Map-reading geometry and homeomorphisms

In *Sirsa nadi* we saw how terrain subdivided into many equal maps can be read cohesively, with focus on any one of them, in an open rectangle U^2 just three times the size of a map. More generally given an n-cuboidal tiling of $\mathbb{R}^n$, there is a similar map-reading geometry on an open n-box U^n just three times the size of a tile, of which the n-torus T^n is a geometric quotient.

This evoked Sullivan's *Hyperbolic geometry and homeomorphisms* (HGH) in which hyperbolic geometry on an open n-ball $\mathcal{U}^n$ is used to show that *outside dimension four any manifold has a unique lipschitz structure.* Now for $n > 1$ the n-torus cannot arise as a geometric quotient, but–*this uses étale homotopy*–the hyperbolic n-ball does have compact torus-like quotients $\mathcal{T}^n$ in the sense that minus any point they can be immersed in n-space. This enables him to imitate Kirby's torus unfurling with lipschitz control.

The *point* being that a hyperbolic quasi-isometry of $\mathcal{U}^n$ a bounded hyperbolic distance from the identity gives a euclidean quasi-isometry of the closed n-ball $\overline{\mathcal{U}}^n$ which is the identity on its boundary. Likewise *a map-reading quasi-isometry of U^n a bounded map-reading distance from the identity gives a euclidean quasi-isometry of the closed n-box $\overline{U}^n$ which is the identity on its boundary* :-

The image of any point is a bounded number N of tiles away, so if the map-reading distance between images of nearby points of U^n is at most L times, the distortion of the euclidean distance is at most $2^{N+1}L$; and a *perturbation* of this basic map-reading geometry ensures also the second part.

Also the decomposition $U^n = U^k \times U^{n-k}$ of map-reading geometries enables us to do the k-handle case even more simply than in HGH. Everything else, in particular furling infinitely repeated to fill a deleted n-ball, remains the same since no hyperbolic geometry was used in these steps.

Notes

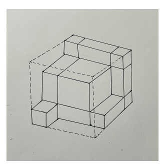
1. Since there is a tgtbt feel to the above, so to lighten the mood and ensure that this paper will have value even in the worst case scenario, shown below is a concurrent but unrelated empirical discovery of mine which, despite its everyday nature, may also be new:-

That is, during my evening walk up and down a number of times on the left side of a fixed trajectory between two stony paths, I observed that, irrespective of my direction, the grass is greener on the other side!

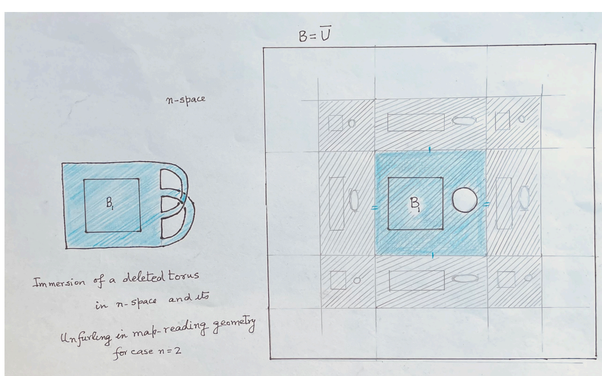
2. About the proof of that 'point' in HGH after correcting typos, viz., "This follows easily since the conformal factor is essentially the Euclidean distance to the boundary, namely, *(hyperbolic element)≈(Euclidean element)÷(Euclidean distance to boundary)"*, we note that more exactly $dh = \frac{2de}{1-r^2}$ for an open n-ball of radius $r = 1$, so $\frac{dh}{de} = \frac{1}{1-r} + \frac{1}{1+r} \approx \frac{1}{1-r}$ near boundary.

The hyperbolic distance from the centre $\int_0^r \frac{2}{1-r^2} dr$ is N on $r = \frac{e^N-1}{e^N+1}$, here conformal factor $\frac{1-r^2}{2} = \frac{2e^N}{(e^N+1)^2}$. So if a homeomorphism of the ball distorts hyperbolic distance between nearby points by at most L, and is at hyperbolic distance at most N from the identity, then it distorts euclidean distance between nearby points by at most $\frac{(e^N+1)^2}{2e^N} L \leq 2e^N L$.

