows that this stronger conclusion also holds provided the pair (M^m, X) is *locally unknotted,* i.e. provided $Lk_X x$ unknots in the $(m-1)$-sphere $Lk_{M^m} x$ for all $x \in X$. By Zeeman's theorem regarding p.l. knotting of spheres this is so for all p.l. submanifolds of codimension ≥ 3.

The aforementioned stronger conclusion is also easily seen to be valid if $X^1 \subset M^m$ is any *p.l. arc.* In fact this extendability property characterizes manifolds.

THEOREM 2. *If p.l. space Y has the property that for any p.l. arc $X^1 \subset Y$ one can find a p.l. triangulation L of Y in which X^1 is covered by just one* 1*-simplex, then Y must be a p.l. manifold.*

Proof. We only consider the main case when Y is connected and homogenously m-dimensional, $m \geq 2$. For any 2 points p and q of Y choose a p.l. arc $X^1 = \overline{apqbc}$. Now choose a triangulation L of Y with X^1 covered by a 1-simplex σ. One easily sees that there is a p.l. self-homeomorphism of $\overline{St_L \sigma}$, which is the identity on its boundary, which preserves the arc X^1, and which takes p to q and q to b. By prescribing it to be the identity elsewhere one can also extend it to all of Y. In any case it shows that if p is *non-singular* , i.e. has a neighbourhood p.l. homeomorphic to R^m, then so is q. So all points of Y are non-singular and it is an m-manifold. **q.e.d.**

Alternatively, one can define a point x of Y to be non-singular if $dim_Y x = m$. Here, the *intrinsic dimension* $dim_Y x$, is the largest number r such that there is some triangulation of Y for which x is in the relative interior of some r-dimensional simplex . Any polyhedron Y stratifies naturally into subpolyhedra over which this intrinsic dimension is constant. This stratification constitutes the so-called *intrinsic sekeleton* of Y.

§5. Skeletons.

We feel that for dimensions greater than or equal to the middle dimension, skeletons of manifold-triangulations must display a certain "rigidity" which remains to be defined precisely.

For codimension one this "rigidity" can be expressed precisely as follows, where 2 p.l. embeddings are called *equivalent* if they are related by some p.l. homeomorphism of the ambient space.

THEOREM 1. *If K_{m-1} is the $(m-1)$-skeleton of a p.l.triangulation K of S^m, $m \geq 2$, then any 2 p.l.embeddings of K_{m-1} in S^m are equivalent.*

Proof. We have to show that any p.l.embedding $g : K_{m-1} \to S^m$ extends to a p.l.homeomorphism $g : K \to S^m$.

Let $f_i(K)$ denote the number of i-simplices of K. The number of components of $S^m \setminus g(K_{m-1})$ is the same as those of $K \setminus K_{m-1}$, i.e $f_m(K)$, because, by Alexander duality, either of these numbers is one more than $dim H_{m-1}(K_{m-1}; \mathbb{Z}_2)$.

For each of these components C_j, $1 \leq j \leq f_m(K)$, the mod 2 boundary ∂C_j, being a nonzero $(m-1)$-cycle, must contain at least $m+1$ $(m-1)$-simplices of the topological simplicial complex $\overline{K_{m-1}} = g(K_{m-1})$.[8] On the other hand each $(m-1)$-simplex of $\overline{K}_{m-1}$ occurs in at most 2 boundaries ∂C_j. But $(m+1)f_m(K) = 2f_{m-1}(K)$. So it follows that each ∂C_j has exactly $m+1$ $(m-1)$-simplices of $\overline{K}_{m-1}$, and thus ∂C_j is the g-image of some $\partial(\sigma^m)_j \subseteq K_{m-1}$. Furthermore it follows also that each $(m-1)$-simplex of $\overline{K_{m-1}}$ occurs in precisely two of these boundaries ∂C_j. So C_j must be one of the 2 components of the complement, in S^m, of the $(m-1)$-sphere ∂C_j, and thus closure of each C_j is a closed m-ball with ∂C_j as its boundary $(m-1)$-sphere.

Let L denote the m-dimensional simplicial obtained from K_{m-1} by

[8]For more details see also Chapter [...], §[..], where the same idea is used to prove codimension one Heawood Inequalities for pseudomanifolds.

sticking on an m-simplex $(\sigma^m)_j$ to each of these $\partial(\sigma^m)_j$. Unknottedness of codimension one p.l.(and thus tame) spheres, i.e. Schoenflies' Theorem, now allows us to extend g to a homeomorphism $g : L \to S^m$.

The final step that in fact $L = K$ follows from Dancis' Theorem (see below) and reflects the fact, evident from the above topological description of the C_j's, that the boundaries $\partial(\sigma^m)_j \subset K_{m-1}$ are precisely those whose deletion from K_{m-1} does not disconnect K_{m-1}, a statement independent of the embedding $g : K_{m-1} \to S^m$. **q.e.d.**

The image $\mathcal{S}^{m-2}$ of a locally unknotted embedding $S^{m-2} \to S^m$, which is not equivalent to the standard $S^{m-2} \hookrightarrow S^m$,is called a *knot.* As is well-known, for $m \geq 3$, one does indeed have such codimension 2 knots, thus the existence of triangulations K of the following type is guaranteed by §4.

THEOREM 2. *Let K be a triangulation of the m-sphere S^m , $m \geq 3$, containing the boundary of some $(m-1)$-simplex* (this $(m-1)$-simplex is not in K!)*which covers a knot $\mathcal{S}^{m-2}$. Then its codimension one skeleton K_{m-1} can not embed linearly in $\mathbb{R}^m$.*

Proof. Think of S^m as $\mathbb{R}^m \cup \{\infty\}$. Since linear image of a σ^{m-1}_{m-2} is not a knot, a linear embedding would yield a non-equivalent embedding $K_{m-1} \to S^m$. **q.e.d.**

We expect the aforementioned conjectural "rigidity" to likewise show the non linear embeddability of some skeletons right down to the middle dimensions. In fact we make the following precise

CONJECTURE. *Let K be a triangulation of the m-sphere containing the boundary of some r-simplex, and that of some s- simplex, where $r, s \leq \frac{m}{2} + 1$ and $r + s = m$, which cover 2 disjoint $(r-1)$- and $(s-1)$-dimensional embedded spheres which* link *each other with* linking number ≥ 2. *Then, for any $t \geq \frac{m}{2}$, the t-skeleton K_t can not embed linearly in* $\mathbb{R}^m$.

More evidence of the expected "rigidity" is provided by the result alluded to in the course of the proof of Theorem 1.

DANCIS-PERLES THEOREM. *Up to simplicial isomorphism a triangulation of an m-manifold is determined by its t-skeleton if $t > \frac{m}{2}$. If the middle homology is zero the same is true even for $m = 2t$.*

Proof. Call a $(t+1)$-simplex σ a *hollow simplex* of our simplicial m-manifold K if $\partial\sigma \subseteq K$ but $\sigma \notin K$. We will show that *if $t < \frac{m}{2}$*, *then σ^{t+1} is a hollow simplex of K iff $H_i([K_t \setminus \partial\sigma]) \not\cong H_i(K_t)$ for either $i = m - t$ or $i = m - t - 1$; and, if $t = \frac{m}{2}$ with $H_t(K) = 0$, then iff $H_{t-1}([K_t \setminus \partial\sigma]) \not\cong H_{t-1}(K_t)$.* Clearly this will suffice to determine K from K_t.

