24. *The closed and parallelizable spacetimes that closed Note 23 deserve our close attention.* For all $n \geq 1$ there are discrete subgroups $\mathfrak{G}$ of cayley isometries of our conical spacetime such that the quotient space is compact, and by using instead a suitable subgroup of finite index we can always ensure not only that this quotient is a closed $(n+1)$-manifold, but also that it admits $n+1$ smooth linearly independent tangent vector fields $v_1, \dots, v_n, v_{n+1}$. Further, some such groups $\mathfrak{G}$ are generated by a subgroup Γ of cayley isometries of the ball B^n and a single homothety, i.e., a product of two distinct time reversals.

The closed spacetime is then a circle S^1 times a closed n-manifold B^n/Γ which may not be parallelizable, but B^n/Γ *is parallelizable in the complement of a point.* For, this complement has the homotopy type of an $(n-1)$-dimensional polyhedron. So, on it, the unit vector field w tangent to S^1 is homotopic, via never zero sections of the tangent bundle of the spacetime, to v_{n+1}. Lifting this homotopy we obtain, on this complement, $n+1$ linearly independent vector fields $w_1, \dots, w_n, w$. The first n of these give, under projection parallel to w, the required parallelization of B^n/Γ minus a point. $\square$

A smooth n-manifold without boundary *immerses in n-space iff it is open and parallelizable.* Here 'only if' is easy and 'if' is nowadays an existence theorem of flexible p.d.e. theory. However, even for the punctured n-torus $\mathbb{R}^n/\mathbb{Z}^n \setminus \{\mathrm{pt}\}$, an explicit immersion is not easy, and for its relativistic analogues $B^n/\Gamma \setminus \{\mathrm{pt}\}$, we know in general nothing about the discrete groups $\mathfrak{G}$ and Γ beyond what we asserted above without proof about their existence. These existence proofs are very pretty – especially an étale homotopy argument which shows why the obstruction to parallelizability vanishes for a finite cover – but first, let us ponder this painting of that river—Note 20—flowing north out of Africa ...

Sunset on the Nile (Jens, circa 1956)

25. *Where do we come from? What are we? Where are we going?* This is the longish name of a painting by Gauguin. A paper by Gromov starts with that painting, and then, for its title, has *Manifolds* asking these existential questions of themselves. Though surely very few can talk, closed manifolds have always been, for me too, very natural objects. This belief is, I guess, what led me to the results of Note 23, and the many related musings in these notes.

It seems that in recent times physics has returned to its cartesian roots, in particular the dictum that, *matter is but extension, and is differentiated only by its various motions.* Be that as it may, any closed topological manifold is 'cartesian matter' in the sense of this *theorem* : it can be created in finite time as a compact minimal invariant set M^m of some continuous motion of a euclidean space having sufficiently many degrees n of freedom.

A flow of $\mathbb{R}^n$ can probably also have other compact M^m's—all necessarily connected, topologically homogenous and homogenously embedded—but it is manifolds that seem the most natural. Indeed, matter is discrete, so what matter are maybe the *triangulable M^m's : these* are *closed manifolds.* □ This is easy, but the *Bing-Borsuk conjecture,* that *any* locally contractible and topologically homogeneous compactum is a manifold, is still open for $m > 2$. And, for $m = 3$ it would finish another proof of *Poincaré's conjecture,* that a closed 3-manifold with fundamental group $\Gamma = 1$ is the 3-sphere. Also, the unfolding classification of triangulable M^m's for $m \geq 5$ is tied closely to that of *homology 3-spheres,* i.e., closed 3-manifolds with $\Gamma_{ab} = 1$. 'His' homology 3-sphere with Γ finite—the *Miss Universe* of "213, 16A"—was discussed at great length by Poincaré, but who knows, the infinitely many homology 3-spheres which occur as B^3/Γ may be there too in his pioneering and prolific writings on discrete subgroups Γ preserving the geometry of a 3-ball of radius $c < \infty$? The unfolding work on triangulations suggests that the above 'cartesian matter' can be analysed in terms of these 'elementary particles' or 'relativistic crystals' ...

