CYCLIC COHOMOLOGY

This was introduced in Connes' great paper "Non-commutative geometry", but we will look first at his later *Comptes Rendus* version.

Originally he had used a variant of the **Hochschild** coboundary applied directly to algebra cochains which are cyclic. Now he splits up this construction neatly into 2 parts: an algebraical part which can be viewed as an abelianization, and the remaining purely combinatorial part.

The key idea which emerged from this analysis is that **the usual grading with degrees should be replaced by a more exotic grading.** More precisely, instead of working with **simplicial objects**, i.e. functors from the category Δ of numbers, we work with **cyclic objects**, which are functors from the following bigger category Λ.

(A) Cyclic category of numbers Λ. Its objects too are numbers $\mathbf{n}$, only there will be more morphisms than before.

To define these morphisms, lets regard $\mathbf{n}$ *not* as the totally ordered set $\{0,1,..,n\}$, but instead as the following subset of the unit circle of the complex plane.

$$\mathbf{n} = \Lambda_n = \omega_{n+1}\cdot(0,1,..,n,\alpha_n) \subset S^1 \subset \mathbb{C}.$$

Here ω_{n+1} denotes the (n+1)th **root of unity** $\exp(\frac{2\pi\sqrt{-1}}{n+1})$, and α_n is any chosen real between n and n+1. The remaining elements of $\mathbf{n}$ constitute the **cyclic group** of (n+1)th roots of unity which will be denoted by $\mathbb{Z}_{n+1}$.

We equip S^1 with the counterclockwise orientation. Consider now any **increasing continuous degree 1 map**

$$\phi: S^1 \longrightarrow S^1,$$

subject to the condition $\phi(\mathbb{Z}_{n+1}) \subseteq \mathbb{Z}_{m+1}$. The **morphisms**

$$f: \mathbf{n} \longrightarrow \mathbf{m}$$

will be **homotopy classes** $[\phi]$ of such maps ϕ. The restriction of ϕ to $\mathbb{Z}_{n+1}$, which obviously depends only on its homotopy class f, will be denoted by

$$\ddot{f}: \mathbb{Z}_{n+1} \rightarrow \mathbb{Z}_{m+1}.$$

Unless it is a constant map, $\ddot{f}$ *conversely determines* f. This follows because any $\phi \in f$ has to be constant on the circular intervals corresponding to the **cyclic partition**

$$\mathbb{Z}_{n+1} = \bigcup_{j\in\mathbb{Z}_{m+1}} (\ddot{f})^{-1}(j).$$

And, there are exactly n+1 *f's corresponding to a constant* $\ddot{f}$, because now a $\phi \in f$ is constant on all but one of the n+1 equal circular intervals determined by $\mathbb{Z}_{n+1}$, and the image of this exceptional interval is all of S^1.

If $\phi(\alpha_n) = \alpha_m$ then, considered as a map from {0,1,..,n} to {0,1,..,m}, $\ddot{f}$ is **increasing**, and all such increasing maps {0,1,..,n} → {0,1,..,m} occur as such $\ddot{f}$'s. This **subcategory** Δ **of** Λ is thus isomorphic to the category of numbers considered before.

Any f can obviously be represented by the composition of a ϕ satisfying $\phi(\alpha_n) = \alpha_m$, preceded by a **rotation** $\mathbb{Z}_{n+1} \to \mathbb{Z}_{n+1}$. The **subcategory** K **of** Λ will consist of all such rotations [*not* group automorphisms!] i.e. K consists of all **isomorphisms** of the category Λ. Since such a factorization is unique upto homotopy we thus see that

$$\Lambda = \Delta K.$$

Connes also gives a dual splitting of Λ: see comment (1) below.

(B) Cyclic vector spaces. By this we mean any covariant functor functor from Λ to the category of vector spaces over field $\mathbb{F}$. The natural transformations of such functors yield the *category of all cyclic vector spaces over* $\mathbb{F}$.

The example below, which constitutes the aforementioned **abelianization**, will show that cyclic cohomology of algebras is only a special case of the present more general theory.

Each associative $\mathbb{F}$-algebra A yields a cyclic space $A^{\#}$ as follows:

The vector space associated to each number **n** is $A^{\#}_n$ = the *n-fold tensor product of* A, while the linear map

$$f^{\#}: A^{\#}_n \longrightarrow A^{\#}_m$$

associated to an f: **n** → **m** is given by

$$f^{\#}(x_0\otimes x_1\otimes..\otimes x_n) = \otimes_{j=0}^{j=m} (\prod_{i\in(\ddot{f})^{-1}(j)} x_i),$$

where [*most* important!] the algebra product $\prod$ is to be taken in the order dictated by the orientation of the circle. Of course, if A is commutative, this last point is of no importance, and then only $\ddot{f}$ matters.

Being now in an abelian category [of cyclic vector spaces] we can talk of cohomology, and the calculation of the *zeroth cohomology* is very easy.

Proposition 1. *$\mathrm{Hom}_\Lambda(A^\#, \mathbb{F}^\#)$ is isomorphic to the vector space of all traces of* A.

Here Hom is in this category of cyclic vector spaces, and **trace of A** is any symmetric multilinear map from A into $\mathbb{F}$.

In the original construction cyclic cohomology arose as a generalization of traces, so the above suggests the following generalization.

Theorem 1. *The cyclic cohomology of* A *is naturally isomorphic to the cohomology of* $A^\#$.

Here of course we are speaking of **cohomology** *of* $A^\#$ in its category theoretical sense, i.e. as the **derived functors** $\mathrm{Ext}^n_\Lambda(A^\#, \mathbb{F}^\#)$ of $\mathrm{Hom}_\Lambda(A^\#, \mathbb{F}^\#)$.

To prove this theorem Connes will show that these derived functors, of the category of all cyclic vector spaces, can be computed combinatorially in such a way that, for the case of cyclic vector spaces arising from algebras, one recovers the original definition of cyclic cohomology of algebras.

(C) The double complex $\mathcal{C}$. To any pair of integers (n,m) in the closed first quadrant we associate the cyclic vector space C^m spanned by all morphisms of Λ having domain **m**.

Here the cyclic structure of C^m is the obvious one: to each number k is associated its subspace C^m_k spanned by all arrows ending in **k**, and to each f: **k** $\longrightarrow$ **k'** is associated the linear map $C(f): C^m_k \longrightarrow C^m_{k'}$, defined by composition. For example each basic **rotation** $\omega_{m+1}: \mathbf{m} \longrightarrow \mathbf{m}$ gives a linear isomorphism $C(\omega_{m+1})$ of C^m which moves only the summand C^m_m .

For m even, the **horizantal differentials** $(n,m) \xleftarrow{d_1} (n+1,m)$ will be the cyclic vector space maps [in C^m] which equal $\mathrm{Id} - C(\omega_{m+1})$ for n even, and $\mathrm{Id} + C(\omega_{m+1}) + (C(\omega_{m+1}))^2 + \ldots + (C(\omega_{m+1}))^m$ for n odd. For m odd, we replace each $C(\omega_{m+1})$ in this definition by $-C(\omega_{m+1})$.

The d_1*-homology is trivial, except for the first column, at whose* (0,m)*th spot, we get the cokernel of the map* $\mathrm{Id} \mp C(\omega_{m+1})$ *of* C^m.

One can think of this as **cyclically oriented chains**, its dual being the

cyclic cochains.

For n *even*, the **vertical differentials** $\overset{(n,m+1)}{\underset{(n,m)}{\downarrow d_2}}$ will be the cyclic vector space maps given by the alternating sum, *from* 0 *to* m+1, of the cyclic vector space maps $C(F_i)$: $C^{m+1} \rightarrow C^m$, induced by composition from the **face maps** F_i: **m** $\rightarrow$ **m+1**, i.e. the monotone injection which forgets the ith spot of **m+1**. *For* n *odd*, replace $C(F_i)$ by $-C(F_i)$, and also do summation only *from* 0 *to* m.

Thus for n even the vertical differential is the **ordinary coboundary**, while for n odd it is slightly different. The point is that, *with these definitions, one has the requisite relations*,

$$d_1d_1 = d_1d_2 + d_2d_1 = d_2d_2 = 0,$$

of a double complex.

Thus the homology of the double complex coincides with the ordinary homology of the cyclic chains. But this is easily checked to be chain complex of a ball, so one has the following.

Proposition 2. *The above double complex* C *is a projective resolution of the trivial cyclic vector space* $\mathbb{F}^{\#}$.

The **dual double complex** C^* is thus an *injective resolution* of the trivial cyclic vector space.

We can now compute $\mathrm{Ext}_\Lambda(E,\mathbb{F}^{\#})$, for *any* cyclic vector space E, by calculating the cohomology of the double complex $\mathrm{Hom}_\Lambda(E,C^*)$.

In particular this can be done for $E = A^{\#}$, but still, to prove *T*heorem 1 one needs [it seems] to check that *the* **first spectral sequence** *degenerate at the second term in the following way* : the d^1-cohomology of $\mathrm{Hom}_\Lambda(E,C^*)$, $E = A^{\#}$, is trivial outside the first column ? For C^* itself it was so, because the *cyclic action was* **free** : is this so for all E (of type $A^{\#}$)?

(D) Connes' exact sequence. To wind up Connes also formulates a generalization of his famous exact sequence but he works with $\mathrm{Ext}_\Lambda(F^{\#},E)$'s only: is this so because these $\mathrm{Ext}_\Lambda(F^{\#},E)$ and $\mathrm{Ext}_\Lambda(E,F^{\#})$ are dual in some obvious way ?

The **connecting homomorphism** will be the degree 2 map

$$S: \mathrm{Ext}^k_\Lambda(F^\#, E) \longrightarrow \mathrm{Ext}^{k+2}_\Lambda(F^\#, E)$$

available because of the obvious periodicity $(n,m) \longleftrightarrow (n,m+2)$ in the double complex $\mathrm{Hom}_\Lambda(C,E)$ whose cohomology is $\mathrm{Ext}_\Lambda(F^\#,E)$.

Proposition 3. *The* d^2*-cohomology of the double complex* $\mathrm{Hom}_\Lambda(C,E)$ *is zero in all odd columns, and the remaining even columns yield the ordinary cohomology* $\mathrm{Ext}_\Delta(F^\#,E)$ *of* E *regarded as a simplicial vector space.*

In fact, in odd columns, a cochain contraction is induced by the monotone surjection $\mathbf{m+1} \to \mathbf{m}$ which repeats the last number $m+1$. Since the even column's vertical differential is ordinary the result follows.

Thus we have a **second spectral sequence** *whose first term is the ordinary cohomology of* E *and whose final term is the cyclic cohomology of* E.

