Plain Geometry & Relativity, Notes 21 - 23

21. If positions P_1 and P_2 of the same particle occur at times $t_1 < t_2$ of an observer S, and have components $\mathbf{x}_1$ and $\mathbf{x}_2$ in his euclidean space, then $\overrightarrow{P_1P_2}$ parallel to a ray translates into $\|\mathbf{x}_2 - \mathbf{x}_1\| < c(t_2 - t_1)$. For $c < \infty$ this lipschitz property enables us to get by – note 16 – without extra hypotheses like smooth or p.l. on a motion. The half-space $t > 0$ of S is the *euclidean product* of his ray and n-space $t = 1$ and is preserved only by the reflections of the cone preserving this ray. The cone, the intersection of the half-spaces of all the observers, has however a *hidden product structure* – Figure 5 – given by all the rays and the reeb foliation $\tau =$ constant, which is preserved by all its reflections. If P_1 and P_2 occur at absolute times $\tau_1 < \tau_2$, and we use cayley's distance, then *the lipschitz property can be reformulated thus* : $\widehat{P_1P_2} < c\log(\tau_2/\tau_1)$.

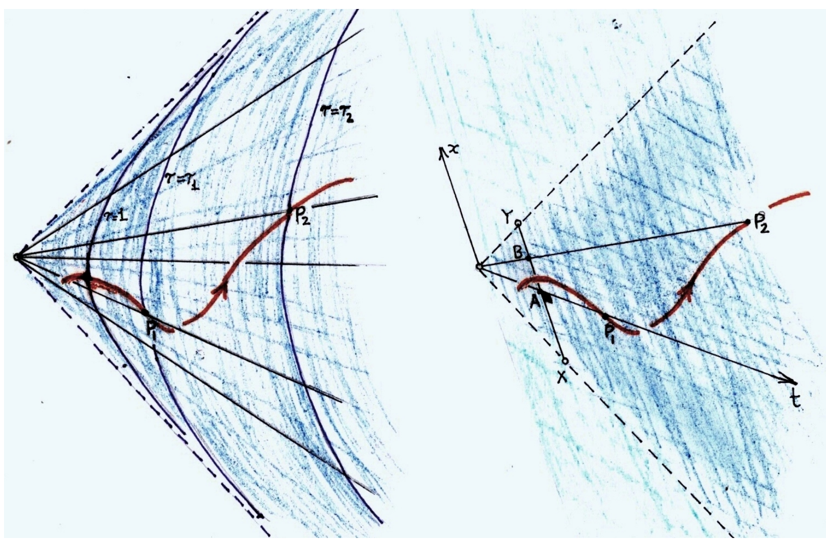

Figure 5

If P_1 and P_2 lie on the same ray the distance $\widehat{P_1P_2}$ between their rays is 0. Otherwise we'll use on their plane the coordinates (t, x) of the observer S whose state of rest is the ray through P_1 with the x axis towards the ray through P_2. So if $P_1 = (l, 0)$ then $P_2 = (l + lu, mlu)$ for some $u > 0$ and $m > 0$. The two rays cut the ball of S in its centre $A = (1, 0)$ and the point $B = (1, \frac{mu}{1+u})$, and AB extended meets the boundary in $X = (1, -c)$ and $Y = (1, c)$. We have $\frac{XB}{XA}\frac{YA}{YB} = \frac{XB}{YB} = \frac{c+(mu/1+u)}{c-(mu/1+u)} = \frac{1+u+mu/c}{1+u-mu/c}$. Also $(\tau_1)^2 = l^2$ and $(\tau_2)^2 = (l + lu)^2 - (lmu/c)^2$, therefore $(\frac{\tau_2}{\tau_1})^2 = (1 + u + mu/c)(1 + u - mu/c)$. Using the definition of $\widehat{P_1P_2}$ in note 14 the above lipschitz property is equivalent to $\frac{XB}{XA}\frac{YA}{YB} < (\frac{\tau_2}{\tau_1})^2$, so it holds iff $1 < (1 + u - mu/c)^2$, i.e., iff $m < c$. $\square$

Since P_1P_2 extended may not intersect the boundary twice, in the above proof we first replaced it by AB. Indeed if $-c \leq m \leq c$ then P_1P_2 extended has an X' on the boundary but Y' is at infinity, and the distance between the two rays is, excepting for $m = \mp c$, *not* given by the limit $\frac{c}{2}\log\frac{X'P_2}{X'P_1}$. This is bigger, for example, for $m = 0$ the two rays coincide, but this expression gives us $\frac{c}{2}\log(\tau_2/\tau_1)$, which is half the *right* side of our inequality!