The exact mod 2 homology sequence of the pair of spaces $(K, K \setminus \partial\sigma)$ gives

$$0 \to H_{m-t}(K \setminus \partial\sigma) \to H_{m-t}(K) \to \mathbb{Z}_2 \\ \to H_{m-t-1}(K \setminus \partial\sigma) \to H_{m-t-1}(K) \to 0,$$

because by Poincaré Duality $H_{m-t}(K, K \setminus \partial\sigma) \cong H^t(\partial\sigma) \cong \mathbb{Z}_2$. By exactness we must have here either $H_{m-t}(K \setminus \partial\sigma) \not\cong H_{m-t}(K)$ or $H_{m-t-1}(K \setminus \partial\sigma) \not\cong H_{m-t-1}(K)$ with the second alternative holding if $H_{m-t}(K) = 0$.

If $\sigma \notin K$, then $\partial\sigma$ is full in K, so $K \setminus \partial\sigma$ has the homotopy type, and thus the same homology, as the complex $[K \setminus \partial\sigma]$. Furthermore, for $t > \frac{m}{2}$ *resp.* $t = \frac{m}{2}$, these simplicial homologies, in dimensions $\leq m - t$ *resp.* $\leq m - t - 1$, coincide with that of the t-skeletons $[K_t \setminus \partial\sigma]$ and K_t.

Conversely if $\sigma \in K$ then $[K \setminus \partial\sigma] = [K \setminus \overline{\sigma}]$ has the same homotopy type as $K \setminus \{pt.\}$ from which it follows quickly that now the homologies in question have to be isomorphic. **q.e.d.**

This implies that in case the last conjecture is false with K_t embedding linearly with image $X \subset \mathbb{R}^m \subset S^m$, then we cannot extend to a triangulation of S^m without introducing some simplices of dimensions $\leq t$ on $S^m \setminus X$.

It is possible that for $t \geq \frac{m}{2}$, and for any triangulation K of S^m, there is (perhaps modulo some orientation reversal) only one $\mathbb{Z}_2$-homotopy class of $\mathbb{Z}_2$-maps from the deleted join $(K_t)_*$ to the antipodal m-sphere.[9] If so, Weber's isotopy theorem would yield the required "rigidity" under appropriate dimensional restrictions. (Do here Mani-Kleinschmidt-Perles also.)

No examples of the type given in Theorem 2 are possible for $m = 2$ because one has the well known (links with rigidity: separate §?)

FÁRY-WAGNER THEOREM. *Any planar graph embeds linearly in $\mathbb{R}^2$.*

Proof. By §4 we can find a triangulation K of S^2 containing the given planar graph G. But by Steinitz Theorem any such K can be realized as

[9] This can be checked when K has very few vertices, so perhaps encouraging here is the result of Chapter [...], §[..], where the van Kampen-Flores Theorem was extended to n-skeletons of all triangulations of S^{2n+2}.

the facial boundary of some 3-polytope. From this the assertion follows easily. **q.e.d.**

The idea of using links also figures in

BREHM'S MÖBIUS STRIP. *There is a 9-vertex triangulation of the Möbius strip which does not embed linearly in* $\mathbb{R}^3$.

Proof. For any Möbius strip embedded in $\mathbb{R}^3$ the bounding circle I links the inside middle circle J (see *fig.1)* . So it links another inside circle J', winding twice, with

linking number ≥ 2 in absolute value. But in the triangulation of *fig. 2* , I and J' (and even J) are just triangles; so this cannot happen if the embedding is linear. **q.e.d.**

So far no simplicial orientable 2-manifold has been shown to be non linearly embeddable in $\mathbb{R}^3$.

However there are *higher-dimensional Möbius strips* analogous to the one above.

THEOREM 3. *Let* $n = 2^r$, $r \geq 1$. *Then the manifold-with- boundary* $M^n = \mathbb{R}P^n \backslash n$*−ball p.l. embeds in* $\mathbb{R}^{2n-1}$, *but admits a triangulation* K *which does not embed linearly in* $\mathbb{R}^{2n-1}$.

Proof. Since $\mathbb{R}P^{n-1}$ embeds in $\mathbb{R}^{2n-2}$, it follows that the trivial line bundle over it embeds in $\mathbb{R}^{2n-1}$. Now we want to obtain an embedding for the twisted line bundle M^n. For this we locally twist the trivial bundle, for each of the $\mathbb{R}^{n-1}$ worth of directions along $\mathbb{R}P^{n-1}$, in the corresponding direction from the $\mathbb{R}^{n-1}$ worth of directions available complementary to the embedded trivial bundle.

For any embedding of M^n *in* $\mathbb{R}^{2n-1}$, *the bounding* $(n-1)$*-sphere must link the "core"* $\mathbb{R}P^{n-1}$*:*

Because, otherwise, there would be a map of $\mathbb{R}P^n$ in $\mathbb{R}^{2n-1}$, with all double points off this core, i.e. in an n-ball. By standard coning constructions of embedding theory (see Ch. II, pp. ...) we would thus be able to embed $\mathbb{R}P^n$ in $\mathbb{R}^{2n-1}$ which, since $n = 2^r$, contradicts Thom's theorem of Ch. II, pp. ...

Thus the self-linking number of the boundary is nonzero and even:

This follows because the boundary can be isotoped (using the aforementioned twisted line bundle), to another $(n-1)$-sphere very close to the core, and it is clear that any general position n-disk with boundary ∂M, will hit this $(n-1)$-sphere twice as many times as it hits the core.

Now we triangulate M^n by a K in such a way that its boundary, and an isotope of the same (of the above kind) are both triangulated as boundaries of n-simplices (not in K). Then this K can not embed linearly in $\mathbb{R}^{2n-1}$ because 2 such $(n-1)$-spheres could then, if at all, link with linking number only $+1$ or -1. **q.e.d.**

This method can be pushed further to give a double-dimensional example thus disproving a conjecture of Grünbaum.

THEOREM 4. *For all $n \geq 3$ there exist n-complexes K^n which embed piecewise linearly, but not linearly, in $\mathbb{R}^{2n}$.*

Proof. We define K^n as follows:

We start with σ_n^{2n+2} and take out an open n-ball from the interior of one of the n-simplices. (The triangulation covering the remaining n-simplices spanned by these original $2n+3$ vertices will remain unchanged in the construction.)

On the other hand we take a cone over $\mathbb{R}P^{n-1}$ and take out an open n-ball from its non-singular part.

We now form the "connected sum" of these complexes. The resulting complex L^n has "boundary" $\mathbb{R}P^{n-1}$. We will denote by P^n the n-complex obtained by coning this boundary over a new point.

Next, we take a higher Möbius n-strip and identify a "core" $\mathbb{R}P^{n-1}$ of the same with the boundary of L^n. This gives K^n. Note that in it the codimension one simplices on $\mathbb{R}P^{n-1}$ have valence 3, and the "boundary" is now the bounding S^{n-1} of the Möbius strip. We take care to triangulate it as a simplicial boundary.

