Plain Geometry & Relativity

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October 26, 2013. *If we are constrained to a bounded region* Ω *of euclidean space* (of dimension $n \geq 2$) *the parallel postulate is no longer true* : a point P not in a codimension-one flat L is plainly in infinitely many flats L_i which do not meet L inside Ω. So, ***non-euclidean geometries are dime-a-dozen,*** for instance – see Figure 1 – that sheet of paper on which we ask schoolchildren to do all those constructions from Euclid has such a geometry; likewise, *any open* n-*ball* B^n *of finite radius* c, and it is this ball geometry that will concern us from here on.

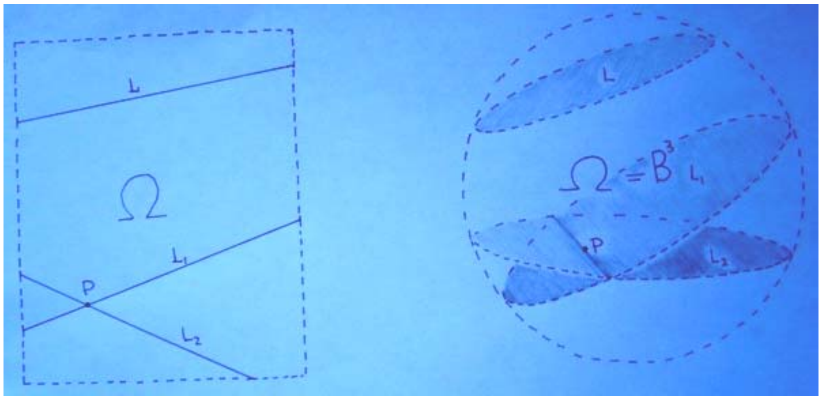

Figure 1

In the euclidean case $c = \infty$, of all the affine **reflections** in a flat, we pick the one in the direction orthogonal to the flat, and call two subsets of n-space congruent iff they are related by a finite sequence of such reflections. The only affine reflections of n-space which preserve a ball B^n of radius $c < \infty$ are the ones in, and orthogonal to, the flats through its center b. However a general and canonical definition emerges if we recall that, to **linearize** the affine transformations of n-space, we should consider it as a flat in *a vector space of one dimension more*—see Figure 2—whose 0 is outside this flat, and identify each point with the *ray* from 0 through that point.

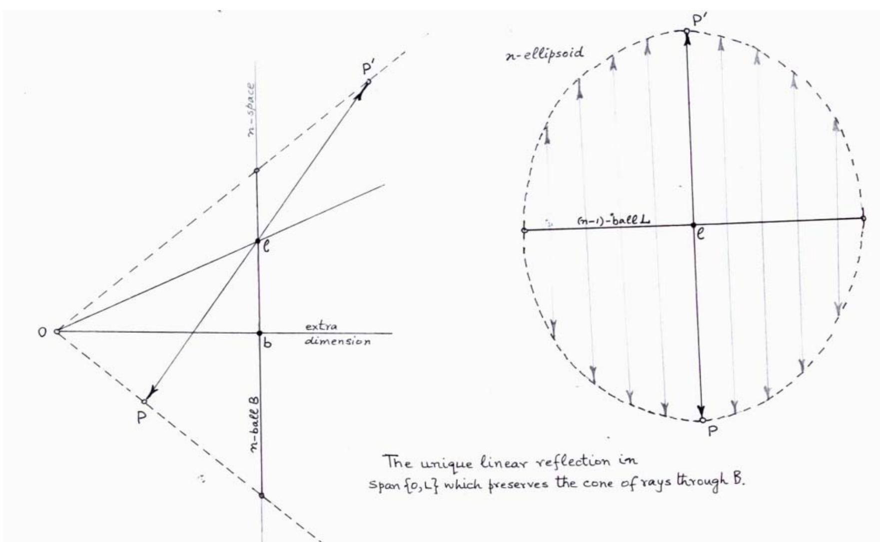

Figure 2

In each flat L ***of the ball*** B ***there is a unique linear reflection which preserves the cone of rays through*** B**.** Using the extra dimension, we have now a one-parameter family of lines orthogonal to L, and passing through its centre ℓ. One of these orthogonal lines has ℓ as the mid-point of its intersection PP' with the cone (note that it is the line such that {0,P,2ℓ, P'} is a parallelogram, which gives an easy construction). The linear map which is the identity on L and which takes P to P' reflects each point of the vector space in span{0,L} in the direction parallel to the segment PP'. Furthermore it restricts, on the flat containing L and PP', to an orthogonal reflection preserving its ellipsoidal intersection—see Figure 2—with the cone of rays through B. So it preserves this cone, and no other linear reflection in L does the same because, the composition of two distinct linear reflections in L restricts to a nonzero translation in any flat parallel to span{0,L}, but our cone does not contain a complete line, q.e.d. So, ***there is a unique linear reflection of the cone which switches the ray through the centre*** b ***with any other ray***, the corresponding flat L of the ball having its centre ℓ suitably between these two rays on the 2-plane determined by them. Also, ***this homogenous but non-euclidean geometry 'explains' euclidean geometry*** : for these reflections approach orthogonal reflections of n-space when c tends to infinity. To wit, the geometry in which subsets of a ball of finite radius are **congruent** iff the sets of rays passing through them are related by a finite sequence of linear reflections preserving the cone.

In physics, ***the rays identify with galilean observers***, to the observer S who considers the ray through the point b of the flat as representing his state of rest, and uses as *time* t *the linear function* which is 1 on this flat, each ray S' represents all particles moving in a fixed direction at the same speed v. Relativity is the dictum that, ***any other observer*** S' ***observes the mirror image of what*** S ***observes***, under the linear reflection switching these rays : *so the time* t' *and the euclidean space* t' = 1 *of* S' ≠ S *must be the transforms of* t *and* t = 1 *under this reflection.* Pragmatic considerations of speed measurement tell us that we should only admit **a ball's worth of observers**, which implies that neither the time nor the space of S' ≠ S are the same as that of S, for example, from Figure 2 it is clear that the mirror image b' of b, which must be in t' = 1, is not in t = 1 : the ***"notions of absolute space and absolute time have no empirical definitions"***, they are only left-overs from the limiting and unrealistic case c = ∞.

The observer S *identifies the points of* span{S,S'} *with their* ***cartesian coordinates*** (t,x) *with respect to* $\overrightarrow{0b}$ *and the unit vector* $\overrightarrow{be}$ *in* t = 1 *towards* S' : so b = (1,0), e = (1,1), ℓ = (1,u) for some u, and S' is the ray through (1,v). The reflection keeps ℓ and all vectors parallel to L fixed. Lemma: *if the sides of a parallelogram have slopes* ±c *then the product of the slopes of its diagonals is* c^2. So PP' has slope c^2/u and b' is on the line through b with this slope such that the mid-point of bb' satisfies x = ut, which gives b' = $((c^2 + u^2)/(c^2 - u^2), 2uc^2/(c^2 - u^2))$. The ray S' passes through b', so v = $2uc^2/c^2+u^2$, and this quadratic in u can be solved, using u < c, to write u in terms of v. Let $\gamma = (c^2 + u^2)/(c^2 - u^2)$, then b' = (γ, γv) and $1/\gamma^2 = 1 - v^2/c^2$. So $\gamma t - (\gamma v/c^2)x = 1$ on ℓ and b', also this linear function is zero on vectors parallel to L, so it is the transform of t : *the observer* S' *has time* $t' = \gamma t - (\gamma v/c^2)x$. His space t' = 1 is the flat spanned by L and b' with the same metric on L, but $\overrightarrow{b'e'}$ is a normal *unit* vector : *to* S', *the "ellipsoidal"* (see Figure 2, but this is a different section of the cone) *image* B' *of* B *is the ball in* t' = 1 *with centre* b' *and radius* c. The observer S' identifies the points of span{S',S} by their coordinates (t',x') with respect to $\overrightarrow{0b'}$ and the unit vector $\overrightarrow{b'e'}$ in t' = 1 towards S. *One has* x' = γvt – γx because $\overrightarrow{be}$ and its image $\overrightarrow{b'e'}$ = u^{-1}(ℓ – b') have (t,x) coordinates (0,1) and $(-\gamma v/c^2, -\gamma)$ respectively. Since the unit vector $\overrightarrow{0b}'$ of S' runs from the t = 0 to the t = γ line, while his unit vector $\overrightarrow{b'e'}$ runs from the x = 0 to the x = – γ line, and γ > 1, S ***deems the clocks of*** S' ***to be slower, and his rulers in the*** x-***direction to be contracted***—the rulers in directions parallel to L are unaffected—***by the factor*** γ, and S' observes the same about the unit vectors $\overrightarrow{0b}$ and $\overrightarrow{be}$ of S in his (t',x') coordinates because, this being a reflection, we also have $t = \gamma t' - (\gamma v/c^2)x'$ and x = γvt' – γx'.

However, the proper time τ ***and space*** τ = 1 ***that we should use are absolute!*** *In analogy with* c = ∞ *this space should consist of all the mirror images* b' *of* b*, with* τ *linear on all rays.* So, ***for*** c < ∞***, proper time is not linear, but now it dictates a distance on the ball which is preserved by all the linear reflections of its cone!*** The (t,x) coordinates of the mirror images b'(v) in any plane through ray S are (γ(v), γ(v)v), but $\gamma(v)^2(1 - v^2/c^2) = 1$, so these points form the hyperbola $t^2 - x^2/c^2 = 1$, t > 0. Therefore, if we decompose each vector parallel to the time t and the euclidean space t = 1 of S, τ^2 *extends to the quadratic form* $(t; \mathbf{x}) \to t^2 - \mathbf{x.x}/c^2$. Since PP' has slope c^2/u, *our linear reflection is orthogonal to its mirror with respect to the associated bilinear form* $(t_1; \mathbf{x_1}) \times (t_2; \mathbf{x_2}) = t_1t_2 - c^{-2}\mathbf{x_1.x_2}$. So, it preserves this form. Also, it replaces t by the time t′ of S′ and t = 1 by the euclidean space t′ = 1 of S′. So *this form is the same as the bilinear form* $(t'_1; \mathbf{x'_1})$ x $(t'_2; \mathbf{x'_2}) = t'_1 t'_2 - c^{-2}\mathbf{x'_1}.\mathbf{x'_2}$ *of* S′. To S′ too, τ = 1 is obtained by revolving $t'^2 - x'^2/c^2 = 1$ around his time axis. On vectors parallel to lines which cut the cone's boundary twice the invariant quadratic form τ^2 is negative. *The positive square root of* $-\,c^2\tau^2(\overrightarrow{b'b''})$ *gives us an invariant distance between points* S′ *and* S″ *of the ball.* The coefficient c^2 ensures that, ***when the radius tends to infinity this non-euclidean distance approaches the euclidean distance*** : to see this it suffices to check that $-\,c^2\tau^2(\overrightarrow{bb'})$ approaches v^2. So this distance gives us the **proper speed** -- i.e. the proper length for the difference $\overrightarrow{b'b''}$ of their 'absolute velocities' $\overrightarrow{0b''}$ and $\overrightarrow{0b'}$ -- separating the observers S″ and S′.

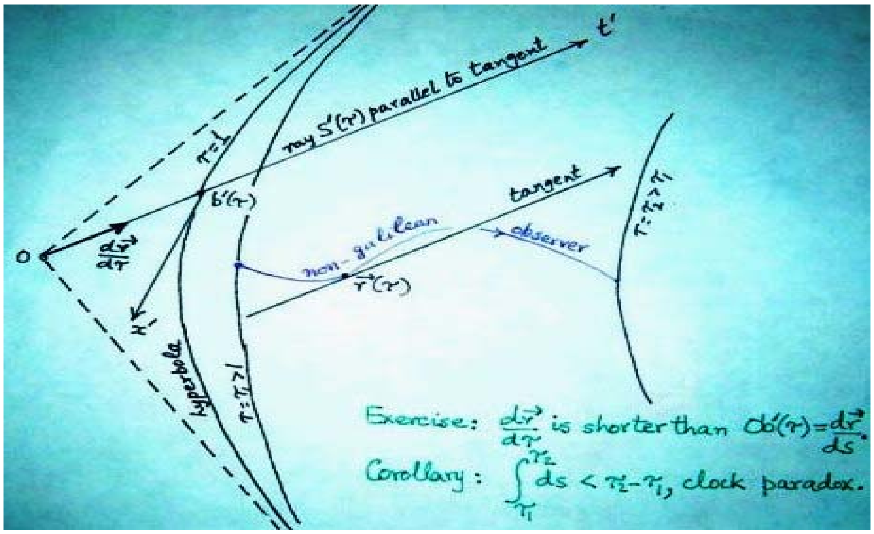

Figure 3

A particle's à priori or ***cartesian absolute motion*** may be any smooth directed arc C with τ strictly increasing, i.e., a smooth vector function $\mathbf{r}(\tau)$, with each $d\mathbf{r}/d\tau$ nonzero along a ray S′(τ). Its **absolute velocity** $d\mathbf{r}/ds$ at proper time τ, i.e., the 'absolute velocity' $\overrightarrow{0b'}$ of S′(τ), is longer – Figure 3 – than $d\mathbf{r}/d\tau$ if 0 is not on tangent, so $ds < d\tau$. Hence the clock paradox: *the **elapsed time*** $\int ds$ *on a clock carried by a non-galilean observer is less than his proper life-time!* Further, the newtonian "the rate of change of momentum is equal to the force" suggests $d/ds(m\, d\mathbf{r}/ds) = \mathbf{F}(s)$, where the constant m is called the **proper mass** of the particle, and the right hand side, the **absolute force** on the particle.

An observer S will write $\mathbf{r} = (t; \mathbf{x})$ to split the last equation into temporal and spatial components : $d/dt(m\, dt/ds) = T\, ds/dt$ and $d/dt(m\, dt/ds\, \mathbf{v}(t)) = ds/dt\, \mathbf{X}$, using $d\mathbf{x}/dt = \mathbf{v}(t)$ and $\mathbf{F} = (T; \mathbf{X})$. Taking the spatial equation as Newton's law, ***the observer*** S ***deems*** $ds/dt\, \mathbf{X}$ ***as the force on the particle, and*** $m\, dt/ds$ ***as its varying mass*** m(t)! Since $d\mathbf{r}/dt = (1; \mathbf{v}(t))$ is the vector from 0 to (1, v(t)) in the plane of S and S′(τ), and γ(v(t)) times this vector is $\overrightarrow{0b'} = d\mathbf{r}/ds$, ***we have*** $dt/ds = \gamma(v(t))$ ***on the arc*** C. Also differentiating $d\mathbf{r}/ds$ x $d\mathbf{r}/ds \equiv 1$ we see that, ***the absolute acceleration is always orthogonal to the absolute velocity***, i.e., $d^2\mathbf{r}/ds^2$ x $d\mathbf{r}/ds = 0$, i.e., $d^2t/ds^2\, dt/ds - (d^2\mathbf{x}/ds^2.d\mathbf{x}/ds)/c^2 = 0$ in the components of S. Multiplying by mc^2 we obtain $c^2T = \mathbf{X}.\mathbf{v}$, therefore $T\, ds/dt\, c^2 = ds/dt\, \mathbf{X}.\mathbf{v}$ = the rate of working of the force on the particle = the rate of change of its energy. So the temporal equation of motion shows S that, ***the energy of the particle is*** $E = m\, dt/ds\, c^2 = m(t)c^2 = m(1 - v(t)^2/c^2)^{-1/2}c^2 = mc^2 + ½\, mv(t)^2 + \ldots$!

But, $m(t)\mathbf{v}(t)$ plus a nonzero vector **a** also satisfies the spatial equation, why not this rest-momentum? Because it violates mirror-relativity, the opposite direction is just as good. And, $m(t)c^2$ plus a nonzero scalar k satisfies the temporal equation, but energy is additive in mass, this would add k/c^2 to m(t), which we have accepted. Excepting some other similar points, that one can debate endlessly, we have now covered all the basics, and our treatment was *plain* indeed : all our key arguments used only *plane* geometry, though the results are quite general!

Notes

1. The non-euclidean geometry of a polytopal and possibly non-convex region Ω of euclidean n-space is also worth pursuing, especially because of its uncanny ***galois symmetries***, cf. Sullivan, M.I.T. Notes (1970).

2. The word group makes us forget that the basic symmetry is that of Euclid's *pons asinorum*, viz., a reflection, for example for Galois it was a transposition of roots, and we know now that practically all non-abelian **finite simple groups** can be realized by using the linear reflections of the cone of a 2-ball, cf. *How I learnt some well-known folklore* (2010). Besides, though a reflection takes us abruptly into the oppositely oriented mirror world, it is ***magic*** that an even number of such hops will do the same job as any continuous motion which preserves our geometry!

3. Linearization by adding extra dimension(s) is like adding 0 to the positive numbers ... we all use it ... I used it (1992, 1997, 2000) to look at some problems of convex geometry with only partial success ... maybe because I'd turned away from number fields which had given me the initial idea? This thought comes because number fields give many examples of **discrete subgroups Γ of linear transformations of the cone** with B^n/Γ a closed n-manifold, and the much finer use—which I've still not quite understood—of galois symmetry in Deligne and Sullivan, *Fibrés vectoriel complexes à groupes structurel discret,* C. R. Acad. Sci. Paris 281 (1975) 1081-1083, shows further that there are hordes of these groups for which this closed manifold is almost parallelizable.

4. Appollonius knew from the sun-dial in his garden in Perga (Turkey) that ***the shadow of a sphere is an ellipse***, the point F common to the sphere and the table being a focus; the other focus F' is the point of PP' – see Figure 2 – on the *other* circle in the cone which touches the sides of 0PP'; but in his treatise he used a 2-dimensional but less natural definition of an ellipse which unfortunately is the only one that is taught in schools today.