The unrestricted use of Cayley's formula—i.e. $\widetilde{AB} := \frac{c}{2}\log(\frac{XB}{XA}\frac{YA}{YB})$ always with the limit to be used if X or Y is at infinity—*is however natural and gives us more.* It gives *a metric* as against a pseudometric. We have $\widetilde{AB} > 0$ because the complete line containing the points $A \neq B$ is not in our cone, and $\widetilde{AB} + \widetilde{BC} \geq \widetilde{AC}$ follows by taking the limit of the triangle inequality – see note 14 – for the cone truncated by a flat on the right. A linear isomorphism preserves the ratios of segments of a line, so $\widehat{AB}$ and $\widetilde{AB}$ are invariant under *all* linear isomorphisms of the cone. Further this metric is well-behaved with respect to the hidden product structure. On each leaf $\tau =$ constant it coincides with the cayley distance between rays. On each ray it coincides with $\frac{c}{2}\log(\tau_2/\tau_1)$, so by analogy this expression will be called *the cayley distance between the leaves* on which P_1 and P_2 lie. We recall that the factor $\frac{c}{2}$ was put only to get the coincidence, of the riemannian metric of the particular leaf $\tau = 1$, with the cayley distance between rays. With this artificial factor now gone, the lipschitz property becomes: *for any pair of subsequent points on an absolute motion the cayley distance between rays is less than twice the cayley distance between leaves.* Since 0 and 1 are the only whole numbers less than two, the thought arises that the other side of the $c < \infty$ coin – see note 12 – is this discrete micro reality: *at the next instant of absolute time a 'particle' is either at the same or at one of the adjacent spots of absolute space?*

22. In mechanics one also considers motions of two, three or more particles, even of fluids and plasmas with uncountably many, and collisions, fusion and fission of particles too, but all in a still space. The 'particles' at the end of the last note are different, they are not things *in* space, but things revealed by the motions *of* space. Since space stays put, these motions are via bijections which induce bijections of open sets, viz., *homeomorphisms* ϕ_τ of the absolute space of all rays or of $\tau = 1$, parametrized continuously by absolute times $\tau > 0$. Also, as before, the absolute motion P_τ of each point P of $\tau = 1$ shall be strictly increasing with respect to the partial order of the cone, i.e., if $P_1 = P_{\tau_1}$ and $P_2 = P_{\tau_2}$ are the points of $\tau = \tau_1$ and $\tau = \tau_2$ on the rays through $\phi_{\tau_1}(P)$ and $\phi_{\tau_2}(P)$ for any $\tau_1 < \tau_2$, then $\overrightarrow{P_1P_2}$ is parallel to a ray of the cone.

The corresponding motion ϕ_τ of the ball B^n of any observer S extends to homeomorphisms $\overline{\phi}_\tau$ of his euclidean space identity outside B^n. The lipschitz inequality of note 21 applies to the absolute motion of any point, so $\phi = \phi_\tau$ is at a bounded cayley distance $A = c|\log\tau|$ from the identity, i.e., $\widehat{P\phi(P)} < A \ \forall\ P$. So this homeomorphism ϕ of B^n maps its centre into the concentric open ball of radius a where $\frac{c}{2}\log\frac{c+a}{c-a} = A$, i.e., $a = c\tanh(\frac{A}{c})$. More generally ϕ maps any $P \in B^n$ into the *cayley ball of radius A* around P, i.e., all points at cayley distance less than A from P. The extension $\overline{\phi}$ is a homeomorphism because, *in the euclidean metric of S, these cayley balls become arbitrarily small when P approaches the boundary of B^n.*

