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A ONE-DIMENSIONAL WHITNEY TRICK AND KURATOWSKI'S GRAPH PLANARITY CRITERION

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ABSTRACT

For $n \geq 3$, the ordinary Whitney trick shows that a simplicial complex K^n having van Kampen obstruction class $\mathfrak{o}(K^n) = 0$ embeds in $\mathbf{R}^{2n}$. We give a one-dimensional version of the Whitney trick, by means of which any graph K^1 satisfying $\mathfrak{o}(K^1) = 0$ can be, step by step, embedded in $\mathbf{R}^2$. We then deduce some other planarity criteria, including Kuratowski's, from this result. As a byproduct we also obtain a fascinating description of the mod 2 homology of the deleted product of a graph.

§1. Introduction.

The well-known graph planarity criterion, which now bears his name, was published by Kuratowski [5] in 1930.

Somewhat less well known is the contemporaneous paper of van Kampen [11], 1932, in which is defined, for any $n \geq 1$, an obstruction $\tilde{\mathfrak{o}}(K^n)$, which measures the non-embeddability of an n-dimensional simplicial complex K^n in $\mathbf{R}^{2n}$. One of the results of [11] is that $\tilde{\mathfrak{o}}(K^n) \neq 0$ when $K^n = \sigma_n^{2n+2}$, the n-skeleton of a $(2n+2)$-simplex; thus this n-complex does not embed in $\mathbf{R}^{2n}$. More significantly, van Kampen also outlined, in the converse direction, a general procedure by which any K^n satisfying $\tilde{\mathfrak{o}}(K^n) = 0$ could be embedded in $\mathbf{R}^{2n}$. However, this method — which was made precise only much later, and independently of each other, by Wu [15] and Shapiro [10] — makes essential use of the well-known Whitney trick [14], 1944, and thus works only for $n \geq 3$.

The main object of this note is to show that van Kampen's method extends also to the case $n = 1$; in fact *there is a one-dimensional version of Whitney's trick by*

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means of which any graph K^1 satisfying $\mathfrak{o}(K^1) = 0 \bmod 2$ *can be, step by step, embedded in* $\mathbf{R}^2$. Furthermore, from this planarity criterion, we will directly deduce some others, including Kuratowski's. The latter, in fact, fits into van Kampen Theory very nicely, being a homological reformulation of the cohomological condition $\mathfrak{o}(K^1) = 0$.

As a byproduct we also obtain a fairly complete and fascinating description of the mod 2 homology of the deleted product of K^1.

§2. Planarity of graphs

(2.1) Definition of $\mathfrak{o}(K)$. Our *graphs* will be without loops and multiple edges, i.e. will be one-dimensional simplicial complexes K^1.

The *deleted product* K_* of a graph K is the sub cell-complex of $K \times K$ defined by $K_* = \{\sigma \times \theta \mid \sigma \cap \theta = \phi\}$. We equip it with the free $\mathbf{Z}_2$-action $\sigma \times \theta \mapsto \theta \times \sigma$. From now on we only consider *symmetric* cochains of K_*, i.e., those which are preserved by this action. Furthermore, it will suffice to use cochains with *$\mathbf{Z}_2$-coefficients* only. (For $n > 1$, van Kampen Theory needs integer coefficients.†) The ith symmetric cohomology of K_* with mod 2 coefficients is denoted $H_s^i(K_*;\mathbf{Z}_2)$.

Van Kampen [11] defined his *obstruction class* $\mathfrak{o}(K) \in H_s^2(K_*;\mathbf{Z}_2)$ as follows: Take any (semilinear or simplexwise smooth) general position map $f: K \to \mathbf{R}^2$. Such an f has only finitely many double points, all contained in the interiors of the edges. For any two disjoint edges α, β of K let $\mathfrak{o}_f(\alpha,\beta) = |f(\alpha) \cap f(\beta)| \bmod 2$. Then $\mathfrak{o}(K)$ is the symmetric cohomology class of this symmetric two-dimensional cocycle $\mathfrak{o}_f$ of K_*.

