When is the Locally Nilpotent Part a Direct Summand?

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Abstract. The *locally nilpotent part* of a linear transformation f of a vector space V is its restriction to the subspace $K_f := \bigcup_n f^{-n}(0)$, and it is called a *direct summand* of f if there is a subspace C of V such that $f(C) \subseteq C$, $K_f \cap C = 0$ and $V = K_f + C$. We show that such a C exists only if a whole slew of non-trivial necessary conditions hold, one for each limit ordinal, and that these conditions are independent of each other. Moreover, our examples are *simplest possible*—there is a basis in which the matrix of f has all entries zero excepting at most one in each row—and amongst these transformations we characterize those whose locally nilpotent part is a direct summand.

§ 0. This paper is a compendium of examples from infinite dimensional linear algebra, starting with Example 1 (2.3)—circulated in June, 2005—which settled the following problem of Dinesh Khurana in the negative: *given a linear map* $f: V \to V$*, let* $K_f = \bigcup_n \ker(f^n)$, *can one always find a subspace* C *such that* $f(C) \subseteq C$ *and* $V = K_f \oplus C$?

This example was found interesting[1], in particular, the necessary condition (*) of (2.1) whose failure had been used to detect the non-existence of C. This has transfinite analogues $(*)_\alpha$ which are given in (3.3). Here α is any limit ordinal, so *there is a trans-infinity of necessary conditions for the existence of* C. We show by means of extra examples (3.4)—I knew of all the conditions $(*)_\alpha$ in June, 2005, however these additional examples were observed much later—*that all these necessary conditions are non-trivial and quite independent of each other.*

These conditions are topological, they say (2.2) that certain vector subspaces are closed in suitable adic topologies, and are tied with the *stability* of the transfinite completion $f^\alpha(V)$ of the sequence of iterated images. So we begin with some remarks on stability, then give a construction in (1.5)—with transfinite generalization later in (3.1) and (3.2)—that shows that this sequence may not stabilize till any ordinal whatsoever. The examples mentioned so far are all particular cases of this construction.

The search for more examples is driven by the harder and still open problem: *when exactly can one find a subspace* C *such that* $f(C) \subseteq C$ *and* $V = K_f \oplus C$? In all the examples mentioned so far, the space $f^\omega(V) = \bigcap_n f^n(V)$ is nonzero, put otherwise, the adic topology is non-hausdorff. The suspicion that this might always be so was laid to rest by the quite different Example 2 (2.4)—found in mid-August, 2005 very soon after I was laid up in bed for a couple of months by a cycling accident, the 'types theorem' below was also discovered then—which has $f^\omega(V) = 0$, yet a C of the desired kind may not exist, the detection is however done again by using the same condition (*).

This multiplicity of examples is subsumed in § 4 in a single theorem. Treating its vertices as a vector space basis we associate to a *directed graph* G *with at most one*

[1] For instance by two well-known algebraists who had misproofs showing that the answer to Dinesh's question was "yes", indeed I too had a misproof of my own, an analysis of a gap in it led to Example 1

arrow coming out of each vertex the linear map f_G defined by these arrows. We show that *the locally nilpotent part of* f_G *is a direct summand iff the conditions* $(*)_\alpha$ *hold,* indeed this Theorem (4.2) gives a full classification of such graphs: *the components of any* G *are of the five mutually exclusive types described in* (4.1), *and the locally nilpotent part is a direct summand if and only if only* Types 1, 3 *and* 5 *are present.*

Such maps f_G are the *simplest possible* (5.2): each row of the matrix of f_G with respect to the basis of vertices has at most one nonzero entry. However our methods hold promise for any f, notably they show that the existence of C puts serious restrictions on the dynamics, i.e., the nature of the *saturated orbits* (5.3) of the linear transformation f.

The questions treated in this paper could have been considered by Hamel et al. even in the early 1900's, a perusal of the literature however revealed nothing of that vintage.[2] We found out however that Ulm [3] had already used Cantor's transfinite completion in linear algebra to work out the structure of locally algebraic linear transformations of countable dimensional vector spaces in the 1930's, and I suspect he knew the answer to Dinesh's question. My secondary evidence is Exercise 33(a) on page 32 of Kaplansky's wonderful 1954 book [2]. This exercise is about infinite abelian groups, i.e., **Z**-modules, however later, in § 12 of this book, its author switches to modules remarking that many previous results hold for modules over principle ideal domains. Since linear transformations identify with **F**[X]-modules, I presume Kaplansky was aware of the linear transformation analogue of Exercise 33(a). However, he does not take credit for the exercise, and its context – the exercise is used to show that the countability hypothesis cannot be dropped from Ulm's theorem – leads us to suspect that Ulm knew. Anyway, it is quite certain that the answer to Dinesh's question was known – but obviously not *well* known! – when he posed it to me in April, 2005. However this 'known example' – see (5.1) – is inferior to Example 1 in one important respect, it requires a much bigger uncountable dimensional vector space. In fact, all these 'Kaplansky type examples' fit into our scheme of things as follows: they pertain to the linear transformations φ_G *dual* to f_G.

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§1. A subspace C of a vector space V (the field of coefficients **F** is arbitrary) is a *complement* of another subspace K if $K \cap C = 0$ and $V = K + C$, the last two equations are often rolled into one: $V = K \oplus C$. The study of a linear transformation $f: V \to V$ leads one to focus on subspaces W that are f-*invariant,* i.e., such that $f(W) \subseteq W$. Of particular interest are the following two monotonic sequences of invariant subspaces, namely, the *iterated kernels* $\ker f^n = f^{-n}(0)$ and the *iterated images* $\operatorname{im} f^n = f^n(V)$ of f :

$$0 \subseteq \ker f \subseteq \ldots \subseteq \ker f^n \subseteq \ker f^{n+1} \subseteq \ldots,$$

$$V \supseteq \operatorname{im} f \supseteq \ldots \supseteq \operatorname{im} f^n \supseteq \operatorname{im} f^{n+1} \supseteq \ldots .$$

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[2] *Added July 12, 2006:* From a footnote in a paper of Baer in the Annals of 1936, see p. 767, it seems that the group theoretic Exercise 33(a) of [2] might even be traced back to a 1919 thesis of F. Levi!

We shall often denote the union $\bigcup_n f^{-n}(0)$ and the intersection $\bigcap_n f^n(V)$ of the terms of these two sequences by K_f and I_f respectively.

(1.1) A term of either sequence is called *stable* if it equals the next, then it equals all the subsequent terms: if $f^{-(n+1)}(0) = f^{-n}(0)$ then $f^{-(n+2)}(0) = f^{-1}(f^{-(n+1)}(0)) = f^{-1}(f^{-n}(0)) = f^{-(n+1)}(0)$, and likewise if $f^{n+1}(V) = f^n(V)$ then $f^{n+2}(V) = f(f^{n+1}(V)) = f(f^n(V)) = f^{n+1}(V)$.