If S uses orthogonal coordinates $(t; x, \mathbf{y})$ in which $P = (1; v, \mathbf{0}), v > 0$, the linear reflection of the cone switching the rays through the centre and P – see fifth para of text – is $(t; x, \mathbf{y}) \mapsto (\gamma t - \frac{\gamma v}{c^2} x; \gamma v t - \gamma x, \mathbf{y})$ where $\frac{1}{\gamma(v)} := \sqrt{1 - \frac{v^2}{c^2}}$. The rays through the boundary of the cayley ball around the centre constitute $x^2 + \mathbf{y}^2 = a^2 t^2$. So the boundary of the cayley ball around P is given by putting $t = 1$ in $(\gamma v t - \gamma x)^2 + \mathbf{y}^2 = a^2(\gamma t - \frac{\gamma v}{c^2} x)^2$, i.e. $(\gamma^2 - \gamma^2 \frac{a^2 v^2}{c^4}) x^2 - 2(\gamma^2 v - \frac{\gamma^2 a^2 v}{c^2}) x + \mathbf{y}^2 = \gamma^2 a^2 - \gamma^2 v^2$. Completing a square this can be written as $\frac{\gamma^2}{\delta^2}(x - w)^2 + \mathbf{y}^2 = \beta^2$, where $\frac{1}{\delta(a,v)} := \sqrt{1 - \frac{a^2}{c^2} \frac{v^2}{c^2}}$. So this is an ellipsoid – Figure 6 – with centre $Q = (1; w, \mathbf{0})$, with all semi-axes β, except that along the diameter on which P lies, this semi-axis $\alpha = \frac{\delta}{\gamma} \beta$ is smaller. Further, segment of the diameter through P intercepted by the ellipsoid is bounded by the reflections T', U' of the rays through $(1; \pm a, \mathbf{0})$, viz., the rays through $(1; \frac{v \mp a}{1 \mp \frac{av}{c^2}}, \mathbf{0})$. So $2\alpha = \frac{v+a}{1+\frac{av}{c^2}} - \frac{v-a}{1-\frac{av}{c^2}}$ and $2w = \frac{v+a}{1+\frac{av}{c^2}} + \frac{v-a}{1-\frac{av}{c^2}}$ which give $\alpha = \frac{\delta^2(a,v)}{\gamma^2(v)} a$ and $w = \frac{\delta^2(a,v)}{\gamma^2(a)} v$. Since $w < v$ we see that Q is nearer to the centre of the ball than P; also that the semi-axes $\beta = \frac{\delta}{\gamma} a$ and $\alpha = \frac{\delta^2}{\gamma^2} a$ of the ellipsoid approach 0 when $v \to c$. $\square$

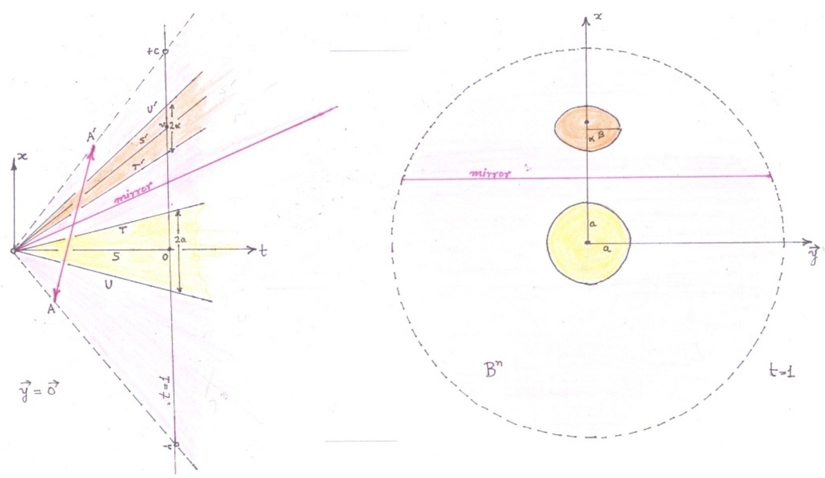

Figure 6

We can't resist remarking once again how *plain* it all is ! The linear reflection $A \leftrightarrow A'$ of the *plane* of S and S' gives us the smaller axis of the ellipsoid, so in particular the factor by which a shrinks in this direction, in all orthogonal directions it only shrinks by the square root of this amount. The *infinitesimal cayley ball at* P is even easier to keep in mind : the ellipsoid with centre P with the radius of the central ball shrunk in these directions by the factors $1 - \frac{v^2}{c^2}$ and $\sqrt{1 - \frac{v^2}{c^2}}$ respectively, where v denotes the euclidean distance of P from the centre of B^n. This because $\delta(a, v)$ and $\gamma(a)$ approach 1 when $a \to 0$. That is, the cayley distance of the ball B^n arises from a riemannian metric which coincides at its centre with the euclidean metric, and at all other points P stretches the tangent vectors in these directions by the reciprocals of these factors.