That $\mathfrak{o}(K)$ is indeed an invariant of K follows from the fact that the *obstruction cocycle* $\mathfrak{o}_f$ depends on f only up to an equivariant coboundary: see [11], Hilfsatz 4. This is also clear from Fig. 1, which shows how a perturbation of a general position linear map f, which moves a vertex v to the other side of an edge β, adds to $\mathfrak{o}_f$ the coboundary of the *elementary* cochain $[v,\beta]$, viz. that which is 1 on $v \times \beta$ and $\beta \times v$, and 0 elsewhere.

From its definition it is clear that for all planar graphs K one has $\mathfrak{o}(K) = 0$. Conversely,

(2.2) *If* $\mathfrak{o}(K) = 0$ *then any simplexwise smooth general position map* $f: K \to \mathbf{R}^2$ *can be changed to an embedding as follows*:

†Modulo remarks enclosed within such brackets, we restrict ourselves exclusively to the graph theoretical case $n = 1$.

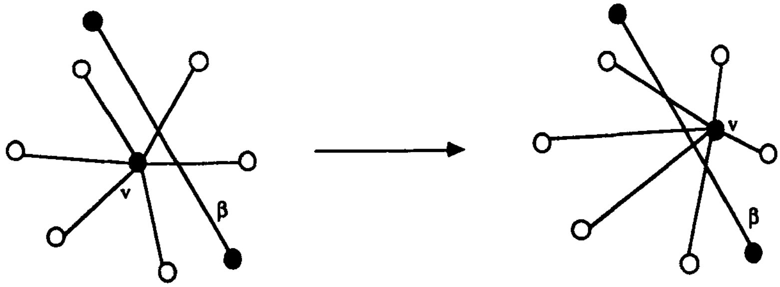


Fig. 1.

(2.2.1) *Step 1.* As in [11], Hilfsatz 3, we start by noting that any elementary coboundary $\delta[v,\beta]$ can be added to the obstruction cocycle by changing β to β', as in Fig. 2. The point to note here, and also in many other pictures below, is that *"modulo" some thin even tubing β' is the "same" as β.* So any general position edge which was cut by β an even number of times, will also be cut by β' an even number of times.

Since $\mathfrak{o}(K) = 0$ we can thus, without loss of generality, assume that

(A) *the f-images of any two disjoint edges have an even number of intersections.*

Self-intersections, if any, of edges are removed easily (see Fig. 3).

Adjacent edges, too, can be made to cut an even number of times. To do this,

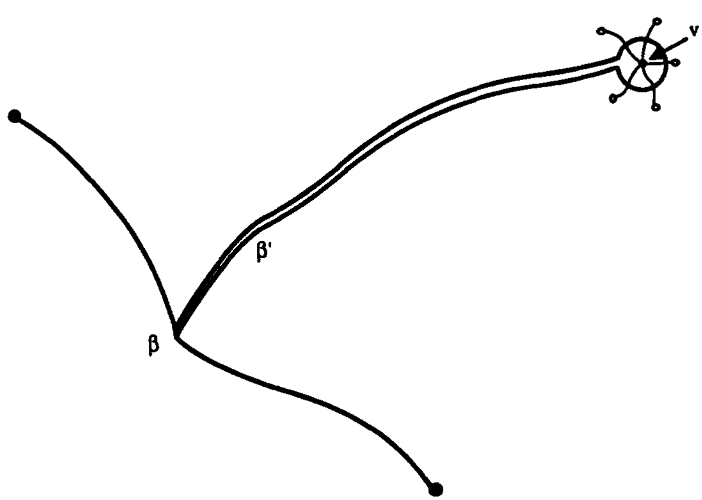


Fig. 2.

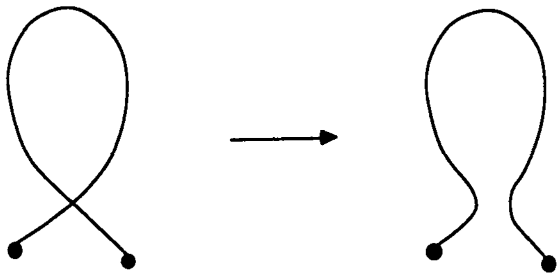


Fig. 3.