5. My scientific journey started with Henry Thomas and Dana Lee Thomas, *Living biographies of great scientists* (1959), my First Prize in Aggregate during my first year, 1960-61, in Government College, Chandigarh. Its last biography led me to mail order a popular text-book on relativity from Bombay, from which I was sharp enough to deduce that the prize-winning "mathematics" I knew was not mathematics at all, which led me in turn to mail order, from Varanasi this time, that classic of Goursat's which I used again in *Straight to Mecca*. I should mention also that my mirror-formulation of relativity is ***slightly stronger***, there is no preferred orientation on spacetime.

6. Again, my use of the phrase proper time is not standard, often it means elapsed time along C. That τ defines a proper speed between galilean observers is nice, but ***this distance only satisfies the triangle inequality infinitesimally*:** at each point b' of the smooth submanifold $\tau = 1$ it gives us the euclidean metric of its tangent hyperplane $t' = 1$. So by integrating this infinitesimal we can assign a length to any smooth curve of this ***riemannian manifold*** between two given points, and the infimum of these lengths gives us another invariant distance satisfying this inequality.

7. The haloed, but nevertheless arbitrary, newtonian tradition of considering only motions given by second order differential equations seems even more arbitrary now : ***for*** $c < \infty$ ***we have a nice and complete classification of generic cartesian absolute motions parametrized by elapsed time!*** This is suggested by the final paragraphs, where we stuck to this tradition, but what gave $E = mc^2$ was the cartesian orthogonality $d^2\mathbf{r}/ds^2 \times d\mathbf{r}/ds = 0$ with respect to our non-degenerate bilinear form. There seems no reason to stop at two, we can keep on normalizing and differentiating à la Frenet and Serret till we have a full complement of n+1 orthonormal vectors all along our generic C, and then using this frame define the n curvatures which will characterize C up to a composition of orthogonal reflections in flats of spacetime. A quick recap of the classical classification of space curves is in my old class notes on *Differential geometry* (1982?), and in the book by Klingenberg, *A course in differential geometry* (1978), it is shown that everything works just as well for any positive definite quadratic form over the reals.

(*contd.*)

8. *The three things that we left to the reader in the text are also easy to check.* The Lemma holds because the product of the slopes of the diagonals of the parallelogram $\{(0,0),(t,-ct),(t',ct'),(t+t',-ct+ct')\}$ is $\frac{ct'+ct}{t'-t} \times \frac{-ct+ct'}{t+t'} = c^2$. Again, $-c^2\tau^2(\overrightarrow{bb'}) = -c^2\tau^2(\gamma-1,\gamma v) = -c^2((\gamma-1)^2-\gamma^2v^2/c^2) = -c^2((\gamma-1)^2+\gamma^2(1/\gamma^2-1)) = -c^2(2-2\gamma) = -2c^2(1-(1-v^2/c^2)^{-1/2}) = -2c^2(-v^2/2c^2+\cdots)$ approaches v^2 as $c \to \infty$. And, for the Exercise in Figure 3 note that $d\overrightarrow{r}/d\tau$ along C at the point of tangency $(t',a), a \neq 0$ – using coordinates (t',x') in the plane containing the tangent line and the parallel S' – equals the same quantity along the tangent line at $\overrightarrow{r} = (t',a)$. So it is equal to the vector $(1,0) = \overrightarrow{0b'}$ times $dt'/d\tau$ along the tangent line at this point, and from $\tau^2 = t'^2 - a^2/c^2$ we see that this derivative is equal to $\tau/t' = (1-a^2/(c^2t'))^{1/2} < 1$. Also, *our clock paradox implies the one usually stated,* because, if the cartesian motion C begins and ends on any ray S, then $\tau_2 - \tau_1 = t_2 - t_1$.

9. *Velocity addition formula.* Given a cartesian motion C, the time t of any observer S increases strictly on it, so it has equation $\mathbf{r}(t) = (t;\mathbf{x}(t))$ and $d\mathbf{x}/dt$ is its varying velocity as observed by S. For example S can use an orthogonal basis of his euclidean space $t = 1$ with respect to which $\mathbf{x}(t)$ has cartesian coordinates $(x_1(t), x_2(t), \ldots, x_n(t))$ and measure the n components $(dx_1/dt, \ldots dx_n/dt)$ of $d\mathbf{x}/dt$. Likewise, the observed velocity $d\mathbf{x}'/dt'$ of C as measured by another *framed observer* S', that is one equipped, besides his unit time vector $\overrightarrow{0b'}$, with an orthogonal basis for his euclidean space $t' = 1$, gives us another n-tuple $(dx'_1/dt', \ldots dx'_n/dt')$. The two bases of the $(n+1)$-dimensional vector space used by S and S' are related by a *matrix* $A = [a_{ij}], 0 \leq i,j, \leq n$, i.e.,

$$\begin{gathered} t = a_{00}t' + a_{01}x'_1 + \cdots + a_{0n}x'_n, \\ x_1 = a_{10}t' + a_{11}x'_1 + \cdots + a_{1n}x'_n, \\ \cdots\cdots\cdots \\ x_n = a_{n0}t' + a_{n1}x'_1 + \cdots + a_{nn}x'_n; \end{gathered}$$

from which we get, for all $1 \leq i \leq n$,

$$\frac{dx_i}{dt} = \frac{a_{i0} + a_{i1}dx'_1/dt' + \cdots + a_{in}dx'_n/dt'}{a_{00} + a_{01}dx'_1/dt' + \cdots + a_{0n}dx'_n/dt'},$$

a formula relating the velocity components of the same motion C as measured by two framed observers. *Often framed observers are called observers, but for us an observer*—that is a ray per the fourth paragraph of the text—*has an* $O(n)$ *worth of frames.* Components of an observed velocity depend on the frame, for example, if we reverse a basis vector, that component changes its sign. Given any other observer S', the central observer S has an $O(n-1)$ worth of frames in which the observed velocity of S' is $(v,0,\ldots,0)$ with v positive. If S uses one of these, and S' the reflected frame as in the fifth paragraph of the text, then $t = \gamma(v)(t' - v/c^2x'_1), x_1 = \gamma(v)(vt' - x'_1), x_2 = x'_2, \ldots, x_n = x'_n$. But S' can, if he wants, 'correct' his reversed orientation by reversing his x'_1-axis, when A is given instead by $t = \gamma(v)(t' + v/c^2 x'_1), x_1 = \gamma(v)(vt' + x'_1), x_2 = x'_2, \ldots, x_n = x'_n$ (and it is usually this case only of the above formula, with C galilean, which is called the velocity addition formula), and more generally, S' can transform the reflected frame by any orthogonal transformation he likes of $t' = 1$.

10. Though in the proof in the third paragraph of the text we temporarily assumed the central ray orthogonal to the euclidean n-flat, we emphasize that the $(n+1)$-dimensional vector space is itself not euclidean. This proof gave us all the linear reflections preserving the cone, there is one and only one in each flat of B^n. Therefore, *the linear reflections preserving the cone form a smooth manifold diffeomorphic to the canonical line bundle of* $\mathbb{R}P^{n-1}$: the flats through b constitute the $\mathbb{R}P^{n-1}$; and for the flats L constituting any small open subset U of this manifold we can choose a continuous normal direction; so, identifying the other flats of the ball parallel to these L's with their centres ℓ on these directed diameters, we obtain local trivializations $U \times (-c, +c)$. We note also that this diffeomorphism type stays put even for $c = \infty$, i.e., when we are talking of the space of all orthogonal reflections of the euclidean n-flat. The next result of this paragraph can also be sharpened: *there is a unique linear reflection of the cone which switches any given pair of distinct rays S' and S''.* For, if neither ray is the central ray S, conjugation with g, the reflection of the cone switching S' and S, gives us a bijective correspondence between reflections switching S' and S'' and those switching S and gS'', but the latter set is a singleton. And, regarding our definition of congruence for n-ball geometry which concluded this paragraph, we note that, *up to a homothety, any linear isomorphism of the cone is a composition of at most $n+1$ linear reflections of the cone*. For, if the isomorphism maps the central ray S to S', then by composing it, if need be, with the reflection switching $S \neq S'$ and a homothety we obtain a linear isomorphism of the cone which is the identity on S. It maps any diameter PQ of the n-ball to a line segment $P'Q'$ having b as its mid-point and with P' and Q' on the boundary of the cone, which is possible only if $P'Q'$ is also a diameter of the n-ball. So our map restricts to an isometry of the euclidean n-flat mapping b to itself, hence it is a composition of at most n orthogonal reflections. Also, *the linear reflections of the cone are conjugate to each other in the group $G(n)$ of all their compositions*: for, if $\ell \neq b$, then conjugation with the linear reflection of the cone which switches the rays through these two points gives us a reflection whose flat passes through b, etc.

11. Using notes 9 and 10, $G(n)$ *is isomorphic to the group of matrices A relating ordered pairs of framed observers.* The 'dictum' of the fourth paragraph says that if observers $S' \neq S$ are equipped with mirror image frames under the linear reflection of the cone switching them, then their measurements must be related by the corresponding matrix A. Since this is obviously true also if the same observer replaces the frame he is using by any orthogonal reflection, *the measurements made by any ordered pair of framed observers are related by the corresponding matrix A.* Usually the orientations of the framed observers are compatible with each other, so only those matrices A come into play whose determinants are positive, equivalently, only the subgroup $G_+(n)$ of all compositions of any *even number* of reflections is admitted. Therefore, as we mentioned before in note 5, mirror relativity is slightly stronger. Notably, *an invariant vector* $\mathbf{a} = \mathbf{0}$ *even for* $n = 1$, which is false if an orientation is preferred: the concluding step in the argument that we used to obtain the mass-energy formula is then valid only for $n \geq 2$. The subgroup $G_+(1)$ of translations is abelian, but the group $G(1)$ of all motions of a 1-ball (the real line has isomorphic groups) is not commutative. This homogenous geometry of a 1-ball, i.e., a bounded open interval, is however not non-euclidean per our usage of this adjective, because the parallel postulate is trivially true. Sometimes this adjective is used *only* for homogenous geometries, then of course it is not at all true that non-euclidean geometries are 'dime-a-dozen' for $n \geq 2$. As for the 'pragmatic considerations' of the fourth paragraph, *these objections were* (imho) *raised to his geometry by practical fellow Egyptians even in Euclid's lifetime!* A down-to-earth person is none too impressed by lines that don't end, or a parallelism of line segments that is not experimentally decidable: a bounded subset of Euclid's plane, above all a disk of a possibly large but finite radius around him, is eminently more reasonable to him. The quotation in this paragraph is from a paper by Arnol'd which is available on my website. It alerts us that, *it is the individual times* t' *and the euclidean spaces* $t' = 1$ *of our 'ball's worth' of observers* S' *that are basic,* what frame an experimenter uses to make his measurements, or a theoretician to do his calculations, is only of secondary importance.

12. A modicum of calculations, in the remainder of our 3-page essay, then gave us time dilation and length contraction by $\gamma(v) = (1-v^2/c^2)^{-1/2}$, the clock paradox, and the mass-energy formula $E = mc^2$. Also we saw that, *relativity is a hidden variables theory* : all the mirror images of any point of the cone form *an absolute but curved space,* and the homogenous function which is one on it *an absolute nonlinear time* τ; but to each observer S' this curved hidden space appears flat, a ball B' of radius c around him, on which his linear time $t' = 1$. The cone is the same for all the observers, but only its hidden foliation $\tau =$ constant is preserved by the full group $G(n)$ generated by all the linear reflections of the cone, its observed foliation into parallel balls $t' =$ constant is preserved only by the subgroup $O(n)$ generated by the reflections preserving the observer S'. That a space consists of all the mirror images of any of its points is nice, but, *infinite divisibilty is not pragmatic* : one may object to Euclid's plane also on the grounds that it can be tiled by an *arbitrarily small* square! Magically, this new objection is also taken care of if we confine ourselves to an n-ball, $n \geq 2$, of radius $c < \infty$: *if a polytope of rays tiles absolute space, its riemannian volume is bigger than a positive constant depending inversely on* c. So the volumes of the closed manifolds B^n/Γ of note 3 are all more than this constant. An observer hears the hidden shapes of these 'particles' as proper values of Γ-periodic 'waves' on the covering space $\tau = 1$ whose differential equation can be written in his coordinates. Historically, *the theories of the subatomic world also arose from* $c < \infty$, but no one seems to understand this other side of the relativistic coin really well. Anyway, some of what little I myself have been able to understand about these quotients is in *Hyperbolic Manifolds* (2012), which will be available from my website as soon as I can write a prefatory note explaining what I was up to in this unfinished paper.

13. The cone is all the spacetime one really needs. Like positive numbers on the real line (cf. note 3) it is closed under $P+Q$, but all its differences $P-Q$ form the full $(n+1)$-dimensional vector space. The *partial order* defined by, $P > Q$ iff $P - Q$ is in cone, is quite basic : *cartesian absolute motions are precisely all the directed and strictly increasing smooth arcs in the cone.* For $P > Q$ implies $\tau(P) > \tau(Q)$ – the converse is not true for $c < \infty$ – and that extra condition '$d\mathbf{r}/d\tau$ nonzero along a ray $S'(\tau)$' on the smooth arc is equivalent to saying that if P comes after Q, then QP is parallel to a ray of the cone, i.e., $P > Q$. Also, *it is true that* $P > Q \iff \tau(P+R) > \tau(Q+R)\ \forall\ R$, but in our set-up parallel motions are deemed to be the same, therefore, *if we admit only the irreversibility of time, then these are all the possible smooth motions.*

14. Considering what all had gone into that definition – see note 6 – of the riemannian metric on $\tau = 1$, it is a miracle that *the associated pseudometric on the cone makes sense for any bounded open convex subset* Ω^n *of affine n-space!* The distance $\widehat{AB}$ between the rays through A and B is equal to $\frac{c}{2}\log(\frac{XB}{XA}\frac{YA}{YB})$ if AB extended meets the boundary in X and Y. In this *two-line Ph. D. thesis*—as Littlewood dubbed this discovery—*of Cayley's,* $c > 0$ is arbitrary, but if Ω^n is an open ball of a norm $\|\cdot\|$ on n-space, e.g. $\Omega^n = B^n$, the best choice is its radius. For then this definition also gives us a distance between rays through B^n which is preserved by all the reflections of the cone, *and* which approaches the euclidean distance for $c \to \infty$, so it coincides with the riemannian metric. To check this we'll again, as in Figure 2 and the subsequent paragraph, temporarily think of the $(n+1)$-dimensional vector space as euclidean.

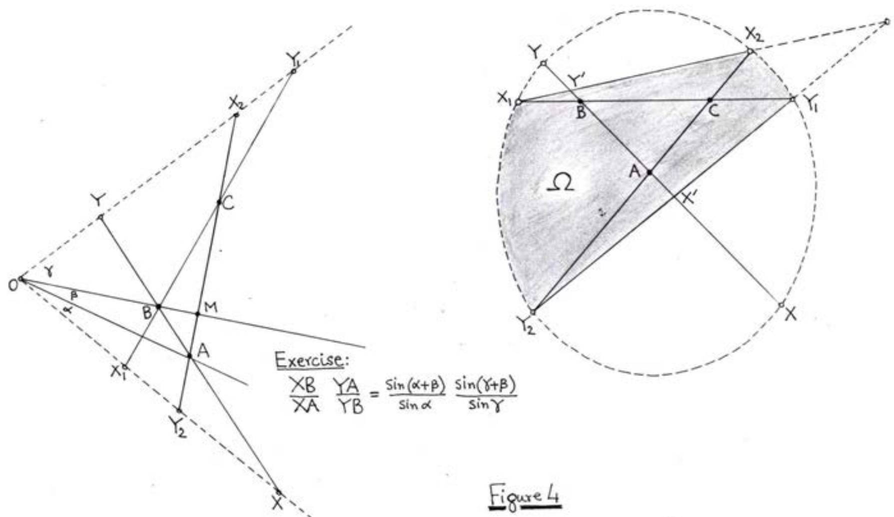

Figure 4

Though the ratios $\frac{XB}{XA}$ and $\frac{YA}{YB}$ depend on the line cutting four given coplanar and coincident lines in X, A, B and Y (unless they are parallel, i.e., coincide at infinity) their product $\frac{XB}{XA}\frac{YA}{YB}$ *is an invariant*. Indeed, using the sine law for triangles one can check – Figure 4, Exercise – that in this *biratio* one can replace each length by the sine of the subtended angle. Using this invariance we'll now prove the *triangle inequality* $\widehat{AC} + \widehat{CB} \geq \widehat{AB}$ whenever extending each side gives us two points on the boundary. When ABC is in a plane through the origin 0, one side is in fact *equal* to the sum of the other two, for example, for the triangle drawn in Figure 4, $\widehat{AB} + \widehat{BC} = \widehat{AM} + \widehat{MC} = \widehat{AC}$. If 0 is not in the plane of ABC, a similar argument gives us $\widetilde{AC} + \widetilde{CB} = \widetilde{AB}$ *for the convex open planar subset* Ω, shown shaded in Figure 4, of the cone between the (possibly parallel) lines X_1X_2 and Y_2Y_1. Which implies the desired inequality, for the left side is the same as $\widehat{AC} + \widehat{CB}$, and we have $\widetilde{AB} \geq \widehat{AB}$ because $\frac{X'A}{X'B} \geq \frac{XA}{XB}$ and $\frac{Y'A}{Y'B} \geq \frac{YA}{YB}$. Any linear reflection preserving the cone preserves its Cayley distance because it does so on the ellipsoidal section – see Figure 2 – on which it coincides with an orthogonal reflection. Finally, we note that a point of the n-ball at euclidean distance r from its centre is at Cayley distance $\frac{c}{2}\log(\frac{c+r}{c-r})$, and this quantity approaches r as $c \to \infty$.