More generally, *even if 'motion' is by bijections* ϕ_τ, but the absolute motion P_τ of each point is as before, our argument shows that, *the extended-by-identity bijections* $\overline{\phi}_\tau$ *of euclidean space are bicontinuous on the boundary of the ball.* When dimension $n \geq 2$ all sorts of fissures can now develop within the n-ball, for example, the points may be stationary till some radius $r_0 > 0$, and for any bigger radius rotating at a small nonzero speed. However continuity in time implies that, *the bijections* ϕ_τ *must be homeomorphisms for* $n = 1$. If the order of two points P and Q of the interval B^1 is reversed under ϕ_τ their flow lines intersect at some $0 < \tau' < \tau$, i.e. $P_{\tau'} = Q_{\tau'}$, contradicting injectivity of $\phi_{\tau'}$. So these bijections are order preserving, and they have no discontinuity either, because any such jump contradicts surjectivity. □

23. It seems that, *any homeomorphism* ϕ *of the* n*-ball at a bounded cayley distance from its identity map can be realized as a* ϕ_τ *of some motion.* So these bounded homeomorphisms are *isotopic to the identity,* however they form a smaller group. For example, any homeomorphism of B^n which is *radial,* i.e., preserves each radius, is isotopic to the identity, but it may not be bounded. Also, we'll see later that, *any strictly increasing curve passing through* P *for* $\tau = 1$ *can be realized as the flow line* P_τ *of some motion of space.*

The projections of the flow lines P_τ on the absolute space of rays or $\tau = 1$ are called the *orbits* $\phi_\tau(P)$ of the motion. Unlike flow lines, orbits can intersect themselves or each other in all sorts of way but, *if* $c < \infty$ *and* $n \geq 2$*, an orbit cannot visit all the points of a nonempty open set.* Using $\widehat{P_1P_2} < 2(\widetilde{\tau_1\tau_2})$—note 21—we see that in any time interval of cayley length $\frac{1}{N}$ the orbit stays in a cayley ball of diameter $\frac{2}{N}$, so over any unit time interval the orbit describes a set which can be covered by N cayley balls of this diameter, but $N(\frac{2}{N})^s \to 0$ as $N \to \infty$ for any $s > 1$, so this set has dimension at most one. □

Here we used *hausdorff dimension* of a metric space, viz., the infimum of all positive real numbers s for which there exists a countable cover such that the sum of the sth powers of the diameters is arbitrarily small. It is easy to see that for submanifolds this is their usual dimension, and an argument similar to the one above shows that, *it is non-increasing under any lipschitz map,* so it is preserved by (bi)lipschitz homeomorphisms of metric spaces.

It follows that that *amazing curve* found by Georg and David while playing dots-and-squares (!) can only be traced by an orbit of a motion with $n = 2$ *and* $c = \infty$. For, it covered a 2-cell with 3 holes, so it can not be lipschitz; but the reader can check that, the euclidean distance between its points at times $t_1 < t_2$ is bounded by a constant multiple of the square root of $t_2 - t_1$. Likewise, *any compact and connected manifold* $M^m \subset \mathbb{R}^n$ *can be traced in finite time as an orbit of some motion of* n*-space,* which moreover satisfies a 'weak lipschitz inequality' involving the mth root of $t_2 - t_1$. □

Here t was the time of an observer S, for $c = \infty$ it is the same as τ. For $c < \infty$ it is not and, *we emphasize that it is the absolute time* τ *which is parametrizing the homeomorphisms* ϕ_τ *of absolute space,* each observer S merely identifies his ball B of radius c in his $t = 1$ with this space of all rays. These ϕ_τ were well-defined because a line in the cone parallel to a ray cuts all the transversals $\tau =$

constant. This is not so, for $c < \infty$, if we use the time t of S : *a transversal* $t =$ constant *may not cut all the flow lines,* for example, if the flow lines are parallel to the same ray. So some points of B may not be on any flow line having a point with a given $t < 1$, and following the flow to a $t > 1$ may give only an injection, not a bijection : we would get a well-defined homeomorphism of B for each $t > 0$ only when all the flow lines arise from the origin.