Then, the **knight moves**, i.e. second term differentials of this spectral sequence, are used to define a map,

$$B: \mathrm{Ext}^k_\Delta(F^\#, E) \longrightarrow \mathrm{Ext}^{k-1}_\Lambda(F^\#, E)$$

which is of degree -1.

Lastly, using the natural inclusion $\Delta \subset \Lambda$, one has the **restriction map**

$$I: \mathrm{Ext}^k_\Lambda(F^\#, E) \longrightarrow \mathrm{Ext}^k_\Delta(F^\#, E).$$

Theorem 2. *The sequence* $\ldots \circ S \circ B \circ I \circ S \circ \ldots$ *is an exact sequence.*

Thus this long exact sequence is like a **Gysin** sequence of above spectral sequence relating ordinary and cyclic cohomologies.

Comments

(1) Connes also checks the important point that the **classifying space of the category Λ is the infinite complex projective space**

$$\left(S^1 * S^1 * S^1 * \ldots\right) \div S^1$$

We don't remember the definition of this classifying space — idea of **Grothendieck**, very well treated in a paper of **Segal** — but its cohomology seems to be $\mathrm{Ext}^*_\Lambda(\mathbb{Z}^\#, \mathbb{Z}^\#)$, which Connes shows is the required polynomial algebra on a degree 2 element.

Note here that all the definitions of this note work even for rings $\mathbb{F}$, e.g. $\mathbb{Z}$, and one now speaks of **cyclic $\mathbb{Z}$-modules** etc.

(2) There is also a **subcategory Δ^* of Λ** which is isomorphic to the **opposite category** of Δ — this is *not* all decreasing maps, which don't

form a category — with respect to which one has the splitting $\Lambda = K\Delta^*$. Thus, *unlike* Δ, *the category* Λ *is isomorphic to its opposite category*. Hence cyclic and cocyclic objects are essentially same.

(3) The dual **cyclic homology** was discovered independently, at about the same time, by **Tsygan**, who also used an analogous double complex. Also, **Kasparov's KK theory**, which came perhaps even before Connes' definition of cyclic cohomology, is closely related: though not explicit in this note, it seems that these KK(A,B), which were originally defined quite differently, are, or are closely related to, $\mathrm{Ext}^n_\Lambda(A^\#, B^\#)$?

(4) *The above double complex* $\mathcal{C}$ *and the* **Smith Richardson** *sequences are very close:* iterating the short SR sequence we get differentials, like the d_1's above, between a sequence of chain complexes, each of which can be written vertically, so d_2 comprises of the boundaries of these complexes. Likewise the long exact sequence of Connes is very close to the long exact SR sequence.

(5) The above suggests that for **deleted joins over** S^1 one should use the rotations of the circle to look at cyclic cochains whose cohomology should contain obstructions to embeddability coming from repeated applications of the connecting homomorphism S ?

CARTAN-EILENBERG

This book deals with **(co)homology** [$= \frac{\ker(d)}{\mathrm{im}(d)}$, where $d: V \longrightarrow V$ obeys $d^2 = 0$], and generalities re this notion, including spectral sequences, are dealt with in Chapters IV and XV.

The main contribution of this classic however was to introduce a special class of (co)homologies, called **derived functors**, *and to show that all the extant (co)homology theories were in fact derived functors.*

Cartan and Eilenberg discovered derived functors in the course of recasting **Kunneth**'s calculations, of the Betti numbers of a [tensor] product, into a group theoretic language. Their first definition — see Chapter III on **satellite** functors — was iterative, as a measure of the original functor's non exactness.

But they soon replaced it by the (now well-known) definition via **resolutions** which is based on the following fundamental fact.

Theorem (A). *Suppose that the following sequences of linear maps are exact.*

$$0 \longleftarrow A \longleftarrow X = 0 \longleftarrow A \longleftarrow X_1 \longleftarrow X_2 \longleftarrow X_3 \longleftarrow \ldots.$$

$$0 \longrightarrow C \longrightarrow Y = 0 \longrightarrow C \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow \ldots.$$

Then the cohomologies obtained, from Hom(X,Y), Hom(X,C), *or* Hom(A,Y), *by* *using the obvious coboundaries, are isomorphic to each other, and depend only on* A *and* C

The above cohomology is denoted Ext(A,C), the "derived functor" of Hom(A,C), with which it coincides in dimension zero:

$$\mathrm{Ext}^0(A,C) = \mathrm{Hom}(A,C).$$

An exact sequence **X**, which "enters" A as above, is called a **projective** resolution of A, while **Y**, which "leaves" C, is called an **injective** resolution of C.

The importance of Ext *stems from the fact that* any *resolutions* **X** *and/or* **Y** *can be used: this makes it eminently computable !*

Likewise Tor(A,C) arises from the tensor product, by taking any projective resolutions **X** and **Y**, of one or both of A and C, and computing the *homology* of **X**⊗**Y**, or **X**⊗C, or A⊗**Y**, with respect to the obvious boundaries.

[Likewise again for other functors .... but the cohomology Ext suffices to see what goes on.]

We now turn to the explicit resolutions and complexes by means of which C-E proved that some well-known combinatorial definitions were particular cases of this categorical notion Ext.

[Throughout this we'll try to restrict to the simple case of vector spaces over a field $\mathbb{F}$, since this already seems to cover most of what is interesting.]

In the following $\mathrm{Ext}_\Omega(W,V)$ will be in the category C_Ω of all vector spaces equipped with a *left* action of an **associative algebra** Ω over $\mathbb{F}$. Here $\mathbb{F}$ will be regarded as an object of the category under a given multiplicative epimorphism or **augmentation** $\Omega \longrightarrow \mathbb{F}$. Mostly we'll be interested in just $\mathrm{Ext}_\Omega(\mathbb{F},V)$.

(B) Koszul complex. *Assume that* Ω *is the* **commutative algebra** *generated by* N *letters* x *over* $\mathbb{F}$. *Then* $\mathrm{Ext}_\Omega(\mathbb{F},V)$ *is the cohomology of*

$$\ldots \longrightarrow C^t(V) \longrightarrow C^{t+1}(V) \longrightarrow \ldots ,$$

Here C(V) *denotes the graded vector space of all skewsymmetric functions from* N *variables* y *to* V, *equipped with the* Ω *action induced from that of* V, *and the coboundary is given by*

$$(\delta f)(y_1, \ldots, y_{t+1}) = \Sigma_{1\le j\le t+1}(-1)^{j-1}\, x_j.f(y_1, \ldots, y_{j-1}, \hat{y}_j, \ldots, y_{t+1}).$$

This follows from the definition of Ext because the augmentation extends to the exact sequence

$$0 \longleftarrow \mathbb{F} \longleftarrow \Omega \longleftarrow \Omega\otimes\Lambda^1 \longleftarrow \Omega\otimes\Lambda^2 \longleftarrow \quad \dots ,$$

where Λ denotes the **exterior algebra** generated by N letters y over $\mathbb{F}$, and the boundary is given by

$$\partial(\omega\otimes y_1\otimes \dots \otimes y_{t+1}) = \Sigma_{1\le j\le t+1}(-1)^{j-1}.(\omega x_j)\otimes y_1\otimes \dots \otimes y_{j-1}\otimes y_{j+1}\otimes \dots \otimes y_{t+1} .$$

[Note that $\Omega\otimes\Lambda$, the tensor product of the *symmetric algebra* Ω over a vector space, with the *skewsymmetric algebra* Λ over another copy of the vector space, is also called a **Weil algebra**, and figures in the theory of connections.]

(C) A change of categories. Each resolution, $0 \longleftarrow \mathbb{F} \longleftarrow \Omega \longleftarrow \mathbf{X}$, of $\mathbb{F}$ in C_Ω, corresponds to a resolution $0 \longleftarrow \mathbb{F} \longleftarrow \Omega^* \longleftarrow \mathbf{X}^*$ of $\mathbb{F}$ in $C_\Omega{}^*$, where Ω^* denotes the **opposite algebra** of Ω, and so, on applying $\Omega\otimes_{\mathbb{F}}(.)$, gives rise to a resolution $0 \longleftarrow \Omega \longleftarrow \Omega\otimes_{\mathbb{F}}\Omega^* \longleftarrow \Omega\otimes_{\mathbb{F}}\mathbf{X}^*$ in $C_{\Omega\otimes_{\mathbb{F}}\Omega}{}^*$.

Conversely, by applying $\mathbb{F}\otimes_\Omega(.)$ to the last resolution, one recovers that of $\mathbb{F}$ in $C_\Omega{}^*$, and so the original one of $\mathbb{F}$ in C_Ω.

One can think of $C_\Omega{}^*$ as the category of vector spaces equipped with a *right* Ω action, and of $C_{\Omega\otimes_{\mathbb{F}}\Omega}{}^*$ as that of vector spaces equipped with *left and right* Ω actions.

Thus, for any Ω, the calculation of $\mathrm{Ext}_\Omega(\mathbb{F},.)$ is equivalent to that of $\mathrm{Ext}_{\Omega\otimes_{\mathbb{F}}\Omega}{}^*(\Omega,.)$, for which one has the following.

(D) Hochschild complex. *For any* Ω, *and any* V *equipped with a two sided action of* Ω, $\mathrm{Ext}_{\Omega\otimes_{\mathbb{F}}\Omega}{}^*(\Omega,V)$ *is the cohomology of*

$$\dots \longrightarrow \mathcal{E}^t(V) \longrightarrow \mathcal{E}^{t+1}(V) \longrightarrow \dots$$

Here $\mathcal{E}(V)$ *denotes all multilinear functions from* Ω *to* V, *and the coboundary is defined by*

$$(\delta f)(y_1,\dots,y_{t+1}) = \Sigma_{1\le j\le t}(-1)^{j-1}f(y_1,\dots,y_{j-1},y_j y_{j+1},y_{j+2},\dots,y_{t+1}).$$

This follows from the definition of $\mathrm{Ext}_{\Omega\otimes_{\mathbb{F}}\Omega}{}^*(\Omega,V)$, because the sequence $0 \longleftarrow \Omega \longleftarrow \Omega\otimes_{\mathbb{F}}\Omega^*$ of $C_{\Omega\otimes_{\mathbb{F}}\Omega}{}^*$ elongates to the exact sequence,

$$0 \longleftarrow \Omega \longleftarrow \Omega\otimes\Omega \longleftarrow \Omega\otimes\Omega\otimes\Omega \longleftarrow \Omega\otimes\Omega\otimes\Omega\otimes\Omega \longleftarrow \dots ,$$

where the left action is on the first factors and the right action on the last factors, and the boundary is defined by

$$d(y_1\otimes\ldots\otimes y_{t+1}) = \Sigma_{1\le j\le t}(-1)^{j-1}y_1\otimes\ldots\otimes y_{j-1}\otimes y_j y_{j+1}\otimes y_{j+2}\otimes\ldots\otimes y_{t+1}.$$

(E) **Homogenous complex.** *In case* Ω *is a* **group algebra** $\mathbb{F}G$, *the above Hochschild complex for calculating* $\mathrm{Ext}_{\mathbb{F}G}(\mathbb{F},V)$ *is isomorphic to that of all* **left invariant** V *valued functions on* **words** *of* G *equipped with the ordinary coboundary.*

This might have been the genesis of the Hochschild complex: **Hopf** e.g. had defined group cohomology by above complex, and later Eilenberg and **Maclane** had replaced it by its *non homogenous* version, which is just the Hochschild complex for this Ω.