15. By a *piecewise linear absolute motion* $P_0P_1 \ldots P_k$ we mean a directed and strictly increasing—in the sense of note 13—broken line in the cone. *The elapsed time for this motion is* $\tau(\overrightarrow{P_0P_1}) + \tau(\overrightarrow{P_1P_2}) + \cdots + \tau(\overrightarrow{P_{k-1}P_k})$. This because, each $\overrightarrow{P_iP_{i+1}}$ is parallel to some ray S', on which ray τ coincides with the linear time t' of this galilean observer, so $\tau(\overrightarrow{P_iP_{i+1}}) := \tau(P_{i+1} - P_i) = t'(P_{i+1} - P_i) = t'(P_{i+1}) - t'(P_i)$ is the time recorded by the moving clock over this segment. However, *since the absolute time τ is non-linear for $c < \infty$,* we can't write $\tau(P_{i+1} - P_i) = \tau(P_{i+1}) - \tau(P_i)$, and then cancel etc., to get $\tau(P_k) - \tau(P_0)$. Instead, *we have the startling clock paradox :* $\tau(\overrightarrow{P_0P_1}) + \tau(\overrightarrow{P_1P_2}) + \cdots + \tau(\overrightarrow{P_{k-1}P_k}) \leq \tau(P_k) - \tau(P_0)$, with equality iff the P_i's are all on the same ray. Equivalently, *the reversed triangle inequality* $\tau(\overrightarrow{AC}) + \tau(\overrightarrow{CB}) \leq \tau(\overrightarrow{AB})$ *holds,* for any three points $0 \leq A < C < B$, with equality iff they are collinear. To see this recall that in the sixth paragraph of the text we showed that $\tau^2(t';\mathbf{x}') = t'.t' - \frac{1}{c^2}\mathbf{x}'.\mathbf{x}'$ if one uses components parallel to the time and the euclidean space of any galilean observer S'. If we take S' parallel to $\overrightarrow{AB}$, then $\mathbf{x}'(A) = \mathbf{x}'(B)$, so the right side is $t'(B) - t'(A)$. Let M be the point on $\overrightarrow{AB}$ such that $t'(M) = t'(C)$. Then $\tau^2(\overrightarrow{AC}) = (t'(M) - t'(A))^2 - \frac{1}{c^2}\overrightarrow{MC}.\overrightarrow{MC} \leq (t'(M) - t'(A))^2$, with equality iff $M = C$. Likewise $\tau(\overrightarrow{CB}) \leq t'(B) - t'(M)$, which completes the proof. We note that, if the above absolute motion is given by the vector function $\mathbf{r}(s)$ of elapsed time, then on the interior of each $\overrightarrow{P_iP_{i+1}}$ we have $ds = dt'$ and $\frac{d\mathbf{r}}{ds} = \overrightarrow{0b'}$, the 'absolute velocity' of the parallel S'. So the above definition of elapsed time is the same as in the seventh paragraph of the text, only then we looked at directed and strictly increasing arcs that are *smooth,* i.e., which are, so to speak, broken lines with infinitely many infinitesimally small links $d\mathbf{r}$. Instead of the above finite sum, it is the analogous riemann integral $\int_{P_0}^{P} \tau(d\mathbf{r})$ taken along the motion that gives us the elapsed time $s(P)$ till any point, so $\tau(d\mathbf{r}) = ds$, i.e., $\tau(\frac{d\mathbf{r}}{ds}) = 1$. That is, the length of $\frac{d\mathbf{r}}{ds}$ is identically 1 with respect to the quadratic form τ^2; so, as in euclidean differential geometry, we'll also call this derivative the *unit tangent vector* $\mathbf{u}(s)$ at any point of the smooth motion.

16. *For $c < \infty$ extra hypotheses like smooth or p.l. are not needed!* A strictly increasing function from an interval into the cone is trapped near each point in the parallel cone, so it is continuous and, *as seen by any observer S in his euclidean space this à priori motion is lipschitz in his time t with constant c,* i.e. $\|\mathbf{x}(t_1) - \mathbf{x}(t_0)\| < c|t_1 - t_0|$, so it is differentiable almost everywhere. The same integral gives the elapsed time $s(P)$, and this motion has a unit tangent vector $\frac{d\mathbf{r}}{ds}$ a.e., but $\frac{d^2\mathbf{r}}{ds^2}$ is only a generalized function or distribution. For example, for a p.l. motion it is supported on the finitely many bends. Nevertheless, *the equations around the mass-energy formula in the eighth paragraph of the text are still valid weakly.* Likewise, the yang-mills formulary, which depends on the special feature of 4-dimensional space that $SO(4)$ is not simple, is valid weakly if we allow all these motions, which suffices to deduce that, *there exist topological 4-manifolds which do not admit any lipschitz structure!* On the other hand in Sullivan, *Hyperbolic geometry and homeomorphisms* (1979), it was shown that, *any topological n-manifold, $n \neq 4$, has a unique lipschitz structure!* At one point in this almost surreal paper – it is available on my website – Sullivan invokes his paper with Deligne that was cited in note 3.

17. *Arbitrarily close to any à priori motion is a piecewise linear absolute motion with the same end points and with elapsed time arbitrarily small!* A riemann sum involves an approximating broken line with almost the same elapsed time; to make this time arbitrarily small use the fact that, *any two points on a ray can be joined by a planar zig-zag of a small amplitude whose links are alternately almost parallel to the two boundary rays.* Smooth absolute motions, even those with a small elapsed time, are likewise dense in à priori motions. The clock paradox is less startling when stated thus: *a journey takes the maximum elapsed time if no force is expended.* The time-stopping oscillations above have impractically big accelerations, perhaps these should be banned too by *a new decree?* We showed in the eighth paragraph of the text $\frac{d^2\mathbf{r}}{ds^2} \times \frac{d\mathbf{r}}{ds} = 0$, i.e., the rate of change of the absolute velocity $\frac{d\mathbf{r}}{ds} = \overrightarrow{0b'}$ is constantly orthogonal to it with respect to the quadratic form τ^2. That is, if we draw an arrow parallel and equal in length to $\frac{d^2\mathbf{r}}{ds^2}$ from b', then it is contained in the euclidean space $t' = 1$. It seems reasonable to us that this arrow should be confined to a ball of a prescribed radius around b'. Which radius, by changing units, we can take once again to be c itself. So, *we can decree that the absolute acceleration $\frac{d^2\mathbf{r}}{ds^2}$ should always remain in the balls B'.* Under this decree, there is a positive lower limit on the elapsed times of journeys between two events.

18. In the sixth paragraph of the text we measured vectors parallel to lines cutting the boundary twice by applying $\tau^* := \sqrt{-c^2\tau^2}$: *it too does not obey the triangle inequality.* If P and Q are two points on any such line, and we draw through them, in the plane containing 0, lines parallel to the two boundary rays, then a path PRQ in this parallelogram and close to its boundary has in fact an arbitrarily small $\tau^*(PR) + \tau^*(RQ) < \tau^*(PQ)$. However, τ^* *on vectors lying in any ball B' gives their euclidean length,* for $-c^2\tau^2(t'; \mathbf{x}') = -c^2 t'.t' + \mathbf{x}'.\mathbf{x}'$ if one uses components parallel to the time and the euclidean space of S'. This is the relative speed—always less than $2c$—between observers as observed by S', and this distance between rays is invariant under reflections of the cone preserving

S'. It is τ^* on point-pairs of *that hidden space* $\tau = 1$ *enveloped by all these balls,* that is $\tau^*(b'b'')$, that gave us an observer-independent and fully invariant proper speed between observers; and $\inf \int_{b'}^{b''} \tau^*(d\mathbf{r})$ over all curves on $\tau = 1$ from b' to b'' gives – note 6 – an invariant distance between rays obeying the triangle inequality. In note 14 we showed that this must be Cayley's distance: *if* S'' *has speed* v *as observed by* S', *then* $\inf \int_{b'}^{b''} \tau^*(d\mathbf{r}) = \frac{c}{2}\log(\frac{c+v}{c-v})$, *with* inf *attained on and only on the curve from* b' *to* b'' *on* $\tau = 1$ *and* $\operatorname{span}\{S', S''\}$. To double-check this let $\frac{c}{2}\log(\frac{c+v}{c-v}) = c\theta$, then $v = c\tanh\theta$, but $\operatorname{span}\{S', S''\} \cap \{\tau = 1\}$ has in the coordinates (t', x') of S' the cartesian equation $-c^2t'^2 + x'^2 = -c^2$ or parametric equations $t' = \cosh\theta, x' = c\sinh\theta$, so over this curve $\int_{b'}^{b''} \tau^*(d\mathbf{r}) = \int_0^\theta \sqrt{-c^2(dt')^2 + (dx')^2} = c\theta$. Also, *the retraction* $P = (t'; \mathbf{x}') \mapsto (t', \|\mathbf{x}'\|) = \overline{P}$ *of the vector space on the half-plane* $x' \geq 0$ *of* $\operatorname{span}\{S', S''\}$ preserves $\tau = 1$ and, for any point-pair on it, since $\|\mathbf{x}'(PQ)\| \geq |\|\mathbf{x}'(P) - \|\mathbf{x}'(Q)\||$, we have $\tau^*(PQ) \geq \tau^*(\overline{PQ})$, with equality iff $\mathbf{x}'(P)$ or $\mathbf{x}'(Q)$ is a non-negative multiple of the other. Hence $\int_{b'}^{b''} \tau^*(d\mathbf{r}) \geq \int_{b'}^{b''} \tau^*(d\overline{\mathbf{r}})$ for any $\mathbf{r}(u)$ on $\tau = 1$ from b' to b'' and its retraction $\overline{\mathbf{r}}(u)$, with equality iff $\mathbf{r}(u) = \overline{\mathbf{r}}(u) \forall u$. *q.e.d.* Here of course, following Riemann, $\int_{b'}^{b''} \tau^*(d\mathbf{r}) := \lim[\tau^*(P_0P_1) + \tau^*(P_1P_2) + \cdots + \tau^*(P_{k-1}P_k)]$, as one takes more and more closely spaced points $b' = P_0, P_1, \ldots, P_k = b''$ in order on $\mathbf{r}(u)$. For the minimizing curve $\mathbf{r}(\theta)$, which is on the plane through 0, $\tau^*(P_iP_{i+1}) = c\sqrt{2\cosh(\theta_{i+1} - \theta_i) - 2} > c(\theta_{i+1} - \theta_i)$, so now its riemann sums are bigger than the integral, but steadily decrease to it under refinement.

19. To elaborate on note 7 we'll switch to $(t_1; \mathbf{x_1}) \star (t_1; \mathbf{x_2}) = -c^2t_1t_2 + \mathbf{x_1}.\mathbf{x_2}$, the bilinear form associated to $-c^2\tau^2$. So, if $\mathbf{r}(s)$ is any smooth à priori motion parametrized by elapsed time and $\mathbf{u}(s) = \frac{d\mathbf{r}}{ds}$ is its unit tangent vector field – note 15 – then $\mathbf{u}(s) \star \mathbf{u}(s) = -c^2$. The $\star$-orthogonal complement of $\mathbf{u}(s)$ is the euclidean space of the parallel galilean observer and on it our bilinear form coincides with its dot product. So one has moving *frames* $\{\mathbf{u}(s); \mathbf{e_1}(s), \ldots, \mathbf{e_n}(s)\}$ of smooth vector fields along the motion such that $\mathbf{e_i}(s) \star \mathbf{e_j}(s) = \delta_{ij}$ and $\mathbf{u}(s) \star \mathbf{e_i}(s) = 0$. An à priori motion is a parallel pencil of directed and strictly increasing arcs – note 13 – in the cone. By perturbing such an arc, in the interior of the smooth manifold $a \leq \tau \leq b$ on whose boundary its end points lie, we can replace it by another which is lipschitz close to it, and which is not only smooth but also *generic,* i.e., its first $n+1$ derivatives are always linearly independent. A smooth generic motions has a *frenet frame* : each unit vector $\mathbf{e_i}(s), 1 \leq i \leq n$, is obtained by multiplying the component of $\frac{d^{i+1}\mathbf{r}}{ds^{i+1}}$ orthogonal to $\operatorname{span}\{\mathbf{u}(s), \mathbf{e_1}(s), \ldots, \mathbf{e_{i-1}}(s)\}$ with the reciprocal of its nonzero length $\kappa_i(s)$. Since adding a vector does not change derivatives these *curvatures* $\kappa_i(s)$ are well-defined, a parallel arc is also generic with the same elapsed times and curvatures at its corresponding points. Moreover, *two smooth generic motions are related by a finite sequence of linear reflections of the cone if and only if they have the same curvature functions* $\kappa_i(s), 1 \leq i \leq n$. Any linear transformation L of the cone preserves τ and $\star$ and maps a smooth generic motion onto another whose derivatives are the images under L of its derivatives; this shows 'only if'. For 'if' represent the two motions by arcs whose initial points are on rays parallel to their initial unit tangent vectors, and then slide one of these arcs along this ray so that τ has the same value on both initial points. Now apply the linear reflection of the cone interchanging these two rays to one of the arcs to get two arcs with the same initial point and the same $\mathbf{u}(0)$ along the ray through this point. Finally, by applying to one of the arcs the orthogonal transformation of this observer which throws the initial unit vectors $\mathbf{e_i}(0)$ of this arc onto those of the other we are reduced to showing that, *there exists a unique smooth generic arc parametrized by elapsed time with a given initial frenet frame* $\{\mathbf{u}(0); \mathbf{e_1}(0), \ldots, \mathbf{e_n}(0)\}$ *and given curvature functions* $\kappa_i(s)$. For this we'll need the *frenet equations* : $\frac{d\mathbf{u}}{ds} = k_1(s)\mathbf{e_1}(s)$, $\frac{d\mathbf{e_i}}{ds} = k_{i+1}(s)\mathbf{e_{i+1}}(s) - k_i(s)\mathbf{e_{i-1}}(s)$ for $1 \leq i < n$ and $\frac{d\mathbf{e_n}}{ds} = -k_n(s)\mathbf{e_{n-1}}(s)$, where $k_1(s) = \kappa_1(s)$, $\mathbf{e_0}(s) = \mathbf{u}(s)$ and $k_{i+1}(s) = \frac{\kappa_{i+1}(s)}{\kappa_i(s)}$. The first equation is 'what gave $E = mc^2$', viz., $\frac{d^2\mathbf{r}}{ds^2} \star \frac{d\mathbf{r}}{ds} = 0$, i.e., derivative of $\mathbf{u}(s) \star \mathbf{u}(s)$ is zero, i.e., $\mathbf{u}(s) \star \mathbf{u}(s)$ is constant. Likewise $\mathbf{e_i}(s) = \frac{1}{\kappa_i(s)} \frac{d^{i+1}\mathbf{r}}{ds^{i+1}}$ + lower order terms, has derivative $\frac{1}{\kappa_i(s)} \frac{d^{i+2}\mathbf{r}}{ds^{i+2}}$ + lower order terms, which for $1 \leq i < n$ is equal to $\frac{\kappa_{i+1}(s)}{\kappa_i(s)}\mathbf{e_{i+1}}(s)$ + a linear combination of $\mathbf{e_j}(s)$ *with* $j \leq i$ *only,* while for $i = n$ even the leading term is missing. Now use $\frac{d\mathbf{e_i}}{ds} \star \mathbf{u}(s) + \frac{d\mathbf{u}}{ds} \star \mathbf{e_i}(s) = 0$ and $\frac{d\mathbf{e_i}}{ds} \star \mathbf{e_j}(s) + \frac{d\mathbf{e_j}}{ds} \star \mathbf{e_i}(s) = 0$ – these express the constancy of $\mathbf{e_i}(s) \star \mathbf{u}(s)$ and $\mathbf{e_i}(s) \star \mathbf{e_j}(s)$ – to obtain the other n equations. These $n+1$ equations can be written more compactly as the *matrix equation,* $\frac{dU}{ds} = K(s)U(s)$, where $U(s)$ is the square matrix of size $n+1$ with row vectors $\mathbf{u}(s), \mathbf{e_1}(s), \ldots, \mathbf{e_n}(s)$, and $K(s)$ is the skewsymmetric matrix of this size whose only nonzero entries are $k_1(s), \ldots, k_n(s)$ immediately above the main diagonal, and their negatives below it. Using the existence and uniqueness theorem for linear ODE's, this equation has one and only one solution $U(s)$ with initial value $U(0)$, and its first row $\mathbf{u}(s)$ determines the motion $\mathbf{r}(s) = \mathbf{r}(0) + \int_0^s \mathbf{u}(s)ds$. This solution $U(s)$ is *not* in general given by $U(s)U(0)^{-1} = \exp \int_0^s K(s)ds$ but this formula is true if the curvatures are constant. However, *any n smooth positive functions can be realized as the curvatures of a smooth generic motion,* unless some new decree – note 17 – on acceleration and higher derivatives is in force. À priori motions also contain another open dense set, of piecewise linear motions with vertices in general position, and, *there is a similar classification of generic p.l. motions.* This frenet theory is akin to Kalai's *algebraic shifting,* a simple but surprisingly useful idea, over which I had mulled for long in the 1990's, but never quite managed to grasp it to my satisfaction ...

