Contents of "Reviews"

There is a **part V** also going up to at least page 165 (but page 161is missing) which concludes review of Wu's book, then takes up Fadell-Neuwirth, then Segal's paper on configuration spaces ...

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REVIEW OF MATERIALS REQUIRED FOR "VAN KAMPEN OBSTRUCTIONS"

SYLVESTER'S THEOREM

A homogenous second degree real polynomial in n variables $X = [x_1, \ldots, x_n]$, or equivalently XAX' where A is a real $n \times n$ symmetric matrix, can always be changed to one of following $n(n+3)/2$ **canonical forms**

$$x_1^2+x_2^2+\ldots+x_p^2-x_{p+1}^2-\ldots-x_r^2, \quad 0 \leq p \leq r \leq n,$$

by means of a suitable non-singular linear substitution X=YP, e.g one obtained via the following

ALGORITHM. In case all diagonal terms of A are zero, but say the equal terms at the (i,j) and (j,i) spots are not, then add jth column and row to the ith column and row, to make double this amount, a nonzero number, appear at (i,i).

Using this, make rest of ith row and column zero. Repeat with smaller matrix obtained by omitting ith row and column. Once new A is diagonal, some interchanges of rows-column pairs, and multiplications of row-column pairs with suitable nonzero scalars, finishes the job.

No two of the above canonical forms are congruent to each other.

If possible suppose that the (p,r) canonical form in the x's becomes a (p',r') canonical form in the y's under a nonsingular linear substitution Y=XP.

Consider first the last n-p' of these substitutional equations Y = XP with the left sides, i.e. the last n-p' y's, replaced by zeros, and, on the right sides, the first p x's replaced by zeros. *If* $p < p'$, the number n-p of the remaining x's is more than the number n-p' of these equations, and we can find a nontrivial solution for these n-p x's.

Now, calculate the first p' y's, using these values of the x's, from the first p' of the equations Y=PX.

We have thus obtained two n-tuples of numbers X_0 and Y_0, related by $Y_0=X_0P$, with the first p coordinates of X_0 all zero and the remaining not all zero, while the last n-p' of the coordinates of Y_0 are all zero.

This is impossible, because the value of the original quadratic form at X_0 is negative, and could not have become non-negative, after this linear substitution, at the corresponding point Y_0.

Thus $p \geq p'$, likewise $p' \leq p$, etc. The invariant r coincides with the rank of A, and p-(r-p)=2p-r is called its **signature**.

Comments

(1) **Skewsymmetric forms**, i.e. skewsymmetric homogenous degree 2 polynomials in n variables, likewise correspond to skewsymmetric n by n matrices.

By a similar algorithm using symmetrical elementary transformations on rows and columns, we now reach one of the following $[n/2]+1$ canonical forms,

$$(x_1x_2-x_2x_1)+\ldots+(x_{2s-1}x_{2s}-x_{2s}x_{2s-1}), \quad 0 \le 2s \le n,$$

of which no two are congruent. Here 2s coincides with the rank r of A, which is thus now necessarily even.

(2) **Bilinear forms** in two sets of indeterminates $X=[x_1, x_2, \ldots, x_n]$ and $Y=[y_1, y_2, \ldots, y_m]$, are real linear combinations of the products $x_i y_j$ with coefficients a_{ij}. Note that any such bilinear form can be written as XAY' where A is the n by m matrix of coefficients.

In case n=m and X=Y, then a bilinear form can be written uniquely as the sum of a symmetric (=quadratic) form and a skewsymmetric form by putting

$$a_{ij}x_ix_j+a_{ji}x_jx_i = 1/2(a_{ij}+a_{ji})(x_ix_j+x_jx_i) + 1/2(a_{ij}-a_{ji})(x_ix_j-x_jx_i)$$

This corresponds to writing $A = 1/2(A+A')+1/2(A-A')$.

But note that even for a bilinear form with n=m and X=Y, it may not be possible to simultaneously reduce both its symmetric and skewsymmetric parts to their canonical forms.

In the set of all bilinear forms it is natural to consider the weaker relation of **equivalence**, i.e. allow all non-singular pairs of linear substitutions X=PW and Y=QZ, with no condition between P and Q.

A similar algorithm now leads to one of the following min(n,m)+1 canonical forms

$$x_1y_1+\ldots+x_ry_r, \quad 0 \le r \le n,$$

with no two of these equivalent to each other, where r = rank(A).

WITTEN'S MORSE THEORY

The following remarks pertain only to a *very small* part of this **** paper of 1982, which is difficult but potentially understandable, and also very important :—

For purposes of calculating the Betti numbers of a closed smooth manifold M, we can obviously replace the de Rham derivative d by

$$d_t = e^{-ht} \circ d \circ e^{ht},$$

where $h: M \longrightarrow \mathbb{R}$ is a smooth function, and $t \in \mathbb{R}$. This **perturbation** is useful because, for $t > 0$, or at least for t large, the spectrum of the **Hamiltonian** (or Laplacian),

$$H_t = d_t \circ d_t^* + d_t^* \circ d_t,$$

is easier to understand, provided h is "good".

For example, if h has only a finite number of non-degenerate critical points p, where the quadratic form approximating h has eigenvalues λ_i, then in the vicinity of each p, H_t is "well-approximated" by the sum of dim(M) one-dimensional **harmonic oscillators**, each with potential $t^2\lambda_i^2$ — so with eigenvalues $t|\lambda_i|$ times an odd integer ≥ 1 — plus dim(M) scalar operators having eigenvalues $+t\lambda_i$ or $-t\lambda_i$.

Using this (see **Henniart** for more details) Witten deduces that the number of zero eigenvalues of H_t, $t > 0$, in dimension r, is at most equal to the number of critical points of index r, i.e. he obtains a revealing analytical proof of the

Morse inequalities. *If smooth function* $h: M \longrightarrow \mathbb{R}$ *has only isolated non-degenerate critical points, then the number of such points of index r is at least equal to the r-th Betti number of the compact manifold* M.

Comments

(1) Witten's paper is a good entré into *physics* : see also **Atiyah** in IHES # 68.

(2) IHES # 68 also contains a chatty talk by **Bott** which is useful to further understand Witten.

(3) Witten in fact gets also the sharp Morse inequalities — i.e. the *Smale-Thom chain complex* of **Milnor**'s book on h-cobordism — by these analytical means, *and much much more*: e.g. the degenerate case, when the critical points of h are submanifolds, is also covered. Then, by a somewhat different perturbation of d, he obtains Hopf-type formulae involving fixed points of vector fields, infinitesimal isometries etc. Finally he considers similar problems for some infinite-dimensional manifolds.

(4) **Laumon** in IHES # 65, in a paper which is un-understandable to us, uses a p-adic version of Witten's perturbation trick to simplify part of Deligne's paper on Weil conjectures

BANCHOFF'S MORSE THEORY

This paper of 1967 is easy, clear and important.

Consider any **linearly embedded cell complex** $K \subseteq \mathbb{R}^N$ and a linear map $h: \mathbb{R}^N \longrightarrow \mathbb{R}$ which assigns distinct values to adjacent vertices. It is easy to see that such maps form an open dense subset U_K of the dual linear space, which, by using the usual metric $\langle \ , \ \rangle$, will be identified with $\mathbb{R}^N$ itself.

We put $A(v,\sigma,h) = 0$, unless v is the vertex at which the maximum of $h|\sigma$ occurs, when we set $A(v,\sigma,h) = 1$. The **index** of h at a vertex v is now defined by

$$\operatorname{ind}(v,h) = \sum_{\sigma \in K} (-1)^{\dim \sigma} A(v,\sigma,h).$$

Since multiplying h by a positive scalar does not alter these indices, we can confine ourselves to linear maps h lying in the dense open subset $u_K = U_K \cap S^{N-1}$ of the unit sphere of $\mathbb{R}^N$.

Critical Point Theorem. *If the linear map* $h: \mathbb{R}^N \longrightarrow \mathbb{R}$ *distinguishes adjacent vertices of the embedded cell complex* $K \subseteq \mathbb{R}^N$, *then the Euler characteristic of* K *is given by*

$$X(K) = \sum_v \operatorname{ind}(v,h).$$

The above follows by a simple computation. Next Banchoff defines the **curvature**, of the linearly embedded cell complex K, at its vertex v, by averaging the index of v over all h, i.e. he sets

$$\operatorname{curv}(v) = \int_{h \in u_K} \operatorname{ind}(v,h)\, dS^{N-1},$$

where dS^{N-1} denotes the usual normalized Lebesgue measure of the unit sphere. So last result implies the

Gauss-Bonnet Formula. *The Euler characteristic of an embedded cell complex equals its* **total curvature**, *i.e.*

$$X(K) = \sum_v \operatorname{curv}(v).$$

The following integral, which involves the choice of some euclidean structure on A^σ, the affine span of a closed cell σ, is easily seen to be independent of the choice of this structure, and is now observed by Banchoff to be an appropriate definition of the **exterior angle** of the cell σ at its vertex v.

$$\mathrm{ang}(v,\sigma) = \int_{h\in u_\sigma} A(v,\sigma,h)\, dS^\sigma.$$

As before, the integration is over all those unit vectors h for which the linear map $\langle h,..\rangle: A^\sigma \longrightarrow \mathbb{R}$ separates the vertices of σ. If v is not a vertex of σ then we'll use $\mathrm{ang}(v,\sigma) = 0$.

Theorema Egregium. *The curvature* curv(v), *of a linearly embedded cell complex* $K \subset \mathbb{R}^N$ *at its vertex* v, *satisfies*

$$\mathrm{curv}(v) = \sum_{\sigma\in K} (-1)^{\dim\sigma}\, \mathrm{ang}(v,\sigma),$$

and is thus independent of the embedding.

I.e. the defining integral for $\mathrm{ang}(v,\sigma)$ can be replaced by an analogous one over $U_K \subseteq S^{N-1}$.

Comments.

(1) The beauty of the above is that it applies to *all* cell complexes. In fact Banchoff also gives, on pp.254-255, a variant of the theory which, at the expense of slightly more involved definitions, even dispenses with the requirement that the linear map h separates adjacent vertices.

(2) When K is an n-manifold, then ind(v,h) is $(-1)^n$ times ind(v,-h), from which Banchoff gets another proof of the well known fact that odd dimensional closed manifolds have zero euler characteristic.

(3) The inspiration for Banchoff was **Kuiper** who had formulated the much more subtle smooth critical point theory into the above format.

KALAI'S ENUMERATION OF ACYCLIC COMPLEXES

Theorem. *Let* $\mathscr{C}$ *denote the class of all* k*-dimensional simplicial complexes on* N *vertices containing the complete* (k-1)*-skeleton which have trivial (reduced) homology over* $\mathbb{Q}$. *Then*

$$\sum_{C\in\mathscr{C}} |H_{k-1}(C)|^2 = N^{\binom{N-2}{k}},$$

where $H_{k-1}(C)$ *denotes the, necessarily finite,* (k-1)*th integral homology group of* C.

For k = 1 we get, because $H_0(C) = 0$ has cardinality 1, the formula of **Cayley** for the number T(N) of trees on N vertices, viz.

$$T(N) = N^{N-2}.$$

Kalai mentions that his more general theorem resulted by analyzing and generalizing the matrix proof of above given in **Biggs** and **Moon**.

Sketch of proof. Let K denote the complete k-dimensional complex on the given N vertices. We assign some total order to the vertices, using this we assign incidence numbers, and shall be interested in the top-most **incidence matrix** I whose rows are indexed by (k-1)-simplices and columns by k-simplices.

The columns of I represent the (k-1)-dimensional elementary coboundaries of K, and it is clear that those stemming from k-simplices containing the first vertex form an integral basis of the column space of I. Moreover, the square submatrix of I determined by these columns, and the rows stemming from (k-1)-simplices not containing the first vertex, is evidently a square matrix of size $\binom{N-1}{k}$ and determinant ± 1.