For $c < \infty$ it is in fact $(B^n/\Gamma) \times S^1$, and more generally any manifold quotient $C^{n+1}/\mathfrak{G}$ of the cone, that is more like a classical crystallographic manifold, for it has a finite parallelizable cover. *These closed spacetimes $C^{n+1}/\mathfrak{G}$ have an induced reeb foliation and transverse line field,* since the cayley isometries of the cone C^{n+1} map leaves and rays to leaves and rays. □ In this context we'll think of C^{n+1} *as the infinite cylinder over the ball B^n of any observer S* :–

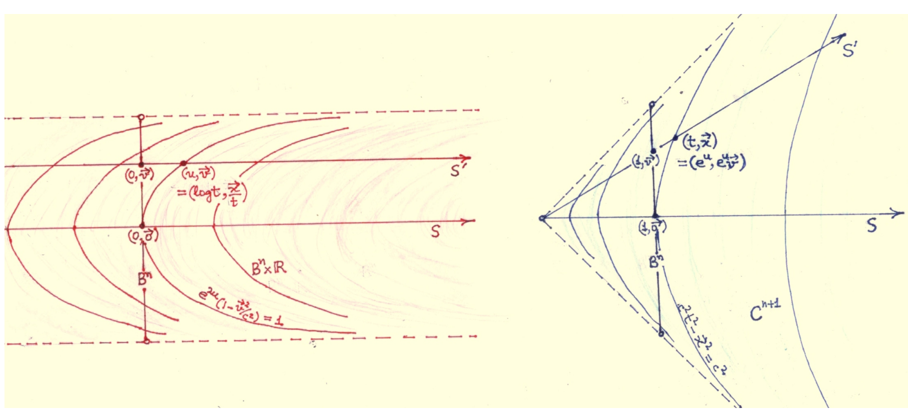


Figure 9

This cylinderical representation is convenient for doing topology, but the geometry gets distorted. The new coordinates are the *logarithmic time* $u = \log t$ of the observer and relative velocity $\mathbf{v}$ with respect to him, so parallels to S represent galilean motions. More generally, *any smooth curve* $(u, \mathbf{v}(u))$ *represents a possible motion iff it obeys the relativistic constraint* $|d\mathbf{v}/du| < c - |\mathbf{v}|$. Moreover, *the lorentz contraction factor* $\gamma(\mathbf{v})$ *is tied intimately with the new equations* $u = \log \gamma(\mathbf{v})$ + constant *of the reeb leaves,* that in the conical picture were simply $c^2t^2 - \mathbf{x}^2$ = constant. The new cylinderical picture is preserved by translations parallel to the axis just like the conical picture was preserved by homotheties, however the cayley isometries of the ball moving its centre are restrictions of nonlinear transformations of the cylinder.

Foliations are 'cartesian' partitions, for example it is likely that, any smoothly foliated closed manifold $(M^m, \mathcal{F})$ can be created in finite time as an invariant set of a smooth relativistic flow of a high dimensional ball B^n of radius $c < \infty$, each leaf of $\mathcal{F}$ a minimal invariant set of this flow. However a parallel generalization of the theorem stated above to all continuously foliated topological manifolds vis-à-vis continuous non-relativistic flows seems more iffy.

This was my cue to revisit my foliations days, doing which I noticed that, *the intermediate partitions used in all those constructions of foliations from that era are most likely 'cartesian' too,* at least as long as everything is smooth. Given below are some other things from this trip back in time.

26. Besides the aforementioned real analytic reeb foliations of the closed spacetimes $C^{n+1}/\mathfrak{G}$ – are there some homology spheres here? – there was that good old *smooth reeb foliation of* S^3 which however I now found myself looking at through the lens of my later deleted joins days :–

Let S^3 be the round 3-sphere of circumference 4 centred on the origin of $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$. Then the spherical distance between the *first and second circles* in which S^3 intersects these summands is 1 and S^3 is the *join* $S^1 \cdot S^1$ of these two circles. That is, any other point of S^3 lies on a unique great circular arc of length 1 from a point x of the first circle to a point y of the second, and can be denoted $(1-\alpha)x + \alpha y$, where α is its distance from the first circle. So, *points at a distance* α *from the first circle are at a distance* $1-\alpha$ *from the second, and form a submanifold* L_α *of* S^3 *diffeomorphic to* $S^1 \times S^1 = \{(x, y) : x \in S^1,\ y \in S^1\}$ *if* $0 < \alpha < 1$*, while* L_0 *and* L_1 *are the first and second circles.*