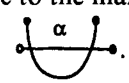
the first intersection of two such edges α and β can be replaced by two intersections, as shown in Fig. 4.

Hence we might, if necessary, by replacing K by an appropriate subdivision, suppose in addition to (A) that

(B) *all double points of* $f: K \to \mathbf{R}^2$ *belong to disjoint pairs of edges.*

(2.2.2) *Step 2.* We now choose a total order < for the set of edges of K. Let α denote the first edge which, under f, cuts any other edge, and out of all such edges cut by α let β be the first edge. So α and β have disjoint vertices and cut each other an even number of times. We plan to reduce the number of these cuts by 2, without introducing any intersections on edges before the αth, and without losing the properties (A) and (B) of f.

If two consecutive cuts of α with β are in the same direction, then they can be removed as shown in Fig. 5.

We now come to the main case, i.e., any two consecutive cuts of α with β form a *Whitney loop* 

. We first ensure that *any other edge intersects this loop*

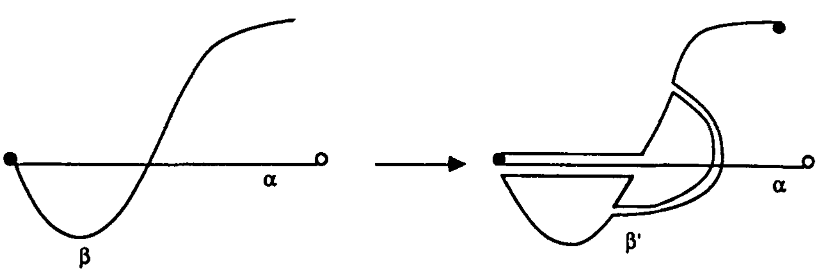


Fig. 4.

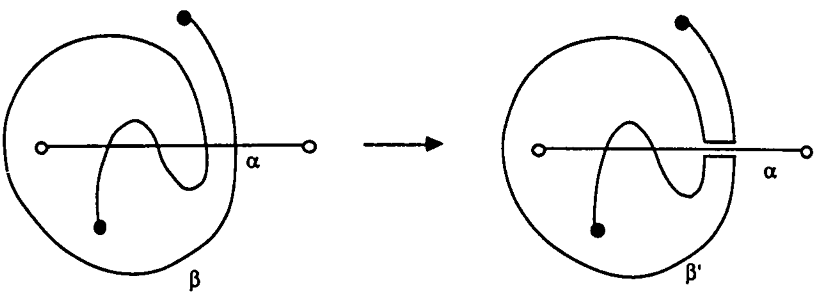


Fig. 5.

only on β or on the right portion of α. To achieve this we push β to the left, as shown, in Fig. 6 in the first step, and then eliminate the new self-intersections of β, as shown in the second step.

For this Whitney loop we can perform the straightforward one-dimensional analogue of the standard Whitney trick [for $n \geq 3$], as shown in Fig. 7, *if and only if there is no edge γ which cuts the bottom portion of β an odd number of times.*

So assume that there is such a γ. One of the segments of γ joining the top and bottom portions of β will cut α an even number of times. Furthermore, by using the construction of Fig. 5 on γ, we can assume here that any two consecutive cuts are in different directions. We use the first and last of these cuts as "inlet" and "outlet" of a tube running parallel to, and to the same side of, the entire right portion of α, to alter the β' of Fig. 7 to β'', as shown in Fig. 8.