3. On its middle third map-reading geometry of U^n is euclidean; then in the first ring of $3^n - 1$ tiles sharing some proper face the scale in all transverse directions is halved; and so on; e.g., shown below is a cube and some of the 26 cubes of the first ring. So map-reading element is at most 2^N times euclidean element in the $(2N+1)^n$ tiles up to the Nth ring, which was what we used for the analogous bound $\leq 2^{N+1}L$.

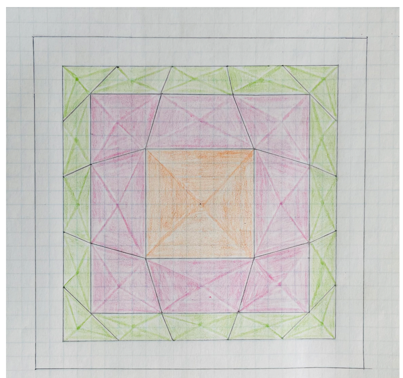


4. So reverting from $\mathcal{T}^n$ − ball of HGH to an immersion of T^n − ball in n-space we lift "a homeomorphism defined near image (T^n − ball) and sufficiently close to the identity to approximately T^n − ball. Thinking of the deleted ball in polar coordination we can furl (see "furling" below) to obtain commutation with a radial homothety and extend over the ball by infinite repetition. We have extended the quasi-isometry defined near B_1 and close to the identity to a global quasi-isometry of T close to the identity".



Then unfurl, i.e., lift this quasi-isometry of T^n to the map-reading open box U^n and apply the point above "to obtain a quasi-isometry of B^n which is the identity at the bdry and containing the original on B_1. This is the 0-handle case required for the construction of isotopies as in Edwards-Kirby". For "identity at the bdry" we use here a *perturbed* map-reading geometry:-

5. As is clear from the last figure for the basic map-reading geometry a bounded homeomorphism of U^2 preserving vertical lines may not converge on bdry to a single-valued function, let alone the identity. So we perturb its defining homeomorphism $U^2 \to \mathbb{R}^2$ to make the corner tiles of each new ring funnel out and compensatingly contract others, so that the diameter of any tile sufficiently close to the boundary is arbitrarily small.



Indeed we can as shown use even quadrilateral tiles such that the outer boundary of any ring is subdivided with all segments equal. Since these segments approach zero as $N \to \infty$ and the sides of any tile are at most of this order their diameter becomes zero on the boundary.

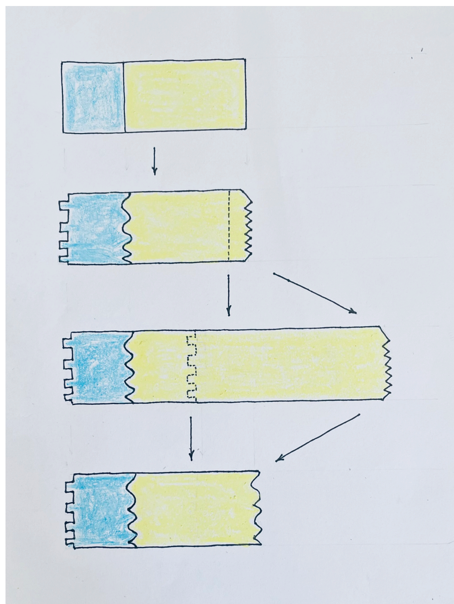
Also we fix a simplicial correspondence between tiles: each tile of a new ring is subdivided into four triangles over the mid-point of the segment joining the barycenters of the shared face (edge or vertex) and the opposite face. The ratios of corresponding segments and angles stay bounded for neighbouring tiles, so a quasi-isometry of U^2 which is bounded in this perturbed map-reading geometry is a quasi-isometry in the euclidean distance of $U^2 \subset \mathbb{R}^2$.

6. More generally, *any bounded pull-back of euclidean space shall be deemed a map-reading geometry,* and some used "in de Rham's 1955 book" for "a smoothing procedure for currents" also work:-

We pull back the euclidean distance of $\mathbb{R}^n$ by a radial (homeo or) diffeomorphism τ to a rotationally invariant distance on $\mathcal{U}^n$ with element $\tau'(r)$ times the euclidean element along any radius. Further, by using a τ *which is not too steep* (note 12), we can ensure that any homeomorphism f of the ball a map-reading distance $\tau(c)$ away from the identity which distorts map-reading distance by at most L is a euclidean quasi-isometry of the ball with distortion at most $\tau'(c)L$. Also, since map-reading balls of radius $\tau(c)$ near bdry of $\mathcal{U}^n$ are very small in the euclidean distance, f converges to the identity on it.