K^n embeds piecewise linearly in $\mathbb{R}^{2n}$:

To see this we first embed L^n. We start with a general position map in which $2n+1$ of the original vertices enclose all others inside their convex hull. This precaution ensures that any pair of double points can be joined by an arc passing through at most one singular point. So we can remove these pair of double points by using [...].

Now we extend this embedding of L^n to a general position map of K^n. By only retaining a portion of the Möbius n-strip near its core we can once again ensure the above at-most-one-singular-point condition.

So another sequence of applications of [...] gives a p.l. embedding of K^n in $\mathbb{R}^{2n}$.

However K^n does not embed linearly in $\mathbb{R}^{2n}$:

To see this we first note that *P^n does not embed piecewise linearly in $\mathbb{R}^{2n}$*. This follows because its deleted join contains a closed invariant $(2n+1)$-pseudomanifold which admits no continuous $\mathbb{Z}/2$-map to S^{2n}.

So, *under any p.l.embedding of K^n, one of the minimal n-sphere from the original $2n+3$ vertices, must link the $\mathbb{R}P^{n-1}$.* Otherwise the embedding of the subspace L^n can be extended to a general position map of P^n with the new cone imaged away from the union of all these n-spheres. But then, using [...] again, we will be able to embed P^n, contradicting the last paragraph.

Reciprocally, *any general position n-disk, thrown across this minimal n-sphere, must hit $\mathbb{R}P^{n-1}$.* This follows by the same homological argument which is used to show that the definition of the linking of two spheres is in fact a reciprocal definition.

Hence it must hit any isotope (within the Möbius collar) of the bounding S^{n-1}, which is sufficiently near this core, an even nonzero number of times. I.e. our minimal n- and $(n-1)$-spheres have linking number ≥ 2 in absolute value. Which is possible only if they have been embedded non-linearly. **q.e.d.**

§6. Combinatorial characteristic cocycles.

We will now identify oriented matroids as "optimal" characteristic cocycles of a free $\mathbb{Z}_2$-simplicial complex. Clearly it suffices to consider the universal free $\mathbb{Z}_2$-complexes U, i.e. *octahedral spheres* on N pairs of vertices.

THEOREM 1. *A set $\mathcal{O}$ of $(m+1)$-simplices of U is a simple oriented matroid if and only if it is a cocycle of $\overline{\mathrm{o}^{m+1}} \in H_s^{m+1}(U;\mathbb{Z}_2)$ having the least number of simplices.*

Proof. If U^{m+1} is an $(m+1)$-dimensional octahedral sphere formed by any $m+2$ pairs of antipodal vertices of U, then it must contain exactly 2 antipodal (and thus disjoint) circuits of the matroid $\mathcal{O}$. Otherwise, some 2 circuits contained in U^{m+1} intersect, and so using axiom **[o.m.]** , one would have a circuit of cardinality less than $m+1$.

To check that the matroid $\mathcal{O}$ forms a mod 2 cocycle we have to verify that any cardinality $m+3$ subset A of U contains an even number of circuits of $\mathcal{O}$. We will show that it in fact must contain either none or else 2 circuits of $\mathcal{O}$.

Suppose circuit $\Sigma \subset A$ with $\{p\} = \{A \setminus \Sigma\}$. Out of all circuits of $\mathcal{O}$ contained in U^A, the $(m+2)$- dimensional octahedral sphere determined by vertices of A and their antipodes, let Θ be a circuit containing p, and having a maximal intersection with Σ. We assert $\Theta \subset A$, i.e. that $\Sigma \setminus \Theta$ has cardinality 1. Otherwise note that the antipodes of all but one of the points of $\Sigma \setminus \Theta$, say q, are in Θ. Since Σ and $\nu(\Theta)$ share these antipodes, axiom **[o.m.]** applied to them gives a circuit Φ containing p, contained in U^A, whose intersection with Σ is bigger than that of Θ, since q too will lie in it.

Now take 2 circuits $\Sigma, \Theta \subset A$ with $\{p\} = A \setminus \Sigma$ and $\{q\} = A \setminus \Theta$. Applying **[o.m.]** to these 2 circuits it follows that the two antipodal circuits of U^A, which miss a given vertex of $\Sigma \cap \Theta$ and its antipode, contain either $\{p, \nu(q)\}$ or $\{\nu(p), q\}$.Thus A contains no circuits other than Σ and Θ.

This $\mathbb{Z}_2$-cocycle $\mathcal{O}$ represents a nonzero $\mathbb{Z}_2$- cohomology class because its $\mathbb{Z}_2$-value $< \mathcal{O}, [U^{m+1}] >$ is 1 on any $(m+1)$-dimensional octahedral $\mathbb{Z}_2$-cycle. Since the real projective space $U/\mathbb{Z}_2$ has the unique nonzero $(m+1)$- dimensional cohomology class $\overline{\mathrm{o}}$, it follows that $\mathcal{O}$ must represent $\overline{\mathrm{o}}$.

Next note that $\mathcal{O}$ has $2\binom{N}{m+2}$ simplices, and any cocycle representing $\overline{\mathrm{o}}$ cannot have a lesser number of simplices, because $< \overline{\mathrm{o}}, [U^{m+1}] > \neq 0 \ \forall \ U^{m+1}$, forces it to have at least one pair of antipodal simplices on each U^{m+1}.

Thus, conversely, an $(m+1)$-dimensional cocycle $\mathcal{O}$ representing $\overline{\mathrm{o}}$, and having $2\binom{N}{m+2}$ simplices, has

(i) exactly one pair of antipodal simplices on each $U^{m+1} \subset U$, and

(ii) has, in any cardinality $m+3$ subset A of U, either none, or else more than one (in fact an even number of) simplices.

Under these conditions it follows from Folkman and Lawrence [F-L], pp.228-231 – (i) and (ii) correspond to (c) and (e) of their p.228 – that $\mathcal{O}$ obeys the required axiom **[o.m.]** of oriented matroids. **q.e.d.**

This result can be further precised by using integer coefficients.

THEOREM 2. *A set $\mathcal{O}$ of $(m+1)$-simplices of U is a simple oriented matroid if and only if it is the support of a cocycle of $\mathrm{o}^{m+1} \in H_{\pm}^{m+1}(U; \mathbb{Z})$ which is nonzero on the least number of simplices. Furthermore this cocycle is unique upto an odd integral factor.*

Proof. Let us define the *Folkman graph* of our oriented matroid $\mathcal{O}$ to be the one whose vertices are the circuits of $\mathcal{O}$ with any 2 joined iff they are contained in a cardinality $m+3$ subset $A \subset U$. In [F-L], p.230, it is shown that this graph is connected; further it can be shown (see below) that it contains no closed loops of odd lengths. Thus upto sign reversal there is a unique way of assigning signs $+$ and $-$ to the vertices of this graph so that adjacent vertices get opposite signs. **q.e.d.**

Before proceeding further we take a quick look back at the standard theory.