(1.2) *The* n*th kernel* $\ker f^n$ *is stable if and only if* $\ker f^n \cap \operatorname{im} f^n = 0$. To see this note that the intersection is nonzero if and only if there is a v with $f^n(v)$ nonzero but $f^{2n}(v)$ zero, that is, if and only if $\ker f^n$ is strictly smaller than $\ker f^{2n}$, now use (1.1).

(1.3) Easy examples show that, in general, the stability of $\ker f^n$ does not imply that of $\operatorname{im} f^n$, however there is one important case in which it does. *For the case* dim(V) *finite dimensional,* $\ker f^n$ *is stable if and only if* $\operatorname{im} f^n$ *is stable, and then each is the unique* f*-invariant complement of the other,* i.e., $V = K_f \oplus I_f$ uniquely.

This follows from (1.2) because now codim ($\ker f^n$) = dim ($\operatorname{im} f^n$) tells us that the dimensions of the kernel and image of any f^n must add up to the finite number dim(V), and for uniqueness check that if C is f-invariant with $V = \ker f^n \oplus C$, respectively $V = C \oplus \operatorname{im} f^n$, then $f^n(V) \subseteq C$, respectively $f^{-n}(0) \subseteq C$.

The locally nilpotent part is known to be a direct summand in many other cases also—but a proof hinging just on the aforementioned can usually be made—however we don't intend to review these arguments, but will now turn to something else.

(1.4) Let us look again at how the above two sequences run. The increasing sequence is obtained, starting from the zero subspace 0, by applying at each step f^{-1} to the previous term, while the decreasing sequence is obtained, starting from the vector space V itself, by applying at each step f to the previous term. The two operations behave differently in one important respect. If we apply f^{-1} to the union of the increasing sequence, that is, to K_f, we get nothing bigger, we only get back K_f. We express this by saying that *the sequence of iterated kernels stabilizes at the first infinite ordinal.* On the other hand, the analogous statement is by no means true for the operation of taking iterated images: applying f to the intersection I_f of the decreasing sequence can give us something smaller, in fact even the following is true.

(1.5) *Any linear map* $g: U \to U$ *can be realized as the restriction to* I_f *of an* $f: V \to V$ *with* $\operatorname{im}(f^n)$ *strictly bigger than* $\operatorname{im}(f^{n+1})$ *for all* $n \geq 0$.

Choose any set of elements v of U such that the union of the (forward) *orbits* $\{g^i v : i \geq 0\}$ spans U and, for each v, a distinct set $\{v_{i,j}\}$ of new linearly independent symbols, $1 \leq j \leq i$. Let V be the linear span of U and these symbols, and define $f: V \to V$ to be the same as g on U, while on the new symbols f is defined thus: it takes any $v_{i,j}$ with $j \geq 2$ to $v_{i,j-1}$ while all the $v_{i,1}$ go to v (cf. the 'tail' of v in Figure 1, which depicts the special case when the forward orbit of v forms a basis of U). This f has the desired properties.

So the second sequence has a non-trivial transfinite completion, however we shall postpone these ordinal-theoretic generalizations to §3, and first present the main idea vis-à-vis the incomplete sequence only.

§2. Though in the following, and its later transfinite analogues, K can be any f-invariant subspace, we'll be using these criteria almost always for $K = K_f$.

(2.1) *Let* $f: V \to V$ *be any linear transformation, then an* f*-invariant subspace* K *of* V *can have an* f*-invariant vector space complement only if*

$$K + \bigcap_n f^n V \;=\; \bigcap_n (K + f^n V). \qquad (*)$$

Proof. If $V = K \oplus C$ with $f(K) \subseteq K$ and $f(C) \subseteq C$, induction shows that one must have $f^n V = f^n K \oplus f^n C$ for all integers $n \geq 0$, so $\bigcap_n f^n V = \bigcap_n f^n K \oplus \bigcap_n f^n C$. It follows that both sides of (*) are then equal to $K \oplus \bigcap_n f^n C$, *q.e.d.*

(2.2) *The* f*-adic topology*, on the underlying vector space V of a linear endomorphism f, is the one whose open sets are all unions of cosets of the type $v + f^n V$, $v \in V$, $n \geq 0$. In this f-adic topology $\bigcap_n (K + f^n V)$ is the closure $\overline{K}$ of any f-invariant subspace K, for example, *the closure* $\overline{0}$ *of the origin is* $I_f = \bigcap_n f^n V$. Thus (*) is the same as saying $K + \overline{0} = \overline{K}$. The f-adic topology is hausdorff if and only if $0 = \overline{0}$, so in this case (*) is just saying that K is closed in V. In general, (*) is saying that K is *quasi-closed* in V, by which we mean that it projects, under the canonical surjection $V \to V/\overline{0}$, to a closed subspace of this associated Hausdorff quotient topological space. We omit the verifications – the later conditions $(*)_\alpha$ also have similar topological reformulations – because no topological arguments are used in this paper.

There is no dearth of invariant subspaces without an invariant complement, for example, the subspace of all (0,y) is such for the transformation $(x,y) \to (0,x)$ of $\mathbf{F}^2$, and in general (*) is not of much use in detecting them, for instance, it obviously holds for all finite dimensional vector spaces V.

However, it is a relatively delicate job to find examples of linear transformations f whose K_f does not have an invariant complement, and we'll see that (*) and its transfinite analogues are quite useful in detecting these. A preliminary remark about this case is the following: (*) *holds for* $K = K_f$ *iff it holds for each direct summand of* f. This is so because the iterated kernels and images of a linear transformation are the direct sums of the corresponding subspaces contributed by its direct summands. This remark applies to the later conditions $(*)_\alpha$ also.

We'll come across the non-triviality of (*) for $K = K_f$ in both the examples worked out below, note that the second has $I_f = 0$, that is, its f-adic topology is hausdorff.

(2.3) EXAMPLE 1. *Let* V *be any countable dimensional vector space, and write a basis of* V *as* $\{v_{i,j} : 0 \le j \le i\}$. *Let* $f : V \to V$ *be the linear map defined by* $f(v_{i,0}) = v_{i+1,0}$, $f(v_{i,1}) = v_{0,0}$, *and* $f(v_{i,j}) = v_{i,j-1}$ *if* $j > 1$ (see Figure 1). *Then the* f-*invariant vector subspace* $K_f = \bigcup_n \ker f^n$ *does not have an* f-*invariant vector space complement in* V.