(F) **De Rham complex.** *In case* $\Omega = \mathfrak{g}^e$, *the* **enveloping algebra** *of* $\mathfrak{g}$, *a* **Lie algebra**, *then the Hochschild complex for calculating* $\mathrm{Ext}_{\mathfrak{g}^e}(\mathbb{F},V)$ *retracts to a subcomplex isomorphic to* $\mathfrak{g}^e\otimes\Lambda(\mathfrak{g})$, *fitted with the de Rham derivative as defined by the familiar Lie brackets formula.*

The "enveloping algebra", as against $\mathfrak{g}$, is associative, and vector spaces with $\mathfrak{g}$-action correspond to those with action of this envelope.

C-E prove the above result by using the **Poincare-Witt theorem**, which gives a convenient basis for the enveloping algebra.

Based on pioneering work of **E.Cartan**, the cohomology theory of Lie algebras was given a finished form by **Chevalley**-Eilenberg.

When $\mathfrak{g}$ arises from a Lie group and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the above complex identifies with the usual de Rham complex of left invariant smooth forms on the group.

Comments

(1) The point of (F) of course is that *in the Lie algebra case* the huge Hochschild complex retracts to the much smaller de Rham complex of left invariant forms.

How does (F) fit in with cyclic complex ?

(2) Likewise the point of (B) seems to be that *commutativity* of Ω also enables one to retract the huge Hochschild complex to the much smaller Koszul complex ?

In particular this shows that $\mathrm{Ext}_{\Omega}(\mathbb{F},V)$ vanishes in dimensions > N : a result subsuming **Hilbert's syzygy theorem**.

We note that there is also a variant of Koszul complex if N is infinite.

(3) It might be even simpler to give a similar small complex for the

anticommutative algebras Ω?

Probably just the algebra itself ? [Cf. a very trivial such example, of exterior algebra with $N = 1$, on p.147 of book.]

(4) The relationship of Koszul/Weil complexes to shifting deserves closer inspection.

(5) **C-E is confusing reading**, and it seems it should be possible to learn this [important and apparently not so hard] material faster and better from somewhere (?) else.

The pages of C-E *relevant for above review are:*

p.107 for definitions of Ext and Tor;
p.182 for above kind of "supplemented algebras" Ω,
which are special kind of "augmented rings" of p.143, 146-147;
pp.150-153, 157 for Koszul complexes and Hilbert syzygy theorem;
pp.149-150, 164, 185-186, 193-194 all this (and more!)for the somewhat confusing (and repetitive !)"change of rings" which we have (over ?) simplified to (C) above;
pp. 174-176 for Hochschild complexes including a "normalized" variant;
pp. 189-190 for homogenous complex for groups;
pp. 271-273 for Poincare-Witt,
and pp. 277-282 for de Rham subcomplex for Lie algebra cohomology.

Of course C-E contains many other things too.

ISOTOPY FUNCTORS

Hu's paper is essentially an exposition of Wu 's "deleted functors", with some new frills, which are as under.

A space X is said to have the same **isotopy type** as a space Y if we can find embeddings $f:X \rightarrow Y$ and $g: Y \rightarrow X$ such that both $g \cdot f: X \rightarrow X$ and $f \cdot g: Y \rightarrow Y$ belong to one-parameter families of embeddings containing the identity mappings.

Thus this relation is intermediate to the relations of being homeomorphic and being of the same homotopy type.

(A) **Residual and enveloping functors.** For any $m \geq 1$, and any space X, $R_m(X)$ will denote the complement of the **diagonal** X of the m-fold product $W = X \times .. \times X$, and $E_m(X)$ the space of all paths of W starting from, but never returning to the diagonal.

Note that $R_1(X) = X$ and $E_1(X)$ is the **tangent space** of X, i.e. the space of all paths of X never returning to their initial point.

Each embedding $f: X \rightarrow Y$ determines embeddings $R_m(f): R_m(X) \rightarrow R_m(Y)$ and $E_m(f): E_m(X) \rightarrow E_m(Y)$, with isotopic embeddings inducing isotopic embeddings.

The (co)homology, homotopy groups, euler characteristics, etc. of $R_m(X)$ and $E_m(X)$, are thus invariants of the isotopy type of X.

(B) More generally whenever $X \subseteq W$, we have the **residual space** $R(W,X) = W\backslash X$, and the **enveloping space** $E(W,X)$, the latter of all paths of W which start from but never return to X.

For *polyhedral pairs* $X \subseteq W$, the second notion ties up with the notion of a **regular neighbourhood** V of X in W as under:

Theorem. V *retracts to* X, *and* $V\backslash X$ *is homotopy equivalent to* $E(W,X)$.

[In fact Hu proves it more generally for all "regular embeddings" $X \subseteq W$.]

Thus the enveloping functor is essentially the "local" deleted functor.

(C) Thanks to the last theorem the triad $(W;V,W\backslash X)$'s exact Mayer-Vietoris sequence gives us a relationship between the homologies of X, W, R(W,X), and E(W,X). For example we have

$$\ldots \rightarrow H^n(X^m) \rightarrow H^n(X)\oplus H^n(R_m(X)) \rightarrow H^n(E_m(X)) \rightarrow H^{n+1}(X^m) \rightarrow \ldots .$$

(D) Hu also gives an exposition of the **Wu triangulation** of $K\times\ldots\times K$, and using it, how $R_m(X)$, $X = |K|$, has the same homotopy type as

$$\{\sigma_1\times \ldots \times\sigma_m \mid \sigma_1\cap \ldots \cap\sigma_m = \emptyset \},$$

and $E_m(X)$ the same homotopy type as the bounding "tube" of the above cell complex.

The last two sections of Hu's paper do some easy computations of these deleted functors for *graphs* K.

CYCLIC COHOMOLOGY OF GROUPS

(A) **Cyclic cohomology of a space X.** We'll consider its definition elsewhere, but we note that an important result, obtained independently by **Goodwillie**, **Dwyer** and others [e.g. see comment (5) below] identifies it with the equivariant cohomology [i.e. of left S^1-invariant cochains] of the **loop space** X^{S^1}.

[*Possibly the cyclic cohomology of* X *can be defined simply by using the "cyclic object" obtained by "rotating" the singular simplices of* X ?]

(B) **Cyclic cohomology of a group G.** In light of the above result this, being the cyclic cohomology of the **classifying space** BG [= G*G* ... divided by the diagonal G-action] of the group, has been identified with the equivariant cohomology of the loop space of BG.

[*The analogous question now is whether this can be defined by using the cyclic object obtained by rotating the left invariant cochains of the group* G ?]

However in the following we review only the earlier computation of **Burghelea**: so he is computing the cyclic cohomology of the **group algebra** $\mathbb{F}G$, as defined originally by **Connes**, i.e. by applying a **Hochschild** type coboundary to cyclic algebra cochains.

However Connes' later *Comptes Rendus* note was now available, and Burghelea uses its "cyclic objects" profusely, thinking of them in the following useful way.

(C) **Cyclic set.** These are sets X_n, indexed by the natural numbers n, equipped with **face maps** $d_n^i: X_n \to X_{n-1}$, **degeneracy maps** $s_n^i: X_n \to X_{n+1}$, and **rotations** $t_n: X_n \to X_n$. These maps are required to satisfy various **commutation rules**. Leaving aside the usual ones involving only face and/or degeneracy maps, the other rules are:

$$(t_n)^{n+1} = 1, \qquad d_n^i \circ t_n = t_{n-1} \circ d_n^{i-1}, \qquad s_n^i \circ t_n = t_{n+1} \circ s_n^{i-1}.$$

For any $\mathbb{F}$, there is [as in Connes' note] a **double complex** associated canonically with X [even columns being ordinary chain complexes, odd columns acyclic "cones" over these, horizantal arrows suitable sums of rotations, etc.] and its total (co)homology is defined to be the **cyclic cohomology of X over** $\mathbb{F}$.

[As before we'll assume that $\mathbb{F}$ *is field of characteristic zero*, but note that Connes definition, and Burghelea's computations, are done also over integers, etc.]

And [as in Connes' note] one has, besides the spectral sequences of this double complex, also the **Connes sequence** [= a "Gysin" reformulation of one of the spectral sequences] which relates the ordinary and cyclic cohomologies of X over $\mathbb{F}$.

(D) **Cyclic set of a group G.** We'll give this definition of Burghelea only for the important case of an **abelian** G.

We'll in fact define below the *cyclic set* X(G,g) *of the pair* (G,g) for each $g \in G$. The required cyclic set X(G) of G is the disjoint union of these X(G,g)'s as g runs over all elements of G.

[For the non-abelian case g runs over representatives of the conjugacy classes of G, and one takes the disjoint union of the $X(G_g, g)$'s, where G_g denotes the centralizer of g in G.]

$X_n(G,g)$ will consist of all words $x_0 \ldots x_n$ of G with product equal to g, and the face, degeneracy, and rotation maps are defined as follows:

$$d_n^i(x_0 \ldots x_n) = x_0 \ldots (x_{i-1} \cdot x_i)x_{i+1} \ldots x_n, \quad 0 \le i \le n.$$

$$s_n^i(x_0 \ldots x_n) = x_0 \ldots x_i \; 1 \; x_{i+1} \ldots x_n, \quad 0 \le i \le n.$$

$$t_n(x_0 \ldots x_n) = x_n x_0 \ldots x_{n-1}.$$

Here, for i = 0, the right side of the first equation is to be interpreted as $(x_n \cdot x_0)x_1 \ldots x_{n-1}$.

(E) **Theorem.** *There is a natural identification of the spectral [and so Connes] sequences of X(G) over $\mathbb{F}$ with those of the group algebra $\mathbb{F}G$.*

Thus these sequences are direct sums of sequences indexed by [conjugacy classes of] elements of G.

