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# Henri Poincaré

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# A Look Back at Poincaré's *Analysis Situs*

§1. *This is a very brief report* on a year-long Topology Seminar which I ran during 1993–94 at Panjab University. The detailed lecture notes of this seminar will be published elsewhere.*

Our object in this seminar was to get an over-all picture of what had been happening in this century's Topology, and with this in mind we had adopted the following strategy.

(i) To understand the mathematics of Poincaré's "*Analysis Situs*" and its five *Compléments* as clearly as possible, and

(ii) to understand the threads connecting Poincaré's ideas to future developements as clearly as possible.

In the course of doing (i) and (ii) we also got

(iii) some new results.

§2. *There is no doubt* that Poincaré's *Analysis Situs* and its five *Compléments* (1953), 1892–1904, constitute a breathtaking, epic, monumental (almost any superlative seems inadequate!) work.

In fact if I were merely to make *a list* of the big ideas which occur one after another in it, I would over-step my time.

Nevertheless, let me at least *start* making such a list: –

– Boundary operator, *Betti numbers, homologies* (using smooth and oriented *singular chains* of a *differentiable manifold:* §§ 1–6). (However we note that Poincaré became aware of torsion only later, in the first *Complément,* while giving another definition of homology via incidence matrices of cell complexes.)

(The extent to which Poincaré's ideas have overshadowed this century's mathematics can perhaps be gauged from this simple little fact:

* A first edition of most of these notes (about 150 pp) is available.

Out of all the Fields Medallists, with the exception of perhaps three or four, every one of them – irrespective of his domain: number theory, algebra, analysis, … – has used some *homology* in his work!!

Perhaps not since the invention of the *calculus* has a single tool so strongly influenced mathematics as *homology.* )

– Periods of indefinite integrals (= differential forms) and (implicitly) *de Rham cohomology* (§7).

– *Intersection matrices* and *Poincaré' Duality* in orientable closed manifolds (§9, with a correct proof only later in §IX of the first *Complément*).

(Again it is remarkable how many fantastic results of this century – going back from Freedman and Donaldson, through Rochlin and Whitehead, to this beautiful duality of Poincaré – are at heart really assertions about the intersection matrix of an $M^{4k}$ !)

– *Triangulability of differentiable manifolds* (assumed in §10, with attempts at proof later in §16 via *quadrillages,* and in §XI of the first *Complément* via a method of *rays*.)

– *Monodromy of integrable linear PDEs* (= flat connection) *on a manifold* and definition of the *fundamental group* $\pi_1(M)$ as the "most general" such group of M (§12).

(This definition of $\pi_1$ was later put on a firm footing, and used in his *de Rham homotopy theory,* by Sullivan (1977).)

– Also the now standard (via *homotopy classes of loops* based at a point) definition of $\pi_1(M)$, and again a third combinatorial definition which gives relations for $\pi_1(M)$ if M is a *CW complex* obtained by pairwise identification of facets of a polyhedron (§§12–13).

– Many *computations* of fundamental groups and homologies, and *a classification theorem for some affine 3-manifolds* (which is perhaps the deepest result of the main paper: §§10–11, 13–14).

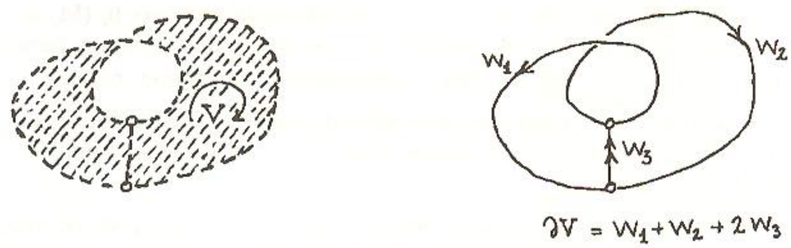
(In Dennis Sullivan's words this result is "*1/8 th of Thurston's theorem*"*:* the latter says roughly that any irreducible 3-manifold can be equipped with one of 8 specified geometries.)

The above is *only the beginning* (based on §§1–14 of the main paper), but let me just stop here, and now tell you some more about the first and last items of the above partial list.