[Figure 1]

Proof. Note that $\bigcap_n f^n V$ is the linear span of $\{v_i : i \ge 0\}$, and that $\ker f$ is the linear span of all differences $v_{i,1} - v_{i',1}$, $\ker f^2$ the linear span of ker f and all differences $v_{i,2} - v_{i',2}$, etc. So K_f consists of all those finite linear combinations of the $v_{i,j}$'s, with $j \ge 1$, such that the sum of the coefficients for each fixed j is zero. Thus $K_f + \bigcap_n f^n V$ is strictly smaller than V. On the other hand, $f^n V$ is the span of all basis elements other than $v_{i,j}$'s with $j > i - n$. For each j, if i is big enough, this inequality is false, so $K_f + f^n V$ equals V. Therefore $\bigcap_n (K_f + f^n V) = V$ and (*) does not hold for $K = K_f$. *q.e.d.*

(2.4) EXAMPLE 2. *Let* V *be any countable dimensional vector space, and write a basis of* V *as* $\{ v_{i,j} : 0 \le j \le mi + c\}$, *where* m *and* c *are two fixed non-negative numbers. Let* $f : V \to V$ *be the linear map defined by* $f(v_{i,0}) = v_{i+1,0}$, *and* $f(v_{i,j}) = v_{i,j-1}$ *if* $j \ge 1$ (see Figure 2 which shows the case $m = 4/3$, $c = 1$). *Then the* f-*invariant vector subspace* $K_f = \bigcup_n \ker f^n$ *has an* f-*invariant complement in* V *if and only if* $m \le 1$.

[Figure 2]

Proof. Note $I_f = 0$ and that K_f is a proper subspace of V because the basis elements are not in it. Any difference of the type $v_{i,i+t} - v_{i',i'+t}$, $i' > i$, i.e., a difference of two basis elements lying on the same parallel to the line $j = i$, is an element of K_f because $f^{i'+t}$ maps both elements to $v_{i',0}$ (these differences constitute a basis of K_f). The subspace $\mathrm{im}\, f^n$ is spanned by all basis elements satisfying $j \le mi + c - n$, i.e., all elements on or under the line parallel to the 'roof' and n units below it. In case $m > 1$, for any t we have $i+t \le mi + c - n$ for all i sufficiently big. This shows that $K_f + \mathrm{im}\, f^n = V$ for all n, so $\bigcap_n (K_f + \mathrm{im}\, f^n) = V$ is bigger than K_f . Thus (*) does not hold for $K = K_f$ and it cannot have an invariant complement.

In case $m \le 1$, there is a t such that $j \le i + t$ for all basis elements $v_{i,j}$, we choose the smallest such t, this is also the biggest t such that there is a basis element of the type $v_{i,i+t}$. Look at the basis elements lying in the orbit of such a $v_{i,i+t}$. For each $c \le t$, there is one and only one element $v_{i,j}$ of this orbit such that $j = i + c$ — i.e., this orbit has one element on each parallel to $j = i$ that contributes the aforementioned differences to K_f —which shows $K_f + C = V$, where C denotes the linear span of the orbit; the linear independence of the orbit's elements ensures the other condition $K_f \cap C = 0$. *q.e.d.*

§ 3. The *transfinite completion* of the sequence $f^n(V)$ of iterated images is defined[3] thus: for the first infinite ordinal ω we'll use $f^{\omega}(V) = I_f = \bigcap_n f^n(V)$, for bigger ordinals α having an immediate predecessor α-1 we shall put $f^{\alpha}(V) = f(f^{\alpha-1}(V))$, while for all limit ordinals α, $f^{\alpha}(V)$ shall be the intersection $\bigcap_{\beta<\alpha} f^{\beta}(V)$.

Using (1.5) we know already that $f^{\omega+1}(V)$ can be strictly smaller than $f^{\omega}(V)$, that a similar statement is true for any $\alpha > \omega$ follows likewise from (3.2) below.

(3.1) *Tails* T_α shall be directed graphs with a unique final vertex which are defined inductively, for all ordinals α, as follows. T_0 is just a vertex. If $\alpha > 0$ has an immediate predecessor α-1, then an α-tail T_α is obtained by drawing an arrow from the final vertex of a $T_{\alpha-1}$ to a new vertex. On the other hand, if $\alpha > 0$ is a limit ordinal, then we take disjoint copies of all the graphs T_β with $\beta < \alpha$, and make T_α by drawing arrows from the final vertices of all these graphs to a single new vertex. So, for n finite, T_n is a *chain* of n arrows, while the subgraph of Figure 1 spanned by v and all preceding vertices is a T_ω.

It is easy to check that two tails are isomorphic iff they correspond to the same ordinal, and that the subgraph of T_α spanned by vertices preceding or equal to a given vertex v is isomorphic to T_β for some $\beta \le \alpha$, we assign to v the *label* β. An easy induction shows that *no vertex of* T_α *has an infinite chain of arrows ending in it*. Note that this is a reformulation of the elementary fact that a strictly decreasing sequence of ordinals is necessarily finite for it has a smallest term, indeed one might think of tails as a graphical construction and definition of the ordinals themselves.

(3.2) THEOREM. *Given any linear map* $g: U \to U$, *and any ordinal* α, *we can realize* g *as the restriction to* $f^{\alpha}(V)$ *of some linear map* $f: V \to V$ *having* $f^{\beta}(V)$ *strictly bigger than* $f^{\gamma}(V)$ *whenever* $\beta < \gamma \le \alpha$.

Proof. Just as in (1.5) we choose any set of elements v of U such that the union of their orbits $\{g^i v : i \ge 0\}$ spans U, but now, we'll make each of these v's the final vertex of a disjoint copy of T_α, the new vertices of all these tails being treated as linearly independent symbols. Let V be the linear span of U and these symbols, and define $f: V \to V$ to be the same as g on U, while on the new symbols it is defined by the arrows of the tails themselves. Then, for any $\beta \le \alpha$, $f^{\beta}(V)$ is the span of U and all new vertices having labels $\gamma \ge \beta$, and so this map f has the desired properties, *q.e.d.*

We now generalize (2.1) to obtain a trans-infinity of necessary conditions $(*)_\alpha$ for an invariant subspace to have an invariant complement, with previous $(*) = (*)_\omega$.