More generally, any of our $\mathbb{Q}$-acyclic C's is determined by a choice of its $\binom{N-1}{k}$ k-dimensional simplices, and the square matrix of this size, determined by the same rows as above, and these columns, has determinant equal to the cardinality of the finite group $H_{k-1}(C)$.

Let I_r be the submatrix of I determined by the aforementioned rows, i.e. those corresponding to (k-1)-simplices not containing the first vertex. By the **Cauchy-Binet theorem**, the determinant of the bigger square matrix $I_r \cdot (I_r)'$ equals the sum of the squares of the determinants of all these smaller square matrices.

Using this, and the fact that $I_r \cdot (I_r)' = N^{\binom{N-2}{k}}$, gives Kalai the aforementioned formula.■

The binomial duality $\binom{N-2}{k} = \binom{N-2}{N-k-2}$ prompts Kalai to also consider, for any complex C on the given set of vertices, the **dual simplicial complex** defined by

$$C^* = \{ \theta : \theta^c \notin C \}.$$

Besides $C^{**} = C$ etc., he notes also that **Alexander duality** shows that C is $\mathbb{Q}$-acyclic iff C^* is, and that then the finite groups $H_{k-1}(C)$ and $H_{N-k-3}(C^*)$ are isomorphic.

He ends by explicitly enumerating the $\mathbb{Q}$-acyclic 2-complexes on 6 vertices, and is especially intrigued by the subclass of $\mathbb{Q}$-acyclic **self-dual complexes**, i.e. those for which

$$C^* = C.$$

He gives the numbers of the $\mathbb{R}P^2_6$'s and other $\mathbb{Q}$-acyclic self-dual 2-complexes which occur on 6 vertices, and poses the problem of finding an enumerating formula for these in general. He refers to **Tutte** for some such formula for the k = 1 case.

Comments.

(1) Kalai also gives a formula for the subclass of $\mathscr{C}$ for which one has prescribed numbers of k-simplices incident to each of the N vertices. Also, using above formula, he checks that "most" $\mathbb{Q}$-acyclic complexes are not $\mathbb{Z}$-acyclic.

(2) For other aspects of self-dual complexes see also papers of **Sarkaria**, **Schild** and **Bier**, as well as of **Brehm** and **Kuhnel**.

THOM'S DIFFERENTIAL FORMS

In his 1975 paper **Swan** gives this 1957 construction of Thom:

Consider the standard geometrical realization $K \subset \mathbb{R}^N$ where N = number of vertices of K. More to the point, consider the associated union $\mathcal{K}$ of the affine planes A_σ determined by the simplices σ of K.

A *differential form on* K will then mean a collection of forms ω_σ on these A_σ such that we have

$$\omega_\sigma | A_\theta = \omega_\theta \text{ whenever } \theta \subseteq \sigma.$$

Here, Thom assumed that each ω_σ is *smooth* and was working over $\mathbb{R}$, while Swan works over $\mathbb{Q}$ and only assumes that these forms have *polynomials* over $\mathbb{Q}$ as coefficients when we look at them in the obvious local coordinates.

Theorem. *Under exterior product, and the de Rham derivative* d, *the aforementioned differential forms on* K *constitute a graded commutative differential algebra, whose cohomology ring coincides with that of* K *over* $\mathbb{R}$ *or* $\mathbb{Q}$ *as the case may be.*

Comments

(1) A different construction of a similar DGA over $\mathbb{Q}$ was given in the interim also by **Quillen**. Thom was to some extent anticipated by **Whitney** whose book inspired **Sullivan** too to above result, and also to much much more.

(2) Swan in fact works more generally and defines *forms on a semi-simplicial complex*,

$$\mathcal{K}: \mathcal{N}umbers \longrightarrow \mathcal{S}ets,$$

as all "natural transformations" (i.e. functions $\mathcal{K}(\mathbf{p}) \longrightarrow \Omega(\mathbf{p})$ obeying obvious commutativity conditions) from this contravariant functor to

$$\Omega: \mathcal{N}umbers \longrightarrow \mathbb{Q}\text{-}\mathcal{M}odules,$$

which associates to a natural number **p** the totality of rational forms on the affine plane $t_0+t_1+ \ldots +t_p = 1$.

(3) Swan mentions as well known the fact that the above theorem does not extend to fields of nonzero characteristics.

(4) It seems that the ordinary simplicial coboundary — which obeys the formula $\omega \longmapsto$ (sum of vertices)$\wedge\omega$ — is the **symbol** of the differential operator d. It should be interesting to formulate the above theorem in this form, i.e. one should also go back from the de Rham complex to this symbolic complex, also taking care of products, and prove that an isomorphism of cohomology rings is induced.

KATCHALSKI-PERLES INEQUALITY

The inequality. *If the nerve* $N(\mathcal{K})$ *of a family* $\mathcal{K}$ *of* t *convex subsets of* d*-dimensional euclidean space has dimension less than* d+r, *then its face numbers are bounded above as follows:*

$$f_{k-1}(N(\mathcal{K})) \leq \sum_{i=0}^{d} \binom{r}{k-i}\binom{t-r}{i} .$$

This inequality is sharp. In fact it is easy to spot an **arrangement of affine subspaces** $\mathcal{K}$ at which the above upper bound, which was conjectured by Katchalski and Perles, is attained.

The conjecture was established independently, for *all* complexes satisfying the conclusion of the following key result, by **Eckhoff** and **Kalai**.

Wegner's Theorem (Kalai's reformulation). $N(\mathcal{K})$ *retracts to a simplicial complex of dimension less than* d *via a sequence of operations in which one removes open stars of free faces of dimension* d-1.

Here by a *free face* we mean a simplex which is a proper face of a unique maximal simplex of the complex.

Idea of Kalai's proof. Arranging the convex sets $\{e_1, \ldots, e_t\}$ in order identify each simplex $\sigma \subseteq \{e_1, \ldots, e_t\}$ with the corresponding exterior monomial of the free **exterior algebra** Λ over $\mathbb{R}$ generated by these t variables.

Then the left side of the inequality equals the dimension of the vector subspace L_k of Λ spanned by all cardinality k simplices of $N(\mathcal{K})$.

On the other hand the right side interprets, with respect to *any* basis of Λ given by exterior monomials in $\{f_1, \ldots, f_t\}$, as the number of such degree k monomials which have at least k-d of their constituent f's in $\{f_1, \ldots, f_r\}$.

Thus the inequality would follow if we can choose these f_i's in such a way that L_k has trivial intersection with the vector subspace R_k of Λ consisting of all degree k elements which are annhilated when we take interior product with any degree k-d exterior monomial in $\{f_1, \ldots, f_r\}$.

This is so e.g. when the f's are related to the e's by an orthogonal t×t matrix for which all square submatrices, and thus also the exterior powers of these submatrices, are non-singular.

In fact if some non-trivial linear combination ω of degree k simplices of $N(\mathcal{K})$ were in R_k, then we would have, for each cardinality k-d subset T of $\{1,2, \ldots, r\}$, an equation $\langle e_\theta, f_T \llcorner \omega \rangle = 0$.

Kalai rules this out by choosing θ to be the first degree d free face, of a "Wegner collapsing" of $N(\mathcal{K})$, which occurs in ω, for then these equations interpret as saying that the columns of the (k-d)th exterior power of a submatrix are linearly dependent.

Comments.

(1) *There is some "duality" between convex polytopes and arrangements of affine subspaces which we don't understand fully.*

For instance, the "dual" of Wegner's theorem is the theorem — of **Steiner** and **Brugesser-Mani** — which tells us that the former can be *shelled*, and the "dual" of the Katchalski-Perles inequality is the *upper bound conjecture* of **Motzkin**, which was established by **McMullen**, and later generalized to all *Cohen-Macaulay complexes* (e.g. simplicial spheres) by **Stanley**.

(2) Kalai's proof was important because it led to **exterior shifting** — see reviews below — which enabled him to in fact establish Eckhoff's conjectured characterization of the f-vectors of such *d-collapsible complexes*. Still later, he was able to show that this characterization holds even for *d-Leray complexes*, a notion "dual" to that of Cohen-Macaulay complexes.

The "dual" characterization of f-vectors of simplicial polytopes was conjectured by McMullen and proved by Stanley. The McMullen-Stanley characterization remains open for f-vectors of simplicial spheres.

We note also that there is an obvious numerical duality between the inequalities of Eckhoff-Kalai and those of McMullen-Stanley.

(3) Regarding arrangements of affine subspaces, especially the topology of their complements, see also the **Goresky-MacPherson** book, and

Ziegler-Zivjaljevic 's, and other older, papers. Also see literature on *oriented matroids*: these in turn figure in characteristic classes for embeddability obstructions.

CATEGORY OF NATURAL NUMBERS

Associating to each ordered pair p, q of natural numbers the set of all monotone (i.e. non-decreasing) maps from $\mathbf{p} = \{0,1,..,p\}$ into $\mathbf{q} = \{0,1,..,q\}$ we get the category $\mathcal{N}umbers$.

A contravariant functor

$$\mathcal{K} : \mathcal{N}umbers \longrightarrow \mathcal{S}ets$$

is called a (complete) **semi-simplicial complex**.

Example 1. If K is a simplicial complex, let $\mathcal{K}(\mathbf{q})$ be the set of all sequences of vertices of length q+1 supported on simplices of K, and let $\mathcal{K}(\mathbf{p} \rightarrow \mathbf{q})$ map such a sequence $\mathbf{q} \longrightarrow \text{vert}(K)$ to the composite $\mathbf{p} \longrightarrow \mathbf{q} \longrightarrow \text{vert}(K)$. Note that K can be recovered from this semi-simplicial complex as the set of all supports of these sequences.

Example 2. In case vert(K) is equipped with some total order, one also has the smaller semi-simplicial complex $\mathcal{K}_o$ consisting of non-decreasing sequences of vertices supported on simplices of K.

Structure of a monotone map. Any non-decreasing map $\mu : \mathbf{p} \longrightarrow \mathbf{q}$ is determined by its **critical points** $\{c_1,...,c_s\} \subseteq \mathbf{p}$ (i.e. all c's such that $\mu(c) = \mu(c+1)$) and the subset $\{j_1,...,j_t\} \subseteq \mathbf{q}$ of **jumps** (i.e. j's not in the image of μ). In fact μ has the factorization

$$\mu = \mathbf{j}_t \circ \cdots \circ \mathbf{j}_1 \circ \mathbf{c}_1 \circ \cdots \circ \mathbf{c}_s ,$$

into injective maps $\mathbf{j}$ having a single jump, and surjective maps $\mathbf{c}$ having a single critical point.

It is of interest also to consider, on the category of numbers, functors into other categories. For example, each $\mathcal{K}$ determines, by taking the free abelian group C(S) generated by each set S, a **simplicial abelian group**, i.e. a contravariant functor

$$C(\mathcal{K}) : \mathcal{N}umbers \longrightarrow \mathcal{A}belian .$$

This generalizes the notion of a **chain complex**, for, amongst the maps of $\mathcal{A}belian$, which occur as integral linear combinations of maps $C(\mathcal{K})(\mu)$, $\mu \in \mathcal{N}umbers$, we have in particular the usual **boundary**

$$\partial : C(\mathcal{K})(\mathbf{p}) \longrightarrow C(\mathcal{K})(\mathbf{p-1}),$$

which occurs as an alternating sum of the images ∂_i of some $\mathbf{j}_i$'s.

Comments.

(1) Originally **Eilenberg-Zilber** had defined semi-simplicial complexes as $\mathbb{N}$-graded sets, equipped with some degree +1 and degree -1 operators, obeying a given list of axioms. Their definition was recast into the above neat form by **Kan**. Now these **face and degeneracy operators** of Eilenberg-Zilber coincide with the operators $\mathcal{K}(j)$ and $\mathcal{K}(c)$.

We prefer to say "semi-simplicial complex" instead of "simplicial set", because the latter will often mean a set of simplices.