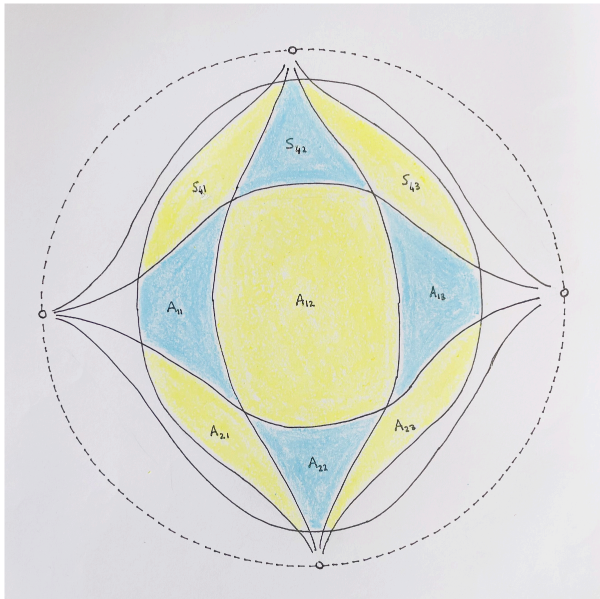
In my thesis, *The de Rham cohomology of foliated manifolds,* which predates HGH by a few years, there are some nice smoothing operators on foliated manifolds and bundles. A non-euclidean geometry on $\mathcal{U}$ seems needed for "a smoothing procedure with conformal symmetry properties" that Sullivan mentions; but for the main results of HGH such a *map-reading geometry of de Rham* should to be the geometry of choice.

7. "Furling ... is ingenious but only two lines": in cylindrical coordination linear stretching and curvilinear shrinking an "almost vertical" homeomorphism of $S^{n-1} \times I$ gives commutation with a translation.

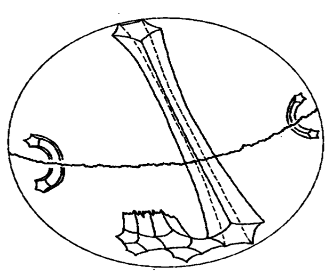


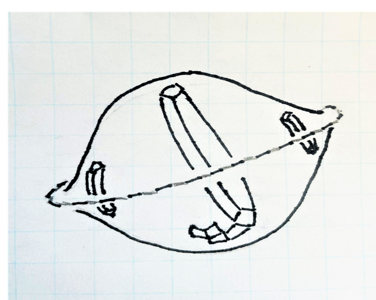
8. *Instead of immersing a deleted* T^n *an imbedding of* T^{n-1} *in n-space* can be used to pick up a homeomorphism close to the identity to an almost vertical homeomorphism of $T^{n-1} \times I$, which furled as above gives a homeomorphism of the n-torus obtained by identifying the ends of the yellow collar. Full details are in §8 of Edwards-Kirby (1971), indeed our figure is an uncluttered version of the one given there. This torus furling idea was used first in Novikov's 1965 proof of the topological invariance of rational Pontryagin classes, which led to Sullivan's thesis of 1966 and a partial proof of the Hauptvermutung, that is, but for a possible 3-dimensional cohomological obstruction. Siebenmann showed in 1969 that this obstruction is actual, by an example tied to the fact that after commutation with a radial homothety we cannot piecewise linearly "extend over the ball by infinite repetition"; his remark that this can obviously be done in the lipschitz context led in turn to HGH.