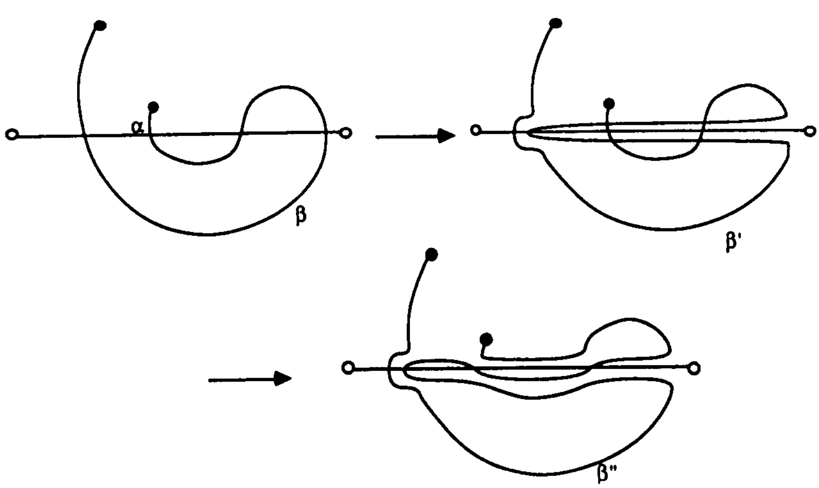


Fig. 6.

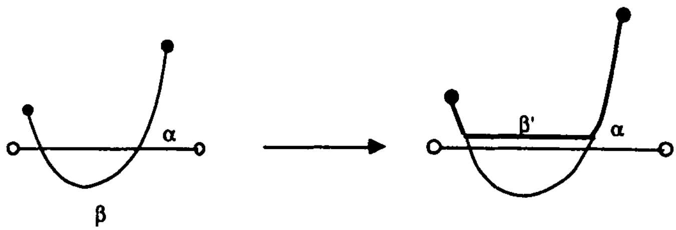


Fig. 7.

Note that $|\beta'' \cap \gamma|$ is even and, once again, since β'' is the "same" as β "modulo" even tubing, the number of cuts with any edge other than γ also remains even. Further, any self-intersections which may be present in β'' can be removed, as in Fig. 3.

The number of cuts of α with β has thus been reduced by 2 in the desired manner. Continuing this process, f eventually becomes an embedding of K in $\mathbf{R}^2$.

§3. Variations

(3.1) Integer Coefficients. Let $f: K \to \mathbf{R}^2$ be a general position map. Fix an orientation of $\mathbf{R}^2$, and, for any two disjoint oriented edges σ and θ of K, count an intersection where the orientation of $f(\sigma)$ followed by that of $f(\theta)$ agrees with that of $\mathbf{R}^2$ as $+1$, and -1 otherwise. Then $\tilde{\mathfrak{o}}(K) \in H_s^2(K_*;\mathbf{Z})$ is the class of the cocy-

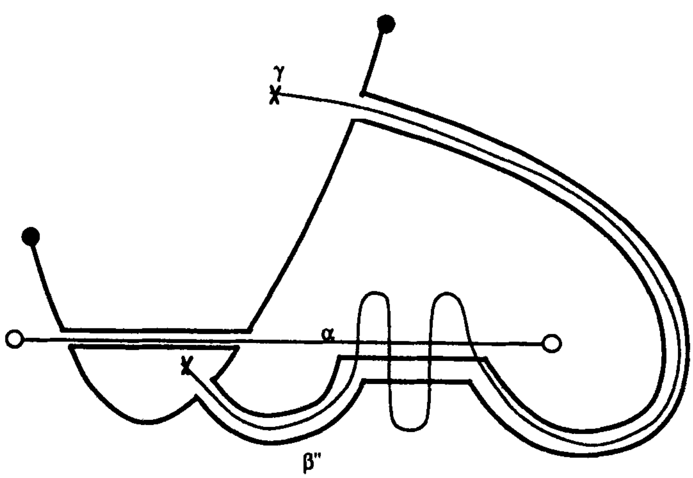


Fig. 8.

cle $\tilde{\mathfrak{o}}_f(\sigma,\theta)$ which counts the intersections of $f(\sigma)$ and $f(\theta)$ algebraically in this fashion.

From the mod 2 planarity criterion of §2 it follows *a fortiori* that *K is planar iff* $\tilde{\mathfrak{o}}(K)=0$. [For $n \geq 3$ the van Kampen method uses integer coefficients, and thus only establishes $K^n \subsetneq \mathbf{R}^{2n} \Leftrightarrow \tilde{\mathfrak{o}}(K^n)=0$; however, I do not know of an explicit K^n, $n \geq 2$, for which $\tilde{\mathfrak{o}}(K^n) \neq 0$ and $\mathfrak{o}(K^n)=0$.]