A REVIEW OF CHARACTERISTIC CLASSES

¶1. Additional (e.g. a differentiable) structure on a space B is often given in terms of a (real) *vector bundle* ξ, which assigns to each $x \in B$ an n-dimensional subspace E_x of $\mathbb{R}^\infty$, the real vector space of eventually zero sequences of real numbers.

Under this *Gauss map* ξ, from B to the *Grassmanian* $G_n(\mathbb{R}^\infty)$ – or $BO(n)$ – of n-dimensional subspaces of $\mathbb{R}^\infty$, the cohomology of the latter pulls back to the ring of *characteristic classes* of ξ.

¶2. Obviously the cohomology of the base space B is isomorphic, under the projection $E_x \mapsto x$, to that of the *total space* $E = \cup_x E_x$. Moreover there is an n-dimensional orientation or *Thom class* u $(= \phi(1))$ along the fibers of $(E, E \setminus B)$[10] , multiplication with which induces an additive isomorphism $\phi : H^*(B) \cong H^*(E) \to H^{*+n}(E, E \setminus B)$. Here the coefficients are $\mathbb{Z}/2$ unless the fibers are compatibly oriented when they are $\mathbb{Z}$.

¶3. The *Stiefel-Whitney classes* $w(\xi) \in H^*(B; \mathbb{Z}/2)$ of ξ can now be defined by *Thom's formula* $w(\xi) = \phi^{-1} Sq\, u$, where Sq denotes the Steenrod squaring operations of Chapter II.

Note that, in the oriented case, and for *i odd or = n*, this recipe also defines integral *obstruction classes* $\mathfrak{o}_i(\xi)$. For $i < n$ these are of order 2 and just Bockstein images of $w_{i-1}(\xi)$. However, the highest dimensional one of these, which is given by $\phi^{-1}u^2$, is more interesting, and is called the *Euler class* $e(\xi)$.

¶4. The functor $\xi \mapsto w(\xi)$, which is determined by its values on the tautological or *universal n-plane bundle* γ_n over $G_n(\mathbb{R}^\infty)$, is non-trivial, and is characterized by the fact that it obeys *Whitney's formula* $w(\xi \oplus \eta) = w(\xi)w(\eta)$.

Moreover there is no universal algebraic relation amongst the w_i's as can be seen e.g. by considering cartesian products of the tautological line bundle of $\mathbb{R}P^2$.

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[10] We identify B with the zero section.

¶**5.** On the other hand the number of monomials ${w_1}^{i_1}\cdots {w_k}^{i_k}$ of degree r is easily seen to equal the number of *partitions* of r into at most n parts. But this is also the number of r-cells in the following *Ehresmann subdivision* of $G_n(\mathbb{R}^\infty)$ into a CW complex.

Here, for each *Schubert symbol* $\sigma = 1 \leq \sigma_1 < \cdots < \sigma_n$, there is the $[(\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n)]$-dimensional open cell Σ : it consists of all n-dimensional subspaces U of $\mathbb{R}^\infty$ whose intersections with the standard flag $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \cdots$ have their n dimensional jumps of 1 at $U \cap \mathbb{R}^{\sigma_1}, U \cap \mathbb{R}^{\sigma_2}, \ldots$.

So it follows that the *mod 2 cohomology of the Grassmannian* G_n is a free polynomial algebra over the classes $w_i(\gamma_n)$.

¶**6.** The highest characteristic class can also be obtained simply by *restricting* the Thom class to B.

Hence, using tubular neighbourhoods, it follows that the highest, i.e. kth, class of the normal bundle of a manifold M^n smoothly embedded in the cohomologically trivial $\mathbb{R}^{n+k}$, must be zero. Whence one has *Whitney's embeddability criterion* $(w^{-1})_k(M^n) = 0$ for the tangent bundle of the manifold.

¶**7.** Also, for the tangent bundle of a smooth oriented manifold, the Euler class equals *Euler characteristic* times the orienting generator of $H^n(M^n; \mathbb{Z})$.

This formula is proved in [...] by using the fact that *the tangent bundle is canonically isomorphic to the normal bundle of the diagonal of* $M \times M$. So what is required is a computation of a restriction of the Thom class of the latter.

The same basic idea gives the more (for $\mathbb{Z}/2$ coefficients) general *Wu's formula* $w_k = \sum_{i+j=k} Sq^i(v_j)$ where $v = 1 + v_1 + \cdots$ is the cohomology class, multiplication by which, gives the squaring operation $x \mapsto Sq\, x$.

¶**8.** The original *homotopy theoretic definition* of the obstruction classes $\mathfrak{o}_k(\xi)$ was as the primary obstruction to finding $n-k+1$ linearly independent sections, i.e a section of the associated bundle of $(n-k+1)$-frames.

Its fiber, the *Stiefel manifold* $V_{n-k+1}(\mathbb{R}^n)$ is $n-(n-k+1)-1$ i.e. $(k-2)$-connected. So there is no obstruction to finding a section over the $(k-1)$-skeleton, and the primary obstruction is in the kth cohomology group of B with local coefficients $\{\pi_{k-1}(V_{n-k+1}(E_x)\}$, which, for k odd or $k = n$, are twisted integers, and for k even, integers mod 2.

§7. Equiangular polygons.

A convex polygon with all angles equal, and sides of lengths $\ell_0, \ell_1, \ldots, \ell_{q-1}$, gives rise to a q-tuple of evenly spaced concurrent vectors $(\vec{\ell_0}, \vec{\ell_1}, \ldots, \vec{\ell}_{q-1})$, $\vec{\ell_i} = \ell_i e^{\frac{2\pi\sqrt{-1}}{q}}$, having vector sum $\vec{\ell_0} + \vec{\ell_1} + \cdots + \vec{\ell}_{q-1} = \vec{0}$ (see *fig.1*).

Allowing these real numbers ℓ_i to be possibly zero or negative, we will refer to any member of $\mathcal{P}oly_q(\mathbb{R})$, the real vector space of all such q-tuples $(\vec{\ell_0}, \vec{\ell_1}, \ldots, \vec{\ell}_{q-1})$, as a (possibly degenerate) real *equiangular q-gon.*

Likewise, if the lengths ℓ_i are restricted to be rational or integral, then we have the $\mathbb{Q}$-vector space $\mathcal{P}oly_q(\mathbb{Q})$ or $\mathbb{Z}$-module $\mathcal{P}oly_q(\mathbb{Z})$ of rational or integral equiangular q-gons.

If a (degenerate) equiangular q-gon has exactly r nonzero lengths, all equal to 1, then we will refer to it as an *inscribed* regular r-gon. Note that r must divide q and that there are $\frac{q}{r}$ such inscribed polygons.

SCHOENBERG'S THEOREM. *The $\mathbb{R}$-vector space $\mathcal{P}oly_q(\mathbb{R})$, of real equiangular q-gons,$q \geq 3$, has dimension $q-2$. On the other hand, the $\mathbb{Q}$-vector space $\mathcal{P}oly_q(\mathbb{Q})$, of rational equiangular q-gons, has dimension only $q - \varphi(q)$. Furthermore, the $\mathbb{Z}$-module $\mathcal{P}oly_q(\mathbb{Z})$, of integral equiangular q-gons, is generated by its inscribed regular p-gons, as p runs over all prime factors of q.*

Here φ denotes *Euler's phi function* , i.e. $\varphi(q)$ is the number of positive integers less than and relatively prime to q.