(2) A **Kan complex** is a s.s.c. for which any n n-simplices (i.e. members of $\mathcal{K}(n)$) occur as n of the n+1 n-faces of some (n+1)-simplex iff ∂ coincides with some ∂_k on their sum.

(3) See **May** 's book for an exposition of the **combinatorial homotopy theory** of Kan complexes. In particular, for the *singular complex* of a topological space, Kan's combinatorial homotopy groups coincide with the usual ones.

BJORNER-KALAI INEQUALITIES

For any simplicial complex K, let $f_i(K)$ be the number of its i-dimensional simplices, and $\beta_i(K)$ the dimension of its ith reduced homology over some field $\mathbb{F}$.

This paper is concerned with relations, between these two sequences of numbers, which hold in general for any K. One such relation is provided by the well known **Euler-Poincare theorem** which tells us that the alternating sum $(f_0 - \beta_0) - (f_1 - \beta_1) + \ldots$ equals 1.

The authors discovered that the smaller alternating sums,

$$\chi_i(K) = (f_{i+1} - \beta_{i+1}) - (f_{i+2} - \beta_{i+2}) + \ldots, \quad i \geq 0,$$

are also not arbitrary.

Theorem. *For any simplicial complex* K, *one can find another simplicial complex* E *which has, for each* $i \geq 0$, $\chi_i + \beta_i$ *simplices in dimension* i, *of which precisely* β_i *are maximal simplices.*

Thus, by applying the *Kruskal-Katona theorem* — see below — to the simplicial complex obtained from E by deleting maximal simplices of dimensions less than i, one obtains the inequality

$$\partial_i(\chi_i + \beta_i) \leq \chi_{i-1},$$

for each $i \geq 1$.

Proof. By **Kalai's theorem** there is a *shifted simplicial complex* $\Delta(K)$ having the same face and Betti numbers as K. Let X be the link of its first vertex.

The required E is obtained from $\Delta(K)$ by deleting the open star of its first vertex, and is thus the disjoint union of X and β_i maximal simplices in each dimension i.

Since the cone C(X) of X over the first vertex has $f_r - \beta_r$ simplices in dimension r, it follows that

$$f_{r-1}(X) + f_r(X) = f_r - \beta_r .$$

Taking the alternating sum of these equations over all $r \geq i+1$ we see that $f_i(X) = \chi_i$. So $f_i(E) = \chi_i + \beta_i$. *q.e.d.*

Comments.

(1) The Kruskal-Theorem implies also that *there are in general no further relations between the sequences* f_i *and* β_i.

To see this use the available $\partial_i(\chi_i + \beta_i) \leq \chi_{i-1} + \beta_{i-1}$ and $\partial_i(\chi_i) \leq \chi_{i-1}$ to find a compressed complex E with face numbers $\chi_i + \beta_i$, having a compressed subcomplex X with face numbers χ_i. Adding the cone of X over a new vertex, we obtain a shifted simplicial complex Δ having face numbers f_i and Betti numbers β_i.

(2) *For a given f-vector the Betti numbers are maximum at the compressed complex having that f-vector.* This result of **Sarkaria** was obtained independently by Björner and Kalai as follows:

As in the above proof it suffices to maximize over shifted complexes Δ having the given f-vector. For Δ shifted we obviously have

$$\beta_i = f_i((\mathrm{St}_\Delta 1)^c) - f_{i+1}(\mathrm{St}_\Delta 1) .$$

But an argument implicit in a paper of **Frankl** shows that *the f-vector of the star of the first vertex is minimum when the shifted complex* Δ *is compressed*. (In fact this is equivalent to the Kruskal-Katona theorem). This implies the required result.

Sarkaria proved above result without recourse to shifting. His inductive argument uses homology exact sequences. Björner and Kalai later gave another proof of their inequalities using such an argument. Its advantage is that it applies to a much larger class than the class of simplicial complexes.

(3) *For a compressed complex one has* $\partial_i(\chi_i + \beta_i) = \chi_{i-1}$ *for all* $i \geq 0$. This follows by an easy calculation and shows that the Björner-Kalai

inequalities are sharp.

Any finite sequence of non-negative integers is the β-sequence of *some* complex, e.g. a bouquet of spheres. But, if we want to find a *smallest* such complex, then we can solve the above equations downwards to obtain all the χ_i's, and thus all the f_i's. It is easy to check that any complex having the given Betti numbers has at least as many simplices as the compressed complex with this f-vector.

(4) A **clutter** (or "anti-chain" or "Sperner system") on N vertices is a family of subsets without any proper inclusion relations. (E.g. the set of circuits of a matroid.)

A given sequence of non-negative integers is the β-sequence of some simplicial complex having $\le$ N+1 vertices iff it is the face vector of a clutter on N vertices.

The necessity follows because (with notation of above proof) $E \setminus X$ is a clutter on N vertices. Conversely given a clutter on N vertices we can close it to a simplicial complex E, and then cone the new part X over a new vertex, to get a complex having β-vector equal to the face vector of the clutter.

(5) The above observation is useful, because commencing with a classic result of **Sperner**, a complete classification of the f-vectors of clutters has been obtained by **Clements** and **Daykin-Godfrey-Hilton**. Thus all this applies to Betti sequences.

For example, for N even, the aforementioned theorem of Sperner says that any clutter on N points has at most $\binom{N}{N/2}$ members, with maximum attained only at the clutter of all cardinality N/2 subsets.

Using this it follows that *for simplicial complexes on* 2n+3 *vertices the sum of the Betti numbers is maximum at the irreducible* **Kuratowski complex** σ_n^{2n+2} .

A very similar result holds also when the number of vertices is odd. There is also given a more complicated result for the minimum of the sum of the Betti numbers of complexes having a given f-vector.

KRUSKAL-KATONA THEOREM

This ubiquitous and multi-faceted result is all-important and has spawned many analogues and generalizations. It is not quite clear what is (are) the most conceptual way(s) of looking at it. So we'll review it from many angles.

Numerical function $\partial_k(t)$. This is the number of (t-1)-dimensional simplice contained in the simplicial complex t_k generated by the t lexicographically first k-dimensional simplices with vertices in $\mathbb{N}$.

Theorem A. *Any simplicial complex having* t *simplices of dimension* k *has at least* $\partial_k(t)$ *simplices of dimension* k-1.

An inductive approach to this result, innaugarated by **Katona**, and polished by **Eckhoff** and **Wegner**, reduces it to the following result regarding the above numerical function, which is given explicitly by the **binomial formula**,

$$\partial_k(t) = \binom{a_{k+1}}{k} + \binom{a_k}{k-1} + \dots + \binom{a_i}{i-1},$$

where the a_j's are the natural numbers, decreasing with j, such that

$$t = \binom{a_{k+1}}{k+1} + \binom{a_k}{k} + \dots + \binom{a_i}{i}.$$

Theorem B. *The above numerical function obeys*

$$\partial_k(s+t) \le \max(\partial_k(s),t) + \partial_{k-1}(t).$$

Another (order-theoretic) approach focusses on **compressed complexes**, i.e. simplicial complexes T, on some totally ordered set (say $\mathbb{N}$), such that if $\sigma \in T$ and θ is lexicographically less than σ, then θ too should be in T. It is clear that there is at most one compressed complex with a given face vector. Now the Kruskal-Katona theorem reduces to the following guise.

Theorem B. *An integer sequence is the face vector of some simplicial complex iff it is the face vector of some compressed simplical complex.*

Combinatorial shifting. Given any simplicial complex K on $\mathbb{N}$ and numbers $i < j$, denote by $\Delta_{ij}(K)$ the simplicial complex of the same size obtained by replacing j by i whenever possible. This construction found many uses with **Erdos** and **Rado**.

If the transpositions (i j) generate the symmetric group of all permutations of the set of vertices of K, then, by doing the above operations Δ_{ij} in any order one gets a **shifted complex**, i.e. one which is closed with respect to the product partial orders on equicardinal subsets of $\mathbb{N}$.

The class of shifted complexes is bigger and more interesting than of compressed complex. But, as far as proving Kruskal-Katona is concerned, **Frankl** showed that one is still left with the job of establishing the following.

Theorem C. *Amongst shifted complexes on* $\mathbb{N}$ *having a given face vector, the compressed complex has the least number of simplices in* St(1).

Why not soup-up the above shifting process to a **compressing** process? **Bollobas-Leader** do this via some operations Δ_{IJ} , involving replacement of subset J by the lexicographically smaller I. These operations preserve closure under inclusion, and, when suitably iterated, lead to the compressed complex, thus yielding Theorem B directly.

Comments.

(1) Even the paternity of K-K is many faceted: apparently **Schutzenberger**, **Harper**, and probably some others, also independently obtained it.

(2) The inductive numerical approach extends the domain of validity of the K-K inequalities way beyond simplicial complexes. For more on these *abstract complexes* — of **Lefschetz** and others — see below. In fact *cubical complexes* were considered even by **Kruskal** himself.

Also, the homological analogue of K-K was proved in this vein by **Sarkaria**, and likewise **Björner** and **Kalai** extended their inequalities way beyond simplicial complexes using this approach.

(3) Combinatorial shifting can be replaced by Kalai's more elegant *exterior shifting*, but still, the aforementioned Frankl residue remains.

An analogy : inductive approach ⟷ exact sequence approach in **Morse theory** while shifting approach ⟷ **Witten** approach. However, in our setting, we don't know how to obtain explicit "models" for complexes other than simplicial ones.

(4) Besides the analogues and generalizations of K-K alluded to above, one also has
the older **Macaulay theorem** which characterizes face vectors of *commutative semi-simplicial complexes*,
the characterization of f-vectors of *antichains* (or "clutters" or "**Sperner** systems") obtained by **Daykin** et al.,
the **Clements-Lindstrom theorem** which subsumed both K-K (i.e. face characterization of the "fermionic", or "anti-commutative semi-simplicial", complexes) and its "bosonic" analogue, i.e. Macaulay's theorem,
the **Heawood-Sarkaria inequality** for least valences, and so chromatic numbers, of simplicial pseudomanifolds,
the **McMullen-Stanley** inequalities for convex simplicial polytopes, which in "VKO" will be extended to all simplicial spheres,
the **Eckhoff-Kalai inequalities** for nerves of arrangements of affine subspaces,
and *much, much more !!*

In fact Chapter V of VKO could also be called **"Kruskal-Katona Theory"**, but the title chosen, **"Heawood Inequalities"**, is more appropriate from the point of view of *embeddability*, the point of view taken by both **Kempe** and **Heawood**, who, much before K-K-etc., were already looking at some such inequalities.

Furthermore the title is justified because this embeddability viewpoint will be shown in "VKO" to yield a strengthened form of the deepest of these "K-K theorems", viz. the McMullen-Stanley inequalities.

HEAWOOD'S "MAP-COLOUR THEOREM"

This **** paper of 1889 will be the starting point of our Chapter V. We will review its contents, but instead of (geographical) "maps", "divisions", and number of their "contacts", we will speak of **graphs** (i.e. one-dimensional simplicial complexes), vertices, and their valence.

Theorem A. *If a graph with* N *vertices embeds in* S^2 *then the average valence of its vertices is at most* $6 - 12/N$.

Heawood checks only that *equality* holds provided one can come down to the complete graph on 4 vertices by a sequence of steps in which the star of a vertex is replaced by joining one of its neighbours, already joined to two other neighbours, to all other neighbours.

He recognizes also that this equality is equivalent to Euler's formula for a simplicial 2-sphere.

For the remaining "cases of degeneracy" he simply asserts that valence can be no higher than this number.

Theorem B. *More generally, if a graph with* N *vertices embeds in* M^2 *then the average valence of its vertices is at most* $6(1 + k/N)$ *where* k *depends only on the closed 2-manifold under consideration.*

Again the argument is as above and the troublesome "degenerate cases" get only a passing mention.

Theorem C. *Let* x *denote the smallest integer bigger than* $6(1 + k/N)$ *for all* N, *or, for surfaces with positive* k, *even such that* $6(1 + k/x+1) < x$. *Then* the *above graph has* **chromatic number** *at most equal to* x.