(3.3) THEOREM. *Let* $f: V \to V$ *be a vector space endomorphism, then an* f-*invariant subspace* K *of* V *can have an* f-*invariant vector space complement only if*

[3] This idea is old, in fact from pp. 32, 36 of P.E.B. Jourdain's "Introduction" to Cantor's papers [1] it seems that the ordinals were perhaps discovered thus in 1870-71: Cantor, while generalizing a theorem of Riemann on trigonometric series, noticed that the decreasing sequence $A^{(n)}$, $n \ge 1$, of *derived sets* of a set A of real numbers, can often be continued non-trivially past infinity in just this way.

$$K + \bigcap_{\beta<\alpha} f^{\beta}(V) \quad = \quad \bigcap_{\beta<\alpha} (K + f^{\beta}(V)), \qquad\qquad (*)_{\alpha}$$

where α *is any ordinal whatsoever.*

Proof. For $\alpha = 0$ the intersections on both sides are vacuous, so equal V per the usual convention, and *for an ordinal* α *having an immediate predecessor* $(*)_{\alpha}$ *is trivially true,* since both its sides are then equal to $K + f^{\alpha-1}(V)$.

For other, that is, limit ordinals α, we use the given hypothesis, viz., that there is a direct sum decomposition $V = K \oplus C$ into f-invariant subspaces. This implies—as can be verified by a straightforward induction—that $f^{\beta}(V) = f^{\beta}(K) \oplus f^{\beta}(C)$ for all ordinals β, which in turn implies that both sides of $(*)_{\alpha}$ are equal to $K \oplus f^{\alpha}(C)$, *q.e.d*

Note that *if* K *has an* f-*invariant complement in* V, *then the above conditions also hold if* V *is replaced by any* f-*invariant subspace* W *containing* K, for intersection with W provides us also with an f-invariant complement of K in W.

We ask next whether for $K = K_f$ the conditions $(*)_{\alpha}$ are non-trivial and new for limit ordinals $\alpha > \omega$? At first sight the obvious generalization of Figure 1—identify v_0 with the final vertex v of an α-tail—seems inadequate, for surely (*) will fail for this new tree also since it is so much more complicated ? Luckily, this is not so—now $(*) = (*)_{\omega}$ holds, only $(*)_{\alpha}$ fails!—and is the key point of the the next argument.

(3.4) THEOREM. *Given any set* Ω *of limit ordinals, there exists a linear map* $f : V \to V$ *for which* $(*)_{\alpha}$ *fails for* $K = K_f$ *if and only if* α *belongs to* Ω.

Proof. Let G_{α} denote the directed graph obtained by identifying the initial vertex of the *infinite chain* $v_0 \to v_1 \to v_2 \to v_3 \to \ldots$ with the final vertex v of a disjoint tail T_{α} (so Figure 1 is a G_{ω}). Treating the vertices of G_{α} as linearly independent we let V be their linear span and define $f : V \to V$ by the arrows of this graph.

Being a particular instance of the construction used in (3.2), $f^{\beta}(V)$, $\beta \le \alpha$, is spanned by $\{v = v_0 , v_1 , v_2 , v_3 , \ldots\}$ and those vertices of T_{α} that have labels $\ge \beta$, in particular, $f^{\alpha}(V)$ is the linear span $\langle v = v_0 , v_1 , v_2 , v_3 , \ldots \rangle$.

We note next that $K_f + f^{\alpha}(V)$ is strictly smaller than V, it cannot contain any vertex w of the tail other than v. For, if such a w were equal to an element u of K_f plus an element of $\langle v = v_0 , v_1 , v_2 , v_3 , \ldots \rangle$, then applying f enough times to this equation we'll get a non-trivial linear combination between the v_i's.

On the other hand for any $\beta < \alpha$ it is true that $K_f + f^{\beta}(V) = V$. To see this take any vertex w of the tail, with say $f^t w = v$. The ordinals $\beta+1, \beta+2, \ldots , \beta+t-1$, are strictly between β and the limit ordinal α. Choose an arrow of the tail ending at its final label β vertex v and starting from a vertex $v_{\beta+t-1}$ with label $(\beta+t-1)$-tail, then an arrow ending at $v_{\beta+t-1}$ and starting from a vertex $v_{\beta+t-1}$ with label $(\beta+t-2)$-tail, etc., finally an arrow ending at $v_{\beta+1}$ and starting from a vertex v_β with label β. So $w = (w-v_\beta) + v_\beta$ where $f^t(w-v_\beta) = 0$ and $v_\beta \in f^\beta(V)$ which shows that $w \in K_f + f^\beta(V)$.

It follows that both sides of $(*)_\beta$ equal V for $\beta < \alpha$, however $(*)_\alpha$ fails since its right side is V, but its left side is strictly smaller. Finally, using $f^{\alpha+n}(V) = \langle f^n v, f^{n+1}v, \ldots\rangle$ and $f^\beta(V) = 0$ for $\beta \geq \alpha + \omega$, we see that both sides of $(*)_\beta$ are equal to K_f for all $\beta \geq \alpha + \omega$, so these conditions also hold.

Since $(*)_\alpha$ holds for $K = K_f$ if and only if it holds for each of its direct summands, it follows that the direct sum of the maps constructed above, one for each $\alpha \in \Omega$, will have the desired properties, *q.e.d.*

(3.5) *Given an* f: V → V *we have stability* $f^{\alpha+1}(V) = f^\alpha(V)$ *for some* α *with* card(α) ≤ dim(V). To see this recall that the cardinality of the set of ordinals having cardinalities ≤ ℵ is strictly bigger than ℵ. So, if the assertion were false, by choosing for each α an element e_α of $f^\alpha(V)$ not in $f(f^\alpha(V))$ we would get a set of linearly independent elements with cardinality strictly bigger than dim(V), a contradiction.

The smallest ordinal α at which $f^{\alpha+1}(V) = f^\alpha(V)$ occurs is called the *length* λ(f) of the linear transformation f. It is true that, *if* λ(f) *is finite, then the locally nilpotent part is a direct summand* (note λ(f) = 0 iff f onto), accordingly the counterexamples above had lengths ≥ ω, but equally, *for each* α ≥ ω *there is an example having length* α *and locally nilpotent part a direct summand.* Indeed, if we let H_α—cf. G_α of (3.4)—be the directed graph obtained by identifying the final vertex of T_α to any vertex of a *doubly infinite chain* $\ldots v_{-3} \to v_{-2} \to v_{-1} \to v_0 \to v_1 \to v_2 \to v_3 \to \ldots$ then the linear map f defined by using H_α has length α and $C = \langle v_i : i \in \mathbf{Z}\rangle$ is an invariant complement of K_f.

However, we'll now cease further analyses of particular examples, and proceed to a general classification valid for all linear maps defined by one-way graphs.