The foliation provided by all the flow lines has another interpretation when we use the product structure of the half space of S – see Figure 5 – instead of the hidden product structure of the cone : *it is what S would observe if his own euclidean space were undergoing a motion.* This because, up to any time $t > 0$, he can discern only the motions of those points of his space which are at distance less than ct from him—that is why he'll plot only a cone full of flow lines in his half space—and the observed positions of any point at times $t_1 < t_2$ are subject to the condition $\|\mathbf{x}(t_2) - \mathbf{x}(t_1)\| < c(t_2 - t_1)$.

The homeomorphisms $\overline{\phi}_\tau$ of the euclidean n-space $t = 1$ of S, identity outside his ball B^n, give *orientation-preserving homeomorphisms of the n-sphere* having an extra point at infinity. Only that about this hidden motion is heard which persists under perturbations : so, for $n \neq 4$, the observer S can assume that these homeomorphisms are lipschitz! Indeed, for $n > 4$ we'll construct later, *an almost radial homeomorphism, identity outside the ball, which conjugates the motion to one which is lipschitz.* Spherically bending the flat mirrors of B^n maximally inwards ensures that a lipschitz inequality holds if one of the points is on the boundary. Within the ball we'll make the homeomorphisms piecewise linear. We'll start with a simplicial approximation of the motion. This may have some singularities, but for $n > 4$ these singularities can be engulfed away, essentially because a simple closed curve on the already good part can be coned away from it, cf. *Embedding and unknotting of some polyhedra* (1987). For $n \leq 4$ this does not work, and the result is in fact false for $n = 4$, but for $n < 4$ there are other constructions which show that the result is again true.

Even for $c = \infty$—now the cone is a half-space and $\tau = t$—*the hidden product structure is different from that of any observer* : all the flats $t =$ constant with all the rays from the origin, instead of all the parallels to a ray S. Once again it is this observer-independent hidden product structure only that we'll use to define the homeomorphisms ϕ_t of the absolute space $t = 1$ from any continuous flow of the same for all absolute times $t > 0$. However for $c = \infty$ the continuous flow lines P_t may not be lipschitz, and these homeomorphisms of euclidean n-space may not be at a bounded distance from its identity map, nor can S assume on à priori grounds for $n \neq 4$ that they are lipschitz. These distinctions show that, *there is no time and* order-preserving *homeomorphism from the half space onto the cone of rays through a ball B^n of finite radius.* $\square$ On the other hand any homeomorphism of $\mathbb{R}^n$ onto B^n determines and is determined by a time preserving homeomorphism which maps rays to rays.

Given a flow of the space its *invariant subsets* are those on which the homeomorphisms ϕ_τ restrict to homeomorphisms, i.e., subsets A such that if $P \in A$ then the entire orbit $\phi_\tau(P)$ is contained in A. *The minimal invariant sets of a flow partition the space into topologically homogeneous parts.* That these sets are disjoint or equal is clear. At points P and $\phi_\tau(P)$ of any invariant set A the topology is the same because ϕ_τ restricts to a homeomorphism of A. If A is minimal then its points are related by finite sequences of points each on some orbit through the preceding.□ Topologically homogenous spaces are nice, nicest being connected manifolds, so we ask: what manifolds are born in flows?

Example. *There is a smooth motion of n-space, $n \geq 2$, with minimal sets parallel* 2*-planes.* Let the flow lines be tangent to the vector field on the half-space whose component along the ray through that point is t, and whose components parallel to a fixed frame of the n-space are $(t \cos \log t, t \sin \log t, 0, \ldots, 0)$. Then the orbits, i.e., the projections from the origin of these flow lines on $t = 1$, are all circles of radius 1 parallel to the first two vectors of the frame. □ A similar construction works also for $c < \infty$, and though the *invariant partition* of a flow is seldom a foliation as in this example, it seems that such constructions put together will suffice to establish that, *any smooth connected manifold occurs as a minimal invariant set of some flow with $c < \infty$.*