(3.2) Continuous $\mathbf{Z}_2$-Maps. Let S^1 denote the unit circle of $\mathbf{R}^2$ equipped with the antipodal $\mathbf{Z}_2$-action. Since the integer $\tilde{\mathfrak{o}}_f(\sigma,\theta)$ can be identified with the degree of the map $F:\partial(\sigma\times\theta)\to S^1$ defined by

$$F(x,y)=\frac{f(x)-f(y)}{\|f(x)-f(y)\|},$$

it follows that $\tilde{\mathfrak{o}}(K)=0$, i.e., that *K is planar iff there exists a continuous* $\mathbf{Z}_2$*-map* $K_*\to S^1$. (This homotopy-theoretic criterion has a higher dimensional generalization even more extensive than that of the cohomological one. It can be shown that, for $2m \geq 3(n+1)$, $K^n \subsetneq R^m$ iff there exists a continuous $\mathbf{Z}_2$*-map* $K_* \to S^{m-1}$; see Weber [12].)

An alternative formulation using the *deleted join* $K_{\#}$ is sometimes more useful. Recall that $K_{\#}$ is obtained from the join $K.K$ of two disjoint copies of K by deleting all simplices of the type $\sigma.\theta$, $\sigma\cap\theta\neq\phi$. A similar proof shows that *K is planar iff there exists a continuous* $\mathbf{Z}_2$*-map* $K_{\#}\to S^2$.

(3.3) Kuratowski's Criterion. The mod 2 symmetric cochains a, and symmetric chains $c=\sum_\theta c_\theta\theta$ of K_*, are dual to each other under $\langle a,c\rangle=\sum_\theta c_\theta a(\theta)$, where in the summation only one θ is to be chosen from each antipodal pair of cells. One has $\langle\delta a,c\rangle=\langle a,\partial c\rangle$, so there is also an induced duality $\langle\ ,\ \rangle$ between symmetric cohomology $H_s^2(K_*;\mathbf{Z}_2)$ and symmetric homology $H_2^s(K_*;\mathbf{Z}_2)$.

The $\subseteq$-minimal nonzero cycles $z\in H_s^2(K_*;\mathbf{Z}_2)$, or *symmetric circuits* of K, determine a matroid on the set of 2-cells of K_*. Obviously they constitute a set of generators of the $\mathbf{Z}_2$-vector space $H_s^2(K_*;\mathbf{Z}_2)$. Thus the class $\mathfrak{o}(K)$ can be nonzero iff there is a symmetric circuit z with $\langle\mathfrak{o}(K),z\rangle\neq 0$. Let $K_z\subseteq K$ denote the *support* of z, i.e. the subgraph determined by the edges which occur as one of the factors of the 2-cells of z.

(3.3.1) *Such a* K_z *is homeomorphic to one of the Kuratowski graphs,* σ_1^4 *or* $\sigma_0^2\cdot\sigma_0^2$.

First note that K_z is a minimal graph whose deleted product contains z, and that the restriction $\mathfrak{o}(K_z)\in H_s^2((K_z)_*;\mathbf{Z}_2)$ of $\mathfrak{o}(K)\in H_s^2(K_*;\mathbf{Z}_2)$ satisfies $\langle\mathfrak{o}(K_z),z\rangle=\langle\mathfrak{o}(K),z\rangle\neq 0$.

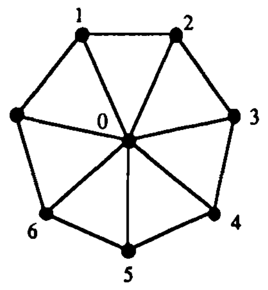


Fig. 9.

We will see later that $(K_z)_*$ contains no symmetric circuit other than z. So the graph K_z cannot possibly contain two disjoint 1-circuits c_1 and c_2, because then we would have $z = (c_1 \times c_2) + (c_2 \times c_1)$, and so $K_z = c_1 \cup c_2$, for which $\mathfrak{o}(K_z) = 0$.