§3. *Poincaré's first definition of homology*. He starts off *Analysis Situs* by defining what we would now perhaps call a differentiable quasi-affine non-singular complete intersection $V^n \subset \mathbb{R}^N$, i. e. a clean intersection of N-n smooth hypersurfaces of an open set of $\mathbb{R}^N$ defined by so many smooth equations.

The aforementioned open set is assumed defined by some inequalities. He now starts replacing, one-by-one, these inequalities by equations, and by adding these, one-by-one, to the N-n defining equations of V, gets the *complete boundary* of V. Then the *boundary* $\partial V$ of V is defined by dropping further the singularities: so e. g.


Each V is (transversely) *oriented* by ordering its defining N-n equations (so a transposition of 2 of these equations gives not V but –V), and the boundary components are oriented by placing the new equation in the end (this is what gives the arrows in the above picture). Poincaré realizes (unlike Betti before perhaps?) that "varieties" can repeat in ∂V, e. g.

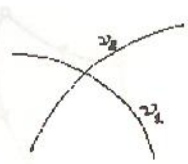



Whenever an integral linear combination of r-dimensional "varieties" equals a boundary ∂V he writes

$$c_1 W_1 + c_2 W_2 + \ldots \simeq 0$$

Starting with these primitive relations (with all $W_i \subseteq M$) he now generates all *homologies of a manifold* M (see below for his definition of manifold) by "treating them just like equations": i. e. by allowing such relations to be added, and terms taken to the other side if one changes sign, or multiplied by integers (and occasionally – and this of course makes a big difference! – even division by nonzero integers).

*We break here for Poincaré's definition of a manifold M*: First he sort of retreats and considers more restrictive *parametrized varieties* i. e. $v^n \subset \mathbb{R}^N$ with a 1–1 onto θ from an open subset of $\mathbb{R}^N$ to $v^n$ given. But next he generalizes enormously via the idea of *continuation:* $v_1$ and $v_2$ are called continuations of each other if $v_1 \cap v_2$ is nonempty *and* is also a parametrized n-variety, e. g.



not continuations

continuations

He defines M as a graph ( = *réseau connexe*) whose vertices i are parametrized varieties $v_i$ (of $\mathbb{R}^N$) with edges $\{i,j\}$ corresponding to pairs of varieties which are continuations of each other. In modern terms he has defined the notion of an abstract manifold M together with an immersion into some $\mathbb{R}^N$ (but for him the latter is always extra baggage, and this is quite explicit as one reads on the paper).

*Returning to homologies*, we now see Poincaré defining *Betti numbers* $b_r$ (M) as the cardinality of a maximal set of linearly independent (i. e. no non-trivial homology between them) and closed (i. e. with $\partial c = 0$) combinations of r-dimensional subvarieties of M.

(*Betti's numbers* on the other hand had been defined restricting the coefficients $c_i$ to be always $\{-1, 0, +1\}$: so Betti was in fact talking of the *least number of generators required to generate* $H_r$ (M) – see below.)

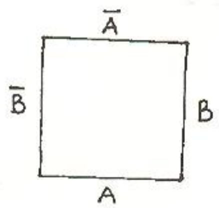
*In modern terms, Poincaré's definition re-interprets as follows*: Let $C_r(M)$ be integral combinations of (oriented) "r-varieties" of M, generalize $\partial$ by linearity to all these to get $\partial$: $C_r(M) \to C_{r-1}(M)$ and since $\partial \circ \partial = 0$ define $H_*(M) = \ker\partial / \mathrm{Im}\partial$. Then Poincaré's $b_r$ (M) is the $\mathbb{Z}$-dimension of $H_r$ (M) mod torsion. (As mentioned before Poincaré did become aware of torsion too, but later.)

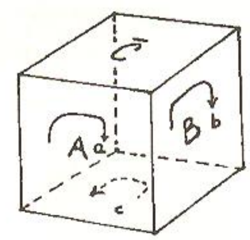
*Relationship of this definition with singular homology*. There are essentially 2 differences. If we use all *continuous* (instead of just smooth) oriented ν's we get the definition of singular homology as given by Lefschetz (1933). If we further use ordered (instead of oriented) ν's, we get the current definition of Eilenberg (1944). We note finally that standard techniques – cf. Eilenberg (1947) – show that, for the case of smooth manifolds M, the aforementioned Poincaré homology groups $H_*(M)$ coincide with the singular homology groups of M.