§ 4. A *one-way graph* G is a *directed graph*—i.e., a set S of *vertices,* and a set A ⊆ S × S of *arrows*—with at most one arrow (v,w) ∈ A 'starting at' any v ∈ S (arrows are also denoted v → w, any number of arrows (u,v) can 'end at' a vertex v). Considering the vertices as linearly independent symbols, we denote by V_G the vector space spanned by S, and by $f_G: V_G \to V_G$ the linear transformation such that f(v) = w if (v,w) is the unique arrow starting at v, and f(v) = 0 if no arrow starts at v. The classification theorem (4.2) explicitly determines all one-way graphs G for which the locally nilpotent part of f_G is a direct summand, all the relevant definitions are given first in (4.1) below.

(4.1) The elements of V_G are linear combinations $\Sigma c_s s$, $s \in S$, $c_s \in \mathbf{F}$, with only finitely many of the c_s nonzero. This implies that f_G is the direct sum of the linear maps associated to the *components* of G, i.e., the maximal connected subgraphs of G, where as usual, *connected* means that, given any two vertices u and v, there is a finite sequence $u_0 = u, u_1, \ldots , u_t = v$ of vertices such that, for each $1 \leq i \leq t$, either (u_{i-1}, u_i) or (u_i, u_{i-1}) is an arrow. Since G is one-way there is in fact at most one switch from the first to the second choice as we proceed along the sequence, i.e., *a one-way* G *is connected if and only if any two vertices* u *and* v *have a common successor* w—alternatively that u and v are both predecessors of the same vertex w—that is, there is a finite chain of arrows going from u to w, and another finite chain of arrows going from v to w. We'll now partition the set of all connected one-way graphs into five mutually exclusive types.

Type 1. A connected one-way graph belongs to this type iff it has a *loop,* that is, either a vertex v_0 with no (case n = 0) arrow from it, or a finite chain of n arrows (v_0, v_1), (v_1, v_2), … , (v_{n-1}, v_0), $n \geq 1$, with the last arrow ending at the vertex from which the first started (so for n = 1, the single arrow starts and ends at v_0). It is easily seen that a Type 1 graph has a unique (upto a cyclic permutation) loop and it is the *sink* of G, that is, starting from any vertex a finite chain of arrows brings us to a vertex of the loop. A one-way G is of Type 1 if and only if it does not contain an *infinite chain* $v_0 \to v_1 \to v_2 \to v_3 \to \dots$ of distinct arrows. For example, the tails T_α are Type 1 with n = 0.

Type 2. A loopless connected one-way graph belongs to this type iff it has a vertex v of *infinite height*, i.e., there are arbitrarily long finite chains of arrows ending at v, this is denoted by $ht(v) = \infty$, but it has no *doubly infinite chain* $\dots v_{-3} \to v_{-2} \to v_{-1} \to v_0 \to v_1 \to v_2 \to v_3 \to \dots$ of arrows. Of course there might also be vertices w of *finite height*, i.e., such that there is a chain of n arrows, but no longer chain, which ends in w, then we write ht(w) = n. The graphs G_α of (3.4) are Type 2.

Type 3. A loopless connected one-way graph belongs to this type iff it contains a doubly infinite chain of arrows. The graphs H_α of (3.5) are Type 3.

Type 4. A loopless connected one-way graph belongs to this type iff it has no vertex of infinite height, and along an infinite chain $v_0 \to v_1 \to v_2 \to v_3 \to \dots$ one has $ht(v_{i+1}) > ht(v_i) + 1$ infinitely often. Note that $ht(v_{i+1}) \geq ht(v_i) + 1$ always, thus we are demanding that height should jump by more than 1 infinitely often. Since two infinite chains merge after finitely many steps, this condition is then true for any infinite chain. The graphs of (2.4) are Type 4 for slope m bigger than 1.

Type 5. A loopless connected one-way graph belongs to this type iff it has no element of infinite height, and along one, and so all, infinite chains of arrows $v_0 \to v_1 \to v_2 \to v_3 \to \dots$ one has $ht(v_{i+1}) = ht(v_i) + 1$ for all sufficiently large i. For example, the graphs of (2.4) are Type 5 if slope m is 1 or lesser.

(4.2) THEOREM. *The locally nilpotent part of* $f = f_G$ *is a direct summand if and only if all the conditions* $(*)_\alpha$ *hold for* $K = K_f$, *and this happens if and only if each component of the one-way graph* G *is of Type 1, 3 or 5.*

Furthermore, if a component is of Type 2 or 4 one and only one of the conditions $(*)_\alpha$ fails for the corresponding summand of f, with α to be described below.

Proof. *Case G connected, Type 1*: we assert that now K_f, $f = f_G$, has codimension n, the length of the unique loop, and the f-invariant space C spanned by the n vertices of the loop is a vector space complement of K_f. The empty loop case n = 0 is trivial, for then obviously $K_f = V$, so assume $n \geq 1$. Since f is one-one on C, we have $K_f \cap C = 0$. Take any vertex w of G and suppose that starting from w we need t arrows to come to a vertex v of the loop. Let t = nq + r where q and r are non-negative numbers with r less than n, and let u be the vertex of the loop r arrows behind v. Since w = (w –u) + u, and $f^t(w-u) = v - v = 0$, $u \in C$, we see that $w \in K_f + C$, therefore $K_f + C = V$, *q.e.d.*

Case G connected, Type 3: now the invariant subspace C spanned by the vertices of a doubly infinite chain complements K_f. Obviously $C \cap K_f = 0$. To see $V = K_f + C$, we express any vertex w as the sum of an element of K_f and an element of C. Suppose t arrows bring us from w to a vertex, say v_i, of the doubly infinite chain. Then $w = (w - v_{i-t}) + v_{i-t}$ is the required expression, *q.e.d.*

Case G connected, Type 4. This case will be ruled out by showing that $(*) = (*)_\omega$ does not hold for $K = K_f$. Its left side is K_f because $\bigcap_n f^n(V) = 0$, there being no element of infinite height, and K_f is of course smaller than V because no vertex of G is killed by any power of f. On the other hand we assert that each vertex v is in the right side of (*) and so the right side is equals all of V. To see this consider the infinite chain that issues out of this vertex $v = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \ldots$. Given any n we can choose t so large that height of these vertices has jumped by more than 1 at least n times, i.e., we have a t such that $ht(v_t) \geq ht(v) + t + n$. So we can find a vertex w such that $f^{n+t}(w) = v_t$, so $v = (v - f^n(w)) + f^n(w)$ with $v - f^n(w)$ in K_f because f^t kills it, while $f^n(w) \in f^n(V)$. Thus v lies in $K_f + f^n(V)$ for all n, i.e., it lies in the right side of (*), *q.e.d.*