20. From the geographical distribution of some traits in the mitochondrial DNA sequence it has been deduced that, *all women are the iterated daughters of just one, who lived in Africa 150,000 years ago!* So once upon a time, whatever mathematics there was, was in Africa only. The recorded history of our subject is shorter, but Africa looms large in it too, in particular, the school of mathematics that flourished in Alexandria—for more than six hundred years!—from Euclid to Pappus. Practically everything above is rooted in those books of the former, and the invariance of the biratio – see note 14 – is only one of the many things about projective geometry that can be found in the prolific writings of the latter. The log put by Cayley before it merely converted this multiplicative distance into an additive one, and may even turn out to be a retrograde step, when we switch to other fields to understand the galois symmetries alluded to in notes 1 and 3. The sine law for triangles, and much else from plane and spherical trigonometry, was known to Ptolemy if not Heron, both of Alexandria. Also, the latter knew that light travels so that minimum time is taken, and had used this to prove the isoperimetric inequality : in fact you'll recall that he once gave a colloquium talk (!) on this very topic even in imperial Rome, see *Extracts* from my Notebooks (2008). Mathematics teaches us humility : much repetition and reworking, sometimes stretching over centuries, is often needed before a key, but in hindsight obvious, idea sinks in. It is then mostly a matter of personal choice as to which mathematician, or a set of mathematicians, one wants to credit with this idea. Though the definition of 'his' distance is all there in a rambling 1859 paper of Cayley's, the crisp 2-line format owes much to Beltrami and Klein, and one presumes that during these years it became 'folklore' that it assigns a length to the segments of any bounded open Ω^n, but this came out only when a letter from Hilbert to Klein was published in 1895. Hilbert was trying to make Euclid's formal presentation of geometry more rigorous, and he tells Klein excitedly that there is a bounded open convex set—to wit the shaded region in Figure 4—with more than one geodesic between two points! *Cayley's*—or if you prefer Pappus's or Lobatchevsky's or Riemann's or Beltrami's or Klein's or Hilbert's ...—*distance breathes life into all the 'dime-a-dozen' geometries that we mentioned at the very outset,* and then again in note 1. Some more water has flown north out of Africa past Alexandria since Hilbert included in his famous list of problems *two* that were closely related to this letter. So much more is known now, for example, Benoist has characterized those convex Ω^n whose dime-a-dozen geometry is hyperbolic in the sense of Gromov. But much still remains to be done, even for convex polytopes ... Moreover, *each of these geometries comes with a concomitant linearization or relativity theory.* Indeed, the definition of an unparametrized cartesian absolute motion is already clear from note 13, and absolute space can be defined once again to be the envelope of all the images of Ω^n under the—now possibly very few—linear symmetries of its cone, so absolute time, et cetera. The paucity of symmetries can be converted from a handicap to a boon, for example, one can focus better on some subgroups of G by replacing the ball itself by a symmetric polytope, and it seems galois symmetries will make up for some loss of symmetry too ...

K S Sarkaria

(*contd.*)

21. If positions P_1 and P_2 of the same particle occur at times $t_1 < t_2$ of an observer S, and have components $\mathbf{x}_1$ and $\mathbf{x}_2$ in his euclidean space, then $\overrightarrow{P_1P_2}$ parallel to a ray translates into $\|\mathbf{x}_2 - \mathbf{x}_1\| < c(t_2 - t_1)$. For $c < \infty$ this lipschitz property enables us to get by – note 16 – without extra hypotheses like smooth or p.l. on a motion. The half-space $t > 0$ of S is the *euclidean product* of his ray and n-space $t = 1$ and is preserved only by the reflections of the cone preserving this ray. The cone, the intersection of the half-spaces of all the observers, has however a *hidden product structure* – Figure 5 – given by all the rays and the reeb foliation $\tau =$ constant, which is preserved by all its reflections. If P_1 and P_2 occur at absolute times $\tau_1 < \tau_2$, and we use cayley's distance, then *the lipschitz property can be reformulated thus* : $\widehat{P_1P_2} < c\log(\tau_2/\tau_1)$.

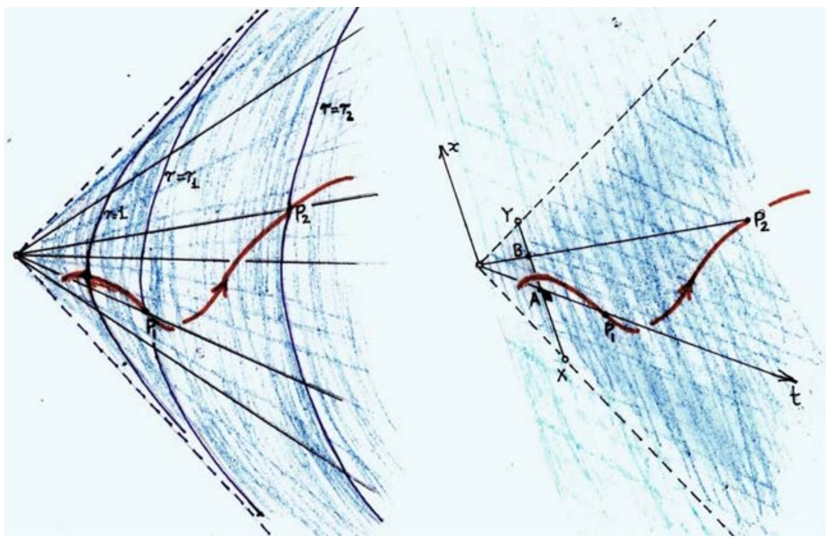

Figure 5

If P_1 and P_2 lie on the same ray the distance $\widehat{P_1P_2}$ between their rays is 0. Otherwise we'll use on their plane the coordinates (t, x) of the observer S whose state of rest is the ray through P_1 with the x axis towards the ray through P_2. So if $P_1 = (l, 0)$ then $P_2 = (l + lu, mlu)$ for some $u > 0$ and $m > 0$. The two rays cut the ball of S in its centre $A = (1, 0)$ and the point $B = (1, \frac{mu}{1+u})$, and AB extended meets the boundary in $X = (1, -c)$ and $Y = (1, c)$. We have $\frac{XB}{XA}\frac{YA}{YB} = \frac{XB}{YB} = \frac{c+(mu/1+u)}{c-(mu/1+u)} = \frac{1+u+mu/c}{1+u-mu/c}$. Also $(\tau_1)^2 = l^2$ and $(\tau_2)^2 = (l + lu)^2 - (lmu/c)^2$, therefore $(\frac{\tau_2}{\tau_1})^2 = (1 + u + mu/c)(1 + u - mu/c)$. Using the definition of $\widehat{P_1P_2}$ in note 14 the above lipschitz property is equivalent to $\frac{XB}{XA}\frac{YA}{YB} < (\frac{\tau_2}{\tau_1})^2$, so it holds iff $1 < (1 + u - mu/c)^2$, i.e., iff $m < c$. $\square$

Since P_1P_2 extended may not intersect the boundary twice, in the above proof we first replaced it by AB. Indeed if $-c \leq m \leq c$ then P_1P_2 extended has an X' on the boundary but Y' is at infinity, and the distance between the two rays is, excepting for $m = \mp c$, *not* given by the limit $\frac{c}{2} \log \frac{X'P_2}{X'P_1}$. This is bigger, for example, for $m = 0$ the two rays coincide, but this expression gives us $\frac{c}{2} \log(\tau_2/\tau_1)$, which is half the *right* side of our inequality!

The unrestricted use of Cayley's formula—i.e. $\widetilde{AB} := \frac{c}{2} \log(\frac{XB}{XA} \frac{YA}{YB})$ always with the limit to be used if X or Y is at infinity—*is however natural and gives us more.* It gives *a metric* as against a pseudometric. We have $\widetilde{AB} > 0$ because the complete line containing the points $A \neq B$ is not in our cone, and $\widetilde{AB} + \widetilde{BC} \geq \widetilde{AC}$ follows by taking the limit of the triangle inequality – see note 14 – for the cone truncated by a flat on the right. A linear isomorphism preserves the ratios of segments of a line, so $\widehat{AB}$ and $\widetilde{AB}$ are invariant under *all* linear isomorphisms of the cone. Further this metric is well-behaved with respect to the hidden product structure. On each leaf τ = constant it coincides with the cayley distance between rays. On each ray it coincides with $\frac{c}{2} \log(\tau_2/\tau_1)$, so by analogy this expression will be called *the cayley distance between the leaves* on which P_1 and P_2 lie. We recall that the factor $\frac{c}{2}$ was put only to get the coincidence, of the riemannian metric of the particular leaf $\tau = 1$, with the cayley distance between rays. With this artificial factor now gone, the lipschitz property becomes: *for any pair of subsequent points on an absolute motion the cayley distance between rays is less than twice the cayley distance between leaves.* Since 0 and 1 are the only whole numbers less than two, the thought arises that the other side of the $c < \infty$ coin – see note 12 – is this discrete micro reality: *at the next instant of absolute time a 'particle' is either at the same or at one of the adjacent spots of absolute space?*

22. In mechanics one also considers motions of two, three or more particles, even of fluids and plasmas with uncountably many, and collisions, fusion and fission of particles too, but all in a still space. The 'particles' at the end of the last note are different, they are not things *in* space, but things revealed by the motions *of* space. Since space stays put, these motions are via bijections which induce bijections of open sets, viz., *homeomorphisms* ϕ_τ of the absolute space of all rays or of $\tau = 1$, parametrized continuously by absolute times $\tau > 0$. Also, as before, the absolute motion P_τ of each point P of $\tau = 1$ shall be strictly increasing with respect to the partial order of the cone, i.e., if $P_1 = P_{\tau_1}$ and $P_2 = P_{\tau_2}$ are the points of $\tau = \tau_1$ and $\tau = \tau_2$ on the rays through $\phi_{\tau_1}(P)$ and $\phi_{\tau_2}(P)$ for any $\tau_1 < \tau_2$, then $\overrightarrow{P_1P_2}$ is parallel to a ray of the cone.

The corresponding motion ϕ_τ of the ball B^n of any observer S extends to homeomorphisms $\overline{\phi}_\tau$ of his euclidean space identity outside B^n. The lipschitz inequality of note 21 applies to the absolute motion of any point, so $\phi = \phi_\tau$ is at a bounded cayley distance $A = c|\log \tau|$ from the identity, i.e., $\widehat{P\phi(P)} < A \ \forall\ P$. So this homeomorphism ϕ of B^n maps its centre into the concentric open ball of radius a where $\frac{c}{2} \log \frac{c+a}{c-a} = A$, i.e., $a = c \tanh(\frac{A}{c})$. More generally ϕ maps any $P \in B^n$ into the *cayley ball of radius A* around P, i.e., all points at cayley distance less than A from P. The extension $\overline{\phi}$ is a homeomorphism because, *in the euclidean metric of S, these cayley balls become arbitrarily small when P approaches the boundary of B^n.*

If S uses orthogonal coordinates $(t; x, \mathbf{y})$ in which $P = (1; v, \mathbf{0}), v > 0$, the linear reflection of the cone switching the rays through the centre and P – see fifth para of text – is $(t; x, \mathbf{y}) \mapsto (\gamma t - \frac{\gamma v}{c^2} x; \gamma v t - \gamma x, \mathbf{y})$ where $\frac{1}{\gamma(v)} := \sqrt{1 - \frac{v^2}{c^2}}$. The rays through the boundary of the cayley ball around the centre constitute $x^2 + \mathbf{y}^2 = a^2 t^2$. So the boundary of the cayley ball around P is given by putting $t = 1$ in $(\gamma v t - \gamma x)^2 + \mathbf{y}^2 = a^2(\gamma t - \frac{\gamma v}{c^2} x)^2$, i.e. $(\gamma^2 - \gamma^2 \frac{a^2 v^2}{c^4})x^2 - 2(\gamma^2 v - \frac{\gamma^2 a^2 v}{c^2})x + \mathbf{y}^2 = \gamma^2 a^2 - \gamma^2 v^2$. Completing a square this can be written as $\frac{\gamma^2}{\delta^2}(x - w)^2 + \mathbf{y}^2 = \beta^2$, where $\frac{1}{\delta(a,v)} := \sqrt{1 - \frac{a^2}{c^2}\frac{v^2}{c^2}}$. So this is an ellipsoid – Figure 6 – with centre $Q = (1; w, \mathbf{0})$, with all semi-axes β, except that along the diameter on which P lies, this semi-axis $\alpha = \frac{\delta}{\gamma}\beta$ is smaller. Further, segment of the diameter through P intercepted by the ellipsoid is bounded by the reflections T', U' of the rays through $(1; \pm a, \mathbf{0})$, viz., the rays through $(1; \frac{v \mp a}{1 \mp \frac{av}{c^2}}, \mathbf{0})$. So $2\alpha = \frac{v+a}{1+\frac{av}{c^2}} - \frac{v-a}{1-\frac{av}{c^2}}$ and $2w = \frac{v+a}{1+\frac{av}{c^2}} + \frac{v-a}{1-\frac{av}{c^2}}$ which give $\alpha = \frac{\delta^2(a,v)}{\gamma^2(v)} a$ and $w = \frac{\delta^2(a,v)}{\gamma^2(a)} v$. Since $w < v$ we see that Q is nearer to the centre of the ball than P; also that the semi-axes $\beta = \frac{\delta}{\gamma} a$ and $\alpha = \frac{\delta^2}{\gamma^2} a$ of the ellipsoid approach 0 when $v \to c$. $\square$

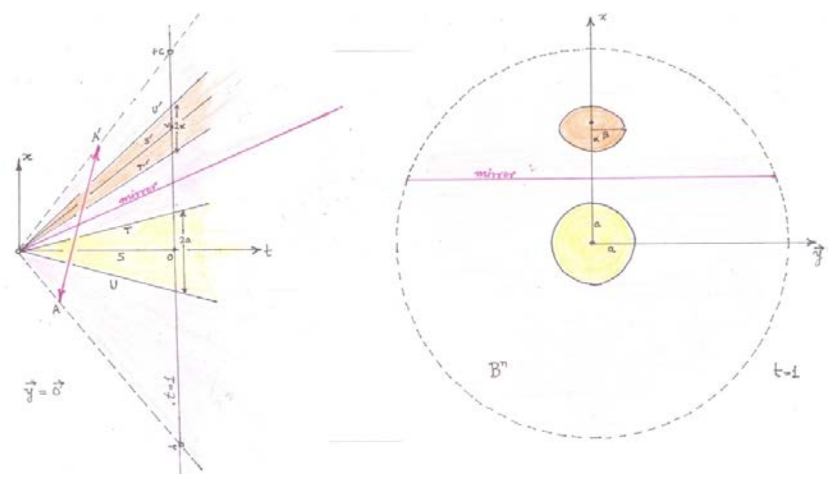

Figure 6

We can't resist remarking once again how *plain* it all is ! The linear reflection $A \leftrightarrow A'$ of the *plane* of S and S' gives us the smaller axis of the ellipsoid, so in particular the factor by which a shrinks in this direction, in all orthogonal directions it only shrinks by the square root of this amount. The *infinitesimal cayley ball at* P is even easier to keep in mind : the ellipsoid with centre P with the radius of the central ball shrunk in these directions by the factors $1 - \frac{v^2}{c^2}$ and $\sqrt{1 - \frac{v^2}{c^2}}$ respectively, where v denotes the euclidean distance of P from the centre of B^n. This because $\delta(a, v)$ and $\gamma(a)$ approach 1 when $a \to 0$. That is, the cayley distance of the ball B^n arises from a riemannian metric which coincides at its centre with the euclidean metric, and at all other points P stretches the tangent vectors in these directions by the reciprocals of these factors.

More generally, *even if 'motion' is by bijections* ϕ_τ, but the absolute motion P_τ of each point is as before, our argument shows that, *the extended-by-identity bijections* $\overline{\phi}_\tau$ *of euclidean space are bicontinuous on the boundary of the ball.* When dimension $n \geq 2$ all sorts of fissures can now develop within the n-ball, for example, the points may be stationary till some radius $r_0 > 0$, and for any bigger radius rotating at a small nonzero speed. However continuity in time implies that, *the bijections* ϕ_τ *must be homeomorphisms for* $n = 1$. If the order of two points P and Q of the interval B^1 is reversed under ϕ_τ their flow lines intersect at some $0 < \tau' < \tau$, i.e. $P_{\tau'} = Q_{\tau'}$, contradicting injectivity of $\phi_{\tau'}$. So these bijections are order preserving, and they have no discontinuity either, because any such jump contradicts surjectivity. $\square$

23. It seems that, *any homeomorphism* ϕ *of the* n*-ball at a bounded cayley distance from its identity map can be realized as a* ϕ_τ *of some motion.* So these bounded homeomorphisms are *isotopic to the identity,* however they form a smaller group. For example, any homeomorphism of B^n which is *radial,* i.e., preserves each radius, is isotopic to the identity, but it may not be bounded. Also, we'll see later that, *any strictly increasing curve passing through* P *for* $\tau = 1$ *can be realized as the flow line* P_τ *of some motion of space.*

The projections of the flow lines P_τ on the absolute space of rays or $\tau = 1$ are called the *orbits* $\phi_\tau(P)$ of the motion. Unlike flow lines, orbits can intersect themselves or each other in all sorts of way but, *if* $c < \infty$ *and* $n \geq 2$*, an orbit cannot visit all the points of a nonempty open set.* Using $\widehat{P_1P_2} < 2(\widehat{\tau_1\tau_2})$—note 21—we see that in any time interval of cayley length $\frac{1}{N}$ the orbit stays in a cayley ball of diameter $\frac{2}{N}$, so over any unit time interval the orbit describes a set which can be covered by N cayley balls of this diameter, but $N(\frac{2}{N})^s \to 0$ as $N \to \infty$ for any $s > 1$, so this set has dimension at most one. $\square$

Here we used *hausdorff dimension* of a metric space, viz., the infimum of all positive real numbers s for which there exists a countable cover such that the sum of the sth powers of the diameters is arbitrarily small. It is easy to see that for submanifolds this is their usual dimension, and an argument similar to the one above shows that, *it is non-increasing under any lipschitz map,* so it is preserved by (bi)lipschitz homeomorphisms of metric spaces.

