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# A Topological Paradox of Motion

*A basic premise of the mechanics of continuous matter, and one invariably found stated in some form in all books of hydrodynamics—often even before the equations of motion are derived—is the hypothesis of continuity, that is, the doctrine, going back to Anaxagoras, that matter and motion are continu-*ous. This is interpreted mathematically as implying at least that, at each time $t$, there is a "particle" (of course hypothetical; we are ignoring the actual molecular nature of matter and are talking say of the *spermata* of antiquity!) at every point of a region $R_t$ of 3-dimensional space $\mathbb{R}^3$, and that, following the motion of these particles, one gets continuous surjections $m_{s,t}:R_t \rightarrow R_s$, $t < s$, varying continuously with $t$ and $s$, and obeying $m_{u,s} \circ m_{s,t} = m_{u,t}$. Though apparently quite reasonable, this implies some funny things, including the following:

*One cannot completely empty a tyre-tube filled with water into a bucket in any finite length of time.*

For this would mean we could continuously deform any homotopically nontrivial loop $C_1$, of the space $X \subset \mathbb{R}^3$ occupied by the apparatus of Figure 1, within $X$, to a trivial loop $C_2$ of $X$. Here, the existence of a non-trivial $C_1$ is ensured by the fact that $X$, which is homeomorphic to the region $R_0$ initially occupied by the fluid, has the homotopy type of a circle, so its fundamental group is $\mathbb{Z}$, and the triviality of $C_2$ follows because the bottom part of $X$ is contractible.

The above notwithstanding, it is customary in all books of hydrodynamics to assume even more: that there is always a *unique* fluid particle at each point of $R_s$, thus it is understood that motion occurs via homeomorphisms $m_{s,t}$. So even the topological type of $R_t$ cannot change with time.

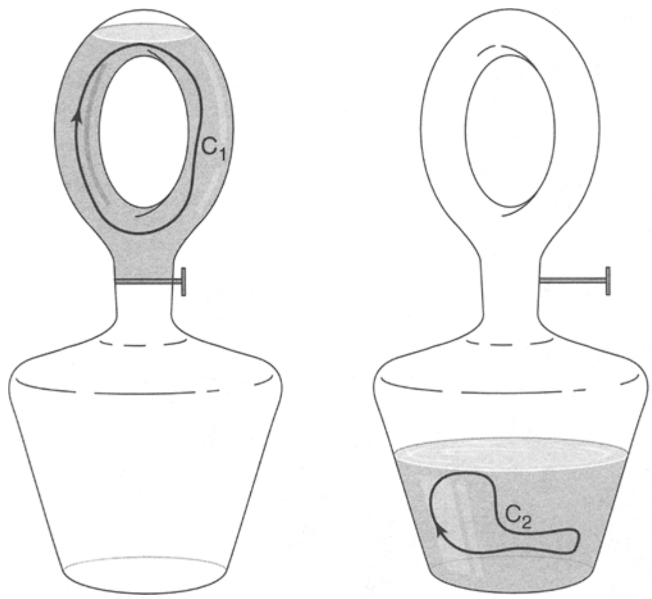


**Figure 1**

This implies, conversely, that a bucketful of water cannot be transferred completely into a tyre-tube of the same volume in any finite length of time.

I emphasize that the above is not a refutation of the science of hydrodynamics but a vivid reminder that the boundaries of its domain of applicability are encountered even in simple everyday situations. The topological contradiction should alert us to the fact that there is something amiss in the mathematical model used, as indeed there is: As soon as local forces in excess of the cohesive limits of the fluid appear near the upper inner portion of Figure 1, the hypothesis of continuity is inapplicable to the flow in that region.

The doctrine of Anaxagoras was very much in keeping with the spirit of his time. After the resolution of the Pythagorean conundrum by means of irrationalities, physical *space* was almost universally regarded as continuous; then, to resolve the well-known paradox of Zeno regarding Achilles and the tortoise, it became necessary to give up the notion of finitely many *moments* between any two events, and *time*, too, came to be regarded as continuous.

However, Democritus, a contemporary of Anaxagoras, was of the view that matter, unlike space, is discrete. Four centuries later, it was this *atomic hypothesis* which was championed by the Roman poet Lucretius, who claimed—see [4], p. 14—that motion would become impossible if we were to believe with Anaxagoras that all of space is full of matter:

There's place intangible, a void and room.
  For were it not, things could in nowise move;
Since body's property to block and check
  Would work on all and at all times the same.
Thus naught could evermore push forth and go.
  Since naught elsewhere would yield a starting place.

Here Lucretius seems to overlook the possibility of rotational motion, i.e., of *vortices*, which (much later) became all the rage with René Descartes, and briefly again in the nineteenth century when Lord Kelvin (William Thompson) made a beautiful attempt to understand atoms via vortices. For more on this, the reader can probably do no better than start with James Clerk Maxwell [5].