However not all topological manifolds are relativistic : *if a closed M^m occurs as a minimal invariant set in a motion with $c < \infty$ then it admits a lipschitz structure.* We can assume $m > 3$ and so $n > 4$, but then S can perturb the motion to a conjugate motion whose ϕ_τ's are lipschitz homeomorphisms of his ball B^n; their restrictions to the perturbed copy of M^m give the desired lipschitz structure.□ We recall – see note 16 – that this only excludes some 4-dimensional manifolds. Nevertheless it seems likely that, *outside these wild* 4*-manifolds, any closed connected topological manifold can be realized as a minimal invariant set in a flow with $c < \infty$,* and that, *for the limiting non-relativistic case $c = \infty$, even these exceptions can be thus realized.*

A motion of space is *steady* in time if the flow lines through any ray are positive multiples of each other, so *$\tau \mapsto \phi_\tau$ is a group homomorphism* $\phi_{\tau_1 \tau_2} = \phi_{\tau_1} \circ \phi_{\tau_2}$: see Figure 7.□ For a steady motion, the minimal invariant sets have just one orbit each. Further, if a point returns to its position, it must repeat its journey, therefore : *each orbit is homeomorphic to an open interval, a circle, or a single point.* So the minimal invariant sets of a steady motion are very simple; only, if $n = 3$, some of these circles may be *knotted* in B^3. On the other hand, it may well be that *any* smooth closed connected submanifold M^m of an n-ball B^n is a minimal invariant set of some *unsteady* motion?

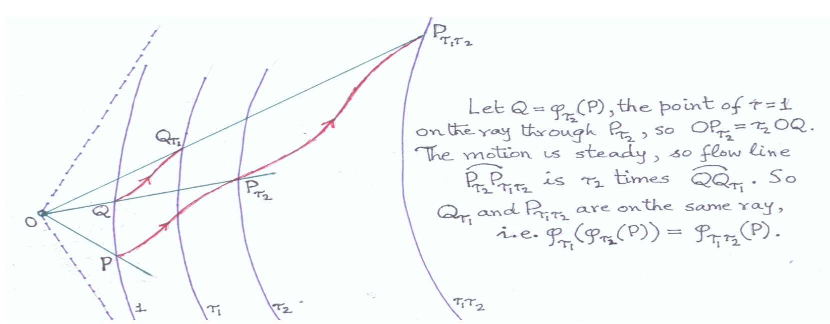

Figure 7

The *homeomorphisms of the cone* $\Phi_\tau(P) := P_\tau$ (the point on the same flow line with proper time τ times) map leaves to leaves. They map—Figure 7—rays to rays iff the motion is steady. These motions preserve the metric $\widetilde{AB}$ of the rays. However, *when $c < \infty$ and $n \geq 2$, no motion other than absolute rest preserves the metric $\widetilde{AB}$ of the leaves* ! If ϕ preserves the orientation and cayley distance of B^n it is a composition of an even number $\leq n+1$ of linear reflections of the cone. If ϕ is not the identity map, and $n \geq 2$, there is a line L whose image $\phi(L)$—also a line by linearity—is distinct from it. Since the cayley balls of any finite radius become arbitrarily small – see note 22 – near the boundary of B^n, the second line is not wholly within a bounded cayley distance of the first line. So a cayley distance preserving ϕ can occur as a ϕ_τ of some absolute motion of space only if it is the identity map of B^n. $\square$ On the other hand the geometry of the infinite n-ball or the finite interval is not rigid : any orientaion and distance preserving ϕ occurs as a ϕ_τ of some motion.

The cayley isometries of the cone are given by the compositions of its linear reflections and time reversals $\tau \mapsto a^2/\tau$. If the homeomorphisms Φ_τ commute with a group $\mathfrak{G}$ of these isometries the motion is called $\mathfrak{G}$-*periodic.* Especially alluring are the discrete subgroups $\mathfrak{G}$ with compact quotients, for example, *in all dimensions there are groups $\mathfrak{G}$ under which the conical spacetime covers a closed and parallelizable manifold* ! The n-ball, held taut at its boundary in his euclidean n-space, and vibrating $\mathfrak{G}$-periodically, enables the observer to hear to some extent the topology of this quotient. This discretization of spacetime is available also for $c = \infty$ and in this non-relativistic schrödinger theory examples of such discrete subgroups are easier to give.

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(*contd.*)