It follows that that *amazing curve* found by Georg and David while playing dots-and-squares (!) can only be traced by an orbit of a motion with $n = 2$ *and* $c = \infty$. For, it covered a 2-cell with 3 holes, so it can not be lipschitz; but the reader can check that, the euclidean distance between its points at times $t_1 < t_2$ is bounded by a constant multiple of the square root of $t_2 - t_1$. Likewise, *any compact and connected manifold* $M^m \subset \mathbb{R}^n$ *can be traced in finite time as an orbit of some motion of* n*-space,* which moreover satisfies a 'weak lipschitz inequality' involving the mth root of $t_2 - t_1$. $\square$

Here t was the time of an observer S, for $c = \infty$ it is the same as τ. For $c < \infty$ it is not and, *we emphasize that it is the absolute time* τ *which is parametrizing the homeomorphisms* ϕ_τ *of absolute space,* each observer S merely identifies his ball B of radius c in his $t = 1$ with this space of all rays. These ϕ_τ were well-defined because a line in the cone parallel to a ray cuts all the transversals $\tau =$

constant. This is not so, for $c < \infty$, if we use the time t of S : *a transversal* $t =$ constant *may not cut all the flow lines,* for example, if the flow lines are parallel to the same ray. So some points of B may not be on any flow line having a point with a given $t < 1$, and following the flow to a $t > 1$ may give only an injection, not a bijection : we would get a well-defined homeomorphism of B for each $t > 0$ only when all the flow lines arise from the origin.

The foliation provided by all the flow lines has another interpretation when we use the product structure of the half space of S – see Figure 5 – instead of the hidden product structure of the cone : *it is what S would observe if his own euclidean space were undergoing a motion.* This because, up to any time $t > 0$, he can discern only the motions of those points of his space which are at distance less than ct from him—that is why he'll plot only a cone full of flow lines in his half space—and the observed positions of any point at times $t_1 < t_2$ are subject to the condition $\|\mathbf{x}(t_2) - \mathbf{x}(t_1)\| < c(t_2 - t_1)$.

The homeomorphisms $\overline{\phi}_\tau$ of the euclidean n-space $t = 1$ of S, identity outside his ball B^n, give *orientation-preserving homeomorphisms of the n-sphere* having an extra point at infinity. Only that about this hidden motion is heard which persists under perturbations : so, for $n \neq 4$, the observer S can assume that these homeomorphisms are lipschitz! Indeed, for $n > 4$ we'll construct later, *an almost radial homeomorphism, identity outside the ball, which conjugates the motion to one which is lipschitz.* Spherically bending the flat mirrors of B^n maximally inwards ensures that a lipschitz inequality holds if one of the points is on the boundary. Within the ball we'll make the homeomorphisms piecewise linear. We'll start with a simplicial approximation of the motion. This may have some singularities, but for $n > 4$ these singularities can be engulfed away, essentially because a simple closed curve on the already good part can be coned away from it, cf. *Embedding and unknotting of some polyhedra* (1987). For $n \leq 4$ this does not work, and the result is in fact false for $n = 4$, but for $n < 4$ there are other constructions which show that the result is again true.

Even for $c = \infty$—now the cone is a half-space and $\tau = t$—*the hidden product structure is different from that of any observer* : all the flats $t =$ constant with all the rays from the origin, instead of all the parallels to a ray S. Once again it is this observer-independent hidden product structure only that we'll use to define the homeomorphisms ϕ_t of the absolute space $t = 1$ from any continuous flow of the same for all absolute times $t > 0$. However for $c = \infty$ the continuous flow lines P_t may not be lipschitz, and these homeomorphisms of euclidean n-space may not be at a bounded distance from its identity map, nor can S assume on à priori grounds for $n \neq 4$ that they are lipschitz. These distinctions show that, *there is no time and* order-preserving *homeomorphism from the half space onto the cone of rays through a ball B^n of finite radius.* $\square$ On the other hand any homeomorphism of $\mathbb{R}^n$ onto B^n determines and is determined by a time preserving homeomorphism which maps rays to rays.

Given a flow of the space its *invariant subsets* are those on which the homeomorphisms ϕ_τ restrict to homeomorphisms, i.e., subsets A such that if $P \in A$ then the entire orbit $\phi_\tau(P)$ is contained in A. *The minimal invariant sets of a flow partition the space into topologically homogeneous parts.* That these sets are disjoint or equal is clear. At points P and $\phi_\tau(P)$ of any invariant set A the topology is the same because ϕ_τ restricts to a homeomorphism of A. If A is minimal then its points are related by finite sequences of points each on some orbit through the preceding.□ Topologically homogenous spaces are nice, nicest being connected manifolds, so we ask: what manifolds are born in flows?

Example. *There is a smooth motion of n-space, $n \geq 2$, with minimal sets parallel 2-planes.* Let the flow lines be tangent to the vector field on the half-space whose component along the ray through that point is t, and whose components parallel to a fixed frame of the n-space are $(t \cos \log t, t \sin \log t, 0, \ldots, 0)$. Then the orbits, i.e., the projections from the origin of these flow lines on $t = 1$, are all circles of radius 1 parallel to the first two vectors of the frame. □ A similar construction works also for $c < \infty$, and though the *invariant partition* of a flow is seldom a foliation as in this example, it seems that such constructions put together will suffice to establish that, *any smooth connected manifold occurs as a minimal invariant set of some flow with $c < \infty$.*

However not all topological manifolds are relativistic : *if a closed M^m occurs as a minimal invariant set in a motion with $c < \infty$ then it admits a lipschitz structure.* We can assume $m > 3$ and so $n > 4$, but then S can perturb the motion to a conjugate motion whose ϕ_τ's are lipschitz homeomorphisms of his ball B^n; their restrictions to the perturbed copy of M^m give the desired lipschitz structure.□ We recall – see note 16 – that this only excludes some 4-dimensional manifolds. Nevertheless it seems likely that, *outside these wild 4-manifolds, any closed connected topological manifold can be realized as a minimal invariant set in a flow with $c < \infty$,* and that, *for the limiting non-relativistic case $c = \infty$, even these exceptions can be thus realized.*

A motion of space is *steady* in time if the flow lines through any ray are positive multiples of each other, so *$\tau \mapsto \phi_\tau$ is a group homomorphism* $\phi_{\tau_1\tau_2} = \phi_{\tau_1} \circ \phi_{\tau_2}$: see Figure 7.□ For a steady motion, the minimal invariant sets have just one orbit each. Further, if a point returns to its position, it must repeat its journey, therefore : *each orbit is homeomorphic to an open interval, a circle, or a single point.* So the minimal invariant sets of a steady motion are very simple; only, if $n = 3$, some of these circles may be *knotted* in B^3. On the other hand, it may well be that *any* smooth closed connected submanifold M^m of an n-ball B^n is a minimal invariant set of some *unsteady* motion?

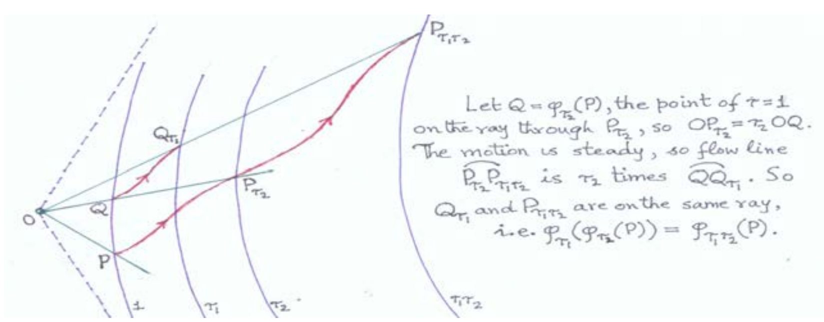

Figure 7

The *homeomorphisms of the cone* $\Phi_\tau(P) := P_\tau$ (the point on the same flow line with proper time τ times) map leaves to leaves. They map—Figure 7—rays to rays iff the motion is steady. These motions preserve the metric $\widetilde{AB}$ of the rays. However, *when $c < \infty$ and $n \geq 2$, no motion other than absolute rest preserves the metric $\widetilde{AB}$ of the leaves* ! If ϕ preserves the orientation and cayley distance of B^n it is a composition of an even number $\leq n+1$ of linear reflections of the cone. If ϕ is not the identity map, and $n \geq 2$, there is a line L whose image $\phi(L)$—also a line by linearity—is distinct from it. Since the cayley balls of any finite radius become arbitrarily small – see note 22 – near the boundary of B^n, the second line is not wholly within a bounded cayley distance of the first line. So a cayley distance preserving ϕ can occur as a ϕ_τ of some absolute motion of space only if it is the identity map of B^n. $\square$ On the other hand the geometry of the infinite n-ball or the finite interval is not rigid : any orientaion and distance preserving ϕ occurs as a ϕ_τ of some motion.

The cayley isometries of the cone are given by the compositions of its linear reflections and time reversals $\tau \mapsto a^2/\tau$. If the homeomorphisms Φ_τ commute with a group $\mathfrak{G}$ of these isometries the motion is called $\mathfrak{G}$-*periodic.* Especially alluring are the discrete subgroups $\mathfrak{G}$ with compact quotients, for example, *in all dimensions there are groups $\mathfrak{G}$ under which the conical spacetime covers a closed and parallelizable manifold* ! The n-ball, held taut at its boundary in his euclidean n-space, and vibrating $\mathfrak{G}$-periodically, enables the observer to hear to some extent the topology of this quotient. This discretization of spacetime is available also for $c = \infty$ and in this non-relativistic schrödinger theory examples of such discrete subgroups are easier to give.

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(*contd.*)

24. *The closed and parallelizable spacetimes that closed* Note 23 *deserve our close attention.* For all $n \geq 1$ there are discrete subgroups $\mathfrak{G}$ of cayley isometries of our conical spacetime such that the quotient space is compact, and by using instead a suitable subgroup of finite index we can always ensure not only that this quotient is a closed $(n+1)$-manifold, but also that it admits $n+1$ smooth linearly independent tangent vector fields $v_1, \ldots, v_n, v_{n+1}$. Further, some such groups $\mathfrak{G}$ are generated by a subgroup Γ of cayley isometries of the ball B^n and a single homothety, i.e., a product of two distinct time reversals.

The closed spacetime is then a circle S^1 times a closed n-manifold B^n/Γ which may not be parallelizable, but B^n/Γ *is parallelizable in the complement of a point.* For, this complement has the homotopy type of an $(n-1)$-dimensional polyhedron. So, on it, the unit vector field w tangent to S^1 is homotopic, via never zero sections of the tangent bundle of the spacetime, to v_{n+1}. Lifting this homotopy we obtain, on this complement, $n+1$ linearly independent vector fields $w_1, \ldots, w_n, w$. The first n of these give, under projection parallel to w, the required parallelization of B^n/Γ minus a point. $\square$

A smooth n-manifold without boundary *immerses in n-space iff it is open and parallelizable.* Here 'only if' is easy and 'if' is nowadays an existence theorem of flexible p.d.e. theory. However, even for the punctured n-torus $\mathbb{R}^n/\mathbb{Z}^n \setminus \{\mathrm{pt}\}$, an explicit immersion is not easy, and for its relativistic analogues $B^n/\Gamma \setminus \{\mathrm{pt}\}$, we know in general nothing about the discrete groups $\mathfrak{G}$ and Γ beyond what we asserted above without proof about their existence. These existence proofs are very pretty – especially an étale homotopy argument which shows why the obstruction to parallelizability vanishes for a finite cover – but first, let us ponder this painting of that river—Note 20—flowing north out of Africa ...

Sunset on the Nile (Jens, circa 1956)

25. *Where do we come from? What are we? Where are we going?* This is the longish name of a painting by Gauguin. A paper by Gromov starts with that painting, and then, for its title, has *Manifolds* asking these existential questions of themselves. Though surely very few can talk, closed manifolds have always been, for me too, very natural objects. This belief is, I guess, what led me to the results of Note 23, and the many related musings in these notes.

It seems that in recent times physics has returned to its cartesian roots, in particular the dictum that, *matter is but extension, and is differentiated only by its various motions.* Be that as it may, any closed topological manifold is 'cartesian matter' in the sense of this *theorem* : it can be created in finite time as a compact minimal invariant set M^m of some continuous motion of a euclidean space having sufficiently many degrees n of freedom.

A flow of $\mathbb{R}^n$ can probably also have other compact M^m's—all necessarily connected, topologically homogenous and homogenously embedded—but it is manifolds that seem the most natural. Indeed, matter is discrete, so what matter are maybe the *triangulable M^m's : these* are *closed manifolds.* □ This is easy, but the *Bing-Borsuk conjecture,* that *any* locally contractible and topologically homogeneous compactum is a manifold, is still open for $m > 2$. And, for $m = 3$ it would finish another proof of *Poincaré's conjecture,* that a closed 3-manifold with fundamental group $\Gamma = 1$ is the 3-sphere. Also, the unfolding classification of triangulable M^m's for $m \geq 5$ is tied closely to that of *homology 3-spheres,* i.e., closed 3-manifolds with $\Gamma_{ab} = 1$. 'His' homology 3-sphere with Γ finite—the *Miss Universe* of "213, 16A"—was discussed at great length by Poincaré, but who knows, the infinitely many homology 3-spheres which occur as B^3/Γ may be there too in his pioneering and prolific writings on discrete subgroups Γ preserving the geometry of a 3-ball of radius $c < \infty$? The unfolding work on triangulations suggests that the above 'cartesian matter' can be analysed in terms of these 'elementary particles' or 'relativistic crystals' ...

For $c < \infty$ it is in fact $(B^n/\Gamma)\times S^1$, and more generally any manifold quotient $C^{n+1}/\mathfrak{G}$ of the cone, that is more like a classical crystallographic manifold, for it has a finite parallelizable cover. *These closed spacetimes $C^{n+1}/\mathfrak{G}$ have an induced reeb foliation and transverse line field,* since the cayley isometries of the cone C^{n+1} map leaves and rays to leaves and rays. □ In this context we'll think of C^{n+1} *as the infinite cylinder over the ball B^n of any observer S* :–

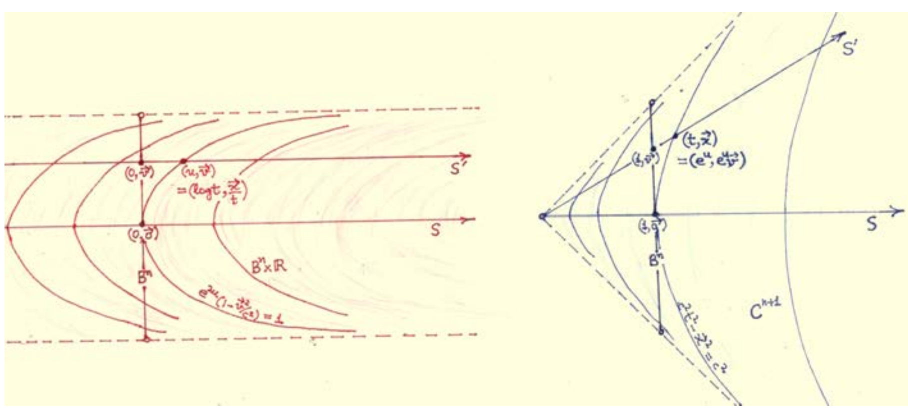


Figure 9

This cylinderical representation is convenient for doing topology, but the geometry gets distorted. The new coordinates are the *logarithmic time* $u = \log t$ of the observer and relative velocity $\mathbf{v}$ with respect to him, so parallels to S represent galilean motions. More generally, *any smooth curve* $(u, \mathbf{v}(u))$ *represents a possible motion iff it obeys the relativistic constraint* $|d\mathbf{v}/du| < c - |\mathbf{v}|$. Moreover, *the lorentz contraction factor* $\gamma(\mathbf{v})$ *is tied intimately with the new equations* $u = \log\gamma(\mathbf{v})$ + constant *of the reeb leaves,* that in the conical picture were simply $c^2t^2 - \mathbf{x}^2$ = constant. The new cylinderical picture is preserved by translations parallel to the axis just like the conical picture was preserved by homotheties, however the cayley isometries of the ball moving its centre are restrictions of nonlinear transformations of the cylinder.

Foliations are 'cartesian' partitions, for example it is likely that, any smoothly foliated closed manifold $(M^m, \mathcal{F})$ can be created in finite time as an invariant set of a smooth relativistic flow of a high dimensional ball B^n of radius $c < \infty$, each leaf of $\mathcal{F}$ a minimal invariant set of this flow. However a parallel generalization of the theorem stated above to all continuously foliated topological manifolds vis-à-vis continuous non-relativistic flows seems more iffy.

This was my cue to revisit my foliations days, doing which I noticed that, *the intermediate partitions used in all those constructions of foliations from that era are most likely 'cartesian' too,* at least as long as everything is smooth. Given below are some other things from this trip back in time.