We now use the foliation of $B^2 \times S^1$, obtained by dividing the infinite cylinder of Figure 9 by a translation, to desingularize this *foliation-with-singularities* of S^3 : that is we plug in a copy to refoliate the diffeomorphic neighbourhood of all points within a certain distance less than one of each circle, smoothness on the bounding toral leaf of this neighbourhood then follows from the fact that $\log \gamma(v)$ and all its derivatives approach infinity when $v \to \pm c$. Also, we can desingularize symmetrically with respect to the switching $\mathbb{Z}/2$-action on $S^3 = S^1 \cdot S^1$, and if we choose the 'certain distance' to be $1/2$ for both circles we would be left with just one toral leaf $L_{1/2}$. $\square$

Likewise, *joining* $q+1$ *spheres gives a foliation-with-singularities of a sphere,* with generic leaf product of all these spheres, so it has *codimension* q*,* but there are also some singular leaves that are products of only some of the $q+1$ spheres, e.g., the $(q+1)$-fold join S^{2q+1} of S^1 has such a 'foliation' with generic leaves $(q+1)$-tori. *When exactly can the join of $q+1$ spheres be desingularized to get a codimension q foliation?* Obviously this sphere should admit a codimension q tangent plane field, for example, the join of two spheres can be desingularized only if is odd dimensional, but this condition is not sufficient.

The join of two spheres can be desingularized iff they are odd dimensional with one a circle. Given the above foliation-with-singularities of $S^i \cdot S^j$, we want to refoliate two disjoint open saturated neighbourhoods $S^i \times B^{j+1}$ and $B^{i+1} \times S^j$, of the singular leaves S^i and S^j, so that the new leaves approach the boundary leaves $S^i \times S^j$. If $i > 1$ and $j > 1$ then $S^i \times S^j$ is simply connected; therefore by Poincaré's original definition of the fundamental group, the global monodromy of any multiple valued function defined on it is trivial; so that given by nearby leaves of these refoliations would be trivial; which rules out approaching leaves. So, because $i + j + 1$ is odd, i and j are odd with one 1.

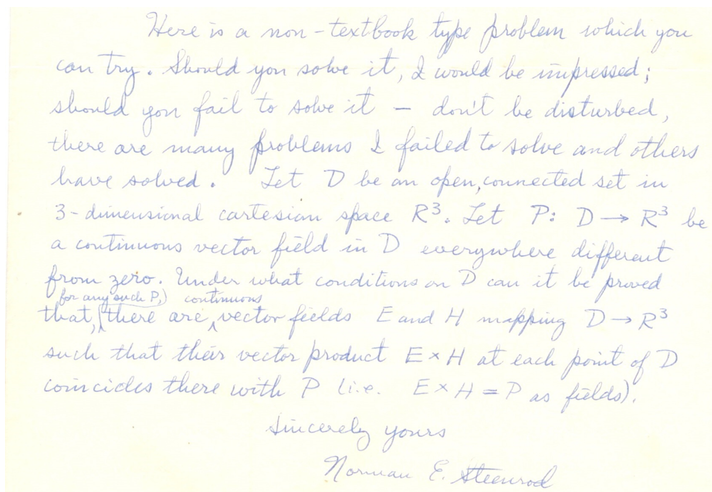
Conversely, Figure 9 modulo a translation refoliates the neighbourhood of a singular circle, but refoliating $S^i \times B^2$ when i is odd but bigger than 1 is much harder. However this neighbourhood obviously admits a smooth nonzero vector field normal to its boundary – also we can ensure that it coincides with a given nonzero vector field on the central S^i – and it is known that the existence of such a vector field implies that of the required refoliation. □

27. This end of the year note is being typed nine months after the one above, but of course I had once again looked long and wistfully at Thurston's *"Existence of codimension one foliations"* (1976). The number of people who got it was nonzero then, but now – four decades later! – it is (imho) even less than those who dig Mochizuki's *"Inter-universal Teichmüller theory"* (2012). This classic characterized manifolds possibly with boundary that admit a smooth foliation having boundary components as leaves. This is done using an explicit local construction which spreads the required foliation steadily, and always transverse to a given vector field normal to the boundary components, till it covers the entire manifold ... so I imagine we should in fact be able to refoliate our $S^i \times B^2$ in such a way that this new foliation cuts the central lower dimensional odd sphere S^i in a given codimension one foliation?