K_z can have no vertex v of valence 1, because the subgraph L obtained by omitting v and the edge incident to it still has $\mathfrak{o}(L) \neq 0$ and so $z \subseteq (L)_*$, which is not possible because K_z is the support of z.

The vertices v of valence 2 can be eliminated one by one as follows. If v is incident only to vw_1 and vw_2, then the edge w_1w_2 cannot be in K_z: otherwise $\mathfrak{o}(L) \neq 0$ and so $z \subseteq (L)_*$ for $L = K_z \setminus \{w_1w_2\}$. So K_z is a subdivision of $M = (K_z \setminus \{v, vw_1, vw_2\}) \cup \{w_1w_2\}$, which has one less vertex of valence 2, and is the support of the unique symmetric circuit contained in M_*.

So without loss of generality we can assume that all vertices of K_z have valence ≥ 3. A theorem of Dirac [3],† of which a purely combinatorial and elementary proof is given in Lovász [6] (see §10, ex. 4, and pp. 377–378), now tells us that K_z is either (i) σ_1^4, or (ii) a graph containing $\sigma_0^2 \cdot \sigma_0^2$, or else (iii) a *wheel* as in Fig. 9. In case (ii) note that $(\sigma_0^2 \cdot \sigma_0^2)_*$ is itself a symmetric 2-cycle. So, by using the minimality of K_z, we must have $K_z = \sigma_0^2 \cdot \sigma_0^2$. The third possibility is ruled out because then $\mathfrak{o}(K_z) = 0$.

If $(K_z)_*$ were to contain a symmetric circuit w other than z, then we could choose a cell $\alpha \times \beta$ of w not in z, and obtain a graph L from K_z by *identifying* an interior point $\hat{\alpha}$ of α with an interior point $\hat{\beta}$ of β, and replacing both α and β by two edges each. It is easily verified that L_* contains a $\bar{z}$ corresponding to z on which $\mathfrak{o}(L)$ is nonzero, and that L is the support of $\bar{z}$. But L_* has less symmetric circuits than $(K_z)_*$. So, by above, we might as well assume that L is homeomorphic to a Kuratowski graph. Since the identification point $\hat{\alpha} = \hat{\beta}$ has valence 4, L must be homeormorphic to a σ_1^4 having $\hat{\alpha} = \hat{\beta}$ as one of its five nodes. This

†Which classifies all graphs having vertices of valence ≥ 3 and not containing any 2 disjoint circuits.

is not possible, because then K is homeomorphic to a σ_1^3 with two edges duplicated, contradicting $\mathfrak{o}(K) \neq 0$. So no such w exists.

(3.3.2) *The planarity criteria* "$\mathfrak{o}(K) = 0$" *and* "$\tilde{\mathfrak{o}}(K) = 0$" *were known previously but* only *as corollaries of Kuratowski's criterion.*

Such a proof—given in Wu [16], p. 210—of "$\mathfrak{o}(K) = 0$" must have been known even to van Kampen, because from his results $\mathfrak{o}(\sigma_1^4) \neq 0$, $\mathfrak{o}(\sigma_0^2 \cdot \sigma_0^2) \neq 0$, it follows at once that $\mathfrak{o}(K) = 0$ only if K does not contain a homeomorph of σ_1^4 or $\sigma_0^2 \cdot \sigma_0^2$.

Flores [4], 1933, established the criterion that "there exists a continuous $\mathbf{Z}_2$-map $(K^1)_\# \to S^2$"—or, equivalently, "$\tilde{\mathfrak{o}}(K^1) = 0$"—by observing that for $K = \sigma_1^4$ or $\sigma_0^2 \cdot \sigma_0^2$, $K_\#$ is $\mathbf{Z}_2$-homeomorphic to the antipodal 3-sphere. So, for any non-planar K, one can have no continuous $\mathbf{Z}_2$-map from $K_\#$ to S^2. [Flores is using the Borsuk-Ulam Theorem [1], 1933, which had just become available: "There exists no continuous $\mathbf{Z}_2$-map $S^{2n+1} \to S^{2n}$". It is amusing to note that this itself follows easily from the result, $\mathfrak{o}(K^n) \neq 0$ for $K^n = \sigma_0^2 \cdot \ldots \cdot \sigma_0^2$, contained in van Kampen [11], 1932, because the deleted join $(\sigma_0^2 \cdot \ldots \cdot \sigma_0^2)_\#$ is $\mathbf{Z}_2$-isomorphic to the $(n+1)$-fold join S^{2n+1} of the circle $(\sigma_0^2)_\#$.]