Proof. For $q \geq 3$, the $\mathbb{R}$-linear map $\mathbb{R}^q \to \mathbb{C}$ given by $(\ell_0, \ell_1, \ldots, \ell_{q-1}) \mapsto \vec{\ell_0} + \vec{\ell_1} + \cdots + \vec{\ell}_{q-1}$ is surjective. So its kernel, $\mathcal{P}oly_q(\mathbb{R})$, has $\mathbb{R}$-dimension $q-2$.

Likewise, the $\mathbb{Q}$-linear map $\mathbb{Q}^q \to \mathbb{C}$ given by $(\ell_0, \ell_1, \ldots, \ell_{q-1}) \mapsto \vec{\ell_0} + \vec{\ell_1} + \cdots + \vec{\ell}_{q-1}$, $\vec{\ell_i} = \ell_i e^{\frac{2\pi\sqrt{-1}}{q}}$, has image $\mathbb{Q}(e^{\frac{2\pi\sqrt{-1}}{q}})$, a *cyclotomic field extension* of $\mathbb{Q}$, which is known (cf. e.g. [xxx], p. xxx) to have $\mathbb{Q}$-dimension $\varphi(q)$. So its kernel, $\mathcal{P}oly_q(\mathbb{Q})$, has $\mathbb{Q}$-dimension $q - \varphi(q)$.

Note that *equiangular q-gons* $(\vec{\ell_0}, \vec{\ell_1}, \ldots, \vec{\ell}_{q-1})$ *identify with polynomials* $\ell(X) = \ell_0 + \ell_1 X + \cdots + \ell_{q-1} X^{q-1}$ *of degrees less than q having value* $\ell(e^{\frac{2\pi\sqrt{-1}}{q}}) = 0$, and that the maps used above were restrictions of the evaluation map $\mathbb{C}[X] \to \mathbb{C}$, given by $\ell(X) \mapsto \ell(e^{\frac{2\pi\sqrt{-1}}{q}})$, to subspaces of $\mathbb{R}[X]$ and $\mathbb{Q}[X]$ consisting of polynomials of degrees less than q.

A complete verification of the last assertion can be found in [xxx] where use is made of the fact that it is obviously equivalent to saying that for any $\ell(X) \in \mathcal{P}oly_q(\mathbb{Z}) \subset \mathbb{Z}[X]$ one has

$$\ell(X) = \sum_{p|q} c_\ell(X)(1 + X^{\frac{q}{p}} + \cdots + X^{\frac{q(p-1)}{p}})$$

for a suitable choice of the coefficients $c_\ell(X) \in \mathbb{Z}[X]$. **q.e.d.**

Since $\varphi(p) = p - 1$ for p prime, one has the following striking special case of the second part of the above theorem, which can also be proved directly by using the irreducibility (see e.g. [xxx], p. xxx) of the polynomial $1 + X + \cdots + X^{p-1} \in \mathbb{Z}[X]$.

COROLLARY 1. *For p prime an equiangular rational p-gon* $(\vec{\ell}_0, \vec{\ell}_1, \ldots, \vec{\ell}_{p-1})$ *must be* regular , *i.e.* $\ell_0 = \ell_1 = \cdots = \ell_{p-1}$.

As against this, for primes $p > 3$, there are many *real* non-regular, but equiangular, p-gons. For example, by parallel displacement of a side of a regular pentagon (see *fig.2*) one gets a one parameter family of real equiangular pentagons.

COROLLARY 2. *For all primes $p > 3$, the diagonals of a regular p-gon, p prime, are incommensurable with respect to its side.*

We omit the details, but note that for $p = 5$ this follows by Cor. 1 because the one-parameter family of equiangular pentagons of *fig.2* degenerates into a triangle having the diagonal as one of its sides.

BEIJING POLYGON. *There is an equiangular* 1990*-gon with sides of distinct lengths from* $\{1^2, 2^2, \ldots, 1990^2\}$.

Proof. Since 2 distinct natural numbers can not yield the same 2 remainders after division by the 2 primes 5 and 199, it follows that the equiangular 995-gon corresponding to the polynomial

$$\ell(X) = (0 + 5X + \cdots + 5 \cdot 198X^{198})(1 + X^{199} + \cdots + X^{4\cdot199}) + (1 + 2X + 3X^2 + 4X^3 + 5X^4)(1 + X^5 + \cdots + X^{198\cdot5})$$

has sides of distinct lengths from $\{1, 2, \ldots, 995\}$.

By doubling these vectors, and then reducing the length of each by 1, we get an equiangular 995-gon whose sides have distinct lengths from $\{1, 3, 5, \ldots, 1989\}$. Now, in the direction having length $1990 - (2i - 1)$, $1 \leq i \leq 995$, take a vector of length $(1991 - i)^2$, and, in the opposite direction, take a vector of length i^2. Since the difference of these 2 numbers is always 1991 times $1990 - (2i - 1)$, it follows that these 1990 vectors also have vector sum zero, and so give an equiangular 1990-gon of the required kind. **q.e.d.**

The same reasoning shows more generally that *for any even q having 2 distinct odd prime factors, there exist equiangular q-gons having sides of distinct lengths from* $\{1^2, 2^2, \ldots, q^2\}$. It would be interesting to precise Schoenberg's theorem further to obtain an explicit characterisation of integer q-sequences (or even of sets of integers) which arise from integral equiangular q-gons.

§8. Carathéodory's Theorem.

The notion of convexity makes sense for a vector space over *any* totally ordered field, and it is useful sometimes to consider it in such a generality: e.g. the following result (with $\mathsf{F} = \mathbb{Q}$) will lead to a more conceptual proof of the theorem of Tverberg considered in §2.

BÁRÁNY'S THEOREM. *Suppose that inside A, a d-dimensional affine space over a totally ordered field F, there are given $d + 1$ subsets S_i, each containing the point v inside its convex hull. Then these $d + 1$ sets have a* section T, *i.e. a set having one element from each S_i, such that $v \in conv(T)$.*

Proof. It is more convenient to pass to a $(d + 1)$-dimensional vector space (cf. p.6) $V \supset A$ and to prove the equivalent *linear version* : if $v \in pos(S_i)$, the positive cone generated by each of $d+1$ subsets $S_i \subseteq V$, then $v \in pos(T)$, for a suitable section T of $\{S_i\}$.

We first treat *the case* $\mathsf{F} = \mathbb{R}$ using the well known theorem (see below) of Carathéodory, which in fact can also be viewed as a special case of the result being proved.

We can obviously assume that each S_i is finite, and thus that there are only finitely many sections $T = \{v_1, \ldots, v_i, \ldots, v_{d+1}\}$, $v_i \in S_i$. Hence it suffices to show that whenever the point v's distance from some $pos(T)$ is positive, then its distance from at least one other $pos(T')$ is smaller:

Let $[vx]$, $x \in pos(T)$, be the segment realizing this positive distance. The hyperplane H_x through x perpendicular to $[vx]$ is a codimension one subspace which either (i) supports, or (ii) contains, the cone $pos(T) \subset V$. In either case – for (ii), by Carathéodory's theorem – there is a $v_i \in T$ such that $x \in pos(T \setminus \{v_i\})$. Further, since $v \in pos(S_i)$, there must also be a $v_i' \in S_i$ on the same side of H_x as v. Replacing v_i by v_i' we get another section T' whose $pos(T')$ is nearer to v.