9. A hyperplane of euclidean n-space a height C above the origin, pulled back by $\tau \approx \tan$ as in note **6** to a ball $\mathcal{U}^n$ of radius $\pi/2$, becomes the hypersurface obtained by revolving $\tan r \sin\theta = C$, a curve which tapers down from a height of $\tan^{-1} C$ to 0 on the boundary; e.g., in this de Rham geometry the nine equal rectangular maps of *Sirsa nadi* are roughly:-



10. For the k-handle case we choose a discrete action of $\mathbb{Z}^{n-k}$ on $\mathbb{R}^{n-k}$ and extend it by perpendicularity to $\mathbb{R}^k \times \mathbb{R}^{n-k}$. This action confined to a *slab* $\{(u,v) : |u| \leq C\}$ and pulled back by τ becomes a $\mathbb{Z}^{n-k}$ action, on a closed subset with horizon S^{n-k-1} of an open n-ball, with fundamental domains $D^k \times D^{n-k}$. We'll use this *walnut* ($k = 1, n = 3$ is shown) instead of the lens of HGH (its "figure 2" is rescaled to interpret it as a closed subset of the open 3-ball between two spheres passing through the circular horizon).





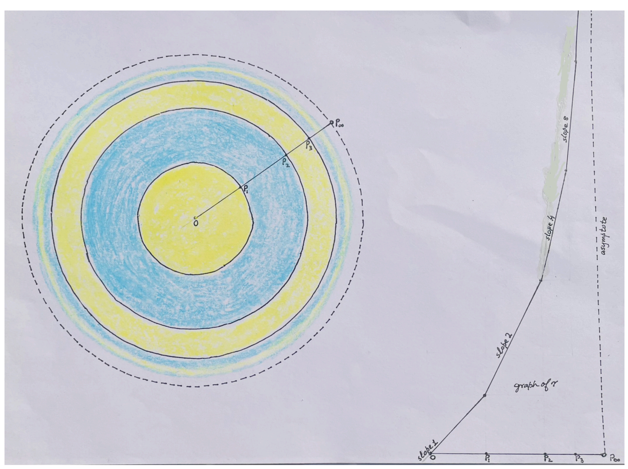
"Then we treat a homeomorphism defined near the core of a k-handle which is close to the identity and equal to the identity near the bdry." Use torus immersion "to obtain an extension to the (fundamental domain - $(n-k)$ ball) $\times D^k$ compatible with" the $\mathbb{Z}^{n-k}$ action. "Then since we have the identity near the bdry we again have an n-dimensional hole which can be filled in by furling and infinite repetition." Now unfurl to walnut "and apply the bounded quasi-isometry remark to obtain a quasi-isometry of B^n which is the identity at the bdry and agrees with the original near the core of the k-handle."

11. Using above and induction on "a fine handle decomposition ... we find in the context of quasi-isometries ... of any smooth manifold ... the Cernavskii, Kirby-Edwards isotopy theory ... So we have ... the Schoenflies theorem, the Annulus theorem, and the component problem in all dimensions." *Thus all these, and likewise the results below of HGH, can be proved using map-reading geometry instead of hyperbolic geometry and étale homotopy.*

Sullivan switches on bottom of page 548 of HGH to: "We want to discuss approximating arbitrary homeomorphism by Lipschitz homeomorphism. For dimensions less than 4 there is a good classical theory (Moise). Dimension 4 is unknown and remains so. For dimensions greater than 4 we get full positive results using Connell's radial engulfing and Kirby's Annulus theorem." Most striking is this upshot: "Topological manifolds of dimension $\neq 4$ have Lipschitz coordinate systems. Such locally Euclidean Lipschitz structures are unique up to homeomorphism close to the identity."

As we'll see $n > 4$ *is used in radial engulfing,* the topological annulus theorem was later proved also for $n = 4$ but: "Quinn doesn't help ..." [1]

12. Indeed we can use *basic round map-reading geometry* which depicts $\mathbb{R}^n$, subdivided into an n-ball and n-annuli around it, all of radial width equal to its diameter, in a $\mathcal{U}^n$ just three times the size of this ball, by successively halving the radial scale of each ring; i.e., we pull back the euclidean distance of n-space by the radial homeomorphism $\tau : \mathcal{U}^n \to \mathbb{R}^n$ shown below.



Let f be any bijection of $\mathcal{U}^n$ relating points $\{P, P' = f(P)\}$ at most N (≥ 2) rings apart, which is such that $\frac{\tau(A')\tau(B')}{\tau(A)\tau(B)}$ and $\frac{\tau(A)\tau(B)}{\tau(A')\tau(B')}$ are both less than L for any segment AB. Then $\frac{A'B'}{AB}$ and $\frac{AB}{A'B'}$ are less than $2^{N+1}L$:-

[1]This tip I owe to a 2010 email from Professor Sullivan, also the next note was sparked by a resumption of this correspondence soon after this paper was first posted.