§4. *Poincaré and 3-manifold theory*. Extrapolating from the case of 2-manifolds (also from his experience with fundamental domains of some Kleinian groups) Poincaré assumes the triangulability of closed 3-manifolds, i. e. that they can be obtained from a 3-polyhedron by a pairwise identification of its facets.

Since analysis of similar identifications had led to a classification of 2-manifolds, Poincaré now quite naturally wants to make lists of the 3-dimensional ones the same way.

*Like always he starts off from something very simple.* He points out that the 2-torus is a *square*, with opposite sides identified, and it is the only orientable 2-manifold obtained this way. So what can we say about the parallel 3-dimensional case of the *cube?* (Note also that any 3-polyhedron is a *subdivided cube,* so the undivided cube can serve as a starting point as one scans for *all* closed 3-manifolds: e. g. the famous homology sphere $P^3$ of Poincaré would be encountered at the "next" level in such a scan because the dodecahedron is combinatorially a very simple subdivided cube.)




*The cube.* Even if we only allow its *opposite* facets to be identified, we have much more leeway now than for a square: we are allowed (see fig. above) to first rotate a facet (through 0, $\pi/2$, $\pi$, or $3\pi/2$) and *then* identify with the opposite one. (We disallow *reflections* because we want *orientable* 3-manifolds only.)

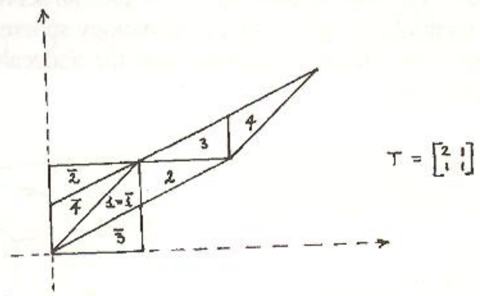
Accordingly let us adopt the notation *abc*, $0 \leq a,b,c \leq 3$, to denote the cell complex obtained by rotating three compatibly oriented adjacent facets of the cube through these multiples of $\pi/2$, and then identifying with the opposite facets.

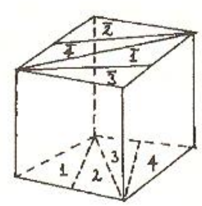
*Poincaré's first five examples* are (in the above notation) *000* (the 3-torus), *113* (a non-manifold), *111* (quaternionic space), *001* (a twisted 3-torus), and *222* (projective space, which he defines a little differently by using an octahedron).

In each case he tests for non-singularity by computing the Euler characteristic of the *links* at the vertices of the cell complex (the other points are obviously non-singular) and computes (for the four manifolds he gets) the fundamental group and the Betti numbers to show that they are topologically distinct.

During our seminar we checked that there are exactly *three more manifolds* of this kind: *002*, *022*, and *122* (Poincaré was certainly aware of at least the first of these because it belongs to the series below).

*Poincaré's series 00T*, $T \in SL(2,\mathbb{Z})$. These manifolds are defined combinatorially as follows. Each integral matrix T with $\det(T) = 1$ determines in a natural way (see fig. below) a *T-subdivision* of the unit square. Use this to subdivide the top of the cube, and analogously subdivide the bottom using the inverse matrix, and leave the vertical faces of the cube unsubdivided. Then *00T* is obtained by identifying opposite pairs of vertical faces without doing any preliminary rotation, and by identifying each piece of the subdivided bottom with the corresponding piece of the subdivided top.




Alternatively *OOT* is also defined *group-theoretically* (Poincaré uses this for all his computations) as the quotient of 3-space by the *discontinuous group* generated by the three *affine motions*

$$(x,y,z) \mapsto (x+1,y,z), \quad (x,y,z) \mapsto (x,y+1,z), \text{ and}$$
$$((x,y),z) \mapsto (T(x,y),z+1).$$

(These manifolds play a bis rôle in the third and fourth *Compléments* – which deal wlth monodromy, etc., of algebraic surfaces – where Poincaré thinks of *OOT* as a *torus bundle* over the circle, viz. the *mapping torus* of the toral automorphism T: $\mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$: more generally he also considers surface bundles over the circle.)