To deal with *subfields* $\mathbb{F}$ *of* $\mathbb{R}$ we will now suppose that the above V was a completion of an $\mathbb{F}$-vector space V_0, and that the point v and the sets S_i were all in V_0. From a section T such that $v \in pos_{\mathbb{R}}(T)$ we now extract a minimal subset $\hat{T} = \{v_{i_1}, \ldots, v_{i_t}\}$ such that $v \in pos_{\mathbb{R}}(\hat{T})$. By Carathéodory's theorem the $\mathbb{R}$-vector space W generated by $\hat{T}$ – which is the completion of the $\mathbb{F}$-vector space W_0 generated by $\hat{T}$ – is t-dimensional with $\hat{T}$ as a basis. Thus the positive real numbers a_i occuring in an expression $v = a_1 v_{i_1} + \cdots + a_t v_{i_t}$ are in fact the nonzero coordinates of the point $v \in V_0 \subset V$ with respect to any $\mathbb{F}$-basis of V_0 (so also $\mathbb{R}$-basis of V) which contains $\hat{T}$. It follows that these a_i are in $\mathbb{F}$, i.e. $v \in pos_{\mathbb{F}}(\hat{T})$, and thus $v \in pos_{\mathbb{F}}(T)$.

Likewise, for *any totally ordered* $\mathbb{F}$, it is enough to consider its real-closure $\overline{\mathbb{F}}$, when an argument similar to the case $\mathbb{F} = \mathbb{R}$ works, provided one uses a "distance" defined in terms of the valuation (see §10) associated to the new total order. **q.e.d.**

Putting $S_i = S$ for all i, one sees that Bárány's result includes

CARATHÉODORY'S THEOREM. *If v is contained in the convex hull of S, a subset of affine d-space A, then it must also be contained in the convex hull of a subset T of S having cardinality $\leq d+1$.*

Proof. Once again, we prefer to verify the equivalent *linear version* in a $(d+1)$-dimensional vector space $V \supset A$. Since $v \in pos(S)$, the smallest positive cone containing S, we know that v is a positive linear combination $\sum a_i v_i$ of the points v_i of some finite subset T of S. If T has more than $d+1$ elements, they must also satisfy a linear relation $\sum b_i v_i = 0$ with some coefficient positive. Amongst all indices i for which b_i is positive, choose that, say j, for which a_i/b_i is minimum. Then, by subtracting a_j/b_j times this linear relation, we see that v is also a positive linear combination of the points of the smaller set $T \setminus \{v_j\}$. **q.e.d.**

This classical result implies that *if f is a linear map of a simplicial* *complex K into $\mathbb{R}^d$, then the f-image of K coincides with that of its d-skeleton K_d.* Likewise, the more general theorem of Bárány can be globalized as follows.

THEOREM 1. *Let $\mathbf{f} = f_1 \cdot \ldots \cdot f_t : \mathbf{K} = {}^1K \cdot \ldots \cdot {}^tK \to \mathbb{R}^d \cdot \ldots \cdot \mathbb{R}^d$ be the join of any t linear maps $f_i : {}^iK \to \mathbb{R}^d$. Then all points of the diagonal, which are contained in the $\mathbf{f}$-image of $\mathbf{K}$, must also be contained in the $\mathbf{f}$-image of its d-skeleton $\mathbf{K_d}$.*

Proof. ... ... **q.e.d.** (Repairs here)

It is time now to look again at a result of §2:

Another proof of Tverberg's Theorem. It is more appropriate to consider the given points as being in *affine* , rather than linear m-space. So let $\Omega = \{v_0, v_1, \ldots, v_{(m+1)(q-1)}\} \subset \mathbb{R}A^m \subset \mathbb{R}^{m+1}$, where $\mathbb{R}A^m$ is given by $x_1 + \cdots + x_{m+1} = 1$.

Also, it is easy to see that there would be no loss of generality in supposing that these points have *rational* coordinates. So we suppose that they in fact lie in rational affine m-space $\mathbb{Q}A^m \subset \mathbb{Q}^{m+1}$.

Case q prime. We treat this case separately because now the argument is easier to visualize geometrically.

For each of the points $v_i \in \mathbb{Q}^{m+1} \subset \mathbb{R}^{m+1} \subset \mathbb{C}^{m+1}$ we construct a set $\{{}^rv_i : 0 \leq r < q\}$ of q points of $\mathbb{C}^{m+1}$, viz. those obtained by *rotating* every component of v_i by the same multiple of $\frac{2\pi}{q}$. So ${}^rv_i = e^{r\frac{2\pi\sqrt{-1}}{q}} v_i$.

Note that these $(m+1)(q-1)+1$ orbits of this rotation group lie in fact in $(\mathbb{Q}(e^{\frac{2\pi\sqrt{-1}}{q}}))^{m+1}$, a $\mathbb{Q}$-vector space having $\mathbb{Q}$-dimension exactly $d = (m+1)(q-1)$ because q is a prime.

So Bárány's theorem applies, and we obtain non-negative rationals t_i having sum $\sum_{i=0}^{(m+1)(q-1)} t_i = 1$, and integers n_i from $[0, q)$, such that

$$\sum_{i=0}^{(m+1)(q-1)} t_i e^{n_i \frac{2\pi\sqrt{-1}}{q}} v_i = 0. \tag{1}$$

This equation of $(m+1)$-tuples is equivalent to $m+1$ scalar equations. Besides, since the sum of the $m+1$ coordinates of each v_i is 1, one also has an additional equation

$$\sum_{i=0}^{(m+1)(q-1)} t_i e^{n_i \frac{2\pi\sqrt{-1}}{q}} = 0. \tag{1a}$$

By Cor.1 of §7 the $m+2$ equiangular q-gons, represented by the $m+2$ equations just mentioned, must all be *regular* polygons. So the q vectors

$$(2) \qquad \sum_{\substack{0 \le i \le (m+1)(q-1) \\ n_i = r}} t_i v_i \ , \ 0 \le r < q,$$

are equal to each other, and likewise the q (necessarily positive) scalars

$$(2a) \qquad \sum_{\substack{0 \le i \le (m+1)(q-1) \\ n_i = r}} t_i \ , \ 0 \le r < q$$

are also equal to each other.

The point of $\mathbb{Q}A^m$, obtained by scalarly multiplying the reciprocal of this scalar with this vector, is contained in the convex hull of each of the q pairwise disjoint subsets of Ω given by

$$(3) \qquad \Omega_r = \{v_i : n_i = r\} \ , \ 0 \le r < q.$$

So $conv(\Omega_0) \cap \cdots \cap conv(\Omega_{q-1}) \neq \emptyset$.

General case. We first choose, e.g. by using Eisenstein's criterion, some irreducible polynomial $a_0 + a_1 X + \cdots + a_{q-1} X^{q-1} \in \mathbb{Z}[X]$, with all a_i's positive integers.