For k positive average valence is a decreasing function of x, so the weaker condition ensures that x is bigger than it provided N is bigger than x. This suffices to do the (by now) standard inductive argument.

Theorem D. *If a complete graph on* y *vertices embeds in* M^2 *then we must have* $(y-1) \leq 6(1 + k/y)$. *For the torus this is best possible.*

The first part follows at once from Th. B. For the second he exhibits a toral "map" with 7 "divisions" each in "contact" with any other.

E. Heawood asserts that in fact the inequality of Theorem B is best possible for all surfaces with $k \geq 1$ pointing out that argument given is complete

" — *apart from the verification figure, which we have indeed given only for the case of an anchor ring, but for more highly connected surfaces* *it will be observed that there are generally contacts enough and to spare for the above number of divisions each to touch each."*

F. Map colour theorem. *For* $k \geq 1$, *the numbers* x *and* y *coincide, and this or the next lowest integer is the chromatic number of the surface.*

This follows by putting together Theorems C and E. He also writes out the (now) famous **square root formula** for this number.

He treats now the problem in which some prescribed "counties", i.e. unions of some "divisions", are required to have the same color. Obviously this falls into the programme of finding chromatic numbers of 2-dimensional **pseudomanifolds** and we'll state the results in this language.

Theorem G. *If a graph embeds in a pseudomanifold obtained from* S^2 *by identifying some disjoint pairs of vertices, then its chromatic number is at most* 12, *and moreover this bound is best possible.*

The first part follows because the valence of each vertex (which, up in S^2 is a pair) is at most 12. The hard part is the second which he proves by giving the following example "obtained with much difficulty".

H. Example. *There is a* 24*-vertex planar graph whose vertices can be split into* 12 *pairs in such a way that any two pairs have neighbourly representatives.*

He suggests that the "curious problem" of finding all such graphs might be of interest.

I. Then, *"assuming the verification figure"*, he proves that if one allows each "county" to have at most r "divisions", then the chromatic number is 6r. And also, with a similar proviso, a square root formula is proved for the analogous problem for surfaces with $k \geq 1$.

J. Five colour theorem. *If a graph embeds in* S^2 *then its chromatic number is at most* 5.

The "map"argument given amounts to the (now) standard book proof: delete a 5-valent star, then join and contract 2 of its neighbours which were not already neighbours, apply induction, etc.

K. Critique of Kempe's proof. He accepts as valid the following points of Kempe's 1879 proof of the Four Color Theorem.

(K1) *In a (well) four coloured graph consider a connected component of the subgraph determined by any two colors* **a** *and* **b**. *The transposition of* **a** *and* **b** *in such an* **a-b region** *yields another four colouring.*

(K2) *If neighbours* 1,2,3,4,5 *have colours* **r,b,r,g,y** *respectively, with* 2,5 *in a* **b-y** *region, and* 2,4 *in a* **b-g** *region, then* 1,4 *cannot be in the same* **r-g** *region, and neither can* 3,5 *be in the same* **r-y** *region.*

So far so good. But now Kempe assumed that a transposition in 1's **r-g** region and in 3's r-y region will remove both the reds. Heawood shows by an example that either transposition can prevent the other from being of any avail and so Kempe's assumption was wrong.

Theorem L. *If a graph having all valences even embeds in* S^2 *then its chromatic number is at most three.*

Heawood ends his paper — which is only 7 pages long ! — by stating the above result, saying that its proof is "*not difficult, but it appears to shed no light*" on the 4 colour theorem.

Comments.

(1) The best way of doing the proofs of (A) and (B) is via *exact homology sequences*. This overcomes the bothersome "cases of degeneracy" neatly.

Using this method, **Sarkaria** generalized these square root chromatic bounds to complexes embedding in higher dimensional pseudomanifolds, but only upto codimension 2 valences.

(2) However generalization of (A) exists even upto the middle-dimensional valences. This key result will be proved in our book using **Van Kampen Theory**, and will in turn extend the **McMullen-Stanley** inequalities to all simplicial spheres.

In fact, Van Kampen Theory will allow us to establish, for *all* compact polyhedra, such chromatic upper bounds, in terms of the **minimal number of vertices** required to triangulate a polyhedron.

(3) The "verification figures" left out by Heawood are now all there — see the book of **Ringel** — so proof of (E), and thus of the Map Colour Theorem, is complete.

The above took quite a while, but **Brehm** opines that discovering these explicit *minimal simplicial 2-manifolds* (like discovering explicit transcendental numbers ?) took so long only because there were *too many* of them !!

So there might be hope still of giving a short account of the proof of (E) ?

(4) That would of course still leave onerous tasks like (I)! But it would be interesting anyway to at least look at the **Heawood Graph** (H) more closely: maybe it relates to some known graph, or to the icosahedron, or $\mathbb{RP}^2_6$?

(5) Regarding (K), there is now the **Appel-Haken** computer-assisted completion of Kempe's proof. Also note that **Kempe theory**, i.e. the study of (good) 4-colorings of graphs, has been pursued most by **Fisk** who has some interesting new results here.

KALAI'S "DIAMETERS ..."

This **** paper introduces **commutative shifting**, defined analogously to **anticommutative shifting**, which Kalai had used previously to establish **Eckhoff's inequalities** for face vectors of nerves of affine arrangements. He shows how commutative shifting yields the "dual" **McMullen's inequalities** for face vectors of simplicial polytopes.

This insightful reformulation of **Stanley**'s proof then leads him on to establish (more general) higher codimensional **Heawood inequalities** for simplicial complexes embedded *convexly* in euclidean spaces.

(A) Generic monomial bases. The commutative algebra A generated by vertices over $\mathbb{C}$ is the linear span of the set M of all commutative monomials in the vertices. Using a generic (same definition as before) graded **algebra automorphism** X of A, we replace its vector space basis M by a new basis X(M).

If K is a simplicial complex on these vertices, M(K) will denote monomials supported on K. Projecting onto the linear span A(K) of M(K), X(M) becomes a spanning set of this subspace. We use any total **order**, of the vertices v, and thus of the **letters** $x = X(v)$, to select, from this spanning set, the lexicographically smallest basis $\Delta(M(K))$ of A(K).

Correction. Kalai's definition of $<_P$ should be only for monomials of same degree, i.e. this is the product partial order, so as before we get the following.

(B) $\Delta(M(K))$ *is a shifted order ideal of monomials.*

Let us say that a monomial of degree r is a **pushout** if it contains no letter less than the *r*th. These arise from degree r **simplices**, i.e. monomials without repeated letters, by pushing out their letters r-1, r-2, ... steps to the right, thus e.g.

$$x_1x_3x_4x_7 \longmapsto x_4x_5x_5x_7 \quad , \quad x_4x_6x_7 \longmapsto x_6x_7x_7 \ .$$

It follows from (B) that the subset of *all* simplices of $\Delta(M(K))$ is a shifted simplicial complex. However its size is usually bigger than that of K, so Kalai concentrates on a shifted subcomplex of this which he defines as follows.

(C) Structure theorem. *The simplices whose pushouts are in* $\Delta(M(K))$ *constitute a shifted simplicial complex* $\Delta(K) \subset \Delta(M(K))$ *which has the same size as* K. *Furthermore, a monomial is in* $\Delta(M(K))$ *if and only if it is the product of some degree* r *pushout of* $\Delta(M(K))$ *and some monomial in the first* r *letters.*

Note that the subset $P(K) \subset \Delta(M(K))$ of pushouts is a (non-shifted) order ideal of monomials.

Regarding its proof, we know only how to deduce (C) starting from the following weaker statement.

(c) *If* $deg(K) = d$, *then* $\Delta(M(K))$ *does not contain any pushout of degree* $d+1$.

Proof of (C) *assuming* (c). Let degree $K = d$ and let L be the subcomplex of K obtained by removing its degree d simplices.

Clearly $M(L)$ is a subset of $M(K)$ which coincides with it in degrees less than d. Since $\Delta(M(L))$ is also a subset of $\Delta(M(K))$, it follows that it too coincides with it in degrees less than d.

Assuming (C) inductively for L, its second part shows that the $f_{d-1}(K)$ degree d monomials of $\Delta(M(K))$ which are not in $\Delta(M(L))$ must be pushouts.

Thus, using (c), all new monomials of $\Delta(M(K))$ must be some monomial in the first d letters times one of these f_{d-1} degree d pushouts.

We note now that the number of degree k monomials in $M(K) \setminus M(L)$ equals $f_{d-1}(K)$ times $\binom{k-1}{d}$, the number of degree k monomials in which **all** d given variables (= the vertices of any degree d simplex) occur.

Omitting an occurence of each of the d variables we see that this factor also coincides with the number of degree $k-d$ monomials in which **some or all** of d given variables (= the first d letters) occur.

Thus monomials obtained by multiplying monomials in the first d letters with the $f_{d-1}(K)$ pushouts of $\Delta(M(K))$ are *all* in $\Delta(M(K))$. *q.e.d.*

Correction. Lemma (6.3) (= Theorem (C) without "only if") is not proved completely in paper because Kalai's (more complicated) argument also assumes (c). As far as (c)'s proof goes, we see from the binomial identity,

$$\binom{N}{1}\binom{d}{0} + \binom{N}{2}\binom{d}{1} + \ldots + \binom{N}{d}\binom{d}{d-1} = \binom{N+d}{N-d} - \binom{N}{N-1-d},$$

that *the number of degree* $d+1$ *monomials in* $M(K)$, *is at most equal to the number of degree* $d+1$ *monomials on the* N *letters which are not pushouts*. Also, these generic non-pushouts are obviously lexicographically initial. What is not clear to us is why the projections of these monomials constitute a spanning set of the degree $d+1$ summand of $L(M(K))$?? If this could be argued out — it would suffice to deal with the maximal case when K consists of all simplices of degrees d or less — then it would follow at once that the degree $d+1$ monomials of $\Delta(M(K))$ are all non-pushouts.

Before looking at how Kalai uses his $\Delta(K)$, we will first go over his reformulation of Stanley's proof of McMullen's g-conjecture.

Shifted order ideals $H(K)$, $G(K)$. For any K with $deg(K) = d$ we will denote by $H(K)$, resp. $G(K)$, the subset of $\Delta(M(K))$ consisting of all monomials whose letters are all bigger than the *d*th, resp. $(d+1)$th.

From above it follows that these shifted order ideals are finite.

Recall that a degree d simplicial complex K is called **Cohen-Macaulay** iff link $Lk_K\sigma$ of each simplex $\sigma \in K$ has trivial reduced homology in degrees less than d-deg(σ).

(D) Reisner's theorem (Kalai's formulation). *If* K *is Cohen-Macaulay, then one can choose* $\Delta(M(K))$ *so that a monomial lies in it iff it is a monomial of* H(K) *times some monomial in the first* d *letters.*

[We recall that this required some local cohomology arguments: the first d letters of such a generic basis form a so-called homogenous **system of parameters** for the ring L(M(K)).]

So, for a C-M K, one can calculate the face vector of H(K) in terms of the face vector of K. This turns out to be the **h-vector** of K. Thus, for a C-M K, its h-vector, being the face vector of an order ideal, must obey **Macaulay**'s inequalities.

(E) Stanley's theorem (Kalai's formulation). *If* K *bounds a* d*-polytope, then one can also ensure that multiplications by powers of the* (d+1)*th letter yield bijections* $H_i(K) \cong H_{d-i}(K)$ *for all* $i \leq d/2$.

[We recall that this required the deep **hard Lefschetz theorem** for **toric varieties** which tells us that, for any toric variety stemming from a polytope, its cohomology, which has the same dimensions as |H(K)|, obeys a strong **Poincare duality**, analogous to that of smooth **Kaehler** manifolds: the (d+1)th letter corresponds to the Kaehler form, and the **primitive classes** correspond to G(K).]