We remark that many other useful planarity criteria, including the well-known ones of Whitney [13] and Maclane [7], are known to be easy corollaries of Kuratowski's criterion, but are not quite so easy to prove directly.

(3.4) Homology of K_*. A fairly complete picture of the 2-circuits of K_* results from the above discussion:

(3.4.1) *The minimal nonzero cycles* $z \in H_2(K_*;\mathbf{Z}_2)$ *consist of*

(i) *some TORI* $c_1 \times c_2$, *one for each ordered pair of disjoint circuits of* K,

(ii) *some SURFACES OF GENUS* 4, *one for each homeomorph of* $\sigma_0^2 \cdot \sigma_0^2$ *contained in* K, *and*

(iii) *some SURFACES OF GENUS* 6, *one for each homeomorph of* σ_1^4 *contained in K. The free* $\mathbf{Z}_2$*-action of* K_* *preserves the 2-circuits of types* (ii) *and* (iii) *and pairs each torus* $c_1 \times c_2$ *with its opposite* $c_2 \times c_1$.

To determine the topology of $(\sigma_0^2 \cdot \sigma_0^2)_*$ and $(\sigma_1^4)_*$ we verify, by counting their cells, that they have Euler characteristics -6 and -10, respectively. That they are manifolds follows by checking that each vertex-link is indeed circular. Finally, note that they are orientable because they are embedded in Flores' 3-spheres $(\sigma_0^2 \cdot \sigma_0^2)_\#$ and $(\sigma_1^4)_\#$.

[For $n > 1$ there are infinitely many non-homeomorphic minimal K^n's with just one non-zero $2n$-cycle z in $(K^n)_*$, and this even if we demand that z be a $2n$-manifold. However, we have shown in [9] that there are only finitely many K^n's for which $(K^n)_*$ is *equal* to a $2n$-pseudomanifold, and that in fact any such n-complex must be of the type $\sigma^{2s_1}_{s_1-1} \cdot \sigma^{2s_2}_{s_2-1} \cdot \ldots \cdot \sigma^{2s_k}_{s_k-1}$. Topologically this corresponds to the fact that there are only finitely many n-complexes which are critically – a notion stronger than minimally – non-embeddable in $\mathbf{R}^{2n}$.]

Note that there is a 1–1 correspondence between the 2-circuits of K_* and the *3-circuits of the deleted join* $K_\#$; however, the latter are all topologically the same, viz. 3-spheres. The ones which are preserved by the free $\mathbf{Z}_2$-action of $K_\#$, i.e. those corresponding to types (ii) and (ii), constitute obstructions to planarity.

(3.4.2) *The dimension of $H_2(K_*;\mathbf{Z}_2)$ is either equal to the maximal number of independent toral circuits contained in it, or else one more than this number. The first alternative occurs if and only if K is planar.*

This follows because the toral circuits generate the kernel of the linear functional $\langle \mathfrak{o}(K), \rangle : H^s_2(K_*;\mathbf{Z}_2) \to \mathbf{Z}_2$.

The above result corrects an error in Copeland [2] where it is asserted that the toral circuits always generate $H_2(K_*;\mathbf{Z}_2)$. The zeroth Betti number of K_* can be computed easily as in [2], e.g., if K is connected and has more than two vertices, then K_* is connected. Lastly, an Euler characteristic computation yields $\dim H_1(K_*;\mathbf{Z}_2)$. [For $n > 1$ very little is known about computing $H_i(K_*;\mathbf{Z}_2)$. However, note that the well-known Richardson–Smith Theorem [8] – see also Wu [16], chapter IV – gives a computation for all $n \geq 1$, of the related mod 2 (co)homology of the pair ($K \times K$, diagonal) in terms of the Steenrod squares, or Smith operations, of K.]

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