Now, for each $v_i \in \Omega$, we construct the q vectors ${}^r v_i = \omega^r v_i$, where ω is a root of the above polynomial. The irreducibility of this polynomial shows that $(\mathbb{Q}(\omega))^{m+1}$, in which all these vectors lie, has $\mathbb{Q}$-dimension $d = (m+1)(q-1)$.

Further, since the positive linear combinations $\sum_{r=0}^{q-1} a_r \, {}^r v_i$ vanish, the origin 0 is contained in the convex hull of each of these q-sets, and so Bárány's Theorem again gives (1) and (1a) with $e^{\frac{2\pi\sqrt{-1}}{q}}$ replaced by ω.

This time each component of the q vectors given by (2), and also the q scalars (2a), will be in the same proportion to each other as $\frac{1}{a_0} : \frac{1}{a_1} : \cdots : \frac{1}{a_{q-1}}$.

So scalarly multiplying the reciprocal of any of these q scalars with the corresponding vector we will always get the same point of $\mathbb{Q}A^m$, which thus lies in the convex hull of each of the q pairwise disjoint subsets Ω_r of Ω given by (3). **q.e.d.**

However we have been unable so far to adapt this new proof to establish the following generalization of Tverberg's theorem which has been conjectured by Tverberg-Vrećica [ ].

CONJECTURE. *Let* $\Omega_0, \Omega_1, \ldots, \Omega_k$, $0 \le k \le m-1$ *be finite subsets of* $\mathbb{R}^m$ *with card(*Ω_i*)* $= (q_i-1)(m-k+1)+1$ *for* $i = 0, 1, \ldots, k$. *Then each* Ω_i *can be partitioned into* q_i *pairwise disjoint subsets* Ω_{ij}, $j = 1, \ldots, r_i$ *in such a way that some k-dimensional affine subspace of* $\mathbb{R}^m$ *meets the convex hull of each* Ω_{ij}.

comment

Attempted Proof. The case $k = 0$ is Tverberg's theorem.

If no such k-flat exists, then the same wil be the case for any neighbouring sets. So we can assume that our points have rational coordinates. As before it will be convenient to have them on the rational affine m-space $\mathbb{Q}A^m \subset \mathbb{Q}^{m+1} \subset \mathbb{R}^{m+1}$.

Assume inductively that the sets $\Omega_0, \ldots, \Omega_{k-1}$ have already been partitioned into Ω_{ij}'s whose convex hulls have nonempty intersections with some $(k-1)$-flat $\overline{A}^{k-1}$. By perturbing it a bit we can assume that this flat is in general position with respect to the finite set Ω_k, and that it is *rational* , and as such, the completion of a $\mathbb{Q}$-flat $A^{k-1} \subset \mathbb{Q}A^m$ having the same properties.

Choose in A^{k-1} any set W of k affinely independent points $w_1, \ldots, w_k$, and consider each of them as having *multiplicity* $q_k - 1$.(The case $q_k = 1$ being trivial we are assuming $q_k > 1$.) Adding these *multipoints* to Ω_k we obtain a *multiset* Ω of cardinality exactly $(q_k - 1)(m + 1) + 1$.

Choose now an irreducible polynomial $a_0 + a_1 X + \cdots + a_{q_k-1} X^{q_k-1} \in \mathbb{Z}[X]$ of degree $q_k - 1$ with all coefficients nonzero. Let $\mathbb{F} = \mathbb{Q}(\omega)$ be the field obtained by attaching a root ω of this polynomial to $\mathbb{Q}$. Note that $\mathbb{F}$ has $\mathbb{Q}$-dimension $q_k - 1$, and so $\mathbb{F}^{m+1}$ has $\mathbb{Q}$-dimension $(q_k-1)(m+1)$.

Corresponding to each $v \in \Omega_k$ we now consider the set S_v of the q_k points $\{a_o v, a_1 \omega v, \ldots, a_{q_k-1} \omega^{q_k-1} v\}$ of $\mathbb{F}^{m+1}$. The sum of these points being zero, one has $0 \in conv(S_v)$ for all $v \in \Omega$.

So, using the rational Bárány theorem, we obtain a section $v \mapsto T(v) \in S_v$, and some non-negative rationals t_v with sum 1, such that

$$\sum_{v \in \Omega} t_v \, T(v) \; = \; 0.$$

Define $\Omega^j = \{v : T(v) = a_j \omega^j v\}$. By the irreducibility of our polynomial the q_k vectors $\sum_{v \in \Omega^j} t_v T(v)$ are all equal, and the same is true for the q_k numbers $\sum_{v \in \Omega^j} t_v$. The last because the sum of the coordinates of each $T(v)$ is 1.

Thus there is a point α common to the q_k convex hulls $conv(\Omega^j)$, viz. that obtained by scalarly multiplying the reciprocal of the aforementioned scalar with the aforementioned vector.

By passing if need be to subsets of Ω^j we can assume that these convex sets are geometric simplices. If α were in A^{k-1}[The mistake is here: argument given below is fallacious and this can occur] , then, because of general position of the latter with respect to Ω_k, each of the Ω^j have to contain some point of W. And so in fact α would be in the convex hulls of the sets $W \cap \Omega^j$. But if its barycentric coordinate with respect to say w_r is nonzero, then it can not lie in the convex hull of that $W \cap \Omega^j$ which does not contain w_r: there is indeed such a j because any of the k new points w_i, being of multiplicity $q_k - 1$, is not contained in all the Ω^j's.

So it follows that the k-flat A^k determined by A^{k-1} and α meets the convex hulls of each of the q_k subsets $\Omega_{kj} = \Omega \cap \Omega_k$, which are indeed pairwise disjoint as required. **q.e.d.**

endcomment (class numbers, Bernouilli etc ?)

Also we do not know of a refinement, of the above method of finding "Tverberg partitions", powerful enough to solve the problem of Sierksma mentioned in §2.

§11. Posets of subspaces.

Generalizing the notion of "cell-subdivision" one can speak, for any (finite) poset P, of a *P-filtered space* X, i.e. one equipped with a monotone function $v \mapsto X_v$ from P to the set of closed subspaces of X. The *order complex* $K(P)$ of the poset P, i.e. that whose simplices are totally ordered subsets of P, can still sometimes give a pretty faithful picture of the topology of X. For example this is so for any "linear filtration" of a Euclidean space, thus yielding the following result for the (reduced) homology of the complement of some affine subspaces.

GORESKY-MACPHERSON FORMULA. *Let P be the poset under $\subseteq$ whose elements are all the intersections of a given finite family $\mathcal{A}$ of affine subspaces of $\mathbb{R}^m$. Then*

$$(1) \qquad H_i(\mathbb{R}^m \setminus \cup\mathcal{A}) \cong \bigoplus_v H^{m-\, dim v\, -i-1}(K(P^{>v}), K(P^{>v} \setminus \{\mathbb{R}^m\}))$$

where v runs over all elements of P other than the maximal element $\mathbb{R}^m$.