So, for a polytopal boundary K, the face vector of G(K) can be calculated in terms of the face vector of K. This turns out to be the **g-vector** of K. Thus, G being an order ideal, it follows that the g-vector of a polytopal boundary obeys Macaulay's inequalities.

Now Kalai makes very good use of (D) and (E) to obtain the following striking result.

(F) Convex embeddability theorem. *If* K *occurs within the boundary of a simplicial* d*-polytope with* N *vertices, then* $\Delta(K)$ *is a subcomplex of* $\Delta(C(d,N))$*, where* C(d,N) *denotes the cyclic polytope on* N *vertices. Furthermore, one has equality* $\Delta(K) = \Delta(C(d,N))$ *iff* K *is the boundary of a neighbourly* d*-polytope on* N *vertices.*.

Proof. Since shifting preserves inclusions it suffices to consider the *case* K = *polytopal boundary*.

Using (D), we associate to each of the $h_k(K)$ degree k monomials of H(K) a disjoint set of simplices of $\Delta(K)$ of degree i, $k \leq i \leq d$, as follows: first take the degree k simplex having such a monomial as pushout, and then add to it any i-k of the first d-k letters.

The total number of i-simplices made this way being $f_i(K)$, it follows that the above is in fact a new description of *all* of $\Delta(K)$. In particular we see that $\Delta(K)$ is dimensionally **pure**, its top-dimensional simplices being given explicitly as above with i=d.

But we also have (E), which tells us that each monomial of H(K) is some power of the (d+1)th letter times a monomial of G(K) of degree $\leq$ d/2. Combining this fact with the above explicit description of the top simplices, it follows that these are all **admissible**. By this is meant that if the (k-1)th letter is outside the degree d simplex, then all letters k through (d-k+3)rd are inside the simplex. Thus it remains only to check the following.

$\Delta(C(d,N))$ — *and more generally the commutative shift* $\Delta(K)$ *of any neighbourly* d-*polytopal boundary on* N *vertices — coincides with the shifted simplicial complex generated by all admissible degree* d *simplices on* N *vertices.*

Since the first complex is, by above, a subset of the latter this follows by checking that the two complexes have the same size. *q.e.d.*

(f) A higher codimensional Heawood inequality. *If* K^n *occurs within the boundary of a simplicial* (2n+1)*-polytope, then we must have*

$$f_n(K) < (n+2).f_{n-1}(K).$$

Proof. Otherwise $\Delta(K)$ being shifted it would contain the n-skeleton of a (2n+2)-simplex. The same would be true for $\Delta(L)$ where L is the ambient polytopal boundary. But from this it is easy to check that some top simplex of $\Delta(L)$ is non-admissible, a contradiction. *q.e.d.*

Comments

(1) Theorem (E) for simplicial spheres requires other ideas because now one can't attach a toric variety and use its Hodge theory. Likewise (F) and (f) are true even when K embeds topologically in requisite dimensional sphere or euclidean space but require new methods.

(2) The purely combinatorial results given in the paper from page 18 onwards apply to any shifting operation $\Delta(K)$ with the above behaviour, so conjecturally to anticommutative shifting also.

(3) This information is then used to study the **diameter problem** for the dual **simple polytopes**. Recall that their graph determines their lattice structure. The optimistic conjecture of **Hirsch** is that a simple d-polytope with N facets has diameter $\leq$ N - d. Kalai gets good bounds for the diameter of **dual-to-neighbourly** polytopes by using (F).

(4) This last deduction proceeds via showing that such a graph is a **magnifier** of appropriate order, i.e. that any vertex set of cardinality $\leq$ N/2 always has an appropriate percentage of its neighbours outside it.

(5) He gives references for the curious connections which exist between

the spectrum of the **Laplacian matrix of a graph**, i.e. incidence matrix times its conjugate, and the magnification of the graph.

VAN KAMPEN'S "KOMPLEXE ..."

VKO will include a translation of this **** paper of 1932.

(A) Theorem. *The space of linear embeddings* $K^n \longrightarrow \mathbb{R}^m$, $m \geq 2n+2$, *is path connected.*

Proof. Any direction not parallel to the, at most (2n+1)-dimensional, and finitely many, affine subspaces determined by pairs of simplices of an embedded K, will be called **non-forbidden**.

Given two linear embeddings of K, and a vertex v of K, we join its two images P_1, P_2 to a third point Q such that P_1Q and P_2Q are non-forbidden with respect to the two embeddings.

We can find a family of linear embeddings containing the first — and likewise another family containing the second — embedding, which keeps the antistar of v fixed, and under which v travels from its initial position to Q. To do this extend P_1Q to $P_1'P_1QQ'$ and, move the star as per $P_1'P_1 \longmapsto P_1'Q$, $P_1Q' \longmapsto QQ'$. *q.e.d.*

(B) Whitney trick. *If* $n \geq 3$, *and a general position p.l. map of* K^n *in* $\mathbb{R}^{2n}$ *has a pair of double points, having* **intersection numbers** +1 *and* -1, *and belonging to the same pair of* n-*faces of* K, *then they can be removed without introducing any new singularities.*

To start with, Van Kampen stated this without any intersection number condition. He joins the two double points by an arc, in just one of the faces, and tries to **cone away** its tubular boundary out of harm's way. **Brown** pointed out to him that this does not work: see "Berichtigung", where Van Kampen also states that the intersection number condition is needed, but can't give a proof of the above. Ten years later, Whitney gave a method of removing such a double point pair which used a smooth structure. However it is true that a more elaborate coning argument also works.

(C) More coning. *Also, in above situation, we can remove any double point belonging to the same or adjacent* n-*faces.*

In "Berichtigung" Van Kampen gives a correct proof. In fact in this case, as **Zeeman** and others were to later notice, only a small elaboration of the above erroneous coning argument also works.

(D) Piping. *In above situation, and even for* $n \geq 2$, *we can, by attaching a long thin pipe to any* n-*face, add extra intersections, all with numbers* +1 *or else* -1, *of this* n-*face with all* n-*faces incident to some other* (n-1)-*face.*

(E) Van Kampen Obstruction. *Let* $\mathfrak{B}$ *be the* **vector** *which associates, to any unordered pair of disjoint faces of an* n-*dimensional* K, *their intersection number under a given general position map into* 2n-*space. If this map is changed, then the new vector can be obtained from the old by a sequence of steps, in each of which one adds* +1, *or else always* -1, *whenever one member of the pair is incident to a chosen* (n-1)-*face.*

Using later terminology, this theorem says that the **characteristic cohomology class** [$\mathfrak{B}$] depends only on K, and that its vanishing is a necessary condition for the p.l. (and, as Van Kampen proves, even for the topological) embeddability of K in 2n-space. Also, using (D), and then (B) and (C), he removes all singularities, of any general position map of such a K in 2n-space, i.e., modulo (B), he obtains the following.

(F) Van Kampen-Wu-Shapiro Theorem. *An* n-*complex,* n ≥ 3, *p.l. embeds in twice dimensional euclidean space if and only if its characteristic class* [$\mathfrak{B}$] *vanishes.*

The latter two authors used Whitney's smooth method to establish (B).

Actually above result holds, by virtue of the contemporaneous **Kuratowski** planarity criterion, even for n = 1. For n = 2 it is possible that this condition may suffice only to ensure topological embeddability ?

(G) Embeddability of manifolds. *An* n-*pseudomanifold embeds piecewise - linearly in* 2n-*space.*

This is proved in main paper by checking [$\mathfrak{B}$] = 0 for such a K, in "Berichtigung" directly by using (C). While doing this direct proof, he also checks that identifications in codimensions ≥ 2 do not effect embeddability in twice-dimensional space.

(H) Van Kampen-Flores Theorem. *The n-skeleton of a* (2n+2)-*simplex, and the* (n+1)-*join of* 3 *points, are n-complexes not embeddable in* 2n-*space.*

To see this he checks that for a special g.p. map $\mathfrak{B}$ has only one nonzero coordinate, and that one needs to change two coordinates at a time as one proceeds to any cohomologous cocycle.

PERLES' CONJECTURE

We will give **Kalai**'s simple proof of the following result, which was conjectured by Perles, and proved first by **Blind - Mani**.

Theorem. *A* **simple polytope** K *is determined by its graph* G(K).

Proof. (A) and (B) below will show how K can be recovered from G(K).

By an **acyclic orientation** of a cell complex K we mean a partial ordering of its vertices which has exactly one local maximum on each cell. Its **h-vector** is defined by setting $h_i(K)$ equal to the number of vertices which have exactly i smaller neighbours.

(A) *Acyclic orientations of* K *can be recognized, amongst acyclic orientations of* G(K), *as those which minimize the sum* $\sum_i 2^i . h_i(K)$.

For an acyclic orientation of K the above sum in fact equals the total number of nonempty cells of K. This follows because each cell has a unique maximum vertex, and, K being simple, any t smaller neighbours of a vertex determine an incident t-dimensional cell, in which the vertex is a local, and thus the global maximum.

On the other hand, were we to calculate the sum using an acyclic orientation of G(K) which is not an acyclic orientation of K, some cell would be counted more than once, and we would get a bigger number.

(B) *Graphs* $G(\sigma)$ *of cells* $\sigma \in K$ *can be recognized as those connected,* **full** *and* **regular** *subgraphs of* G(K), *whose vertices are initial with respect to some acyclic orientation of* K.

That all $G(\sigma)$'s are indeed of this type follows because we can separate the vertices of σ from the other vertices of K by a hyperplane and totally order the vertices by using a general position transverse direction.

Conversely, if t-regular subgraph $H \subset G$ has these properties, consider the t-dimensional face σ of our simple K which is determined by the biggest vertex of H and its t neighbours in H. The biggest vertex of H is a local maximum in σ, so must be the biggest vertex of σ, so all vertices of σ must be in the initial set formed by the vertices of H. Thus, H being full, $G(\sigma)$ must be a subgraph of H. But, since it too, like H, is t-regular and connected, it follows that $G(\sigma) = H$. *q.e.d.*

Comments.

(1) The terminology acyclic orientation stems from the fact that for graphs it amounts to an orientation of its edges not giving rise to any 1-cycle.

There is no loss of generality in working with acyclic orientations which are total orderings of the vertices. To see this note that any acyclic orientation necessarily total orders the vertices of any cell, thus any total order, which extends this partial ordering of the vertices, is itself an acyclic orientation.

Though all total orderings of the vertices of a cell complex are not acyclic orientations — consider e.g. the ordering 1324 of a square — this is clearly so for simplicial complexes.

(2) Does every cell complex have an acyclic orientation ? Though we don't know the general answer, it is so for **linear cell complexes**, i.e. those which embed linearly in some euclidean space.

[In this context it is worth remembering the following striking facts:

A simple manifold M^n *embeds linearly in* $\mathbb{R}^{n+1}$ *only if it is a sphere, and* *a simple* $\mathbb{C}P^2$ *does not embed linearly in any* $\mathbb{R}^N$ *!*]

In fact any **Banchoff function**, i.e. a linear functional separating adjacent vertices, gives a total ordering of the vertices of such a cell complex, which obviously has a unique local maximum on each cell.

It seems likely in fact — only the first part of (B) remains — that above proof will extend to show that any linear simple complex is determined by its graph ?

(3) Simplicity — more exactly the fact that for $t \leq d$ any t edges incident to a vertex belong to a t-cell was important in above proof.

For any K equipped with an acyclic orientation it is useful to consider the **in-links** Lk_-v and **out-links** Lk_+v determined by smaller and bigger neighbours of each vertex v.

With this notation note that h_i is the number of vertices v such that $f_0(Lk_-v) = i$. It is known that *for a simple* K *the h-vector is independent of the chosen acyclic orientation*, but in general the f-vector of Lk_-v is not determined by its f_0, and so one should look also at the numbers $f_j(Lk_-v)$.

For example, the Euler characteristic of an in-link coincides with the definition of the **index** of a vertex in Banchoff's Morse theory, and thus his index formula reads $\chi(K) = \sum_v \chi(v)$, where $\chi(v) = \chi(Lk_- v)$. This follows at once from $f_j(K) = \sum_v f_j(v)$, where $f_j(v) = f_{j-1}(Lk_-v)$.