Now he turns to the computation of the cohomology of each summand.

(F) **Theorem.** *If G is abelian and $g \in G$ is of finite order then the cyclic cohomology of X(G,g) is isomorphic to $H(G/g;\mathbb{F}) \otimes H(S^1;\mathbb{F})$. On the other hand if g is of infinite order then it is isomorphic to $H(G/g;\mathbb{F})$. The ordinary cohomology of X(G,g) is, in either case, isommorphic to $H(G;\mathbb{F})$.*

Here $H(\Gamma;\mathbb{F})$ denotes **the cohomology of the group** Γ over $\mathbb{F}$, i.e. the cohomology of its classifying space $B\Gamma$ over $\mathbb{F}$.

Recall also that the classifying space of the circle is is the **infinite dimensional complex projective space**, so the cohomology of S^1 lives only in even dimensions, being one-dimensional in each of these.

Comments

(1) *The relevance of loop spaces to embedding theory:* this is obvious the moment one notices that the loop space of X is the S^1-**fold cartesian product** of X ! There is X sitting in it as its **diagonal**, i.e. as constant maps from the circle to X. So the pair (X^{S^1}, X) is of the same kind as the pairs intervening in **Thom**'s embedding criterion, **Richardson-Smith** theorem etc. [see **Wu**'s book] except that now the group is infinite. The complement of the diagonal, i.e. the non-constant maps from the circle to X, is likewise a **deleted product**, and the smaller space of all one-one maps from the circle to X is a **configuration space**.

(2) It seems that the double cochain complex, and its concomitant spectral sequences, are very natural objects, which should *not* be avoided, e.g. consider the following proposition.

If a morphism of cyclic sets induces an isomorphism in ordinary cohomology, then it also induces an isomorphism in cyclic cohomology.

Proof. The hypothesis says that the morphism is an isomorphism for the second terms of the spectral sequences, so it must be an isomorphism from here on. *q.e.d.*

[Of course the result also follows from the Gysin sequence but surely this spectral argument is more conceptual.]

Recall again that the complex of cyclic cochains also falls out naturally as the first column in the first [and second to last!] term of the other spectral sequence of the double complex.

However, one *can* avoid double cochain complexes and spectral sequences, at the price of using a somewhat contrived [and huge, but single] cochain complex [see pp. 357-358] which also yields the cyclic cohomology of the cyclic set, and fits into a short exact sequence, whose long exact sequence is the Connes sequence.

(3) *For a finite group* G *the characteristic zero group cohomology* $H(G;\mathbb{F})$ *is zero in all dimensions* ≥ 1. This is clear. But note that the **characteristic zero cyclic cohomology of finite groups is non-trivial !** One gets a copy of $H(BS^1;\mathbb{F})$ for each conjugacy class !!

Also we see, in this case, that the much smaller cyclic set X(G,1) already contained all the information got from X(G).

(4) Burghelea defines his X(G) via **groupoids**, but it amounts to the simpler definition given above.

(5) Much more important is the fact, due to Burghelea-**Fiedorowicz**, and used in the proofs of this paper, that **there is a fibration, with fibre and group S^1, over BS^1, canonically associated to each cyclic set.** In fact there is also, conversely, a natural way of thinking of an S^1-space as a cyclic object.

As we'll see in the review of B-F, these constructions generalize Connes result that the classifying space of the cyclic category is BS^1, and lead easily to the identification of cyclic and equivariant cohomologies alluded to in the beginning of this review.

CARTIER'S EXPOSE

This gives very quickly a wealth of information regarding **cyclic (co)homology**.

(A) Traces. For an associative algebra A $[=\Omega^0]$ over $\mathbb{F}$, char$(\mathbb{F}) = 0$, this means any linear form $\tau : A \longrightarrow \mathbb{F}$ obeying $\tau(ab) = \tau(ba)$, and more generally, for an associative *graded* algebra Ω over $\mathbb{F}$, it is any linear form $\Omega \longrightarrow \mathbb{F}$ obeying

$$\tau(ab) = (-1)^{|a||b|}\,\tau(ba).$$

A trace τ of a **differential graded algebra** (Ω,d) is called a **cycle** if it obeys

$$\tau(d\omega) = 0.$$

The terminology is motivated by the fact that, for the graded-commutative **de Rham algebra** $(\Omega(X),d)$ of a smooth manifold X, the trace $\tau = \int_z$, obtained by integrating over a simplicial cycle $z \subseteq X$, satisfies the above condition by virtue of **Stokes' formula**

$$\int_c d\omega = \int_{\partial c} \omega .$$

(B) Universal differential graded algebra $\Omega(A)$ **of an algebra** A, is [somewhat simplistically] defined by

$$\Omega^0(A) = \mathbb{F} \oplus A,\ \ldots,\ \Omega^n(A) = A\otimes..\otimes A \text{ (n times)} \oplus A\otimes..\otimes A \text{ (n+1 times)},$$

the differential $d: \Omega^n(A) \to \Omega^{n+1}(A)$ being the identification of the second summand of the domain with the first summand of the range.

Elements $a_1\otimes..\otimes a_n$ of the first summand of $\Omega^n(A)$ are written $da_1..da_n$, and the elements $a_0a_1\otimes..\otimes a_n$ of the second summand of $\Omega^n(A)$ are written $a_0da_1..da_n$.

Theorem. *The cycles of* $\Omega(A)$ *are in one-one correspondence with cyclic cocycles of* C(A).

Here a $c \in C^n(A)$, i.e. a function of n+1 variables on A, is being called **cyclic** if it skewsymmetric with respect to rotations of the variables, and **cocycles** constitute the kernel of the **Hochschild coboundary** $b: C^n(A) \to C^{n+1}(A)$, defined as the alternating sum of the n+1 "faces" obtained by multiplying some two consecutive variables.

The required isomorphism is obtained by associating to the cycle τ (i.e. a closed graded trace) of $\Omega(A)$ the cochain c of C(A) given by

$$c(a_0,a_1,..,a_n) = \tau(a_0da_1..da_n).$$

(C) Hochschild and cyclic cohomologies. The former H(A,M), for the case of **bimodule** $M = A^*$, where $(afb)(c) = f(bca)$, is defined to be the cohomology of $(C(A),b)$.

The latter HC(A) is defined, by virtue of the

$$\text{"petit miracle"}: b(C_C(A)) \subseteq C_C(A),$$

where $C_C(A)$ denotes the subspace of cyclic cochains, as the cohomology of this subcomplex $(C_C(A),b)$.

(D) Cup product. Given a cycle τ of $\Omega(A)$ and a cycle θ of $\Omega(B)$, $\tau\otimes\theta$ is a cycle of $\Omega(A)\otimes\Omega(B)$. But there is a natural DGA homomorphism $\Omega(A\otimes B) \to \Omega(A)\otimes\Omega(B)$. Composing with it we get a cycle of $\Omega(A\otimes B)$ called the cup-product $\tau\cup\theta$ of τ and θ. This pairing of cyclic cocycles — see (B) — induces the required cup product

$$HC^p(A) \times HC^q(B) \xrightarrow{\ \cup\ } HC^{p+q}(A\otimes B)$$

in cyclic cohomology.

Theorem. *Under cup product the cyclic cohomology* $HC(\mathbb{F})$ *of* $\mathbb{F}$ *is a polynomial algebra generated by the 2-dimensional class represented by the cyclic cocycle* σ *defined by*

$$\sigma(1,1,1) = 1.$$

Furthermore, the cup product $c\cup\sigma$, *of any cyclic cocycle* c *of an* $\mathbb{F}$*-algebra* A, *with this cyclic 2-cocycle* σ *of* $\mathbb{F}$, *is the cyclic cocycle of* A *given by*

$$(c\cup\sigma)(a_0,a_1,\ldots,a_{p+2}) = \Sigma_{1\le j\le p+1}\, c(a_0\cdot a_j\cdot a_{j+1},a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_{p+2}).$$

The **suspension map** $S: HC^p(A) \longrightarrow HC^{p+2}(A)$, defined by $[c] \longmapsto [c\cup\sigma]$, is important for the next definition and for the exact sequence of (F).

(E) De Rham cohomology $H_{DR}(A)$ of A. This is defined (p.129) to be the cokernel of the map $S-1$: $HC^*(A) \longrightarrow H^*(C)$ [and dually, on p.135, as the kernel of the map $1-S$: $HC_*(A) \longrightarrow HC_*(A)$ in cyclic homology].

The general justification for this terminology is that it is essentially — see pp.134-135 — isomorphic to the cohomology of the graded abelianization of the DGA $\Omega(A)$ of (B).

In case A is *commutative* we also have the DGA $(\Lambda(A),d)$ of **exterior differential forms of** A, and $H_{DR}(A)$ is often, e.g. if it is the algebra of smooth polynomial functions on an affine smooth variety X, isomorphic to the cohomology of $(\Lambda(A),d)$: see p.135.

For additional justification see also (G).

(F) Connes' sequence. We have the short exact sequence

$$0 \longrightarrow C_C(A) \xrightarrow{\ \subseteq\ } C(A) \longrightarrow C(A)/C_C(A) \longrightarrow 0$$

of cochain complexes. Its long exact sequence reads

$$\ldots \xrightarrow{\ S\ } HC^p(A) \xrightarrow{\ I\ } H^p(A,A^*) \xrightarrow{\ B\ } HC^{p-1}(A) \xrightarrow{\ S\ } HC^{p+1}(A) \xrightarrow{\ I\ } \ldots$$

by virtue of the following result.

Theorem. $H^p(C(A)/C_C(A))$ *is isomorphic to* $H^{p-1}(C_C(A))$.

This is shown by constructing a cochain epimorphism $C(A)/C_C(A) \longrightarrow C_C(A)$, which lowers degrees by 1, and induces above isomorphism in cohomology. This epimorphism is induced by the epimorphism

$$B: C^p(A) \longrightarrow C_C^{p-1}(A)\ ,\ B := (1 + t_{p-1} + \ldots\ t_{p-1}^{p-1}) \circ s_p \circ (1 - t_p).$$

Here the t's denote signed rotations and s_p replaces last variable by 1.

Also the above epimorphism B induces the map named B in the long sequence. Thus, although the short sequence

$$0 \longrightarrow C_C(A) \xrightarrow{\subseteq} C(A) \xrightarrow{B} C_C(A)[-1] \longrightarrow 0,$$

is only *semi-exact* in the middle, it does nevertheless induce the above long exact cohomology sequence.