Using his group theoretic definition of these manifolds Poincaré now proves the following (which he had announced in his 1892 *Comptes Rendus* note).

*Poincaré's rigidity theorem. OOT* is *homeomorphic to OOU if and only if* T *is conjugate to* U *or its inverse in* GL(2,$\mathbb{Z}$).

We remark that the above is in fact a *corrected* version of the result stated in the paper (the "or its inverse" is necessary, also Poincaré seems to conjugate within SL(2,$\mathbb{Z}$) which won't do). In our seminar we checked that the above is true even if det(T) = –1 = det(U) (when of course these manifolds are non-orientable) and we also obtained the following arithmetical addendum to Poincaré's result.

Enumeration of Poincaré's series. *There are infinitely many topologically distinct manifolds 00T with* tr(T) = ± 2. *However for all* t *other than* ± 2, *the number* P(t) *of such manifolds with* tr(T) = t *is finite, and is given by*

$$P(t) = \frac{h(t) + n_2(t) + 1}{2}$$

*where* h(t) = *number of ideal classes of* $\mathbb{Z}[(t^2-4)^{1/2}]$ *and* $n_2(t)$ = *number of elements of order 2 in this class group.*

A similar rigidity and enumeration result can most probably be established for another infinite series *22T* containing the manifolds *222*, *221*, and *220*. Since these manifolds *22T* are definable "like" *Lens spaces* (starting from $\mathbb{R}P^3$ minus a disk instead of a disk) we see that Poincaré's result is close (in spirit at least!) to the classification of lens spaces given in Reidemeister (1935).

More obviously Poincaré's rigidity theorem is akin to the later rigidity theorems of Bieberbach (1911), and of Mostow (1966). The former deals with conjugacy, by means of affine motions, of discontinuous groups of *Euclidean motions,* and thus is especially close to Poincaré's result, which deals with a similar problem for some discontinuous groups of *affine motions.*

We remark also that some definitive general results on the conjugacy of discontinuous groups of affine motions of 3-space have been proved by Fried-Goldman (1983). For example they show that a closed 3-manifold is affinely flat if and only if it is finitely covered by a torus bundle over the circle (i. e. an *00T*). These authors also show that these are all the closed 3-manifolds which admit three (viz. the ones modelled by the left-invariant metric of a solvable 3-dimensional Lie group) of the eight "geometries" of Thurston (1982).

From the above I think it is amply clear that Poincaré's impressive contributions to 3-manifold theory are by no means limited to the very famous problem about closed 3-manifolds which he left to us (*Is one of them an exotic homotopy sphere?*) or to the enchanting (and ubiquitous!) *exotic homology sphere* $P^3$ which he discovered in the fifth *Complément* of this paper.

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## References

Bieberbach, L.

1911 Über die Bewegungsgruppen der euklidischen Raüme I, II, *Math. Annal. 70,* 297–336 and *72 (1912)*, 400–412.

Eilenberg, S.

1944 Singular homology theory, *Ann. Math. 45,* 407–444

1947 Singular homology in differentiable manifolds, *Ann. Math. 48,* 670–681.

Fried, D. and W. H. Goldman

1983 Three-dimensional affine crystallographic groups, *Adv. Math. 47,* 1–49.

Lefschetz, S.

1933 On singular chains and cycles, *Bull. Am. Math. Soc. 39,* 124–129.

Mostow, G. D.

1966 On the conjugacy of subgroups of semisimple groups, in *Algebraic groups and discontinuous subgroups, Proc. Sympos. Math. A. M. S.,* 413–419.

Poincaré, H.

1953 *Oeuvres, vol. VI,* (Paris: Gauthier-Villars) 189–498; these papers, i. e. a C. R. note, *Analysis Situs,* and its five *Compléments,* appeared first over the years 1892–1904.

Reidemeister, K.

1935 Homotopieringe und Linsenraüme, *Abhand. Sem. Hamburg 11,* 102–109.

Sullivan, D.

1977 Infinitesimal computations in topology, *Pub. I. H. E. S. 47,* 269–332.

Thurston, W. P.

1982 Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Am. Math. Soc. 6,* 357–381.