(4) Simple polytope being determined by their lowermost incidences, the dual simplicial polytopes are determined by their uppermost incidences. Kalai mentions that in fact any lower-half, resp. upper-half, **incidence matrix** of a simple, resp. simplicial, polytope will do the job. It is natural again to ask whether this stronger result also holds for simplicial/simple manifolds ?

A parallel result of Perles says that the lower half **skeleton** of a simplicial polytope determines it. This was extended to simplicial manifolds by **Dancis**.

(5) Some other points:
the directions mentioned in the proof of the first part of (B) **shell** the polytope,
graph-theoretic manifold theory — see e.g. **Cavicciolli** — is close to above,
as is characteristic class theory of **Gelfand** et al.,
and "order-orientable" triangulations might also fit into this scheme ?

SQUEEZED SPHERES

We will work with a **totally ordered** universe U of N letters. A subset S of U will be called **contiguous** if any letter of U which is between two letters of S is itself in S. Also, for any $S \subseteq U$, a maximal contiguous subset of S will be called a **component**.

We denote by $\mathcal{E}(d)$ the set of those cardinality d+1 subsets of U whose components are of *even cardinality* or else contain the first letter x_1 or the last letter x_N of U.

As we'll see below, the simplices of $\mathcal{E}(d)$ together with all their faces form a d-dimensional sphere C(d). Though these **cyclic spheres** were discovered in 1911 by **Caratheodory**, the present description was given much later by **Gale**.

A subset $\mathcal{K} \subseteq \mathcal{E}(d)$ will be called **shifted in** $\mathcal{E}(d)$, respectively **compressed in** $\mathcal{E}(d)$, if it is closed with respect to the product partial order, respectively lexicographic total order, of $\mathcal{E}(d)$.

(A) Theorem. *If $\mathcal{K}$ is a shifted proper subset of $\mathcal{E}(d)$, then the simplices of $\mathcal{K}$ together with all their faces form a d-dimensional shellable ball K.*

The compressed case (as well as (B) below) is due to **Billera-Lee**. Much later **Kalai** observed that their proof applied even under this shifted hypothesis. Such balls K, respectively their boundaries ∂K, were christened **squeezed balls**, respectively **squeezed spheres**, by Kalai.

Proof. Kalai knew already that pure shifted complexes K, i.e. those generated by any $\mathcal{K}$ which is shifted in the set of *all* cardinality d+1 subsets of letters, were shellable. He noticed now that the exact same argument worked even for $\mathcal{K}$'s shifted in $\mathcal{E}(d)$.

As per the definition of **shellabillity** of K, we need to show that $\mathcal{K}$ can be so totally ordered, that each $\sigma \in \mathcal{K}$ shares a face θ with a preceding simplex of $\mathcal{K}$ only if θ is a face of a preceding **adjacent** simplex of $\mathcal{K}$, i.e. one sharing a degree d face with σ .

Such a **shelling order** *is in fact given by any total order which extends the product partial order of $\mathcal{K}$.*

To see this note that, $\mathcal{K}$ being shifted, all **combinatorial shifts** $\Delta_{ab}(\sigma)$ of σ, which are in $\mathcal{E}(d)$, are in $\mathcal{K}$. If a face θ of σ is not in any of these simplices, then it must contain all components of σ not containing the first vertex of U. The assertion follows because a member of $\mathcal{E}(d)$ containing all these components is $\geq \sigma$ in the product partial order.

C(d) *is a pseudomanifold.*

To see this note that any codimension one simplex can have at most one odd component containing neither x_1 nor x_N. If it has, the only incident top faces are those obtained by lengthening it either way. If it has none, then the incident top faces are obtained by adding the letters contiguous to the component of x_1 or x_N. In either case we have exactly 2 possibilities.

So it follows that C(d) is a sphere and all other K's are balls properly contained in it. *q.e.d.*

(B) Theorem. *The* **g-vector** *of a squeezed sphere ∂K is the face vector of an order ideal of monomials.*

We'll sketch the argument only for the more important **even case** d+1 = 2e and that too only when $\mathcal{K} \subseteq \mathcal{E}\langle 2e\rangle$, the set of those cardinality d+1 sets of letters whose components are *all* of even cardinality.

[Note that this subset $\mathcal{E}\langle 2e\rangle$ of $\mathcal{E}(d)$ is closed with respect to the product partial order, thus a $\mathcal{K} \subseteq \mathcal{E}\langle 2e\rangle$ which is shifted in $\mathcal{E}\langle 2e\rangle$ is automatically shifted in $\mathcal{E}(d)$.]

Proof. The sets $\mathcal{E}\langle 2e\rangle$, and {monomials of degree ≤ e in N-2e+1 letters}, are equinumerous.

An explicit **bijection** α is obtained if, out of the letters of any $\sigma \in \mathcal{E}\langle 2e\rangle$ not lying in a maximal contiguous set containing the least vertex of U, we keep only the first, third, fifth ... letters, and decrease them by 1, 3, 5, ... steps respectively. So for example

$$x_2x_3x_5x_6x_7x_8 \longmapsto x_1x_2x_2 \quad \text{and} \quad x_1x_2x_5x_6x_7x_8 \longmapsto x_4x_4 \ .$$

The $\alpha(\mathcal{K})$ just defined is an order ideal of monomials whose face vector coincides with the g-vector of the squeezed sphere ∂K.

Very briefly, B-L verify first that $\alpha(K)$ is an order ideal, and then use the shelling order given above to check that the h-vector of the shellable ball K coincides with the face vector of $\alpha(K)$. The proof is completed by checking that the h-vector of this ball coincides with the g-vector of its boundary ∂K. *q.e.d.*

To complete their celebrated proof of the sufficiency part of **McMullen**'s **g-conjecture** for simplicial polytopes, Billera-Lee were then essentially left with checking the following.

(C) Theorem. *If $\mathcal{K}$ is compressed then the sphere ∂K is polytopal.*

We will look at its proof elsewhere.

Comments.

(1) Odd degree squeezed balls are not very important because they are just cones of even degree squeezed balls.

Also Kalai observes that *each squeezed ball is determined by its bounding squeezed sphere.*

Also Kalai observes that *each squeezed ball is determined by its bounding squeezed sphere.*

(2) *Squeezed balls are* **order-orientable**, *thus their lower-half skeleton is up on their boundary* ∂K. (Cf. Prop. 5.3 of Kalai's paper.)

Stanley has verified the g-conjecture for *all* simplicial spheres obeying the conclusion of the above result.

(3) *The* **Hirsch conjecture** *is true for squeezed spheres.*

Lee proved this for the compressed case only, but once again his inductive argument extends easily to squeezed spheres. It only uses the fact that *for any of our squeezed balls, both the link and the antistar of the smallest vertex are squeezed balls*.

(4) With e fixed, Kalai checks that the log of the number of squeezed (2e-2)-spheres on N vertices is at least of order N^e for large N.

On the other hand **Goodman-Pollack** showed that the log of the number of polytopal (2e-2)-spheres is at most of order N.logN for large N.

[This uses some bounds of **Milnor** on sums of Betti numbers of some algebraic varieties. Later **Alon** generalized the G-P argument to non-simplicial polytopes.]

So *most squeezed spheres are non-polytopal.*

In fact, even though they bound a shellable ball, it is quite possible that squeezed spheres are mostly non-shellable ? [**Pachner** showed that any simplicial sphere, having some subdivision in common with a minimal sphere, bounds a shellable ball.]

The asymptotic upper bound for the log of the total number of simplicial (2e-2)-spheres can be computed from Stanley's upper bound theorem, and is about N.logN times Kalai's lower bound given above. It would be interesting to augment the squeezing construction somehow, say by using some knotting perhaps, so as to bridge this gap.

GOLDSTEIN-TURNER FORMULA

We will deal with a simplicial complex K, whose vertices are equipped with a partial order, which restricts to a total order on each simplex. For example, if K is a **derived**, then it comes with a **natural** partial order of this type, viz. the inclusion order of the original complex.

Following **Steenrod**, a face θ of a simplex σ, of any such K, will be called a **regular face** of σ, if any of its components not containing the last vertex of σ, either is odd and contains the first vertex of σ, or else does not contain this vertex and is even.

Theorem. *If* K *is a mod* 2 **Euler space** *then the mod* 2 *sum of all the* *regular faces of simplices of* K *is a cycle which represents the total* **Stiefel-Whitney class** *of* K.

In case K is a derived, and is equipped with the natural partial ordering of its vertices, then the above cycle is the sum of all the simplices of K.

This case of the above theorem was conjectured by Stiefel, and proved (unpublished) by Whitney, and later, by **Halperin-Toledo** and **Cheeger**.

In fact Goldstein and Turner deduce the above result from this case as follows. Let K' $\longrightarrow$ K be the simplicial map which images each vertex of K' (= simplex of K) to its smallest vertex. Then the induced mod 2 chain map images the sum of the simplices of K' to the mod 2 sum of the regular faces of of the simplices of K.

Comments

(1) The totally ordered set of vertices of each $\sigma \in$ K determines a **cyclic sphere**, one in each dimension less than that of σ. The biggest of these determines the ordinary mod 2 **boundary** $\partial\sigma$. It might be useful to also define the sum of the top faces of all these spheres as the **higher boundary** of σ ?

(2) It seems e.g. that regular faces of σ determine a sequence of **squeezed balls**, one in each dimension upto that of σ, and that these bound the aforementioned cyclic spheres. Thus, by virtue of the fact that the higher boundary of the sum of all the simplices of an Euler space is zero, it follows that the mod 2 sum of the regular faces of all simplices is a cycle.

(3) Steenrod's work came much before that of **Gale** on cyclic polytopes. His regular faces were used to usher in the **Steenrod squares** of cohomology theory, and there is a well-known formula of **Thom** for Stiefel-Whitney classes in terms of these operations.

(4) It is possible that the higher boundaries $\partial_j(\sigma)$ of (1) also shed more light on the Kruskal-Katona theorem and function $\partial_j(n)$?

(5) Goldstein-Turner mention that their result also follows from the work of **Banchoff** and **McCrory**.

KALAI'S "f-VECTORS"

In this ms. Kalai sketches how the local **Cohen-Macaulay** property, on the links of a simplicial complex, is equivalent to a vanishing condition on a global cohomology, defined in terms of anticommutative cochains as follows.

For Λ, the **exterior algebra** generated by our N vertices $v_1, v_2, v_3,$ .. over $\mathbb{F}$, we choose an algebra isomorphism $X:\Lambda \rightarrow \Lambda$ yielding letters x_i =

$X(v_i)$. We note that, for any simplicial complex K, the maps $x_i : L(K) \longrightarrow L(K)$, $1 \le i \le r$, defined by $[\omega] \mapsto [x_i \wedge \omega]$, image into the kernel of the map $x_1 \,..\, x_r : L(K) \longrightarrow L(K)$ defined by $[\omega] \mapsto [x_1 \wedge \,..\, \wedge x_r \wedge \omega]$. The **rth iterated cohomology** of K with respect to X is defined by

$$H[r,X](K) = \frac{\ker(x_1 \,..\, x_r)}{\mathrm{Im}(x_1) + \,..\, + \mathrm{Im}(x_r)} .$$

We denote by $\Delta_X(K)$ the **lexicographically smallest basis** contained in the spanning set of L(K) determined by the set of all exterior monomials in the letters.

We recall that this is a simplicial complex which is **shifted** whenever X is **generic**. We will denote this **exterior shift** simply by $\Delta(K)$ because, upto a simplicial isomorphism, it is independent of the generic X chosen.

(A) Theorem. *Generically* H[r,X](K) *does not depend on* X. *Moreover this generic rth iterated cohomology* H[r](K) *of* K *coincides with that of its exterior shift* $\Delta(K)$.