If the Connes' sequence is considered as an **exact couple**

$$\begin{array}{ccc} & H(A,A^*) & \\ {\swarrow}^{B} & & {\nwarrow}^{I} \\ HC(A) & \xrightarrow{\quad S \quad} & HC(A) \end{array}$$

then it gives rise to **a spectral sequence** whose final term is the de Rham cohomology $H_{DR}(A)$ of A.

Note that the second term of this spectral sequence is the cohomology of $H(A,A^*)$ under $I \circ B$.

(G) Currents or distributional forms, of a smooth manifold X, are *continuous* linear forms $\tau : \Omega(X) \longrightarrow \mathbb{F}$. [E.g. $\tau = \int_c$ where $c \subseteq X$ is any simplicial chain is a current.] The **boundary of currents** $\partial : C(X) \rightarrow C(X)$ is defined by

$$\langle \partial\tau , \omega \rangle = \langle \tau , d\omega \rangle \quad .$$

and the **homology of the manifold** is that of $((C(X), B)$. [Note that Stokes' formula shows that, for $\tau = \int_c$ one has $\partial\tau = \int_{\partial c}$, which justifies the notation ∂ for boundary of currents also.]

Theorem. *When* A *is the algebra of smooth functions of a smooth manifold*

X, *then* $H_{DR}(A)$ *is isomorphic to the cohomology of* $(H(A,A^*), I\circ B)$, *and this complex* $(H(A,A^*), I\circ B)$ *is isomorphic to the complex* $((C(X), \partial)$ *of currents of* X.

Thus in this case the spectral sequence of (F) degenerates at the second term. Note also that, in the above result, it is understood that *continuous* cochains of A are being used.

(H) **Categorical definitions.** Following **Karoubi**'s simplification of Connes' definition, **cyclic objects** are defined as simplicial objects equipped with rotations obeying required rules.

Using cyclic cochains one defines **cyclic (co)homology** HC(E) **for any cyclic vector space** E. When $\mathbb{Q} \subseteq \mathbb{F}$ [the case being considered] it identifies with the (co)homology of the [by now familiar to us] **double complex of Tsygan**.

There is also **another double complex** — nonzero only on the upper half of the first quadrant and with all vertical differentials b and all horizantal differentials B — whose (co)homology is also HC(E).

Topologically, the categorical approach amounts to the following sharpening of **Moore**'s result [= its first part].

Theorem. *Simplicial homology* H(E) *identifies with the homotopy groups of the geometrical realization* $|E|$ *of* E, *while cyclic homology* HC(E) *identifies with the homotopy groups of a cyclic quotient* $|E|_{cyc}$ *of* $|E|$.

Doubtless this connects to work of **Burghelea-Fiedorowicz**, **Jones**, et al.

Homologically, the categorical approach reduces $HC^*(A)$ and $HC_*(A)$ to $\mathrm{Ext}(A^\#, \mathbb{F}^\#)$ and $\mathrm{Tor}(A^\#, \mathbb{F}^\#)$ as explained in Connes' C.R. note.

(I) **Gelfand's theorem.** *The category of locally compact spaces is equivalent to that of commutative* C^***-algebras**:

$$X \longleftrightarrow C_0(X, \mathbb{C}).$$

Here $C_0(X,\mathbb{C})$ denotes all continuous functions $X \to \mathbb{C}$ which vanish at infinity, and of course X is recovered from this commutative algebra A as the space of its maximal ideals

Note that X is compact iff A has identity. Also **Swan's theorem** tells us that the abelian category of complex **vector bundles** on a compactum X is equivalent to that of **projective modules** over A. From this one gets an equivalence of the topological K-theory K(X) with the algebraical K-theory K(A).

These results [as well as (G) and the case of an algebraic variety X] illustrate the dictum :**if an algebra** A **is commutative there is a reasonable space** X; however, in the non-commutative case, the spaces are usually "unreasonable", and one must work with A directly.

(J) Bott periodicity. The groups $K_i(A)$, $i \geq 0$, of any A as defined by **Quillen**, are the homotopy groups of [the + *modification* (?) of] the classifying space BGL(A) of the infinite general linear group of A. Also defined are the groups $K^i(A)$, $i \leq 0$.

For A a *commutative* C^*-algebra, these $K^i(A)$ coincide with $\pi_i(GL(A))$ which are periodic of period 2.

Read topologically, i.e. as $K^i(X)$, where $X \longleftrightarrow A$, this periodicity says that X and $X \times \mathbb{R}^2$ have isomorphic K-theories. Or, if X is compact, that X and its double suspension $S^2(X)$ have isomorphic K-theories.

So, while considering the extraordinary [= dimension axiom not obeyed] cohomology theory $K^i(X,Y)$, we can restrict ourselves to $i = 0$ and $i = -1$. Thus the long exact sequences of this theory are exact hexagons.

Keep in mind: $K(X) = K^0(X)$ is the **Grothendieck** group of the abelian category of complex vector bundles on compactum X, in fact it is a ring under product induced by $\otimes$. And $K^{-1}(X)$ is the same thing for the suspensiom SX of X.

(K) Chern character. *There is a functorial ring isomorphism*

$$ch: K(X)\otimes\mathbb{Q} \longrightarrow H^{even}(X;\mathbb{Q}).$$

For line bundles L this map goes $L \longmapsto \exp(c_1(L))$ where $c_1(L)$ is the two dimensional **characteristic class** of the line bundle.

The K-theoretic interpretation of $H^{odd}(X;\mathbb{Q})$ now follows at once by using SX because we have

$$H^i(X) \cong H^{i+1}(SX).$$

The next three sections show how Connes and Karoubi define *cyclic characteristic classes*, thereby constructing an analogous "Chern character" from the K-theory K(A) of any A into cyclic cohomology HC(A).

(L) Morita equivalence. *The Hochschild and cyclic homologies of any* $\mathbb{F}$*-algebra* A *are naturally isomorphic to those of the algebras* $M_r(A)$ *of* $r \times r$ *matrices over* A.

The isomorphism is induced by the cochain map defined by

$$\underline{c}(F_0\otimes a_0, \,..\, ,F_p\otimes a_p) = Tr(F_0\cdot\ldots\cdot F_p).c(a_0, \,..\, ,a_p),$$

whenever $F_i \in M_r(\mathbb{F})$, $a_i \in A$.

This Morita equivalence will quickly yield — see (M) below — the cyclic characteristic classes for K(A).

However, to construct the Chern character for all $K^*(A)$, or dually for all $K_*(A)$, we need to observe also that, since $GL(A) \subset M(A)$ = algebra of *all* square matrices over A, we have a homomorphism of the **group algebra** $\mathbb{F}[GL(A)]$ into M(A), and thus, by Morita equivalence, a natural homomorphism of the cyclic cohomology of $\mathbb{F}[GL(A)]$ into that of A.

(M) Cyclic characteristic classes of projective A-modules. Given such a module M we can find some $r \geq 1$ such that M is isomorphic to the image of some projection $P: A^r \longrightarrow A^r$, i.e. some **idempotent** $P \in M_r(A)$.

Now consider the m-dimensional cochain c_m of $M_r(A)$ which is zero unless all the m+1 variables are equal to P. This cochain is cyclic, i.e. obeys $c_m(P, \ldots, P) = (-1)^m c_m(P, \ldots, P)$, iff m is *even*. Furthermore, in this case, using $P \cdot P = P$, it follows that the Hochschild coboundary $bc_m = 0$. The cyclic cohomology classes

$$[c_{2k}] \in HC^{2k}(M_r(A)) \cong HC^{2k}(A),$$

can be verified to depend only on [the stable isomorphism class of] the module E and will be called its cyclic characteristic classes.

The homomorphism ch: $K(A) \longrightarrow HC^{even}(A)$ defined by $[E] \longmapsto [c_{even}(E)]$ is the requisite *Chern character* and is compatible with the degree 2 suspension maps S, thus there is also an induced ch: $K(A) \rightarrow H_{DR}(A)$.

(N) Cyclic cohomology of groups. Let $C^m(G)$ denote all $\mathbb{F}$-valued functions of m+1 variables on G, and let $\delta: C^m(G) \rightarrow C^{m+1}(G)$ be the ordinary coboundary.

Theorem. *The cohomology* HC(G) *of the* **cyclic cochain subcomplex** $(C_C(G),\delta)$ *of* $(C(G),\delta)$ *is related to the cohomology* $H_G(G)$ *of the* **left-invariant subcomplex** $(C_G(G),\delta)$ *of* $(C(G),\delta)$ *by*

$$HC^m(G) \cong \Sigma_j \, H_G^{m-2j}(G).$$

In other words the second spectral sequence, of the Tsygan double complex of the cyclic vector space C(G), degenerates at the second term.

Recall now that $H_G(G)$ coincides with the cohomology of the classifying space BG of G with coefficients $\mathbb{F}$, i.e. the so-called **cohomology of the group** G.

Thus we can use the **Hurewicz homomorphism** to go from the homotopy of

BGL(A), which for G = GL(A) is essentially the higher K-theory of A, to the above left-invariant cohomology, which injects by above theorem, into all HC(G)'s of degrees higher by even numbers.

Theorem. HC(G) *is a direct summand of the cyclic cohomology* HC(𝔽G) *of the group algebra of* G.

Starting from π(B(GL(A))) we can thus reach HC(𝔽(GL(A))), from where we finally go to HC(A) using — see (L) — Morita equivalence. This completes the description of the Karoubi character for all of K(A) [improving on **Dennis**, who had earlier given a homomorphism of K(A) into Hochschild cohomology.]

This character commutes with the degree 2 suspensions S, thus inducing a character in $H_{DR}(A)$.

Comments

There are very many interesting applications of cyclic (co)homology, of which the following seem most appealing.

(1) **Index theory and analysis on foliated manifolds.** Our 1974 thesis was motivated by the problem of finding a **leaf-wise index theorem** for foliated manifolds. To do this it was necessary to define appropriate characteristic classes ...

About 6-8 years later Connes succeeded ... the key idea being to ignore the "unreasonable" quotient space, and instead focus on a natural non-commutative C^*-algebra. Cyclic cohomology was discovered by Connes to solve this problem.

This material is best studied from other sources, e.g. Connes own papers and book, and the book by **Moore-Schochet**, but we note that on p.130 Cartier does take a very quick peep, without giving all definitions, at the interesting example of the **irrational flow** on the 2-torus.

(2) **Loop spaces.** We know already that their equivariant cohomology is a cyclic cohomology. Also, the pair (X^{S^1}, X) is important for embedding theory, and, following **Witten** and **Atiyah**, also for index theory ... as well as for **elliptic cohomology** (?)... which uses **formal groups** (?) for each one of which there is the K-theory of **Morava** (?)...and also loop spaces are basic for the **string theory** (?) of physics.