Proof. *Step1. A reformulation of the formula.*

For $x \in \mathbb{R}^m$ and $y \in |K(P)|$, we denote by v_x and σ_y the minimal members of P and $K(P)$, which contain x and y respectively. Now define a P-filtered space $C(P)$ by $(C(P))_v = \{(x, y) : v_x \subseteq min(\sigma_y) \subseteq max(\sigma_y) \subseteq v\}$. Also, equip $\mathbb{R}^m$ and $K(P)$ with the P-filtrations $(\mathbb{R}^m)_v = v$ and $(K(P))_v = \{\sigma : max(\sigma) \subseteq v\}$.

The projections $x \leftarrow\!\!\prec (x,y) \rightarrowtail y$ of $C(P)$ onto $\mathbb{R}^m$ and $|K(P)|$ preserve these P-filtrations. Furthermore, their fibers, $\{x\} \times K(P[v_x, \mathbb{R}^m])$ and $min(\sigma_y) \times \{y\}$ are contractible, and have contractible intersections with all the subspaces $(C(P))_v$.

From this, by working up the skeleta, it follows easily that *$\mathbb{R}^m$ and $K(P)$ have the same P-homotopy type* (i.e. that they are homotopy equivalent via homotopies preserving their P-filtrations) and likewise that, for all non-maximal $v \in P$, $\mathbb{R}^m$ and $K(P^{>v})$ have the same $P^{>v}$-homotopy type.

So (1) is equivalent to saying that the homology of the complement of the family $\mathcal{A} = \{A_0, A_1, \ldots, A_k\}$ is given by

$$H_i(\mathbb{R}^M \setminus \bigcup_{i \geq 0} A_i) \cong \bigoplus_v H^{m-dimv-i-1}(\mathbb{R}^m, \bigcup_{i\geq 0}\{A_i : A_i \supset v\}) .$$

with v running over intersections of all nonempty subsets of $\mathcal{A}$.

Let S_i denote the sphere in $S^m = \mathbb{R}^m \cup \infty$ occuring as the one-point compactification of A_i. Applying Alexander duality in S^m and noting that $\mathbb{R}^m$ is contractible we obtain the equivalent formula

$$(2) \qquad H^q(\bigcup_{i\geq 0} S_i) \cong \bigoplus_v H^{q-1-dimv}(\bigcup_{i \geq 0}\{A_i : A_i \supset v\})$$

where $q = m - i - 1$ and, as before, v runs over intersections of all nonempty subsets of $\mathcal{A}$.

Step II. Requirements for an induction on k.

For $k = 0$ the only term on the right side of (2) is the summand corresponding to $v = A_0$, i.e. the reduced cohomology of the empty set, and so is nonzero (and $= \mathbb{Z}$) only when $q - 1 - dimA_0 = -1$, i.e. $q = dimS_0$, and thus coincides with the left side, which is the cohomology of the sphere S_0.

Now suppose that A_0 is such that this last occurs as a summand in the following direct sum decomposition of the left side of (2),

$$(3) \qquad H^q(\bigcup_{i\geq 0} S_i) \cong H^q(S_0) \oplus H^q(\bigcup_{i\geq 1} S_i) \oplus H^{q+1}(\bigcup_{i \geq 1} S_0 \cap S_i),$$

and also is such that the right side of the formula admits the following direct sum decompositions for the summands contributed by all v such that $v \subset A_0$,

(4)

$$H^p(\bigcup_{i\geq 0}\{A_i : A_i \supset v\}) \cong H^p(\bigcup_{i\geq 1}\{A_i : A_i \supset v\}) \oplus H^{p+1}(\bigcup_{i\geq 1}\{A_0 \cap A_i : A_i \supset v\}),$$

where $p = q - 1 - dimv$.

Linear?
Pts → 1. Counter example to the ind
→ 2. Dyson lemma links-up
→ 3. Fadell Neuwirth paper example for Wu lemma (giving it to preprint)
→ 4. Connection again with oriented matroids

In this case the inductive hypothesis, applied to the smaller affine arrangements, $\{A_1, \ldots, A_k\}$ and $\{A_0 \cap A_1, \ldots, A_0 \cap A_k\}$, immediately implies (2) because of (3) and (4).

Step III. Satisfying the requirement.

If the affine arrangement is in ***sufficiently general position*** , i.e. if any non-empty intersection becomes bigger if one of the A_i's occuring in it is excluded, then the obvious Mayer-Vietoris sequences suggested by (3) and (4) split into 3-term short exact sequences with *all groups free abelian* , from which it follows that the requirement is satisfied by choosing A_0 to be *any* of the subspaces.

For the general case the proof of this last step will be given later. Thus for the moment we have, only for the g.p. case, **q.e.d.** (Zeigler-Ziv best now)

comment

Notes. The Hauptvermutung i.e. that homeomorphic simplicial complexes are piecewise linearly homeomorphic, was disproved by Milnor [M], and, for simplicial manifolds, by R.D.Edwards [E]. The latter gave a simplicial complex homeomorphic to S^5 with no subdivision in common with that of an ordinary triangulation, say σ_5^6.

Hudson's Problem is stated on p.16 of [H]. See also pp. - and Zeeman[Z], pp. - . A very combinatorial treatment of basic p.l. topology using stellar subdivisions was given by Alexander [A].

Radon's Theorem appeared in [R] and Tverberg's Theorem in [T], see also [T2]. An extension to continuous maps was given in [BB] and [BLS] (for p prime), see also [S1]. The globalization via linear embeddings is apparently new. For Sierksma's Problem and some others see Reay [Re]. See also §§ ... of this book.

A basic reference for oriented matroids is [BW]. The relationships with convex polytopes and ... see Connelly and Henderson[CH], ...

Simple oriented matroids as characteristic cocycles is new. See however Gelfand and MacPherson [GM].

For extending triangulations see besides Armstrong [A] also Akin [A_0], $whoalsoisresponsibleforintrinsicdimensionetc. Zeeman'sUnknottingTheoremappearedin[Z2].$ ∎

The Rigidity Theorem, though possibly known, appears here for the first time. Examples similar to that of Theorem 2, §5, were given first by van Kampen [vK5], and were inspired by those of Cairns [C]. On the same line is the example of Grunbaum and Sreedharan [GS]. Dancis' Theorem appeared in [D]. The Fary [F]-Wagner[W] theorem has been rediscovered repeatedly see [S], [Y], ... Brehm's example appeared in[B]. The (in our opinion very optimistic) conjecture that all triangulations of the 2-torus embed linearly in $\mathbb{R}^3$ is attributed to Dick [D] where more can be found re such questions.

endcomment

July 23, 2021 I began hunting for this old (prob. 1991) paper amongst the mass of stuff in my office because its memory came up as I was typing an item for the next installment of my miscellany or ℓλα πα underway. It took quite a few weeks before I refound these old notes! I have scanned (∵ tex file is maybe only a floppy, so readable only by tech savvy, in said stuff) as well the two pages of memos in my hand attached in front because they give an idea of the sweep of this intended "Chapter IV" – I'd totally forgotten this point – of an intended book. More of interest may come to mind (especially if I pursue what I've typed in aforesaid item), if so all this – hopefully including updates on problems and errata – will then go into an accompanying web page (*).