26. Besides the aforementioned real analytic reeb foliations of the closed spacetimes $C^{n+1}/\mathfrak{G}$ – are there some homology spheres here? – there was that good old *smooth reeb foliation of* S^3 which however I now found myself looking at through the lens of my later deleted joins days :–

Let S^3 be the round 3-sphere of circumference 4 centred on the origin of $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$. Then the spherical distance between the *first and second circles* in which S^3 intersects these summands is 1 and S^3 is the *join* $S^1 \cdot S^1$ of these two circles. That is, any other point of S^3 lies on a unique great circular arc of length 1 from a point x of the first circle to a point y of the second, and can be denoted $(1-\alpha)x + \alpha y$, where α is its distance from the first circle. So, *points at a distance* α *from the first circle are at a distance* $1-\alpha$ *from the second, and form a submanifold* L_α *of* S^3 *diffeomorphic to* $S^1 \times S^1 = \{(x, y) : x \in S^1,\ y \in S^1\}$ *if* $0 < \alpha < 1$*, while* L_0 *and* L_1 *are the first and second circles.*

We now use the foliation of $B^2 \times S^1$, obtained by dividing the infinite cylinder of Figure 9 by a translation, to desingularize this *foliation-with-singularities* of S^3 : that is we plug in a copy to refoliate the diffeomorphic neighbourhood of all points within a certain distance less than one of each circle, smoothness on the bounding toral leaf of this neighbourhood then follows from the fact that $\log\gamma(v)$ and all its derivatives approach infinity when $v \to \pm c$. Also, we can desingularize symmetrically with respect to the switching $\mathbb{Z}/2$-action on $S^3 = S^1 \cdot S^1$, and if we choose the 'certain distance' to be $1/2$ for both circles we would be left with just one toral leaf $L_{1/2}$. $\square$

Likewise, *joining* $q+1$ *spheres gives a foliation-with-singularities of a sphere,* with generic leaf product of all these spheres, so it has *codimension* q*,* but there are also some singular leaves that are products of only some of the $q+1$ spheres, e.g., the $(q+1)$-fold join S^{2q+1} of S^1 has such a 'foliation' with generic leaves $(q+1)$-tori. *When exactly can the join of $q+1$ spheres be desingularized to get a codimension q foliation?* Obviously this sphere should admit a codimension q tangent plane field, for example, the join of two spheres can be desingularized only if is odd dimensional, but this condition is not sufficient.

The join of two spheres can be desingularized iff they are odd dimensional with one a circle. Given the above foliation-with-singularities of $S^i \cdot S^j$, we want to refoliate two disjoint open saturated neighbourhoods $S^i \times B^{j+1}$ and $B^{i+1} \times S^j$, of the singular leaves S^i and S^j, so that the new leaves approach the boundary leaves $S^i \times S^j$. If $i > 1$ and $j > 1$ then $S^i \times S^j$ is simply connected; therefore by Poincaré's original definition of the fundamental group, the global monodromy of any multiple valued function defined on it is trivial; so that given by nearby leaves of these refoliations would be trivial; which rules out approaching leaves. So, because $i + j + 1$ is odd, i and j are odd with one 1.

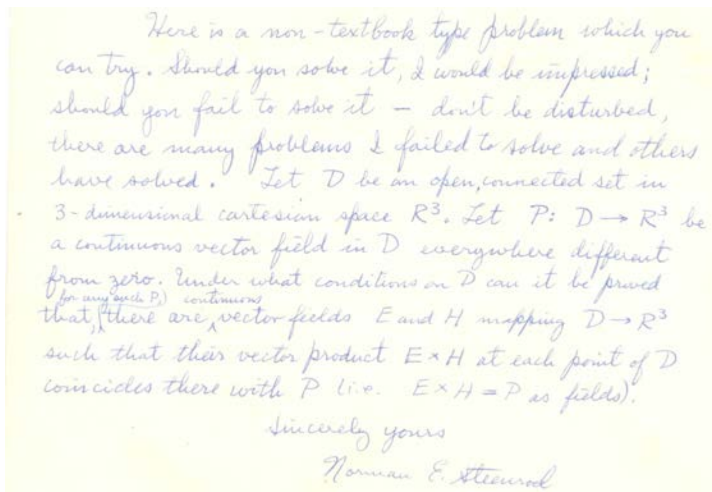
Conversely, Figure 9 modulo a translation refoliates the neighbourhood of a singular circle, but refoliating $S^i \times B^2$ when i is odd but bigger than 1 is much harder. However this neighbourhood obviously admits a smooth nonzero vector field normal to its boundary – also we can ensure that it coincides with a given nonzero vector field on the central S^i – and it is known that the existence of such a vector field implies that of the required refoliation. □

27. This end of the year note is being typed nine months after the one above, but of course I had once again looked long and wistfully at Thurston's *"Existence of codimension one foliations"* (1976). The number of people who got it was nonzero then, but now – four decades later! – it is (imho) even less than those who dig Mochizuki's *"Inter-universal Teichmüller theory"* (2012). This classic characterized manifolds possibly with boundary that admit a smooth foliation having boundary components as leaves. This is done using an explicit local construction which spreads the required foliation steadily, and always transverse to a given vector field normal to the boundary components, till it covers the entire manifold ... so I imagine we should in fact be able to refoliate our $S^i \times B^2$ in such a way that this new foliation cuts the central lower dimensional odd sphere S^i in a given codimension one foliation?

Unlike Thurston my ability 'to see from within' noneuclidean geometries is very limited, may be that is why I've given primacy to familiar n-space only, with indeed the—to my mind just pragmatic, but also called relativistic—restriction that it ought to be of a finite radius $c < \infty$. In this receptacle are born from its own cartesian motions all lipschitz manifolds, and if smooth enough, its cayley distance induces on them a riemannian metric. The usual tools of vector calculus and forms, tensors, etc., are available chart by chart – with an occasional sign ambiguity for orientation dependent quantities – so existence of smooth foliations translates into existence theorems of analysis, for example, *the partial differential equation $\overrightarrow{E} \cdot \mathrm{curl}\overrightarrow{E} = 0$ has an always nonzero solution on any closed 3-manifold,* because this is the same as saying that the 2-dimensional plane field orthogonal to $\overrightarrow{E}$ is tangent to a foliation.

The story in fact began when this problem of vector calculus was posed for S^3 by Hopf in 1935. It was solved by Ehresmann and Reeb in 1944, but then there was an extended drought of interesting foliations, till Lawson got infinitely many odd dimensional spheres in a clever way, but it was Thurston soon after who reached the bottom of the well, his chart by chart approach akin to how Edwards et al were trying to triangulate topological manifolds.

Which reminds me of another problem about vector fields that was posed to me in late 1969 in a very kind letter from Professor Steenrod :-


Here is a non-textbook type problem which you
can try. Should you solve it, I would be impressed;
should you fail to solve it — don't be disturbed,
there are many problems I failed to solve and others
have solved. Let D be an open, connected set in
3-dimensional cartesian space R^3. Let $P: D \to R^3$ be
a continuous vector field in D everywhere different
from zero. Under what conditions on D can it be proved
that (for any such P) there are continuous vector fields E and H mapping $D \to R^3$
such that their vector product $E \times H$ at each point of D
coincides there with P (i.e. $E \times H = P$ as fields).

Sincerely yours

Norman E. Steenrod

Working out by myself what in the topology of D prevents this factorization $\overrightarrow{P} = \overrightarrow{E} \times \overrightarrow{H}$ was a wonderful way of learning some obstruction theory, and so appreciate later on Haefliger's necessary conditions for the existence of foliations in all codimensions, which prepared the ground for Thurston to complete the job from the other end. Also it helped me move to virtually a ring side seat even as this dénouement was about to be played out.

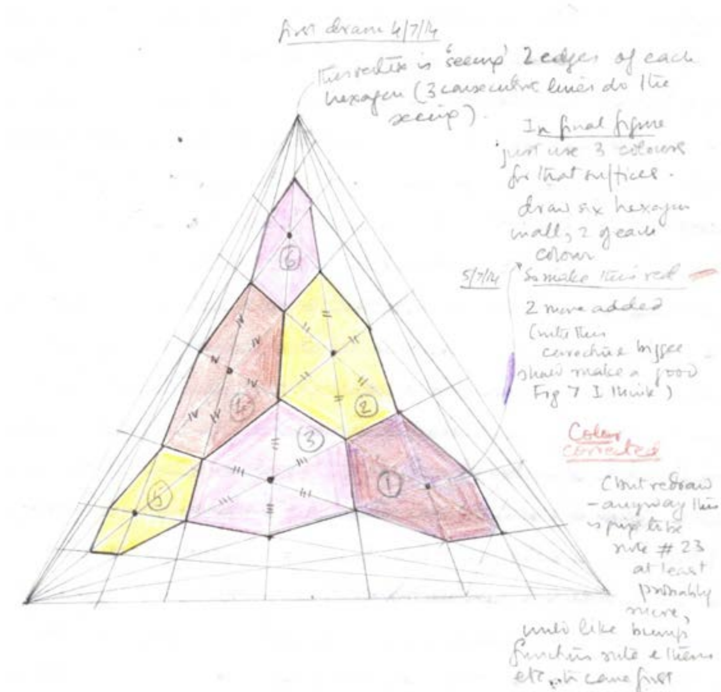
There was much else equally exciting going on then, e.g., the index formulas of Atiyah et al. Trying to make their analysis less messy I stumbled on the smoothing operators in de Rham's *Variétés Différentiables.* If there are enough flows preserving leaves they generalized to foliated manifolds and gave finiteness theorems. *In the cartesian context a smoothing operator comes with the primal motion that gave birth to our manifold.*

The charts lipschitz if $c < \infty$ come too, so *abstract manifolds are natural.* Poincaré's crossword dissection of smooth manifolds, perfected by Cairns, used a *quadrillage* of the ambient space. For topological manifolds we use a grid in each chart, and puzzle out if two overlapping dissections can be made to fit, then three, etc. For lipschitz structures Sullivan played the same game using $1/c > 0$ tilings, so these crosswords or torus tricks quantize manifolds.

The new year is here and I'll return to a sequel to ਉਂਗਲੀਆਂ ਅਤੇ ਟਾਇਲਾਂ – its translation *Fingers and tiles* will be available soon – which I posted in July 2015. This gave four proofs of a cute problem tied to the burgeoning Thurston lore, viz. the first of the ten (!) stories about him that Sullivan relates in the November 2015 issue of the Notices of the A.M.S. I thought this was an auspicious way to start (re)learning the constructions needed to understand better some questions that have arisen in this work. For example the sequel that I am working on dwells on constructions used to study the embeddability of simplicial complexes in double dimensional space.

There is also a considerable backlog of older things that need to be typed up, so I hope to continue this series of notes as well. For example given below is a cute picture, scanned directly from my notebook of 2014 to save on time, exemplifying how deleted joins, to wit Flores' spheres, join the fray as cayley balls if we replace the ball B of radius c by a regular simplex.

K S Sarkaria (*contd.*)



Plain Geometry & Relativity, part V

28. *Matter is but extension, and is differentiated only by its various motions.* `Let us summarize where we are` now in our efforts to answer those three questions—Note 25—of manifolds starting from this dictum of Descartes.

(28.1) We saw `closed manifolds emerge magically`—and it seems in all their diversity—*in euclidean n-spaces from their own periodic motions.*[1] Even simple physical systems have many more degrees of freedom n than visual space, but we can here work in just the one cartesian space of all sequences of real numbers having only finitely many nonzero entries.[2]

(28.2) That pragmatic restriction—any observer's space at his time $t = 1$ is *an n-ball* of radius $c < \infty$ around him—led us to an *absolute time* τ defined on, and preserved by the linear reflections of *the cone* over his n-ball. The *flow lines* of a motion of the n-ball project its moving points on the successive absolute spaces $\tau =$ constant. *Their chords are parallel to rays of the cone,* besides we require that this motion proceeds via homeomorphisms.

(28.21) *For $n > 4$ this requirement implies that there is a perturbation of the motion of the n-ball with lipschitz homeomorphisms.* We sketched in Note 23 why this seems true and gives one half of : `A closed manifold emerges in a periodic relativistic motion if and only if it admits a lipschitz structure.` For the other half we showed that a 2-ball emerges from a smooth motion for any $n \geq 2$, from which it is clear any closed smooth manifold occurs, hoping that similar constructions work for closed lipschitz manifolds.

(28.22) Further, using a `lipschitz yang-mills theory`, Donaldson and Sullivan have shown that some closed 4-manifolds do not admit any lipschitz structure. On the other hand, using `a bieberbach theorem for relativistic crystallography`, the latter had shown long before that outside dimension 4 all manifolds admit a unique lipschitz structure. However I have yet to understand these technologies to my full satisfaction.

(28.3) From that complicated hidden motion we only got `static` closed manifolds in our n-ball of radius $c < \infty$. `What sets them moving` is more pragmatism : *only a compact interval of absolute time for each snapshot.* Depending on the *scale* at which we are discerning the hidden motion, there is say *a number* $1 < s < \infty$ – very big in macrophysics, almost 1 for microphysics – and *the τth frame of this moving picture uses* $[\tau/s, \tau s]$*,* i.e., the actual hidden motion is replaced with the one having this restriction as a period to make this snapshot. Since there is in fact no periodicity the closed manifolds move and occasionally coalesce or bifurcate in this movie : cobordism or intrinsic `homology arises naturally from motion.` Not only that, as our discernment of the hidden motion becomes finer, what was a minimal manifold can get partitioned, say into a foliation, and its compact leaves will be now natural candidates for cartesian matter at this smaller scale s, et cetera.

[1] As `persistent minimal sets` of these motions (more details of this and some other things still to be done) so our manifolds are *connected* but may not be orientable.

[2] This space $\mathbb{R}^\infty$ was used in the last two `parallel notes` उ and ध.

29. `A cartesian motion` is a partition of conical spacetime into infinite arcs, its *flow lines,* that cut each copy $\tau =$ constant of the absolute space once and only once, inducing homeomorphisms between these level surfaces, and have chords parallel to rays. If all flow lines tend to the origin as $\tau \to 0$ it is a `deformation`[3] of the basic example, *cartesian rest,* which has as flow lines all rays of the cone. We'll now discuss why this definition is reasonable.

(29.1) The `continuity` of flow lines does not for $n > 1$ guarantee that the bijections induced between the spaces $\tau =$ constant are bicontinuous, that was a separate condition. To obtain manifolds that are smooth, piecewise linear or lipschitz we'll also use cartesian motions with flow lines and homeomorphisms smooth, piecewise linear or lipschitz. The condition, `chords parallel to rays`, which kicks in for $c < \infty$ implies then that *the flow lines are lipschitz.* This being an open condition it persists under perturbations and so we'll be able to approximate – if $c < \infty$, $n \neq 4$ and the given motion is periodic – by arbitrarily close cartesian motions with homeomorphisms lipschitz.[4]

(29.2) *The* `initial condition` *at* $\tau \to 0$ *ensures the flow lines cut each* $t =$ *constant of any observer once and only once* :- With the continuity of flow lines it gives one cut, and there can't be two because the chords of these n-balls are not parallel to even the rays of the *closed* cone over them. □

In other words, *the time of any observer is strictly increasing and takes all possible values on flow lines.* Conversely for $c < \infty$ this implies the initial condition, and that chords of flow lines are parallel to the rays of the closed cone[5] :- The time t of S takes all values on it, so the flow line must start from O. It can't have a chord $\overrightarrow{P_1P_2}$ which extended on either side exits the cone, for then we can find a nearby chord of the cone which separates O and P_2 from P_1 in the plane of these points, so the time t' of the ray S' through this chord's mid-point would have a lesser value at P_2 than at P_1. □

(29.3) We call cartesian motions preserved by all homotheties *steady*, and those preserved by some homothety other than the identity `periodic in time`. *If* $c < \infty$ *the restriction of a deformation to times* $[t/s, st]$ *of an observer extends to a periodic deformation preserved by multiplication by* s^2 :- For $s > 1$ we must use homothetic patterns of flow lines over the intervals $\ldots, [t/s^3, t/s]$ and $[st, s^3t], \ldots$ which is okay since concatenation preserves continuity, the sum of two vectors parallel to rays is parallel to a ray, and the new flow lines are defined for all $t > 0$, so the initial condition also holds. □ Likewise, we can replace any portion of a motion by a homothetically equivalent portion of another, and concatenation works just as well in the everything lipschitz or piecewise linear

[3]Talking of deformations, I still don't know – see ट – if a `cantorian PG&R` ties up with his IUTT, but, even as Mochizuki finished his 8 talks in Kyoto, I saw it was child's play to win a game on p. 787 of the Notices of the A.M.S. of August 2016 by using deformations of addition (for example, if the ith kid guesses the number which makes the total i mod 10)!

[4]The `deforming homeomorphisms`, that change each ray to the flow line through the same point on $\tau = 1$, are made lipschitz step by step, using as scaffolding a crystallographic tiling of spacetime, $n = 4$ excluded because of those – see ड, ਚ – `switching difficulties`.

[5]So this more general definition permits photon-like instantaneous motion on some intervals of absolute time, but we'll use our open and particle-like definition, on each flow line the `elapsed time` defined by $\int_O^P \tau(d\mathbf{r})$ – see Note 15 – is also strictly increasing.

context, but some sandpapering is needed for smoothness.

Similarly, *the restriction of any cartesian motion to absolute times* $[\tau/s, s\tau]$ *extends periodically, but we may lose the initial condition,* for example, consider cartesian rest for $(0, \tau_0]$ followed by all lines parallel to a ray S. An infinite repetition will play a key rôle again in the torus tricks needed to get also some measure of spatial periodicity into a given cartesian motion.

30. Periodicity of a cartesian motion in fact makes sense with respect to any transformation of spacetime which preserves this notion, say a *linear reflection* of the cone, or else the `time reversal` of all its rays in a curved mirror $\tau = a$, or any composition of these, see Note 23. Clearly, *a homothety is a composition of two time reversals,* but why do these *nonlinear* factors also preserve the notion of cartesian motion and the cayley distance of the cone? We'll see why, also we'll see that cayley distance is born from the age-old definitions of `adding and multiplying segments` that are given in elementary classes.

(30.1) *The above involutions mirror cartesian motions to cartesian motions.* This will follow easily once we have checked the following.

(30.11) *A line parallel to the boundary and cutting the mirror in one point is switched with the other such line on the same plane through the origin* :-

This plane—of the given line L and the origin—is shown below, P being the one point of the mirror on the line. So, *if the mirror is flat,* it cannot contain this plane – in this trivial case the line stays put – it cuts it in the ray S through P. Any point A of the plane reflects to the point A' such that the mid-point of AA' is on S. It follows that the mirror image L' of our line is the other line M of this plane through P which is parallel to the boundary of the cone.