(3) **Cohomology of Lie algebras.** Since this is dual to de Rham the connection with cyclic cohomologies is obvious. In fact Tsygan discovered HC(A) as the **primitive cohomology** of the Lie algebra $\mathfrak{gl}(A)$. By theorem of **Hopf** the cohomology of $\mathfrak{gl}(A)$ is generated freely by these primitive classes.

Realizations and classifying spaces

(A) Given a finite set [or *simplex*] S let S_* denote [cf. **Eilenberg-Steenrod**, Ch.II] the set of all functions

$$x: S \to \mathbb{R},\ \sum_{v\in S} x(v) = 1,\ x(v) \geq 0,$$

equipped with the topology induced by the metric

$$\| x-y \|^2 = \sum_{v\in S}(x(v)-y(v))^2.$$

[Each x is called a *point* of the simplex S, and its value x(v) on the *vertex* v of σ is called its vth *barycentric coordinate*.]

Furthermore, to each function θ: S $\to$ T between finite sets, is associated the continuous map

$$\theta_*: S_* \longrightarrow T_*,\ (\theta_*(x))(v) = \sum \{x(w): w \in \theta^{-1}(v)\}.$$

Thus lower star is a covariant functor from the category $\mathcal{F}in$ of finite sets into the category $\mathcal{T}op$ of (metric) topological spaces.

(B) However we will interest ourselves mainly in the subcategory $\mathcal{N}$ of $\mathcal{F}in$, whose objects are the sets S = {0,1,..,s} of numbers, and whose morphisms θ: S $\to$ T are order preserving mappings of these sets.

By a **simplicial set** A is meant any contravariant functor

$$A: \mathcal{N} \longrightarrow \mathcal{S}ets,$$

from $\mathcal{N}$ into the category of sets. [Likewise *simplicial spaces*, *simplicial groups*, etc., mean contravariant functors from $\mathcal{N}$ into the categories $\mathcal{T}op$, $\mathcal{G}roups$, etc.] We will also denote A(S) and A(θ): A(T) $\to$ A(S) by S^* and $\theta^*: T^* \to S^*$.

We equip the disjoint union of the product topological spaces $S_* \times S^*$, S $\in$ Obj($\mathcal{N}$), with the equivalence relation = defined by

$$(x,\theta^*(y)) = (\theta_*(x),y),\ x \in S_*\ ,\ y \in T^*\ ,\ (\theta: S \to T) \in \mathrm{Morph}(\mathcal{N}).$$

The quotient space of equivalence classes is called the **realization** of the simplicial set A, and will be denoted by $|A|$. [This is **Segal**'s reformulation of **Milnor**'s definition.]

Furthermore, each morphism Γ: A $\to$ B of simplicial sets, i.e. maps Γ(S): A(S) $\to$ B(S), S $\in$ Obj($\mathcal{N}$), obeying the obvious commutation rules, has the *realization* $|\Gamma|: |A| \to |B|$, which images the equivalence class of (x,y), $x \in S_*$, y $\in$ A(S), to that of (x,Γ(S)(y) $\in S_* \times$ B(S).

Thus realization is a covariant functor from the category of simplicial sets to $\mathcal{T}op$.

(C) For example, each **simplicial complex** K, i.e. a finite set of finite sets closed under subsets, defines a simplicial set A(K) as follows:

Associate to each $S \in Obj(\mathcal{N})$, the set S^* of all sequences $a: S = \{0,1,\ldots,s\} \longrightarrow vert(K)$ with $Im(a) \in K$, and to each $(\theta: T \longrightarrow S) \in Morph(\mathcal{N})$, the map $\theta^*: S^* \to T^*$, $\theta^*(a)(t) = a(\theta(t))$.

Theorem. *$|A(K)|$ is homeomorphic to*

$$K_* = \{x: vert(K) \to \mathbb{R},\ \Sigma_{v \in S}\ x(v) = 1,\ x(v) \geq 0,\ supp(x) \in K\},$$

equipped with the topology induced by the metric

$$\| x-y \|^2 = \Sigma_{v \in S}(x(v)-y(v))^2.$$

[In particular we note that $|A(S)| = S_*$, where the S on the left side denotes the simplicial complex consisting of all subsets of the finite set S.]

Remark. Any set K of simplices defines a subcategory of $\mathcal{F}in$, viz. that whose objects are the [nonempty] simplices of K, with Hom(S,T) empty, unless $S \subseteq T$, when it consists of just this inclusion. For this category K, the next construction is akin to taking the *derived* K' of K.

(D) By a **chain in a category** $\mathcal{C}$ we mean either an object of $\mathcal{C}$ [these are the 0-chains] or a sequence $\theta_1\theta_2..\theta_p$ [its p-chains, $p \geq 1$] of its morphisms, with image of each contained in the domain of the one following it.

The **Hochschild faces** of a chain are obtained by composing any two consecutive morphism or dropping the first and last ones [for p = 1 the first face of $\theta: S \to T$ is the object S and the second T]: thus e.g. the first, second, and third faces of of the 2-chain $\theta_1\theta_2$ are θ_2, $\theta_1 \cdot \theta_2$, and θ_1, respectively.

Also, each p-chain gives rise to p+1 (p+1)-chains by inserting an identity morphism at any spot: e.g. $\theta_1\theta_2$, where $\theta_1 \in Hom(S,T)$ and $\theta_2 \in Hom(T,U)$ gives rise to $I_S\theta_1\theta_2$, $\theta_1 I_T \theta_2$, and $\theta_1\theta_2 I_U$.

Using the well-known equivalent definition of a simplicial set by face and degeneracy operators we have thus defined a simplicial set $\mathcal{N} \longrightarrow \mathcal{S}ets$, called the **nerve** $N(\mathcal{C})$ **of the category** $\mathcal{C}$.

The **classifying space** $B(\mathcal{C})$ **of the category** $\mathcal{C}$ is the realization $|N(\mathcal{C})|$ of this simplicial set. [This is a reformulation of **Segal**'s account of **Grothendieck**'s definition.]

Remark. Each p-chain of $\mathcal{C}$ determines a commutative diagram in $\mathcal{C}$, with as many arrows as edges in a p-simplex: to do this simply think of the other arrows as compositions. Thus the nerve of $\mathscr{E}$ can be looked at as the contravariant functor on $\mathscr{N}$ which assigns, to each $s \geq 0$, the set of all such s-dimensional **simplicial commutative diagrams** of $\mathscr{E}$. Looked at this way the Hochschild faces of a chain correspond to the smaller simplicial commutative diagrams obtained by omitting the incident morphisms of any vertex of such a diagram.

(E) For example each group G determines a category $\mathscr{E}$(G) as follows:

Its objects are integers s, and the set Morph(s,t) of morphisms from s to t is empty, resp. = 1, resp. = G, for s > t, resp. s = t, resp. s < t.

Theorem. *The classifying space of the category $\mathscr{E}$(G) coincides with the classifying space*

$$BG = G*G* \,..\, *G* \,...\, /G$$

of the group G.

This classifying space, the quotient space of the **deleted join** over G of the infinite simplex, was introduced by Milnor.

If one uses instead the category with one object * ,and with Morph(*,*) = G, one obtains $G \times G \times \,..\, \times G \times \,..\,/G$.

Comments

(1) The definition of realization $|A|$ given in (B) makes sense for any contravariant functor A, from any subcategory of $\mathscr{F}in$, to the category $\mathscr{S}ets$ [or to the bigger category $\mathscr{T}op$].

For many purposes, e.g. for defining (co)boundaries, it is the much smaller **subcategory** $\mathscr{N}_0 \subseteq \mathscr{N}$ **of monomorphisms** which matters. [Note that, besides this, each category also has the *subcategory of epimorphisms*.]

For example, a total ordering of the set vert(K) of vertices of a simplicial complex K, gives the contravariant functor $\mathscr{N}_0 \longrightarrow \mathscr{S}ets$, which maps each S to the set of ordered s-simplices of K, and the realization of this is again K_*.

Is there a conceptual definition of (co)boundary in terms of the algebra of $\mathscr{N}_0$ *?*

The bigger subcategory $\mathscr{O}rd$ of $\mathscr{F}in$, obtained by equipping each $S \in$ Obj($\mathscr{F}in$) with a total order, and considering only order preserving morphisms $S \to T$, is also useful. However the realization space of each contravariant functor $\mathscr{O}rd \to \mathscr{S}ets$ can be seen to coincide with that of a corresponding contravariant functor $\mathscr{N} \to \mathscr{S}ets$.

(2) **There are many other interesting functors associated to a simplicial complex**, e.g.

(i) Associate to each $S \in \mathrm{Obj}(\mathcal{N}_0)$ the set S^* of all monomorphisms $S \to$ vert(K) with image in K. [No total order on vert(K) is used.] The realization of this functor $\mathcal{N}_0 \to \mathcal{S}ets$ is not K_* but it has some additional top-dimensional "holes": so a different (co)homology.

(ii) Choosing some total order on vert(K) associate to each $S \in \mathrm{Obj}(\mathcal{N})$ all order preserving maps $S \to$ vert(K) with image in K. This $\mathcal{N} \to \mathcal{S}ets$ is the simplicial set most often associated to K, and its realization is K_*.

(iii) Choose some total order on vert(K), and an $r \geq 1$. Associate to each $S \in \mathrm{Obj}(\mathcal{N}_0)$ all order preserving maps $S \to$ vert(K) with image in K, and such that each vertex repeats at most r times. For $r = 1$ this functor $\mathcal{N}_0 \to \mathcal{S}ets$ coincides with the example [from Eilenberg-Steenrod] given in (1).

The realization space of this contravariant functor seems to depend only on the parity of r.

Thus for all odd r we expect the realization space to be K_*. For even r it would be interesting to work out the realization, for it would give a geometrical basis for the second part of the following.

Bier's theorem. *For* r *odd*, *resp.* r *even, the (co)homology of above functor coincides with the (co)homology* [of the link of the empty simplex] *of K, resp. the* **total (co)homology** *of K, i.e. the direct sum of the cohomologies of all the links of K.*

[Functors (i) and (iii) were considered first by Bier.]

(3) Segal also gives very elegant definitions of some **spectral sequences**, e.g. one generalizing that defined by **Leray** for [open etc.] **coverings** of a space. And also another one, generalizing that of **Atiyah-Hirzebruch**, which relates a generalized (co)homology of a complex to its ordinary (co)homology.

(4) Apparently **Quillen**'s "homotopy colimit" (?) is also nothing but the classifying space of a category. See paper of **Ziegler-Zivjaljevic** for this and applications to complements of affine hyperplane arrangements.