Ever since John Dalton and Robert Brown there has been abundant microscopic evidence which favours the atomic hypothesis. Nonetheless, it is contended in all books of fluid mechanics that, for macroscopic purposes, one can still safely assume the hypothesis of continuity. As shown above, one has not only microscopic evidence, but *a priori* arguments from topology (i.e., the mathematics of continuity) which show that even a weakened hypothesis of continuity is untenable, so that matter and motion cannot both be assumed continuous.

Even a gas, confined to the lower bulbous part of $X$ with the top evacuated, would change its topology after the stopcock is opened, which should suggest, independent of any other evidence, that its matter is probably discrete. This, of course, is what the *kinetic theory* assumes, and the equations of motion of hydrodynamics are, as is well known, statistical averages of the Boltzmann transport equation—see, for example, Desloge [3]. But for the case of liquids (as against gases) this approach runs into some unresolved difficulties—see, for example, Batchelor [1]. So, following Jean-Claude St. Venant and George Stokes, it is convenient to invoke the hypothesis of continuity. Unless the approximate nature of this assumption is emphasized, however, this runs the risk of making the equations of hydrodynamics appear more basic than they possibly can be. We recall that Daniel Bernoulli, Claude Navier, Siméon Poisson, and Augustin-Louis Cauchy, all, had sought to understand hydrodynamics starting from various atomic hypotheses. These original attempts need to be perfected, because a natural understanding of turbulence will probably be found only in such statistical foundations.

## Matter and motion cannot both be assumed continuous.

For very small values of time, the flow of Figure 1 does obey the hypothesis of continuity, and the fluid region $R_t$ retains the topology of a solid torus; however, its geometry, which depends on the nature of the fluid and the boundary conditions, changes rapidly, with $R_t$ becoming thinner and thinner at the top (and for a creeping flow, say of treacle, it seems to tend towards a well-defined limiting position). But at the moment when the thin $R_t$ breaks, the continuity hypothesis becomes invalid, and the flow is no longer governed by hydrodynamics.

Similarly, in a swift stream going past an obstacle, water contained in neighbourhoods of homotopically non-trivial loops and surfaces encircling the obstacle, is probably being swept entirely past the obstacle, and so must be breaking up topologically. This failure of the hypothesis of continuity, which we suspect is usually over an open subset, implies that some flows cannot be modelled by *any* smooth velocity vector field, not even a generic one having all sorts of strange attractors. In particular, Jean Leray has suggested the Navier-Stokes equation is probably inadequate for modelling turbulence.

Sometimes matter is assumed to be continuous, but its velocity field is allowed to be discontinuous. An analysis of some such arguments is given in Birkhoff's classic *Hydrodynamics* [2]. For example, in aerofoil theory, one gets around the D'Alembert paradox by guessing a suitable flow topology: wake, dividing stream line, etc. Despite their successes, such *ad hoc* devices can obviously not be deemed to be physical explanations of these phenomena.

I observe next that the "opposing doctrines of the plenum and atom"—as Maxwell [5] calls them—can actually be reconciled with one another, if one believes that space is discrete, and more generally, that *all* physical notions are discrete. From this viewpoint, which is roughly like that of Gottfried Leibniz's *Monadology* (1714), Zeno's paradox arose only because physical space was confused with geometrical space, and, to cover this initial "lie," it became necessary to invent more; for example, that time is continuous. The discrete *monads* are not "in" anything—there being no empty space or vacuum—they are by themselves, forming a discrete plenum! If one wants to proscribe action at a distance, more structure is needed; for example, one may postulate that each monad acts only via some others that are contiguous with it. It is this extra structure, true, or mistakenly imposed by us on reality, which makes it appear continuous: contiguity gives us a simplicial complex, thus a continuous space. In this view, physical motion is only a sequence of monadic permutations, not an arbitrary flow on this geometrical space, and may be bound by some *quantization* rules—say, limitation to those locally irrotational flows whose circulations are integral multiples of Planck's constant. In other words, we are but "hearing" some discrete aspects of the topology—integral homology—of tiny portions of this monadic simplicial complex, via the quantum-mechanical observables of microsystems!

I conclude by recalling that, even more than continuity, the essential doctrine of Anaxagoras was *homoeomaria*, that is, that a part is like the whole, or, as someone raised on fractal graphics would now put it, self-similarity. In analogy with this, Leibniz required that each monad of his discrete plenum be a replica of the entire universe! This property was however dictated more by his teleological predilections and earlier work on ethics, parodied memorably as the absurd Dr. Pangloss of Voltaire's *Candide*.


**REFERENCES**

[1] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge (1970).

[2] G. Birkhoff, *Hydrodynamics: A Study in Logic, Fact, and Similitude*, Princeton University Press, Princeton (1960).

[3] E. A. Desloge, *Statistical Physics*, Holt, Rinehart and Winston, New York (1966).

[4] Lucretius, translator W. E. Leonard, *Of the Nature of Things = De Rerum Natura*, 58 B.C., Dutton, New York (1950).

[5] J. C. Maxwell, *Molecules*, and *Atom*, in, "The Scientific Papers of James Clerk Maxwell", Vol. II, 1890, pp. 361–378, and 445–484, Dover reprint, New York (1952).


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