Unlike Thurston my ability 'to see from within' noneuclidean geometries is very limited, may be that is why I've given primacy to familiar n-space only, with indeed the—to my mind just pragmatic, but also called relativistic—restriction that it ought to be of a finite radius $c < \infty$. In this receptacle are born from its own cartesian motions all lipschitz manifolds, and if smooth enough, its cayley distance induces on them a riemannian metric. The usual tools of vector calculus and forms, tensors, etc., are available chart by chart – with an occasional sign ambiguity for orientation dependent quantities – so existence of smooth foliations translates into existence theorems of analysis, for example, *the partial differential equation $\overrightarrow{E} \cdot \mathrm{curl}\overrightarrow{E} = 0$ has an always nonzero solution on any closed 3-manifold,* because this is the same as saying that the 2-dimensional plane field orthogonal to $\overrightarrow{E}$ is tangent to a foliation.

The story in fact began when this problem of vector calculus was posed for S^3 by Hopf in 1935. It was solved by Ehresmann and Reeb in 1944, but then there was an extended drought of interesting foliations, till Lawson got infinitely many odd dimensional spheres in a clever way, but it was Thurston soon after who reached the bottom of the well, his chart by chart approach akin to how Edwards et al were trying to triangulate topological manifolds.

Which reminds me of another problem about vector fields that was posed to me in late 1969 in a very kind letter from Professor Steenrod :-


Here is a non-textbook type problem which you
can try. Should you solve it, I would be impressed;
should you fail to solve it – don't be disturbed,
there are many problems I failed to solve and others
have solved. Let D be an open, connected set in
3-dimensional cartesian space R^3. Let $P: D \to R^3$ be
a continuous vector field in D everywhere different
from zero. Under what conditions on D can it be proved
that, (for any such P,) there are continuous vector fields E and H mapping $D \to R^3$
such that their vector product $E \times H$ at each point of D
coincides there with P (i.e. $E \times H = P$ as fields).

Sincerely yours

Norman E. Steenrod

Working out by myself what in the topology of D prevents this factorization $\overrightarrow{P} = \overrightarrow{E} \times \overrightarrow{H}$ was a wonderful way of learning some obstruction theory, and so appreciate later on Haefliger's necessary conditions for the existence of foliations in all codimensions, which prepared the ground for Thurston to complete the job from the other end. Also it helped me move to virtually a ring side seat even as this dénouement was about to be played out.

There was much else equally exciting going on then, e.g., the index formulas of Atiyah et al. Trying to make their analysis less messy I stumbled on the smoothing operators in de Rham's *Variétés Différentiables.* If there are enough flows preserving leaves they generalized to foliated manifolds and gave finiteness theorems. *In the cartesian context a smoothing operator comes with the primal motion that gave birth to our manifold.*

The charts lipschitz if $c < \infty$ come too, so *abstract manifolds are natural.* Poincaré's crossword dissection of smooth manifolds, perfected by Cairns, used a *quadrillage* of the ambient space. For topological manifolds we use a grid in each chart, and puzzle out if two overlapping dissections can be made to fit, then three, etc. For lipschitz structures Sullivan played the same game using $1/c > 0$ tilings, so these crosswords or torus tricks quantize manifolds.

The new year is here and I'll return to a sequel to ਉਂਗਲੀਆਂ ਅਤੇ ਟਾਇਲਾਂ – its translation *Fingers and tiles* will be available soon – which I posted in July 2015. This gave four proofs of a cute problem tied to the burgeoning Thurston lore, viz. the first of the ten (!) stories about him that Sullivan relates in the November 2015 issue of the Notices of the A.M.S. I thought this was an auspicious way to start (re)learning the constructions needed to understand better some questions that have arisen in this work. For example the sequel that I am working on dwells on constructions used to study the embeddability of simplicial complexes in double dimensional space.

There is also a considerable backlog of older things that need to be typed up, so I hope to continue this series of notes as well. For example given below is a cute picture, scanned directly from my notebook of 2014 to save on time, exemplifying how deleted joins, to wit Flores' spheres, join the fray as cayley balls if we replace the ball B of radius c by a regular simplex.

K S Sarkaria (*contd.*)