(5) In "Formal theories are acyclic" we used something very close to the simplicial-commutative-diagram description of the classifying space of a category.

CIRCULAR CLASSES

These occur in the very last section (pp.274-282) of **Wu**'s book.

(A) **Definition.** Let $\Omega(M)$ = Emb(D,M) denote **the space of all smooth**

embeddings of the disk $D \subset \mathbb{C}$, in a smooth n-manifold M, topologized as the subspace of C(TD,TM) determined by the *derivative* $f \longmapsto f_*$, and equipped with the *circle action* $(e^{i\theta}f)(z) = f(e^{i\theta}z)$!

The classes in question are the (cup-) powers,

$$o^k(M) \in H^{2k}(\Omega(M)/S^1;\mathbb{Z}),$$

of the characteristic class $o(M) = o^2(M)$ of the associated *principal circle bundle*. Alternatively, these are the usual characteristic classes, in *equivariant cohomology*, of the free S^1-space $\Omega(M)$.

Key observation. *Each smooth embedding* $f:M \to N$ *induces an equivariant map* $\Omega f:\Omega(M) \longrightarrow \Omega(N)$ *under which*

$$(\Omega f)^*(o^k(N)) = o^k(M).$$

So if, e.g., M and N are such that $o^k(N)$ is zero while $o^k(M)$ is not, then M will not embed in N.

Wu turns now to the question of computing these classes.

(B) Equivariant cohomology of $\Omega(\mathbb{R}^m)$**.** Consider the **Stiefel manifold**

$$V(\mathbb{R}^m) = \{(v,w) \in \mathbb{R}^m \times \mathbb{R}^m: \|v\| = 1 = \|w\|,\ \langle v,w\rangle = 0\},$$

of orthonormal 2-frames of $\mathbb{R}^m$. Note that under the free *circle action*,

$$e^{i\theta}(v,w) = (\cos\theta.v + \sin\theta.w,\ -\sin\theta.v + \cos\theta.w),$$

the quotient $V(\mathbb{R}^m)/S^1$ is the **Grassmann manifold** of oriented 2-planes of $\mathbb{R}^m$. Note also that $V(\mathbb{R}^m)$ comes with the *involution* $(v,w) \longleftrightarrow (w,v)$.

Identifying each $(v,w) \in V(\mathbb{R}^m)$ with the restriction to D of the linear embedding $\mathbb{C} \to \mathbb{R}^m$ defined by $0 \longmapsto 0$, $1 \longmapsto v$, and $i \longmapsto w$, we will think of $V(\mathbb{R}^m)$ as an invariant subspace of $\Omega(\mathbb{R}^m)$.

Proposition 1. *There is an equivariant deformation retraction of* $\Omega(\mathbb{R}^m)$ *onto its compact invariant subspace* $V(\mathbb{R}^m)$.

Proof sketch. The first step uses *translation* to retract $\Omega(\mathbb{R}^m)$ onto the subspace $\Omega_0(\mathbb{R}^m)$ of smooth embeddings $D \to \mathbb{R}^m$ which map 0 to 0. Next, for each compactum's worth of embedded 2-disks of $\Omega_0(\mathbb{R}^m)$, we find a common $\epsilon > 0$, such that after *shrinking* by factor ϵ, the flat *projections* of all these 2-disks, on their tangent spaces at 0, are still embeddings. The final step is to *normalize* these flatly embedded 2-disks to linearly embedded 2-disks. ∎

This retraction [which used the fact that the disk embeddings are *smooth*] allows Wu to use the information available about the cohomology of $V(\mathbb{R}^m)$ and $V(\mathbb{R}^m)/S^1$.

Proposition 2. *For the free S^1-space $V(\mathbb{R}^m)$ one has, with either $\mathbb{Z}$ or $\mathbb{Q}$ coefficients,* $o^{m-2} \neq 0$ and $o^{m-1} = 0$. *Also, for m odd and bigger than 2, the rational equivariant cohomology ring of $V(\mathbb{R}^m)$ is generated by o, and the involution of $V(\mathbb{R}^m)$ switches the sign of o.*

The observation of (A) then gives the following vanishing criterion.

Proposition 3. *If the smooth manifold M^n embeds smoothly in $\mathbb{R}^m$ then $o^{m-1}(M) \in H^{2(m-1)}(\Omega(M)/S^1;\mathbb{Z})$ is zero.*

(C) Rational Pontrjagin classes. Let $E \to X$ [here X is any polyhedron] be a *real vector bundle* with *fibre dimension* n, and, with respect to some fibre metric, let V(E) be the associated bundle of ordered orthonormal 2-frames of the fibres, equipped with a fibrewise circle action and involution as in (B).

The *circular class* o(E) of the free S^1-space V(E) can be used to define the rational Pontrjagin classes of $E \to X$ as follows.

Proposition 4. *If n is odd and bigger than 2, then there are classes,*

$$p_k(E) \in H^{4k}(X;\mathbb{Q}),$$

uniquely determined by the equation

$$o^{n-1} - \pi^*(p_1).o^{n-3} + \ldots \pm \pi^*(p_{(n-1)/2}) = 0,$$

satisfied by their pull-backs $\pi^*(p_k)$'s in $H^{2(n-1)}(V(E)/S^1;\mathbb{Q})$.

Proof. Since n is odd and bigger than 2, the cohomology of the fibre $V(\mathbb{R}^n)/S^1$ of $V(E)/S^1$ is a *polynomial algebra* truncated above dimension n-2 by Proposition 2.

So, by the **Leray-Hirsch Theorem**, *any* cohomology class c of $V(E)/S^1$ can be written uniquely as

$$c = \pi^*(c_0) + \pi^*(c_1).o + \ldots + \pi^*(c_{n-2}).o^{n-2}.$$

For the case $c = o^{n-1}$ under consideration, the coefficients of the odd powers of o must be zero in this equation. This follows by applying the involution of V(E) to above equation: o becomes − o, but all coefficients are unchanged, because the involution commutes with π.∎

If n is even and bigger than 2 then $p_k(E)$'s are defined as above by using $E \oplus \mathbb{R}$ instead of E.

The components, of the inverse of the total class $1 + p_1 + p_2 + \ldots$, are called the *dual Pontrjagin classes of* the vector bundle $E \to X$.

Proposition 5. *If the* $(m-1)$*th power of the circular class* $o(E)$ *of a* n*-vector bundle* $E \to X$ *is zero, then its dual Pontrjagin classes vanish in dimensions bigger than* $2(m-n)+1$.

Though we still need to learn its details this follows [Wu, p.281] by "an easy calculation".

[Probably similar to an earlier calculation, which gave vanishing of some dual **Stiefel-Whitney classes**, as a consequence of the vanishing of some mod 2 classes of **Van Kampen**].

Is the converse of the above implication [and likewise of "Van Kampen zero" → "SW zero"] false ?

(D) Smooth embeddability criterion. We now use (C) when X is a smooth n-manifold and E is the *tangent bundle* of M.

Proposition 6. *A smooth* n*-manifold* M *embeds smoothly in* $\mathbb{R}^m$ *only if the* $(m-1)$*th power of the circular class* $o(TM)$ *of its tangent bundle is zero.*

Proof. By using the *exponential map*, in a small neighbourhood of the zero section, we can find an equivariant map

$$V(TM) \longrightarrow \Omega(M).$$

So the result follows from Proposition 3. ■

Now using Proposition 5 one obtains at once **Pontrjagin's embeddability criterion** involving vanishing of the dual tangent classes of M: this also follows from the **Whitney sum formula** which shows that the dual tangent classes are the classes of the *normal bundle* of an embedding.

[In the mod 2 case one had the analogous **Thom embeddability criterion** involving vanishing of the dual Stiefel-Whitney classes of the **local deleted product** of X, i.e. a small deleted neighbourhood of the diagonal of $X \times X$. The analogy is perfect because the tangent bundle TM is nothing but the germ of a neighbourhood of the diagonal of $M \times M$.]

Comments

(1) **A topological version of the above should be very interesting.** For instance, it may be possible [suggestion of **Sullivan**] to arrive at the topological invariance of the rational Pontrjagin classes via these circular classes and cyclic cohomology !

W.-T.Wu confined himself to the smooth case only because **then** there were no tools available to compute [e.g. the equivariant cohomology of the space of all topological embeddings of the 2-disk in a euclidean space] but **now** — with loop spaces everywhere ! — the situation might be very different ?

A natural thing to do in this context might be to use as ΩX the space of all embeddings of the circle in X ? Or, maybe, the space of all non constant maps of the circle into X ? [The latter generalizes notion of deleted product of X, which was all non constant functions $\mathbb{Z}_2 \to X$!]

(2) *The correlation to Wu's* [notation-thick !] *treatment is as follows:*
(A) = pp.274-6 till Theorem 1.
(B) = further till p.277, Corollary [the important retraction of Proposition 1 is on top of p.277].
(C) = further till p.279 end + "easy calculation" remark on p.281 [the casually given Lemma 3 = Proposition 4 is most important: we ignored its generalization, Theorem 3, which is trivial in light of the above retraction].
(D) = remaining [p.280 looks at TM as germ of the diagonal of M×M, and p.281 gives another equivariant map $V(TM) \to \Omega(M)$: also given is the application of Pontrjagin's criterion to embeddability of complex projective spaces in euclidean spaces].

SERRE SPECTRAL SEQUENCES

From **Fuks**' exposition:

(A) BASIC FACTS needed for most applications are the following.

(i) To any **fibration** $F \xrightarrow{\subseteq} E \xrightarrow{\pi} B$ [it suffices to assume that π has the *homotopy lifting property*] one can associate a sequence of **differential groups** [= cochain complexes] (E_i, d_i), $i \geq 0$, with $H(E_i) = E_{i+1}$ for all i, which converge to the additive cohomology H(E) of the total space.

(ii) Each of these *terms* E_i has a **finer grading** than by dimensions m, namely by pairs (p,q) of non-negative integers such that $p+q = m$, and the **differentials** [= coboundaries] d_i, which are pure with respect to this finer grading, have bidegree $(i,-i+1)$.

(iii) The **second term** E_2 is the cohomology of the base space B with coefficients in the cohomologies of the fibres F, so, if B is *simply connected*, one has

$$E_2^{p,q} = H^p(B) \otimes H^q(F).$$

(iv) There is a **product** defined in each E_i, which is graded-commutative with respect to the dimensional grading, and such that each d_i obeys the **product rule** of derivatives.

(v) In E_2 this product is the one determined by the **cup products** of the base and the fibre; however the product of the final term E_∞ can be very different from the cup product of the total space.