(i) Any AB not collinear with O has a component (it is in the plane OAB) tangent to the $(n-1)$-sphere through its mid-point M, and a normal radial component. (ii) For a small *radial* AB, the ratio $\frac{\tau(A)\tau(B)}{AB}$ is the slope of the above graph, so it is 2 if AB is in the first ring, then jumps to 4 in the second, in the third ring it is 8, etc. (iii) For a small *tangential* AB the ratio $\frac{\tau(A)\tau(B)}{AB} = \frac{O\tau(A)}{OA} = \frac{O\tau(B)}{OB}$, the slope of the chord from O of graph(τ), increases continuously but less steeply from $\frac{2i-1}{3-2^{2-i}}$ to $\frac{2i+1}{3-2^{1-i}}$ in the ith ring.

So for any AB small, so also $A'B'$ small at most N rings away, $\frac{\tau(A)\tau(B)}{AB}$ and $\frac{\tau(A')\tau(B')}{A'B'}$ are each at most 2^{N+1} times the other, hence $\frac{A'B'}{AB}$ and $\frac{AB}{A'B'}$ are less than $2^{N+1}L$. Using convexity of $\mathcal{U}^n$ – cf. Federer, *Geometric measure theory,* page 64 – the same is true for any segment AB. $\square$

13. *What if graph*(τ) *has any positive constant slopes* s_i*?* That asymptote exists iff $\frac{1}{s_1} + \cdots + \frac{1}{s_i} + \cdots$ *converges to an* $S < \infty$; then τ pulls back the distance of $\mathbb{R}^n$ to a ball $\mathcal{U}^n$ which is $1+2S$ times the core. But that all-important 'point'–which we checked for the case $s_i = 2^i$–may *not* hold: for it to hold it is necessary and sufficient that $\frac{s_{i+1}}{s_i}$ be bounded above by some K:-

If no such bound, a translation of $\mathbb{R}^n$ pulls back to an isometry $P \to P'$ of the pulled back distance on $\mathcal{U}^n$, but on and near the opposite end of the parallel diameter there is AB with $A'B'$ an arbitrarily big multiple.

If there is such a K we can analyze as before, e.g., now for a tangential AB the conversion ratio $\frac{\tau(A)\tau(B)}{AB}$ increases continuously from $\frac{2i-1}{1+2(S-S_i)}$ to $\frac{2i+1}{1+2(S-S_{i+1})}$ in the ith ring, where $S_t = \frac{1}{s_t} + \frac{1}{s_{t+1}} + \cdots$. So there is hardly any increase for this component of any AB after a while, compared to the discontinuous jumps by ever increasing factors s_i for the radial component. Yet no jump is more than K times the last, so over N rings the euclidean distortion is at most K^{N+1} times the distortion L of the pulled back distance. Where, to ride over an initial superiority of the tangential increase of the above ratio, we might have to stipulate also that N is bigger than a prescribed constant. $\square$

In other words the sum of reciprocals $\frac{1}{s_1} + \cdots + \frac{1}{s_i} + \cdots$ should converge but its convergence should be no faster than geometrical.

14. *What if graph*(τ) *is smooth with a similar shape?* That is slope $\frac{d\tau}{dr}$ is also increasing, and there is an asymptote $r = R$, i.e., $\int_0^\infty \frac{dr}{d\tau} d\tau$ *converges to* R; now the 'point' holds iff convergence is no faster than some $1/K^\tau$.

For example $\tau(r) = \tan(r)$ will do, and $\tau(r) = \log \frac{1+r}{1-r}$ is hunky-dory. This last gives us on the open unit ball a map-reading geometry whose element *radially* is that of hyperbolic geometry. The conversion ratio for the tangential element varies much more slowly so we still have that any homeomorphism of the ball bounded and lipschitz with respect to this pulled back distance is lipschitz also with respect to its euclidean distance. The upside of giving up conformality being that we can now also pull back the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$, so the good old n-torus is again available to prove the main results of HGH in an "elementary" way, that is without using étale homotopy theory.

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