Case G connected, Type 5. Take any infinite chain $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \ldots$ and assume that for all $i \geq t$ one has $ht(v_{i+1}) = ht(v_i) + 1$. Choose a maximal finite chain $w_0 \rightarrow w_1 \rightarrow \ldots \rightarrow w_s = v_t$, $s = ht(v_t)$. We assert that the invariant subspace C spanned by $\{w_0, w_1, \ldots, w_s = v_t, v_{t+1}, v_{t+2}, \ldots\}$ is a complement of K_f. Obviously since f is one-one on C, one has $K_f \cap C = 0$. To see $K_f + C = V$ we shall now check that any vertex v can be written as a sum of an element of K_f and an element of C. Take any r such that $f^r(v) = v_{t+i}$ for some $i \geq 0$, we must have $r \leq s + i$ because $ht(v_{t+i}) = s+i$. Then $v = (v - f^{s+i-r}(w_0)) + f^{s+i-r}(w_0)$ where f^r kills $v - f^{s+i-r}(w_0)$ while $f^{s+i-r}(w_0)$ is in C, *q.e.d.*

Case G connected, Type 2. This remaining case will be ruled out by showing that $(*)_\alpha$ fails for $\alpha = \lambda$, the length of $f = f_G$, when this limit ordinal—now λ is the smallest of the ordinals β such that $f^\beta(S) = \emptyset$—has no immediately smaller limit ordinal. Otherwise, the subgraph of G spanned by $f^{\lambda-\omega}(S)$ is of Type 4 or 5, and accordingly one has $\alpha = \lambda$ or $\lambda - \omega$, the limit ordinal immediately preceding λ.

The graph has merging unique infinite chains of arrows issuing out of each vertex, but the maximal chains ending at each vertex are finite and issue from vertices which have no arrow ending at them. This allows a *labelling* of the set S of vertices: the label 0 is assigned to all vertices having no arrow ending at them, and inductively assuming that all ordinals less than β have already been assigned, we assign the label β to those and only those of the remaining vertices at which only those arrows end whose starting vertices have already been labelled, the process stops when there are no remaining vertices. From this definition we see: *given a vertex* v *having label* β *and any* $\gamma < \beta$ *there is a vertex* w *preceding* v *and having the label* γ. We denote by S_β the subset of S having labels $\geq \beta$, we note that if this is nonempty then so is $S_{\beta+1}$ and is strictly smaller than S_β. This ensures that the process of labelling will stop, and that $S_\lambda = \emptyset$ must occur first at a limit ordinal $\lambda > \omega$, the last because $S_\omega \neq \emptyset$, there being given a vertex of infinite height. To identify λ with the length $\lambda(f)$ of $f = f_G$ note that $S_\alpha = f^\alpha(S)$ where the latter notation means: $f^0(S) = S$, $f^\alpha(S) = f(f^{\alpha-1}(S))$ if α has an immediate predecessor, and $f^\alpha(S) = \bigcap_{\beta<\alpha} f^\beta(S)$ if α is a limit ordinal; so $f^\alpha(V)$ is the subspace of V spanned by S_α , and these subspaces are strictly decreasing till $f^\lambda(V) = 0$.

The left side of $(*)_\lambda$ is thus K_f which is strictly smaller than V since no vertex is killed by any f^n. Let us now compute $K_f + f^\beta(V)$ for $\beta < \lambda$. First assume also $\beta + \omega < \lambda$. Then $S_{\beta+\omega}$ is nonempty, so the infinite chain $v = v_0 \to v_1 \to v_2 \to v_3 \to \dots$ issuing out of any vertex v of G enter $S_{\beta+\omega}$ in finitely many steps, say $v_t \in S_{\beta+\omega}$. Amongst the predecssors of v_t we can find a vertex $v_{\beta+t}$ having label $\beta+t$, then amongst the predecessors of $v_{\beta+t}$ a vertex $v_{\beta+t-1}$ having label $\beta+t-1$, so on, finally amongst the predecessors of $v_{\beta+1}$ a vertex v_β having label β. The infinite chain of arrows issuing out of v_β thus stays in S_β and wends it way to v_t after t or more arrows. We choose in this chain a vertex u which is exactly t arrows behind v_t. Now write $v = (v-u) + u$ and note that f^t kills v-u while $u \in S_\beta$ which implies $v \in K_f + f^\beta(V)$. So for $\beta + \omega < \lambda$ one has $V = K_f + f^\beta(V)$. When λ has no immediately smaller limit ordinal $\beta + \omega < \lambda$ follows from $\beta < \lambda$, so in this case we have shown that $(*)_\lambda$ does not hold.

If $\lambda > \beta \geq \lambda - \omega$, i.e. $\beta = (\lambda - \omega) + n$, where $\lambda - \omega$ denotes the limit ordinal immediately preceding λ, then $S_{\beta+\omega}$ is empty and the above reasoning is not valid. A vertex u of the nonempty $S_{\lambda-\omega}$ cannot be the end point of arbitraily long finite chains of arrows with all vertices in $S_{\lambda-\omega}$, we denote by $ht^\lambda(u)$ the maximum length possible. The infinite chain $v = v_0 \to v_1 \to v_2 \to v_3 \to \dots$ emanating out of any v is eventually in the nonempty $S_{\lambda-\omega}$. If $ht^\lambda(v_{i+1}) > ht^\lambda(v_i) + 1$ infinitely often, we can find (cf. argument of Type 4 above) a v_t in $S_{\lambda-\omega}$ having $ht^\lambda(v_t) > t+n$ So a u in $S_{\lambda-\omega+n}$ such that $f^t(u) = v_t = f^t(v)$. Then $v = (v-u)+u$ again shows $v \in K_f + f^\beta(V)$, so $(*)_\lambda$ does not hold.

If $ht^\lambda(v_{i+1}) = ht^\lambda(v_i) + 1$ eventually, say for $i \geq s$, then $(*)_\lambda$ does hold. If $\beta = (\lambda - \omega) + n$, $K_f + f^\beta(V) = K_f + \langle f^i(w) : i \geq 0 \rangle$, where w is the initial vertex of a maximal length chain, in the subgraph spanned by $S_{\lambda-\omega}$ having v_s as its final point (cf. argument of Type 5 above) from which it follows that the right side $\bigcap_{\beta<\lambda} (K_f + f^\beta(S))$ is also equal to K_f.