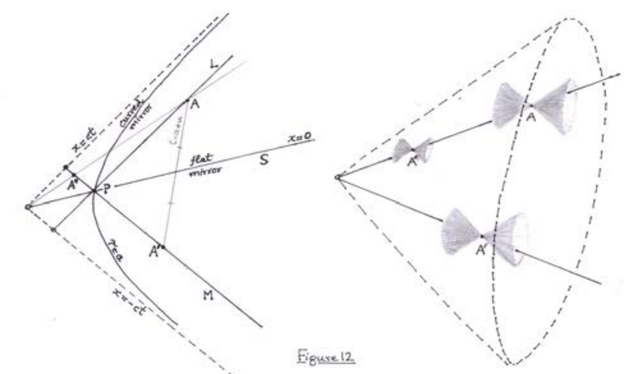


Figure 12

If the mirror is curved, $\tau = a$, then any point A reflects to the point A'' on the same ray such that $\tau(A)\tau(A'') = a^2$. We recall that $\tau^2 = t^2 - \frac{x^2}{c^2}$ in the coordinates (t, x) of S. So if $A = (a + u, cu)$ is on the line L with slope c through $P = (a, 0)$ we have $\tau^2(A) = a^2 + 2au$. Therefore $\tau^2(A'') = \frac{a^4}{a^2+2au}$ and $\frac{\tau(A'')}{\tau(A)} = \frac{a}{a+2u}$. Hence $A'' = (\frac{a(a+u)}{a+2u}, \frac{acu}{a+2u}) = (a - \frac{au}{a+2u}, \frac{cau}{a+2u})$, the point on the ray through A and the line M of slope $-c$ through P. The mirror image L'' is again M, but now as A runs over L linearly in the direction τ increasing, its image A'' describes M non-linearly in the direction τ decreasing. $\square$

So these involutions of spacetime not only preserve its product structure, they map a line parallel to its boundary to another such line. The hypersurface of all such lines through A is mapped to the hypersurface at A' or A''. A segment joining A to B is parallel to a ray iff B is in a component of the complement of this hypersurface through which the ray of A passes. Using this we see that the mirror images of all flow lines obey the chord condition. $\square$

(30.12) In fact, *the above switching property fixes the curved mirrors and so τ up to a constant multiple.* Also, $\tau(B'') < \tau(A'')$ iff $\tau(B) > \tau(A)$ was fine above because our flow lines are *unoriented.* And, as is shown below, the segment $B''A''$ is *not* the reversal of AB unless B is on the ray of A. Hence, *there is no piecewise linear cartesian motion other than rest which is preserved by a time reversal!* Which suggests that, in this unfolding tale about the cartesian genesis of closed manifolds, this non-linear doubling of the symmetries of spacetime will tie up with the `cohomological obstruction` to piecewise linearity.

(30.13) *Any other line R parallel to a ray S reverses to an R'' which is coplanar, strictly convex, tangent to S at the origin, and asymptotic to the line L'' parallel to the boundary whose reversal L has the same end as R* :-

In coordinates (t, x) of the observer S such that R consists of all points $A = (u, b)$, $\frac{b}{c} < u < \infty$, its reversal R'' in the curved mirror $\tau = a$ consists of all points $A'' = \frac{a^2c^2}{c^2u^2-b^2}(u, b)$. As u increases both coordinates of A'' decrease to 0, the second at a faster rate, so the graph of R'' is strictly convex downwards and approaches the origin tangent to S. As $u \to \frac{b}{c}$, R and L approach their common end E, so R'' approaches the line L''. $\square$

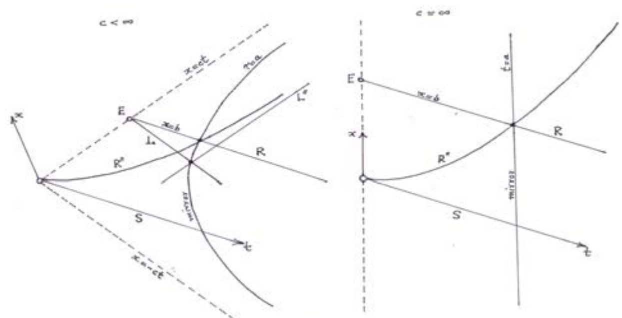


Figure 13

(30.14) Eliminating $u = \frac{bt}{x}$ we see that $A'' = (t, x)$ satisfies $bx^2 + c^2a^2x - c^2bt^2 = 0$, so R'' lies on a hyperbola. *This non-linearity persists in the classical limit $c \to \infty$* : now $A'' = \frac{a^2}{u^2}(u, b), 0 < u < \infty$, so R'' is the $t > 0$ portion of the parabola $a^2x - bt^2 = 0$ in the coordinates of the observer S.[6] So, *the cartesian motion with flow lines straight and parallel to S always reverses to one whose flow lines, other than this ray, are conics tangent to it at the origin.*

[6]From note 23 : *even for $c = \infty$ the hidden product structure is different from that of any observer.* Things are not always easier now, classical notions often have simpler relativistic deformations. From page 2 of PG&R text : *ball geometry 'explains' euclidean geometry,* the classical limit of the naturally defined linear reflections of the cone restrict to the euclidean reflections on the flat $t = 1$ of half-space.

On the other hand, a linear reflection in a mirror containing S preserves this motion, and, *in a flat mirror not containing S, it reflects to a motion with flow lines straight but, unless $c = \infty$, not parallel to each other* : they diverge towards infinity because signals are reaching S' at a finite speed.

(30.15) Indeed, the lines parallel to S form one half of this observer's product structure, which therefore reverses to the arcs above together with the reversals of his balls. *The reversal B'' of any ball $B = \{(\mathbf{x}, t) : t = d\}$ in $\tau = a$ is given by translating $\tau = \frac{a^2}{2d}$ parallel to S by $\frac{a^2}{2d}$* :- A obeys $f(t, \mathbf{x}) = 0$ iff A'' obeys $f(\frac{a^2c^2}{c^2t^2-\mathbf{x}^2}t, \frac{a^2c^2}{c^2t^2-\mathbf{x}^2}\mathbf{x}) = 0$, so B'' is the subset of all $(\mathbf{x}, t)$ such that $\frac{a^2c^2}{c^2t^2-\mathbf{x}^2}t = d$, i.e., $dc^2t^2 - a^2c^2t - d\mathbf{x}^2 = 0$, i.e., $(t - \frac{a^2}{2d})^2 - \frac{\mathbf{x}^2}{c^2} = (\frac{a^2}{2d})^2$. $\square$

(30.16) *So let's extend the principle of mirror relativity to reversals!* The hidden product structure consists of the rays and the level surfaces of absolute time. The point on ray S and surface $\tau = a$ will be denoted S_a. We had called S an observer, we'll now think of each S_a as an `alien` associate who can also reverse in time. So we have, *an n-ball's worth of observers, each a line's worth of aliens.* Mimicking the enunciation we used in PG&R text : *any other alien S_b observes the curved mirror image of what S_a observes, under the time reversal switching these points.* This too is dictated by the aesthetics of Note 2. Also, the composition of two time reversals is a homothety, and our instruments can simulate a species for which time is apparently speeded up or slowed down. The postulate implies that our physical laws, scaled by a suitable factor, coincide with their laws. However, this extension shall really come into play when we come to the cartesian genesis of elementary particles. If we can hear orientation dependent `characteristic numbers` of a closed manifold, it says aliens can hear the same manifold born with the opposite orientation. Further, I've heard said that *we too* can hear these anti-particles! Therefore, but more importantly just for the fun of it, let's reflect some more on reflections.

(30.17) We first recall why, *any observer deems the rulers of another shrunk up to, and his clock slower by, the same factor* :- Any observer S puts himself at the center of a euclidean ball whose radius is increasing in proportion c to the time on his clock : the disjoint union of all these balls, a right cone, is spacetime as he sees it. *Mirror relativity identifies this multitude of right cones, one for each observer, with just one cone* : the unique[7] linear reflection of the right cone of S which switches its axis S with another ray S' transfers the *product structure*[8] *of S* to another representing how S' sees spacetime. For example PQ below represents the distance between S and S' as measured by S at his time OP. It reflects to $P'Q'$ which has the same length as measured by S' at the same time OP' on his clock. However, $P'Q'$ is not in a ball of S, and even its spatial component[9] $P'R$ is more than PQ. Likewise, OP' is not parallel to S, and even its temporal component OR is more than OP. The said factors are the same, i.e., $PQ/P'R = OP/OR$, by similar triangles. $\square$

[7]Since its *flat mirror contains all midpoints of chords of the cone parallel to PP'.*

[8]Each point has its *ball* and *parallel* to S (but a ball and parallel may be disjoint); this structure is preserved by the *many* linear reflections of the cone which preserve S.

[9]In ball of P'; the parallel to S through P' may not cut his ball at time OP.

Calculation of this factor in terms of v, the speed PQ/OP of S' as measured by S, equivalently $P'Q'/OP'$ of S as measured by S' :- We'll use coordinates of S. Let EF be the diameter of the ball of S extending PQ (perpendicular diameters reflect to equal and parallel chords). The parallelogram $\{O, E', 2P', F'\}$ has sides of slope $\pm c$ and one diagonal has slope v, so the other diagonal extending $Q'P'$ *has slope* c^2/v, equivalently $1/\gamma = \sqrt{1 - v^2/c^2}$ in

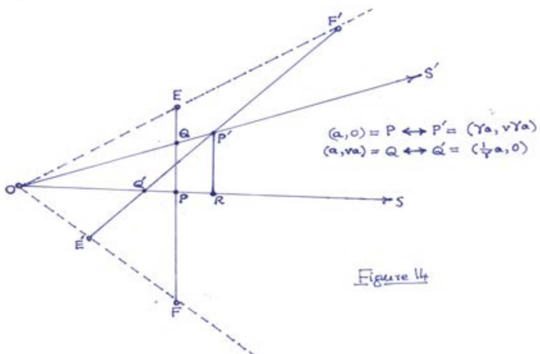


Figure 14

. $\square$

In the same vein, S may deem a point `invisible` to S' if no parallel to S through its ball goes through its mirror image. *This invisible-to-S' subset of a ball of S consists of all points which are not in the ellipsoid with centre on ray S', all perpendicular diameters equal to that of the ball, but the one in this plane is shrunk by* $1/\gamma$:- Linearity and $P \mapsto P', P' \mapsto P$ imply $(t, x, \mathbf{y}) \mapsto (\gamma t - \frac{\gamma v}{c^2}x, v\gamma t - \gamma x, \mathbf{y})$ which preserves $c^2t^2 - x^2 - \mathbf{y}^2$. So $(t, x, \mathbf{y})$ is in the hyperplane of the ball of S' iff $\gamma t - \frac{\gamma v}{c^2}x = a$, i.e. $t = \frac{a}{\gamma} + \frac{v}{c^2}x$, and is the mirror image of the point $(a, \overline{x} := v\gamma[\frac{a}{\gamma} + \frac{v}{c^2}x] - \gamma x = va - \frac{x}{\gamma}, \mathbf{y})$ in the hyperplane of the ball of S, which satisfies $\gamma^2(\overline{x} - va)^2 + \mathbf{y}^2 < a^2c^2$ iff $x^2 + \mathbf{y}^2 < a^2c^2$. $\square$ We note that P is not in this ellipsoid iff $\gamma v \geq c$, i.e., *iff* $\sqrt{2}v \geq c$, *then each observer may think that he is invisible to the other!* $\square$ Finally, here is a construction of P' via the time reversal that keeps P fixed :-

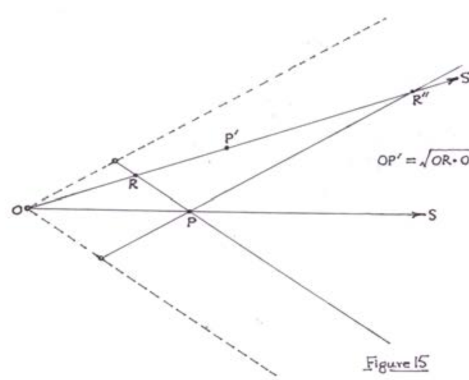


Figure 15

(30.18) *Of lines through P parallel to the boundary two are cut by any other ray, the reversal that keeps P fixed switches these cuts,* see (30.11). So any point

T is mapped to T'' on the same ray such that $OT \cdot OT'' = OR \cdot OR''$ where R and R'' denote these cuts, and its fixed point on this ray, i.e., the point P' on the curved mirror through P, is given by $OP' = \sqrt{OR \cdot OR''}$. Thus, *a single reversal, defined directly in this way for $c < \infty$, gives all the linear reflections of the cone,* and lays bare its hidden product structure. $\square$

Any alien S_a perceives spacetime as a euclidean ball around him whose radius is growing in proportion c to a `two-valued time` on his two-way clock, with both orientations equally natural. (30.16) *identifies all these right cones—one for each observer, two for each alien—with just one cone* : Let the right cone of S be one of the right cones of S_1; the unique linear reflections which switch S with any other ray S' then identify the right cones of the observers S' with one of the right cones of the aliens S'_1; the time reversal in the curved mirror $\tau = 1$ converts these to the other product structure of these aliens; and finally, the unique time reversals which switch $\tau = 1$ and any other $\tau = a$ give us the pairs of right cones of the remaining aliens. *So any oriented alien $\mathcal{A}$ perceives the right conical structure of another, e.g., his alter ego $\mathcal{A}^*$ with the other orientation, as distorted,* and can jump to misconceptions, *even for $c = \infty$* :-

For example, two aliens are related by a time reversal, necessarily unique, iff they are associated to the same observer S. Any point T at time t and a distance r from S in the right cone of $\mathcal{A}$, reverses to a T'' which is at the same time and distance from S in the right conical structure of $\mathcal{A}''$, but $\mathcal{A}$ deems these measurements of $\mathcal{A}''$ to be his own spatial and temporal components for T'', and for $c < \infty$ he may also deem an annulus of his ball through T invisible to $\mathcal{A}''$ in the same sense of this word as used before.[10]

To compute this distortion we think of Figure 14 now as *the right cone of $\mathcal{A}$.*[11] So the point A of S to which $\{\mathcal{A}, \mathcal{A}^*\}$ correspond has time 1, and $\mathcal{A}$ is related to $\mathcal{A}''$ at the point A'' with time a^2 by the reversal in the curved mirror through P. More generally, if P' is the point of this mirror on the ray OT, then T reverses to T'' on this ray such that $\frac{OT}{OT''} = \frac{OT^2}{OT \cdot OT''} = \frac{OT^2}{OP'^2} = \frac{OT^2}{OQ^2}\frac{OQ^2}{OP'^2} = \frac{t^2}{a^2}(1 - \frac{r^2/t^2}{c^2})$[12], *the ratio by which the time and distance-from-S measurements of $\mathcal{A}''$ are deemed off by $\mathcal{A}$.* For $c < \infty$, T is deemed invisible to $\mathcal{A}''$ by $\mathcal{A}$ iff his distance from S to T'' is $\geq ct$, the radius of the ball of T, i.e., iff $r \geq ct\frac{t^2}{a^2}(1 - \frac{r^2/t^2}{c^2})$. The case of equality re-arranged with $g = \frac{ct}{r}$ shows that, *the ratio g of the outer to inner radii of any invisible-to-$\mathcal{A}''$ annulus satisfies $g^2 - \frac{a^2}{t^2}g - 1 = 0$,* in particular at time $t = a$ of $\mathcal{A}$, it is precisely the `golden ratio!` $\square$

(30.19) *As a group all compositions of reversals is the nonabelian double cover of the positive numbers under multiplcation*:- A composition κ of two reversals

[10] *"When I use a word,"* Humpty-Dumpty said, *"it means just what I chose it to mean,"* and Synge has said, in his turn, that relativity has much the same appeal as Alice in Wonderland: aliens and invisibility enhance this fairy tale charm!

[11] Previously, it was *the right cone of the observer S,* in which the time u at A is such that $S_u = \{\mathcal{A}, \mathcal{A}^*\}$. Even if $u = 1$, this may be the right cone of $\mathcal{A}^*$, in which the right cone of $\mathcal{A}$ is awfully distorted; and its about $\tau = 1$ *reversed rays have only a fictitious origin at infinity.* The hidden product structure is preserved by reversals, but the so-called *absolute time τ is invariant only if we limit ourseles to linear reflections of the cone.*

[12] i.e., once again, $T = (t, \mathbf{x}) \mapsto \frac{a^2c^2}{c^2t^2 - \mathbf{x}^2}(t, \mathbf{x}) = T''$, where $\mathbf{x}^2 := r^2$.