Proof. The argument resembles that of the familiar case r = 1:

We check that, in $\Delta_X(K)$, the **rth iterated cocycles**, i.e. things lying in kernel of the map $x_1 \,..\, x_r$, appear as words whose augmentation by $x_1 \,..\, x_r$ is not in $\Delta_X(K)$, and that, out of these, the **rth iterated coboundaries**, i.e. those lying in the sum of the images of the maps $x_1, \,..\, , x_r$, are those which contain at least one of the first r letters. Thus

$$\dim H[r,X](K) = |\{\sigma \in \Delta_X(K) : \sigma \cap (x_1, \ldots, x_r) = \emptyset \;,\; \sigma \cup (x_1, \ldots, x_r) \notin \Delta_X(K)\}| .$$

From the fact that $\Delta_X(K)$ *is shifted it follows that* $\Delta_X(\Delta_X(K)) = \Delta_X(K)$.

Using this, the above formula shows that dim H[r,X](K) = dim $H[r,X](\Delta_X(K))$, i.e. that H[r,X](K) and $H[r,X](\Delta_X(K))$ are isomorphic vector spaces.

Finally, since $\Delta_X(K)$ is independent of the generic X, the above also easily implies that H[r,X](K) too is independent of X. *q.e.d.*

Besides generic X's, it is useful also to consider those whose letters are only in **general position** with respect to the vertices, i.e. are such that the linear expansion of each lexicographically first word $x_1 x_2 x_3 \,..\, x_r$ contains all degree r simplices.

For example, when the first letter of such an X is the sum of all the N vertices, H[1,X](K) coincides with the reduced **ordinary cohomology** H(K) of K. However it follows easily that this is isomorphic to the generic H[1](K). Thus (A) generalizes Kalai's previous result that exterior shifting preserves cohomomology.

(B) Theorem. *A degree* d *simplicial complex* K *is Cohen-Macaulay if and only if the generic iterated cohomology groups* $H^k[d-k](K)$ *vanish for all* $k \geq 0$.

The following is the argument of "only if" sketched in the paper. Kalai also gives a more detailed separate proof of this implication for the special case when K is **shellable**. He gives no proof for "if".

In the *very* special case when K is *shifted*, shellability is equivalent to the Cohen-Macaulay property, or even to the **purity** of K. Now both implications follow trivially from the above explicit description of the iterated cohomology of shifted complexes.

Proof of "only if". The iterated cohomologies were not defined as homologies of chain complex, so it is more convenient to look at the **associated iterated cohomologies** $\underline{H}^k[t](K)$, i.e. the homologies of the following chain complexes:

$$\frac{L(K)}{\ker(x_1 \ldots x_{t-1})} \xrightarrow{x_t} \frac{L(K)}{\ker(x_1 \ldots x_{t-1})} .$$

(B1) $H^k[d-k](K)$ *vanishes iff* $\underline{H}^k[t](K)$ *vanishes for all* $t \leq d-k$.

This follows very easily from the above definition. Using now the standard machinery of exact homology sequences Kalai checks the following.

(B2) *If all proper links of a degree* d *simplicial complex* K *are Cohen-Macaulay, then the cohomology* $\underline{H}^k[t](K)$, $t \leq d-k$, *is unaffected by stellar subdivisions.*

(B3) *In fact this subdivision-invariant cohomology* $\underline{H}^k[t](K)$, $t \leq d-k$, *identifies with the ordinary cohomology* $H^k(K)$ *of* K.

This last step of proof is accomplished by imitating a well-known argument which identifies **de Rham cohomology** with ordinary cohomology:

For each vertex v of K, let K_v denote its closed star in the **derived** K'. Then $\{K_v\}$ is an **acyclic cover**, because the **dual cells** $\cap_v(K_v)$ have trivial iterated cohomology by the given condition on proper links. So, by imitating the aforementioned, the iterated cohomology of K' coincides with the ordinary cohomology of the **nerve** of this cover, which is isomorphic to K. *q.e.d.*

(C) Theorem. *If* K *is Cohen-Macaulay, then its exterior shift* $\Delta(K)$ *is pure.*

This is an immediate corollary of Theorems(A) and (B). Note that this is the anti-commutative analogue of **Reisner's theorem** which is equivalent to saying that the *commutative shift* of a Cohen-Macaulay K is pure. However the current proof of this analogue is quite different and employs some algebraic geometry machinery of **Grothendieck**.

The next result too has a known, but only *partial*, commutative analogue, viz. for polytopal spheres only, whose proof too is quite different, and employs the **hard-Lefchetz** theorem of **toric varieties**.

(D) Theorem. *The exterior shift* $\Delta(K)$, *of any* K *with* N *vertices contained in a degree* d *sphere, is contained in the exterior shift of* **the cyclic sphere** *of degree* d *with* N *vertices.*

Kalai's attempted Proof. Recall that the kth letter is outside a degree d **admissible simplex** iff the next d-2k+2 letters are all inside it. Such simplices determine a shifted simplicial complex A(d,N) of the same size as the cyclic degree d sphere on N vertices.

So the result will follow from $\Delta(K) \subseteq A(d,N)$, and moreover we would have shown that A(d,N) is none other than the exterior shift of the cyclic degree d sphere on N vertices.

(D1) *If a shifted complex is not contained in* A(d,N) *then it must contain some* **Kuratowski complex** T *of the following kind:*

$$T = \sigma_r^r \cdot \sigma_{s-1}^{2s} \quad \text{where} \quad r+2s = d.$$

This follows easily from the explicit description of A(d,N). Now, since T_* is an antipodal d-dimensional sphere, it follows that its d th **van Kampen obstruction** is nonzero. Thus the theorem would follow if we could, à la **Sarkaria**, show the following.

(D2) *If* $\Delta(K)$ *contains the aforementioned* T, *then there is a* $\mathbb{Z}_2$*-cochain map*

$$L(K_*) \longrightarrow L(T_*)$$

which maps 1 *to* 1.

However *the ms. gives no proof of the above*.

Remark. Note here that we cannot expect such a cochain map from $L(K_*)$ into $L((\Delta(K)_*)$, because *there are planar graphs* K *whose* $\Delta(K)$ *is not planar*. (D2) asserts in particular that all such $\Delta(K)$'s must always contain the (3,3)-Kuratowsi graph, never the complete graph T on 5 vertices. Incidentally *there are also non-planar graphs* K *with* $\Delta(K)$ *planar*, which shows that we cannot have a cochain map of above kind from $L((\Delta(K)_*)$ to $L(K_*)$ either.

Even $\Delta(K_*) \supseteq T_*$ *was left open.* But Kalai was able to establish a suggestive analogue for the **exterior square** $K_{\wedge}$ of K, i.e. the simplicial complex of all simplices which are unions of some 2 simplices of K.

(D3) $\Delta(K_{\wedge}) \supseteq (\Delta(K))_{\wedge}$ *and more generally* $\Delta(K \wedge L) \supseteq \Delta(K) \wedge \Delta(L)$.

To prove this Kalai used the interesting dual description of $\Delta(K)$, viz. that it consists of the **first words** in linear expansions of elements of the subspace L(K) of Λ.

We will give in Chapter V of "VKO" a proof of Theorem (D), which uses a new **equivariant shifting**. This process is *noncommutative*, since it is based neither on the anticommutative, nor on the commutative, algebra generated by the vertices, but on another non-graded-commutative graded algebra defined by using the group action.

Comments

(1) The dual problems regarding f-vectors of nerves of affine euclidean **arrangements** had earlier led Kalai to the study of **d-Leray complexes**, and he had already proved an analogue of Theorem (B) for these. In his first paper however this dual result was only proved under the stronger assumption of a **Wegner shellability** which all nerves of above type obey.

(2) Using commutative cochains, **Bier** has defined very simple cohomologies, in terms of which, both the Cohen-Macaulay and Leray properties are equivalent to some vanishing criteria. It would be very nice if we could show, analogously to the above, that commutative shifting preserves these, and then maybe even obtain a simpler proof of Reisner's theorem, and more ?

(3) **McMullen's conjecture.** *For any simplicial degree* d *sphere,* $(h_0, h_1 - h_0, \ldots, h_{[d/2]} - h_{[d/2]-1})$ *is the face vector of an order ideal of monomials.*

Arguments already given by Kalai in "Diameters .." show that the above is a corollary of (C) and (D).

Here the h_i's are coefficients of the **h-polynomial**, which is obtained, from the face or **f-polynomial** of the simplicial complex under consideration, by changing x to x-1, i.e. one has

$$\Sigma\, h_k x^{d-k} = \Sigma\, f_{k-1} (x-1)^{d-k} .$$

For a (d-1)-sphere one has the **Dehn-Sommerville equations** $h_k = h_{d-k}$. This is very easy. In fact the equation for k = 0 is **Euler's equation** of the sphere, and the others follow very easily from (and indeed are equivalent to) the fact that similar Euler's equations hold also for all proper links.

[Kalai notes that the D-S equations also follow, by an induction on d and k, from $\Sigma_v\, h_k(Lk_K v) = (d-k).h_k(K) + (k+1).h_{k+1}(K)$, which is not hard to check directly.]

Thus, by (C) and (D), McMullen's conjecture would follow if we could verify the same assertion for any pure and shifted subcomplex of A(d,N) which obeys the Dehn-Sommerville equations.

This had been already done by Kalai in his "Diameters ..". He gives a more elegant version of this argument in this ms., explicitly constructing the required order ideal of such a shifted complex.

(4) Besides the duality (both of results and methods) between polytopes and arrangements (e.g. the dual of McMullen's conjecture is that of Eckhoff's) we have the uncanny parallelism (of results, not of methods) between commutative and anticommutative shifting. It is possible that perhaps any irreducible representation of the symmetric group likewise determines a useful shifting process? Also, there should be a uniform method, maybe involving de Rham theory or cohomology theory of algebras, which can be applied in every case ?

CP^2_9

(A) *If a simplicial 2-sphere's vertices are all of the same degree, then it must be isomorphic to the 4-vertex tetrahedron* S^2_4*, or the 8-vertex octahedron* S^2_8*, or else the 12-vertex* **icosahedron** S^2_{12}.

This was known even to **Plato**, and follows easily from **Euler**'s formula.

The tetrahedron and the octahedron have obvious higher-dimensional analogues, but not so the icosahedron. However its existence is a very fortunate fact, because **without the icosahedron, mathematics would have been much poorer**. In Chapter IV of "VKO" we'll give many examples based on the icosahedron.

(B) Kuhnel's 9-vertex complex projective plane CP^2_9 *is an example of "icosahedral ubiquity" because it is a "complexification" of* RP^2_6 = $S^2_{12}/\mathbb{Z}_2$*, the* **6-vertex real projective plane**, *in the following sense:*

There is a 12-vertex complex projective plane, obtained by derivng 3 edges of Kuhnel's 9-vertex complex projective plane, which has a simplicial involution, whose fixed-point-set is a subcomplex isomorphic to the 6-vertex real projective plane.

The above, which we don't understand completely, is one of many nice things contained in the paper of Kuhnel-**Banchoff**.

Not so elegant, or even directly connected with the icosahedron, is **Kuhnel's original path to his discovery:**

The icosahedron can be constructed from a **pentagonal prism**, by first deriving its top and bottom faces, and then **perturbing** it slightly so two triangles replace each of the rectangular vertical faces. A similar perturbation of the **triangular prism** gives the octahedron.

Start now with a *solid* triangular prism with base much bigger than the top, and derive the top. Then cone the surface, excluding the bottom, to a new vertex, and perturb to make the rectangular prism faces into tetrahedra. Finally add a tetrahedron. This gives **Bruckner's neighbourly 8-vertex 3-sphere** S^3_8.

Kuhnel constructed his CP^2_9 by coning S^3_8 over a 9th vertex, and then used some permutations of the 9 vertices, forming a group of order 9, to add more simplices.