Now look at the left side $K_f + f^{\lambda-\omega}(S) = K_f + \langle f^i(w) : i \geq 0 \rangle$ of $(*)_{\lambda-\omega}$. This cannot contain any vertex u of G outside $S_{\lambda-\omega}$ for this is the same as saying that u is the sum of an element of K_f and a finite linear combination with nonzero coefficients of some iterated images of w. Applying to this equation a suitable f^t, $t > 0$, iterates these images t more times, kills the element of K_f, and replaces u by a lesser iterated image of w. Thus giving us a non-trivial linear combination in the orbit of w, a contradiction. On the other hand we know that $K_f + f^\beta(S) = V$ for all $\beta < \lambda-\omega$, so the right side of $(*)_{\lambda-\omega}$ is V. Thus now it is $(*)_{\lambda-\omega}$ that fails. We have shown that a connected Type 2 graph fails some $(*)_\alpha$. It is also easily seen that for bigger α this condition holds, in fact one only needs to verify $(*)_\lambda$ because, for any f, the condition $(*)_\alpha$ obviously holds for all $\alpha > \lambda(f)$, *q.e.d.*

General case. If the nilpotent part of $f = f_G$ is a direct summand, then all the conditions $(*)_\alpha$ hold for f, which happens iff these conditions hold for all the direct summands of f contributed by the components H of G. This implies by above that none of these components H is Type 2 or Type 4. Conversely, if the components H are of Types 1, 3 or 5 then by above $K_f \cap H$ has an invariant complement C_H in V_H, and $C = \oplus_H C_H$ gives us an invariant complement of $K_f = \oplus_H(K_f \cap H)$ in $V_G = \oplus_H V_H$, ***q.e.d.***

§ 5. We shall now wrap up this paper with three concluding remarks.

(5.1) One can also associate to G the *dual map* φ_G: $\mathbf{F}^S \to \mathbf{F}^S$ defined by $\varphi_G(v)(s_1) = v(s_2)$ if (s_1 , s_2) is the arrow out of s_1, and = 0 if there is no arrow out of s_1. This dual vector space $\mathbf{F}^S$ of all functions on S is much bigger – its dimension is bigger than the cardinality of S – and is not the direct sum of the function spaces contributed by the components of G, so results are very different now. Even for a disjoint union of finite chains of unbounded lengths the nilpotent part of φ_G is not a direct summand: (*) *does not hold for* $K = K_\varphi$ *where* $\varphi = \varphi_G$ *is the dual map associated to any of the one-way graphs* $G = G_m$, $m > 0$, *with set of vertices* $\{s_{i,j} : 0 \le j \le mi\}$ *and arrows* $(s_{i,j} , s_{i,j+1})$. One can check that $I_\varphi = 0$, so the left side of (*) equals K_φ . Take any positive rational r less than m, and let $v \in \mathbf{F}^S$ be the function which is 0 on all vertices other than those of the type $s_{i,ri}$ on which it takes the value 1. It is easy to check further that v is not in K_φ but it belongs to $K_\varphi + \mathrm{Im}(\varphi^n)$ for all $n \ge 0$, so v is in the right side of (*). We remark that this example is inspired by Exercise 33(a), p.32 of Kaplansky [2] on abelian groups. We won't enter here into the details of the analogous and easier classification for these dual maps, but remark that, for any $\varphi = \varphi_G$, the non-existence of an invariant complement for K_φ is once again reflected in the failure of at least one of the conditions $(*)_\alpha$.

(5.2) The *matrix* $M : S \times S \to \mathbf{F}$ of a linear map f: $V \to V$, with respect to a basis S of V, is defined by $f(v) = \sum_w M(v,w)w$, thus each *row* $M(v, .) : S \to \mathbf{F}$ of M has only finitely many nonzero entries. Alternatively, we can represent any linear map by a directed graph G on the given basis S, with at most finitely many arrows issuing out of each vertex, viz., all (v,w) such that M(v,w) is nonzero, with each arrow assigned the nonzero label M(v,w). It is remarkable that the non-triviality of all the conditions $(*)_\alpha$ was shown by using just the maps f_G because these are the *simplest possible*: they have a matrix in which each row has at most one nonzero entry. In fact, even this nonzero entry was 1, but a minor change in its proof generalizes (4.2) to all simplest possible maps: note now (while calculating K_f) that if two length r chains $u = u_0 \to u_1 \to \ldots \to u_{r-1} \to u_r = w$ and $v = v_0 \to v_1 \to \ldots \to v_{r-1} \to v_r = w$ end in the same vertex, then it is the non-trivial linear combination $(\prod_i M(v_i , v_{i-1}))u - (\prod_i M(u_i , u_{i-1}))v$ that is in $\ker(f^r)$. It seems that an analogous classification, for linear maps admitting matrices that have at most two nonzero entries in each row—using Jordan's normal forms theorem this includes all finite dimensional complex linear maps—would be much more complicated.

(5.3) The special nature of the maps f_G stems from the linear independence of S. If we *allow linear dependencies*—by adding a 0 vertex we can also insist now that there *is* a unique arrow issuing from every vertex—compatible with the arrows then these maps become very general indeed! Indeed every linear map f: V → V is of this kind, now S = V, and the components of our one-way G are described by the restriction of f to the disjoint *saturated orbits* $\cup_n f^{-n}\{f^{\,i}(v): i \geq 0\}$ into which V is partitioned by f. It is not too hard to adjust the proof of (4.2) to show that in one direction it remains true even now. *For any linear map* f: V → V, *if* K_f *admits an invariant complement* C, *or even if all the necessary conditions* $(*)_\alpha$ *for this to happen hold, then each saturated orbit of f must be of* Type 1, 3 *or* 5, where the five possible mutually exclusive types are defined exactly as in (4.1), except that 'has a loop' is replaced by 'orbit of v is linearly dependent' and 'loopless' by 'orbit of v is linearly independent'. Thus the existence of C imposes strong conditions on the dynamics of linear maps of this type. For the restriction of f to the putative C this is clear. This restriction is injective—that is, given any point of C there is at most one arrow starting in C which ends at that point—so it follows at once that *the intersection of any saturated orbit with* C *is a loop, a doubly infinite chain, or a singly infinite chain*. Conversely, one has locally – i.e., within the span of a saturated orbit of Type 1, 3 or 5 – defined invariant complements C_v , however it is not clear whether they can always be glued together to form a global invariant complement C. We hope to give all this, and more, in a sequel to this paper.