$\alpha\circ\beta$ multiplies the right cones of any alien $\{\mathcal{A},\mathcal{A}^*\}$ with the same number and its inverse, viz., *the squared ratio k of his time at the curved mirrors of α and β.* For $\beta\circ\alpha$, these two numbers interchange, our group is nonabelian. Moreover, $(\alpha\circ\beta)\circ(\gamma\circ\delta)$, etc., multiply the two cones by the product of these numbers for $\alpha\circ\beta$, $\gamma\circ\delta$, etc., thereby giving us two isomorphisms of the *index two subgroup of all even compositions* with the group of positive reals under multiplication, which are related to each other by inversion. $\square$

Though the *ratio of times*[13] at an ordered pair of points of a ray is the same up to inversion in all the right conical structures, even a *homothetical oriented alien $\kappa(\mathcal{A})$* can be grossly misunderstood by $\mathcal{A}$:- The time and distance of any point $\kappa(T)$ in the right cone of the oriented alien $\kappa(\mathcal{A})$ are exactly the same[14] as those of the point T in the right cone of $\mathcal{A}$, but, in the same sense as before, $\mathcal{A}$ *deems the time and distance measurements of $\kappa(\mathcal{A})$ to be off by the above factor k, and if $k>1$ he may also deem points at distance ct/k or more in his ball of radius ct to be invisible to $\kappa(\mathcal{A})$!* $\square$

(30.191) *Distortion of an alien $\{\mathcal{B},\mathcal{B}^*\}$ on another ray S'* :- To see this $\mathcal{A}$ can use, after[15] a reversal α about an apt time a, or the homothety κ multiplying his right cone by a^2, the linear reflection f switching S and S'. Therefore, $T=(t,x,\mathbf{y})\mapsto \frac{a^2c^2}{c^2t^2-x^2-\mathbf{y}^2}(t,x,\mathbf{y})\mapsto \frac{a^2c^2}{c^2t^2-x^2-\mathbf{y}^2}(\gamma t-\frac{\gamma v}{c^2}x, v\gamma t-\gamma x,\mathbf{y})=(f\circ\alpha)(T)$, or $T=(t,x,\mathbf{y})\mapsto a^2(t,x,\mathbf{y})\mapsto a^2(\gamma t-\frac{\gamma v}{c^2}x, v\gamma t-\gamma x,\mathbf{y})=(f\circ\kappa)(T)$.[16] So $\mathcal{A}$ deems the time of $\mathcal{B}=(f\circ\alpha)(\mathcal{A})$ to be off by the factor $\frac{t^2}{a^2}(1-\frac{r^2/t^2}{c^2})\gamma$, and his distance-to-ray measurement off by the same factor in the plane of S and S', but only by $\frac{t^2}{a^2}(1-\frac{r^2/t^2}{c^2})$ in directions perpendicular to this plane. Further, he may also deem the point T invisible to $\mathcal{B}$ if the distance of $(f\circ\alpha)(T)$ from S is ct or more, and these invisible-to-$\mathcal{B}$ subsets of his balls can be calculated from the formula above. Likewise, $\mathcal{A}$ deems the time and distance-to-ray measurements of the alter ego $\mathcal{B}^*$ to be off by, and up to the factor $a^2\gamma$, and he may also deem the points of his balls, not in $1/a^2$ times the ellipsoids of (30.17) with centre on S', to be invisible to this oriented alien. $\square$

(30.192) Though $f\circ\alpha$ and $f\circ\kappa$ both map the point A on which the alien $\{\mathcal{A},\mathcal{A}^*\}$ lives, to the point B of $\{\mathcal{B},\mathcal{B}^*\}$, they are very different transformations.

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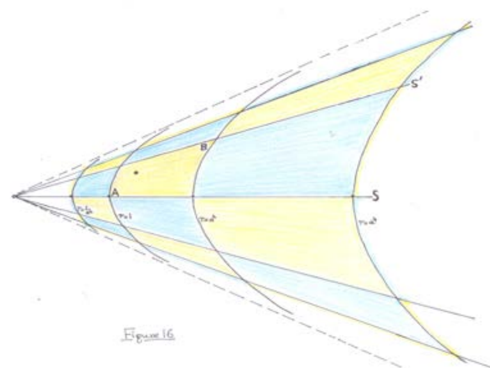
[13]The cayley distance between the two aliens is the log of the $c/2$th power of the *bigger* ratio, while the squares of *both* ratios give $\{k,k^{-1}\}$. Modulo reversals, an alien can explain this rescaling by saying the other is using *different units* of absolute time, the primary *physical quantity* in the sense of *On dimensional analysis* (1999) for cartesian $c<\infty$ physics. With reversals thrown in, this is moot, but we have the discrete and dimensionless topological invariants of the created manifold-matter.

[14]And this, in fact, is so for *any* composition κ of reversals and linear reflections.

[15]Or *before* : $\alpha\circ f=f\circ\alpha$ is true for any function α of the cone to itself which on each ray restricts to the same but *arbitrary function* $\tau\mapsto\alpha(\tau)$ of numbers, because the linear reflection f maps rays to rays and preserves τ. This gives many interesting deformations, for example, the beautiful theorem of Sarkovskii joins the fray, but $\alpha(\tau)=a^2/\tau$ are the only decreasing homeomorphisms of positive numbers that preserve cayley distance.

[16]We are again in, and Figure 16 shows, the right cone of $\mathcal{A}$ with his τ, v is the slope of S', $\gamma(v)$ as in (30.17), and the $\mathbf{x}$ of (30.18) has been split into $(x,\mathbf{y})$ along and perpendicular to the plane of S and S', so $x^2+\mathbf{y}^2=r^2$. The perceived *distortion depends, but only up to an orthogonal transformation of his cone, on how $\mathcal{A}$ is seeing $\{\mathcal{B},\mathcal{B}^*\}$,* for example, the size and shape of the invisible subsets are fixed, but not how they sit in his balls.

The orientation-preserving non-linear `half-turn` $f \circ \alpha$ keeps the `axis` in which the mirrors of f and α intersect[17] fixed, and is its own inverse. The orientation-reversing linear `glide reflection` $f \circ \kappa$ has for $a \neq 1$ no periodic points, but preserves the mirror of f and multiplies it by a^2. A `translation` $f \circ g$, i.e., a composition of two linear reflections of the cone, is identity on the intersection of their flat mirrors and preserves the complementary subspace spanned by the directions in which they reflect.[18] This subspace is 2-dimensional if $f \neq g$, and $f \circ g$ multiplies one of the boundary rays in it by a number bigger than one, and the other by its inverse, for example, if $g(t, x, \mathbf{y}) = (t, -x, \mathbf{y})$, then $f \circ g$ multiplies the boundary rays $(t, ct, \mathbf{0})$ and $(t, -ct, \mathbf{0})$ by $\sqrt{\frac{c+v}{c-v}}$ and $\sqrt{\frac{c-v}{c+v}}$, while, for the inverse translation $g \circ f$, these *proper values* are switched.



(30.193) *As a group all compositions of linear reflections parallel to a given plane is the nonabelian double cover of the positive numbers* :- $f \circ g$ multiplies the two boundary rays of the right cone of any $\mathcal{A}$ parallel to this plane with the same number and its inverse, for $(f \circ g) \circ (h \circ k)$ these proper values multiply, so giving two[19] isomorphisms of the index two subgroup of translations, with the multiplicative group of positive reals, related by inversion. □

(30.194) So $x = 0$ is mapped by the translations $(f \circ g)^i$ to $x = v_i t$, where $\frac{c+v_i}{c-v_i} = (\frac{c+v}{c-v})^i$, while the homotheties κ^j take $\tau = 1$ to $\tau = a^{2j}$. These flat and curved hypersurfaces give a subdivision of the cone of $\mathcal{A}$ which restricts to the deformed `rectangular tiling` of his 2-*cone* $\mathbf{y} = \mathbf{0}$ shown above. Dividing by these symmetries gives a 2-*torus*; and if we divide it by f or the glide reflection

[17]This codimension two axis cuts the plane of S and S' in that *black dot* with $\mathcal{A}$-coordinates $a\sqrt{\frac{2}{1+\gamma}}(\frac{\gamma+1}{2}, \frac{\gamma v}{2})$ inside the tile with vertices $A = (1, 0), A' = (\gamma, \gamma v), B = (a^2\gamma, a^2\gamma v), B' = (a^2, 0)$, it is the point on the ray through the mid-point of AA' with $\tau = a$.

[18]Likewise, the subspace spanned by the directions of *any* number of linear reflections of the cone is the complement of the intersection of their mirrors with respect to the *non-degenerate symmetric form of* τ^2*,* with cone *all* null vectors, etc. This algebra is useful, but, for us, *only that open connected cone is spacetime,* in particular, we don't use the linear reflections, also called 'time reversals', which interchange it with its missing half.

[19] *The proper values of* $f \circ g$ *are also switched for* $\mathcal{A}^*$*,* but, as in (30.91), the log of the cth power of the bigger is a *cayley distance* $\frac{c}{2} \log \frac{c+v}{c-v}$*;* by which the rays parallel to the plane get translated; *for* $c \to \infty$*,* $(f \circ g)(1, x, \mathbf{y}) \to (1, x + v, \mathbf{y})$ *and* $\frac{c}{2} \log \frac{c+v}{c-v} \to v$.

$f \circ \sqrt{\kappa}$ a *möbius strip*[20] or *klein bottle*; while further division by the half-turn $f \circ \alpha$ wraps it twice with a branch point over another torus, etc.

(30.195) The two discrete subgroups of positive numbers coincide iff $\frac{c+v}{c-v} = a^{\pm 2}$, but even for this `square tiling` of the 2-cone, no composition of linear reflections and reversals can interchange adjacent sides [21] of its tiles, so, *its symmetry group is no bigger.* Again, if v is fixed and c is big, then a is almost 1 for squarehood, so, *in the classical limit there is no such tiling*; but, if we make *no* additional demands, we can always keep both v and a fixed, and straighten the limiting subdivision, of the half-space $t > 0$ by flats $x = ivt$ and $t = a^{2j}$, by a suitable $(t, x, \mathbf{y}) \mapsto (\log_C t, \frac{x}{t}, \frac{\mathbf{y}}{t})$ to obtain, on the entire euclidean plane $\mathbf{y} = 0$ of $\mathcal{A}$, *an ordinary tiling by squares of size $v \times v$.*

(30.196) Staring us in the face also from Figure 16 is a magical `stairway to heaven`[22] which is a concomitant of $c < \infty$! The curves suggest points of constant height on the boundary of a cone of one dimension more, which it is natural to put inside the cone over the ball B^{n+1} with the same centre and radius, for, the linear reflections of the cone over B^n not only extend to it, with their rotations they give all of them ... till finally we are in the ball B^∞ of radius $c < \infty$, where this stairway ends, because, we can shift each guest in an infinite hotel to the next room, and put a new arrival in the first.

[20] Ditto if we divide $S^1 \times S^1$ by the involution which switches its factors, so, *a möbius strip is the space of quadratic homogenous equations $ax^2 + bxy + cy^2 = 0$ over $\mathbb{R}$ with $b^2 - 4ac \geq 0$* :- the `quadratic formula` tells us this condition is necessary and sufficient for factorization over $\mathbb{R}$, and $S^1 = \mathbb{R} \cup \infty$. □ Attaching the remaining equations with complex conjugate roots, an open 2-disk, then completes *the manifold $\mathbb{R}P^2$ of all real quadratic equations,* but a like dissection of $\mathbb{R}P^n$ for $n \geq 3$ is more involved. Unlike the circle, `the symmetric powers of a 2-manifold are manifolds` :- a suspect link is the join of a sphere and a circle divided by its antipodal action, but this is a circle too. □ This implies, *the multiplication of n linear equations in x and y over $\mathbb{C}$, an injective map from the nth symmetric power of $S^2 = \mathbb{C} \cup \infty$ to the manifold $\mathbb{C}P^n$ of all degree n equations, is surjective,* that is, `the fundamental theorem of algebra!` The relativistic analogues of this bijection, for 2-manifolds $M^2 = B^2/\Gamma$ – the inverse map is how *'Poincaré had solved any polynomial equation by using automorphic functions'* – are once again 'well-known', but not to me! These old memories had resurfaced after a conversation with Keerti about three months ago; the symmetric powers of 2-manifolds also figure in some attractive *'numerologies'* about particles, going back to Majorana, in which, for example, a beautiful recent paper of Atiyah and Manton also indulges.

[21] Despite the fact that, *there is no cayley isometry of the cone other than these compositions*, and, *all four sides of these tilings do have the same cayley length.* My plan–see page 3–was to go over cartesian motions, cayley distance and segments rather quickly in (30.1), (30.2) and (30.3), but things went truly for a toss after the advent of the aliens in (30.16)! Fearing that it might be some time before I return to these topics, let me remind you that the factor $\frac{c}{2}$ in cayley distance made its classical limit the euclidean distance on $t = 1$, but, because of it, *the cayley distance between distinct times blows up as $c \to \infty$.* Again, the points at a constant cayley distance can be funny, for example, *the inscribed cayley circle of our square tile touches its boundary in segments*; but, *distinct cayley circles of a disk intersect in at most two points,* for this distance is equivalent to its `conformal metric`, which has genuine but eccentric circles. Anyway, from the square tiling of the 2-cone we can make trivalent `bricklaying patterns`, and it is not hard to calculate the cayley diameter of a star of a vertex, so the fourth proof in ਉਂਗਲੀਆਂ ਅਤੇ ਟਾਇਲਾਂ (2015) still works. There is also a `cayley-invariant volume`, and one can probably find for the 2-cone also, *all quadrilaterals for which this area is equal to the product of the average cayley lengths of the opposite sides,* etc.

[22] My name for what is usually *Poincaré extension,* for example, in Beardon's book.

(30.197) So these ball geometries arise plainly, one after the other, from a mere segment $[0, c)$. Then, in Note 23, had dawned on us the realization that, *associated to any cartesian motion there is a canonical* `partition` $\mathcal{F}$ *of the ball* B^n *into path connected subsets* M *that are* `topologically homogenous`.[23] Any M inherits its charts from the motion itself, and if `compact` we had deemed it to be `cartesian matter` provided it is `persistent`, i.e., appears also in any perturbation of the cartesian motion.

(30.198) *Calculus, more precisely lipschitz calculus, is also a child of* $c < \infty$. If n is big enough, or even not four, any cartesian motion can be perturbed over a compact time interval by an arbitrarily small amount to one whose homeomorphisms are lipschitz. The proof needs `a cayley lattice` *of the* $(n+1)$*-cone with quotient a closed parallelizable manifold.* Given this scaffolding, it is like simplicial approximation, except that it is nearby homeomorphisms, not just maps that are sought. *For* $c = \infty$ *any two translations commute,* so n of these with a homothety give one, with quotient an $(n+1)$-dimensional torus, but now that all-important chord condition on the flow lines of the cartesian motion is not available, so the results are different : *it is only for* $c < \infty$ *that cartesian matter is necessarily lipschitz-smooth.*

(30.199) So, just from a segment, a whole world, see Note 28, has sprung up, *but what about fractional dimensions?* It is topological homogeneity that is the basic feature of manifolds, the other, that they should be locally euclidean, is our natural desire to not stray too far from home. There seems no reason why all cartesian matter should be locally euclidean, but about path connected homogenous fractals, I know very little; however the Bing-Borsuk conjecture, see Note 25, suggests there may be surprises. Also: *what dimension is best?* In the lipschitz context we saw that `hausdorff dimension` is natural, but there are certainly other candidates.[24]

We have stayed at home, yet in these n-balls have popped up naturally it seems all manifolds and some other path connected and homogenous compacta. This cartesian matter can be examined, if the birthing motion was smooth enough, using only the elementary tools of the calculus of several variables. Besides we have the option $n = \infty$ to sidestep or delay `knotty questions`, still without giving up the creature comforts of home.[25]

Coming back to that cayley lattice, the natural idea, 'lets make c an integer and search over $\mathbb{Z}$', needs to be finessed, as Fricke pointed out long ago. Due to their quadratic nature the hyperboloids may not have enough integral points, so

[23]For, once again, these `minimal sets` M are the equivalence classes generated by the *binary relation* R on the ball B^n defined by xRy iff x and y are on the same `orbit`, i.e., iff they are the projections of two points on the same flow line, say, at absolute times τ_1 and τ_2, but then the homeomorphism $\phi_{\tau_1\tau_2}$ of B^n given by the motion throws x on y, and besides, it maps each equivalence class to itself, so it preserves any M and its complement $B^n \setminus M$. □

[24]More general and natural might be `von Neumann dimension` : from the defining relation R of $\mathcal{F}$, one should make nice $\mathbb{C}^*$-algebras with involution given maybe by reversal. Also, this reminds me of the `smoothing operators` *for closed foliated manifolds* $(M, \mathcal{F})$ that I had played around a lot with in the 1970s, but it was Connes and Skandalis, a bit later, who had got a full-blown index formula for foliations by using such a dimension.

[25]For an example of these comforts, see *From calculus to cyclic cohomology* (1995).

we search quadratic extensions, and this suffices to find a lattice with quotient a closed manifold; but to get parallelizability the search was extended into the étale or grainy nature of this homotopy type. Maybe this can be avoided, but it would be even nicer if we could understand this 'grainy nature' with the same cartesian clarity. Which brings me to the last note of this year.

(30.199...) The dots indicate I'm keeping my options open in case I want to add another tid-bit before (30.2), but you might object: *dots mean* 9 *recurring from school and that this is* (30.2) *already,* written in a long-winded way. *Cantor opened a wonderful world by simply giving up this school dictat* ! Even with 9 recurring, it was now not (30.2) for him, or even something lesser than it by an infinitesimal, it was just an infinite sequence (with one point) of ten things, and on all such sequences – a power of the cardinality ten set – we have only the product topology. *What can one do with this mere dust? Almost anything!* is the short answer : *not just the real line, all interesting spaces you can think of, for example, all manifolds, are but quotient spaces of this dust.* Besides, it is not hard to show this, and that is precisely the rub, the bewildering number of nice ways[26] in which we can lift reality to this combinatorial dust. Though various criteria have guided what seems 'best' it would be fair to say that what followed from *Hensel's discovery of those wonderful fields in such dust* is so far the clear winner. From (30.196) and (30.197) we know how, starting with a mere segment, we can bring into being naturally manifolds and all, so we ask, *is there an equally natural* `cantorian P G & R` *of which this is only a functorial quotient?* Many possibilities come to mind, but when it comes to reading the mind of God almost, it is best to proceed slowly and in humility ...

K S Sarkaria (*contd.*)

[26]See, for example *"Amazing curves!"* (2010); to all those motifs has recently been added the above mural *"P G & R"* on a verandah roof. December 31, 2016.