(C) CP^2_9 *has an order 54 group of symmetries which acts transitively on the 9 vertices, thus every vertex-link is a Bruckner 3-sphere.*

The isotropy group of each vertex, i.e. group of symmetries of the Bruckner sphere, is of order 6, being in fact $\mathbb{Z}_2 \times \mathbb{Z}_3$, and the quotient by this normal subgroup is $\mathbb{Z}_3 \times \mathbb{Z}_3$.

That the above procedure yields a manifold with the homotopy type of the complex projective plane is not too hard, and at this point **Freedman's** classification theorem can be used.

But the above paper also has a geometric argument which exhibits an explicit homeomorphism.

(D) CP^2_9 *is the only* 3-neighbourly *simplicial 9-vertex 4-manifold.*

This was checked even before it was verified that this manifold is the complex projective plane. To do this Kuhnel-**Lassmann** verified that none of the other neighbourly 8-vertex 3-spheres led to a 9-vertex 4-manifold in the above way.

(E) *The deleted join of the 2-neigbourly* RP^2_6 *is a 4-sphere, while that of the 3-neighbourly* CP^2_9 *is a 7-sphere.*

The neighbourliness is obvious, while the sphericity follows from the fact that these are **self-dual**, or, in **Schild**'s terminology, "nice" complexes.

Note that (E) implies in particular that RP^2_6 and CP^2_9 embed respectively in $\mathbb{R}^4$ and $\mathbb{R}^7$, and that these dimensions 4 and 7 are the best possible.

(F) *The general position linear embedding of* RP^2_6 *in* $\mathbb{R}^5$, *resp. of* CP^2_9 *in* $\mathbb{R}^8$, *is* **tight**, *in the sense that all half-spaces contain a connected, resp. simply connected, part of the complex.*

Furthermore the space of **secants**, *i.e. lines meeting the complex in at* *least two points, is of dimension less than 5, resp. 8.*

The second part is essentially equivalent to saying that the deleted joins are of dimensions less than 4 and 8 respectively. Likewise, the first part corresponds to the 2- and 3-neighbourliness of the complexes.

(G) *Smooth versions of above embeddings are also known.*

For example, consider the map from $\mathbb{R}^3$ to $\mathbb{R}^6$ given by

$$(x,y,z) \longmapsto (xx, yy, zz, \sqrt{2}.xy, \sqrt{2}.yz, \sqrt{2}.zx).$$

This maps the unit sphere S^2 of $\mathbb{R}^3$ into [the sphere S^4 obtained by sectioning the unit sphere of $\mathbb{R}^6$ by] the affine subspace $\mathbb{R}^5 \subset \mathbb{R}^6$ on which the sum of the first three coordinates is 1. Furthermore 2 points of S^2 have the same image iff they are antipodal points.

This smooth **Veronese embedding** of $\mathbb{R}P^2$ in $\mathbb{R}^5$ has to be tight because compact conics are connected. It is known also that its secants form a hypersurface.

The definition of the Veronese embedding of the complex projective plane in 8-space is analogous. These were of course known since long, and **Kuiper** had posed the problem of finding non-smooth embeddings with the same properties vis-à-vis tightness and secants. Possibly it was this question which led Kuhnel to his $\mathbf{CP}^2_9$?

(H) *There is an* **elliptic curve** *in* $\mathbb{C}P^2$, *i.e. a torus determined by a third degree equation, such that the 9 Veronese images of its 9* **inflexion points** *determine a linear embedding of* $\mathbf{CP}^2_9$ *as in* (F) .

This is harder, but K-B give these 9 inflexion points explicitly. Recall here that an elliptic curve is isomorphic to a torus as a group, the group action of the curve being given by collinearity of 3 points. The inflexion points are precisely the order 3 points in this group, and constitute a $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Comments

(1) We lack a really conceptual definition of $\mathbf{CP}^2_9$. The group $\mathbb{Z}_3$ (or its square) seems important, and of course it is within the 3-fold join of a 2-simplex, it would be nice if it could have a quick definition as a particular kind of deleted join ... ?

(2) If K is **t-neighbourly** then any linear map of K in a Euclidean space is **t-tight**, i.e. any half-space intersects it in a t-connected thing.

An interesting problem here is whether a **simplicial manifold** K which is t-connected, $t \leq n/2$, can be retriangulated in a t-neighbourly way ?

(3) If **dimension of deleted join** of K is ≤ s, then the **secant space** of any linear embedding of K has dimension ≤ s. Note this space is made up of **van Kampen**'s "forbidden directions", and by projection along a different direction, K embeds into s-space.

The analogous problem here would roughly be whether a simplicial manifold linearly embeddable in s-space can be retriangulated to make its deleted join of dimension ≤ s ?

SHIFTING AND MATROIDS

Let T be a finite subset of a vector space. Then

(i) M, *the family of all linearly* **independent subsets** *of* T, *is closed with respect to* ⊆, *and any such independent set can always be augmented to a bigger one by adding a suitable element of another given bigger independent subset of* T.

(ii) $\mathcal{M}$, *the family of all* **circuits**, *i.e. minimal linearly dependent subsets, of* T, *has no proper* ⊆ *relations, and if two distinct circuits contain a common element, then their union contains a circuit not containing this element.*

Following **Whitney**, 1935, any family M, resp. $\mathcal{M}$, of finite subsets of a set T, which obeys (i), resp. (ii), is called a **matroid**. Note that the **simplicial complex** M, and the **clutter** $\mathcal{M}$ determine each other.

Kalai's matroid-theoretical generalization of exterior shifting:

(A) Iterated determinants. Given N^2 **variables** X_{ij} with indices $i,j \in \{1,2,...,N\}$, we associate, to any two equicardinal sets $\sigma, \theta \subseteq \{1,2,...,N\}$, the determinant

$$X_{\sigma\theta} = \det\,(X_{ij} : i\in\sigma,\ j\in\theta).$$

Likewise, we associate, to any two (lexicographically ordered) equicardinal families L,K of equicardinal sets, the determinant

$$X_{LK} = \det\,(X_{\sigma\theta} : \sigma\in L,\ \theta\in K).$$

Note that these successive determinants are elements of $\mathbb{Z}\,[X_{ij}]$, the ring of all integral polynomials in the N^2 variables.

(B) Generic matroid. *To any family* K, *of* t-*subsets of* $\{1,2,...,N\}$, *we associate the* **matroid** M(K), *on the set* T *of all* t-*subsets of* $\{1,2,...,N\}$, *whose maximal independent sets are all* L*'s with* X_{LK} *nonzero.*

[More generally given any prime p we can postulate X_{LK} nonzero mod p.]

Let Λ denote the exterior algebra over $\mathbb{F} = \mathbb{Q}(X_{ij})$ generated by $v_1, v_2, \ldots, v_N$, and $C(K) \subseteq \Lambda$, its vector subspace spanned by exterior monomials in the v_i's which are supported on K.

Then the canonical projections of all the degree t exterior monomials in the letters $x_i = \sum_j X_{ij} v_j$ determine a spanning set of C(K) which we will identify with T in the obvious way.

Clearly the generic matroid M(K) of K coincides with the matroid of all linearly independent subsets of the **generic spanning set** $T \subset C(K)$.

(C) *The generic matroid* M(K) *is* **symmetric**, *in the sense each permutation of* (1,2,...,N) *induces a simplicial automorphism of the simplicial complex* M(K).

This is clear from the definition of M(K).

(D) **Lexicographically minimal basis** $\Delta(M)$, *of any symmmetric matroid* M, *on the set* T *of all cardinality* t-*sets of* (1,2,...,N), *is* **shifted**, *i.e. is closed with respect to the product partial order of* t-*sets*.

Here of course "basis" means a maximal independent set and the "lexicographically minimal" basis is obtained by seiving out t-sets depending (in the obvious abstract sense) on lexicographically previous t-sets. It need not be the lexicographically smallest basis.

The above result follows because the shuffle permutation proof generalizes at once to the above abstract situation.

(E) **Growth of shifted families.** *Any shifted family* Δ *of cardinality* t-*sets of* $\mathbb{N}$ *has only finitely many* **generators**, *i.e.* **minimal** *sets* S *whose augmentations by any* t-|S| *bigger numbers are all in* Δ. *Thus shifted families have polynomial growth.*

The first part though surprising is very easy and at once gives the formula

$$|\Delta_N| = \Sigma_S \binom{N-\max(S)}{t-|S|},$$

where Δ_N is the shifted subfamily of Δ with all vertices in (1,2,...,N), and the summatiom is over this fixed finite set of generators S of Δ. This establishes the second part.

(F) **Growth of symmetric matroids.** *Let* M *be a symmetric matroid on the set* T *of all* t-*sets of* $\mathbb{N}$, *and let* M_N *be its restriction to the set* T_N *of all* t-sets of (1,2,...,N). *Then* $rank(M_N)$ *is a polynomial function of* N.

Here "rank" of course means cardinality of any basis of the matroid. The result follows from (E) and (D). But note that in (D) one should use the *reverse lexicographic order* in which T_N is an initial set: so the $\Delta(M_N)$'s will now arise by restricting a shifted family Δ of t-sets of $\mathbb{N}$ to $\{1,2,\ldots,N\}$.

(G) Shifting is a projection. *If* K *is already shifted, then* $\Delta(K) = \Delta(M(K)) = K$. *So* $\Delta(\Delta(K)) = \Delta(K)$.

This follows at once from the next result.

(H) Some determinantal identities. *If* K *and* L *are equicardinal shifted families of* t-*sets of* $\{1,2,..,N\}$, *with determinant* X_{LK} *nonzero, then we must have* $L = K$ *and*

$$X_{KK} = \prod_{i=1}^{N} (X_{\mathbf{ii}})^{(|St_i K| - |St_{i+1} K|)} .$$

Here we use the notation of (A) and $\mathbf{i} = \{1,2,..,i\}$. When K consists of *all* t-sets of $\{1,2,\ldots,N\}$, then above formula is due to **Sylvester**.

This result is proved by the interesting device of *connecting* the generic matrix to the identity matrix by **elementary row/column operations**. The result is true of course for the identity matrix. Shifted hypothesis comes in to show that X_{LK} does'nt change when a row/column is added to another.

(I) Shifted families are well-quasi-ordered by inclusion, *i.e. in any infinite sequence of finite shifted families of* t-*sets, some family must be contained in some succeeding family.*

Kalai observes that this follows immediately from a lemma of **Higman**.

Comments

(1) Kalai's paper has *lots* of additional results and information. For example he gives a complete *characterization of the growth polynomials* of symmetric matroids.

(2) The **blocker** (or *Alexander dual* à la **Bier**) of the basis of a matroid is the basis of the **dual matroid**. Above shifting process is well behaved vis-à-vis this duality.

[Another new word for a known thing is **threshold graph**: which coincides with *shifted graph*. Still another: the **compression** process in **Bollobas**' book is *combinatorial shifting*.]

(3) The w.q.o property of shifted complexes might be useful for the *Kuratowski problem* ?

(4) Consider the 3-step heirarchy: vertices, (semi)simplices = ordered sets (sequences) of vertices, **hyper(semi)simplices** = ordered sets (sequences) of (semi)simplices. Visualizing these hypersemisimplices as **joins** of their constituent (semi)simplices we thus see that the objects **K, L**, in the **4th step** of this heirarchy, i.e. *sets of sets of sets of vertices, are also natural geometric objects*.

[However none of the total orders (lexicographic, reverse lexicographic, etc.) on simplices is quite "natural" with respect to the given total order on the vertices. On the other hand the product partial order is: so maybe one should think of "hypersimplices" as sets of simplices *partially ordered* by this, and then, to visualize them, join any two which are comparable ?]

Shifting theory at 4th step gets much more exciting: e.g. besides *distinctness* of the constituent simplices of a hypersimplex [which suggests using exterior algebra over Λ, i.e. the next level of determinants X_{LK}] we can also consider their *disjointness* [now some other quotient of the tensor algebra over Λ, e.g. the **star algebra**, is required] which leads to **deleted joins**, etc.

(5) The connections between shifting and the theory of **polynomial identities** should also be worth looking into ?