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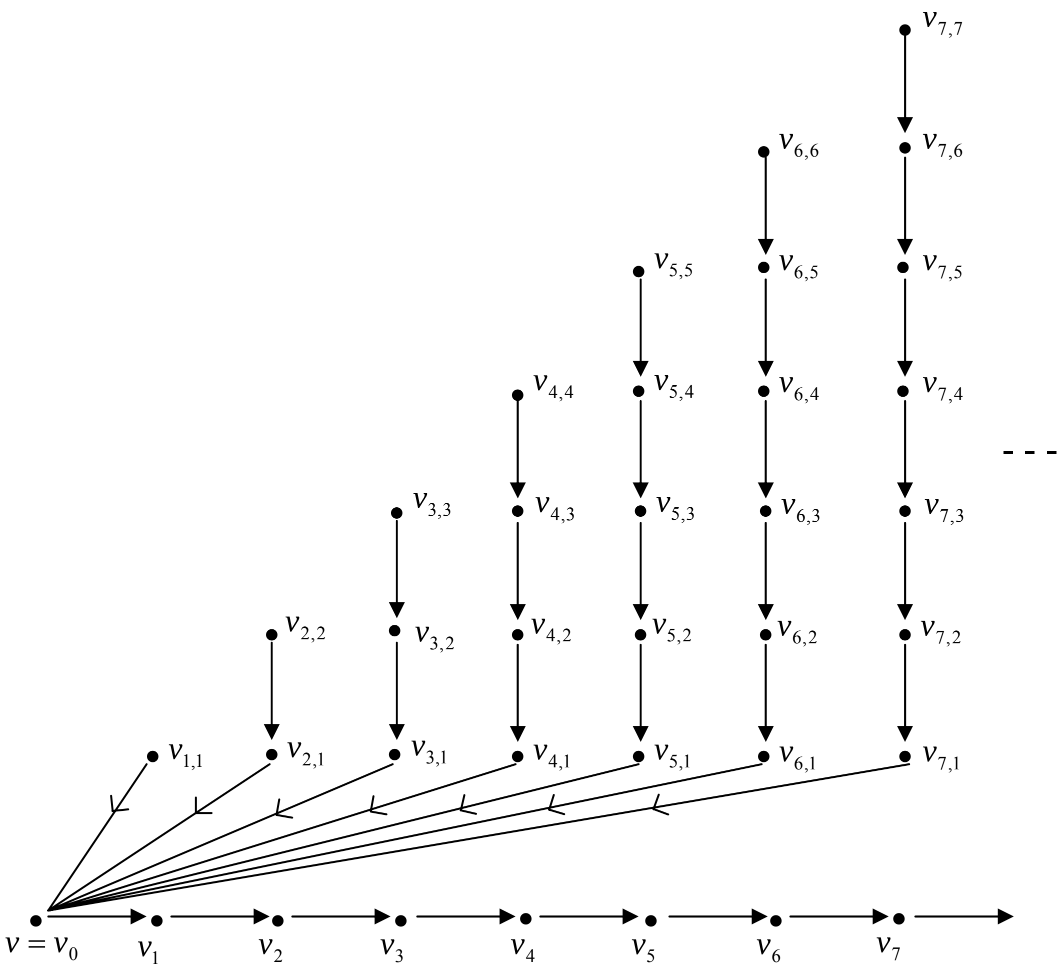
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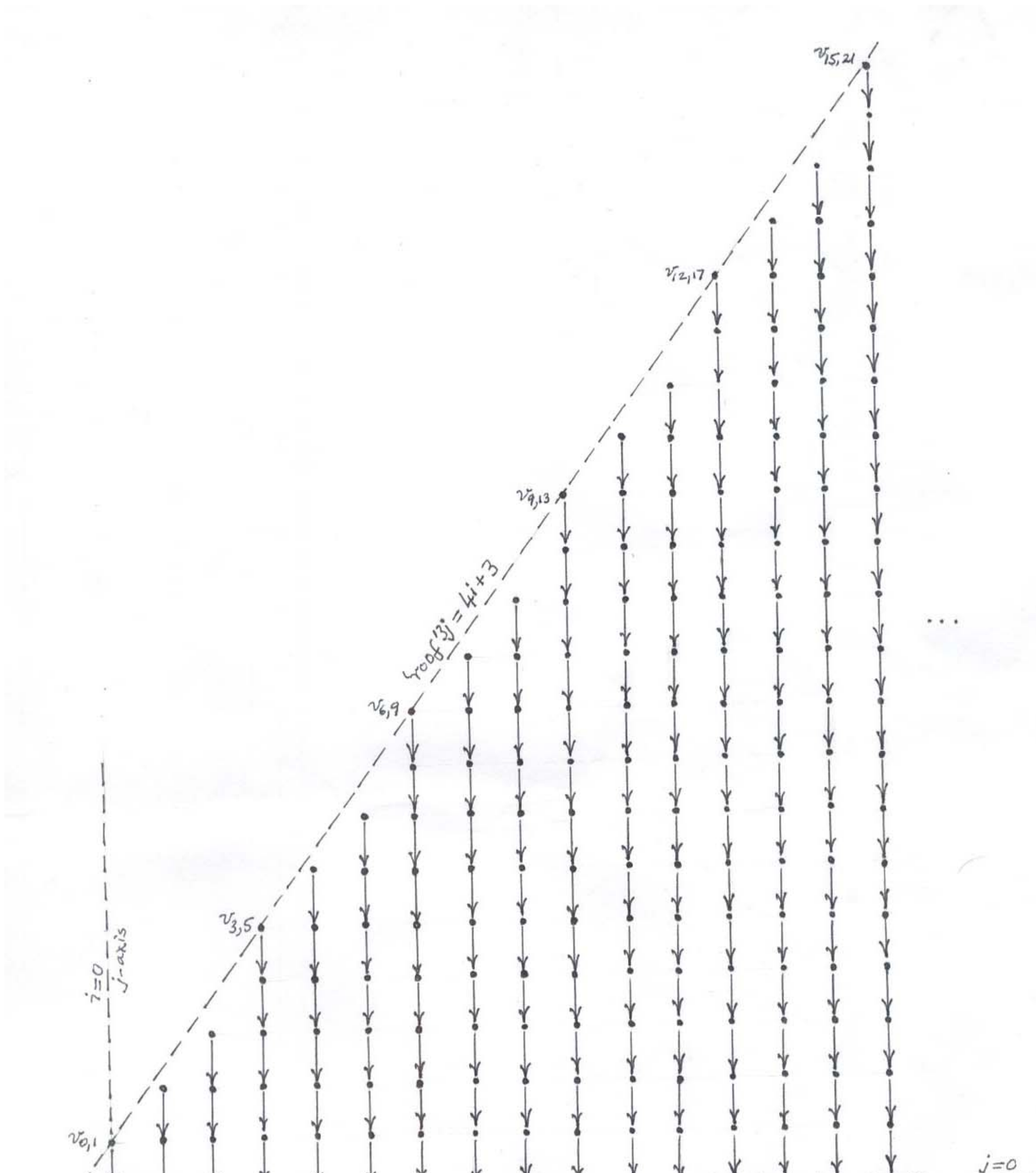
Figure 1



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$v_{3,5}$
$v_{0,1}$
$v_{0,0}$
$v_{15,0}$
roof $3j = 4i+3$
$i=0$
j-axis
$j=0$
i-axis

Figure 2