(vi) The construction is **functorial**, i.e. a fibre morphism from one fibration to another induces a pull-back morphism in their spectral sequences.

(vii) The projection of H(E) on its summand $E_\infty^{0,*}$, followed by the inclusion $E_\infty^{0,*} \subseteq E_2^{0,*} = H(F)$, coincides with the **map** $H(E) \longrightarrow H(F)$ **induced by fibre inclusion** in the total space.

(viii) The natural quotient map $E_2^{*,0} \to E_\infty^{*,0}$, followed by inclusion in H(E), coincides with the **map** $H(B) \longrightarrow H(E)$ **induced by the projection** π. From this it follows that if a fibration with connected fibre has a **section**, then $E_2^{*,0}$ is final.

(ix) The last non-trivial differentials on the 0th column, i.e. the ones which connect it to the 0th row, are called the **transgressions** of the fibration. These are defined only on a subspace of H(F), with values in a quotient space of H(B), i.e. they are "partial many-valued functions" from H(F) to H(B). Viewed as such they coincide with the composition of [a suitable restriction of] the connecting homomorphism $H(F) \to H(E,F)$ with the correspondence inverse to the map $H(B,pt) \to H(E,F)$ induced by the projection.

We'll sketch constructions of such gadgets in (C) and (D), but first we'll play some

(B) **SPECTRAL TIC-TAC-TOE.** Spectral sequence calculations are done best two-dimmensionally. The simple ones below in fact essentially require only two marks: *crosses* "x" to denote the coefficients $\mathbb{F}$ (say $\mathbb{Z}$), and the *circles* "o" (or empty spots) for 0.

(B1) We start the proceedings with the fibration

$$S^1 \xrightarrow{\subseteq} S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n,$$

whose projection π maps the points $(z_1, \dots, z_n)$, $|z_1|^2 + \dots + |z_n|^2 = 1$, of the unit sphere S^{2n+1} of $\mathbb{C}^n$, to the points $[z_1, \dots, z_n]$ of $\mathbb{C}P^n$. [For n = 1 this is the **Hopf fibration** $S^1 \xrightarrow{\subseteq} S^3 \xrightarrow{\pi} S^2$.]

Proposition 1. *The multiplication of* $H^m(\mathbb{C}P^n)$ *is determined by the above fibration.*

Proof. Since $H^m(\mathbb{C}P^n)$ is $\mathbb{Z}$ or 0, depending on whether m is even or odd, we have

$$E_2 = H^*(\mathbb{C}P^n) \otimes H^*(S^1) = \begin{matrix} x\ o\ x\ o\ \dots\ o\ x \\ x\ o\ x\ o\ \dots\ o\ x \end{matrix},$$

with isomorphisms

$$d_2 :\ E_2^{p,1} \xrightarrow{\cong} E_2^{p+2,0}, \quad 1 \le p \le 2n-2,$$

because only then can we get $E_\infty = H^*(S^{2n+1})$. [So $E_3 = E_\infty$.]

Now if e is a basis element of $H^2(\mathbb{C}P^n)$ [$= E_2^{2,0}$], assume inductively that we have checked that e^{j-1}, $1 \le j < n$, is a basis element of $H^{2j-2}(\mathbb{C}P^n)$ [$= E_2^{2j-2,0}$]. Let g be the basis element of $E_2^{0,1}$ [$= H^1(S^1)$] such that $d_2(g) = e$. Then $g.e^j$ is a basis element of $E_2^{2j-2,1}$, and so

$$d_2(g.e^{j-1}) = d_2(g).e^{j-1} - g.d_2(e^{j-1}) = e.e^{j-1} - g.0 = e^j$$

is a basis element of $H^{2j}(\mathbb{C}P^n) = E_2^{2j,0}$. *q.e.d.*

(B2) The **loop space** ΩX of a path connected space X having a distinguished point x_0, i.e. the space [with the compact-open topology] of all maps $S^1 \to X$ such that $1 \mapsto x_0$, enters in the **Serre fibration**

$$\Omega X \xrightarrow{\subseteq} EX \xrightarrow{\pi} X,$$

where EX is the contractible space of all paths starting at x_0, and π maps any such path to its end point.

The **homotopy groups** of X and ΩX are related by

$$\pi_{m-1}(\Omega X) \cong \pi_m(X),$$

which follows from the long exact **homotopy sequence** of the above fibration.

[In fact the equality $\pi_m(X) = \pi_0(\Omega \ldots \Omega X)$ = the number of path components of the mth **iterated loop space**, is frequently chosen to be the definition of $\pi_m(X)$.]

Proposition 2. $H^m(\Omega S^n) \cong \mathbb{Z}$ *if* m *is a multiple of* n-1 *and is zero otherwise.*

Proof. Since the homotopy [and so (co)homology] of ΩS^n is trivial below dimension n-1, we know the two non-trivial and identical columns, of the second term $E_2 = H^*(S^n) \otimes H^*(\Omega S^n)$ of the Serre fibration of S^n, below height n-1:

$$\begin{matrix} ? & & ? \\ 0 & & 0 \\ & & \\ 0 & & 0 \\ \times & & \times \end{matrix}$$

But $E_2 = E_n$ has to collapse to $H^*(ES^n) = \mathbb{Z}$ under d_n: this forces the ? of the first column to be $\times$, and its next n-2 spots to be o's. The other column being identical, we now know both columns below height $2(n-1)$.

Working upwards like this we obtain the entire first column, i.e. $H^*(\Omega S^n)$. *q.e.d.*

A similar argument shows $H^m(\Omega X) \cong H^{m-1}(X)$, $m \leq 2n-2$, for *any* X whose cohomology is trivial below dimension n.

(B3) Cohomology of classical groups. Recall that SU(n) is the compact (2n-1)-dimensional manifold whose points are all n×n matrices A over $\mathbb{C}$ such that $A\bar{A}^t = I$ [i.e. matrices whose rows are mutually perpendicular vectors of $\mathbb{C}^n$ of length 1] and $\det(A) = 1$.

Proposition 3. *The graded algebra* $H^*(SU(n))$ *is isomorphic to the truncated exterior algebra on generators* $e_2, e_3, \ldots, e_n$ *in dimensions* $3, 5, \ldots, 2n-1$, *respectively.*

Proof. We use induction on n, and the spectral sequence of

$$SU(n-1) \xrightarrow{\subseteq} SU(n) \xrightarrow{\pi} S^{2n-1},$$

where the projection π maps each matrix A to its first row. [$SU(1) = \{1\}$, and, for n = 2, the above π is a homeomorphism, so $SU(2) \cong S^3$.]

The second term $E_2 = H^*(SU(n-1)) \otimes H^*(S^{2n-1})$ consists of two identical columns, and it is clear from the inductive hypothesis, that as a ring it is precisely the exterior ring described: the new e_n sits at the (2n-1,0) spot, and $e_i.e_j = -e_j.e_i$ by cup commutativity.

Thus the required result is equivalent to saying that **the spectral sequence of above fibration degenerates at the second term.**

Obviously we do have $E_2 = E_{2n-1}$. What we need to check is that d_{2n-1} is zero on the 0th column. In fact, since it is a derivation, it is enough to check this on its multiplicative generators $e_2, \ldots, e_{n-1}$: but this follows at once by dimensional considerations. *q.e.d.*

Remark 4. Analogous degenerescence results yield the *cohomology* *algebras of* U(n) [one now has an extra 1-dimensional generator e_1] *and* Sp(n) [when the exterior algebra generators are in dimensions 3, 7, ..].

Integral cohomology of SO(n) is harder, but over the field $\mathbb{F}_2$ of two elements, the same argument, slightly modified, yields the following.

Proposition 5. *The graded vector space* $H^*(SO(n);\mathbb{F}_2)$ *is isomorphic to the the underlying vector space of the truncated exterior algebra over* $\mathbb{F}_2$ *on generators* $e_1, e_2, \ldots, e_{n-1}$ *in dimensions* 1,2, ... ,n-1, *respectively.*

For O(n) = union of two copies of SO(n), one also has an additional zero-dimensional generator e_0.

[Q. Does one in fact have here, and in the generalization given in (B5) below, an algebra isomorphism ?]

Again one proceeds inductively employing the fibrations

$$SO(n-1) \xrightarrow{\subseteq} SO(n) \xrightarrow{\pi} S^{n-1},$$

whose spectral sequences do degenerate at E_2. However now the vanishing of d_{n-1}, on the top most generator e_{n-2} of the 0th column, does not follow by a dimensional argument.

Instead one *compares* with the tangent sphere bundle $V_{n,2} \to S^{n-1}$: this is done by mapping each matrix A of SO(n) to its first two rows, which amounts to the total space $V_{n,2}$ of this tangent sphere bundle, etc.

(B4) Gysin / Wang sequences. For a fibration of which the fibre/base is a sphere [e.g. for each of the above fibrations!] it is possible to summarize [almost] all of the information of its spectral sequence in a long exact Gysin / Wang sequence.

To obtain the Gysin sequence of $S^n \xrightarrow{\subseteq} E \xrightarrow{\pi} B$ note that its spectral sequence has only two non-trivial rows, viz. the 0th and nth rows, and each of these is a copy of H(B). So $E_2 = E_{n+1}$, and its cohomology under d_{n+1} is the final term $E_\infty = H^*(E)$ of the sequence.

The two rows of E_∞ give, for each m, a short exact sequence,

$$\operatorname{coker}(d_{n+1}) \longrightarrow H^m(E) \longrightarrow \ker(d_{n+1}),$$

and concatenating these one gets the required long exact sequence

$$\ldots \longrightarrow H^{m-n-1}(B) \longrightarrow H^m(B) \longrightarrow H^m(E) \longrightarrow H^{m-n}(B) \longrightarrow H^{m+1}(B) \longrightarrow \ldots,$$

in which the connecting homomorphisms are d_{n+1}'s.

The construction of the Wang sequence is exactly similar.

(B5) Cohomology of Stiefel manifolds $V_{n,q}(\mathbb{R}) = V_{n,q}$. These consist of all sequences of length q of orthonormal vectors of $\mathbb{R}^n$, thus at one extreme we have $V_{n,n} = O(n)$, and at the other $V_{n,1} = S^{n-1}$.

Proposition 6. *The graded vector space* $H^*(V(n,q),\mathbb{F}_2)$ *is isomorphic to the underlying vector space of the truncated exterior algebra on generators* $e_{n-q},\dots,e_{n-1}$ *in dimensions* $n-q,\dots,n-1$ *respectively.*

This is proved by an induction on q using the spherical fibration

$$S^{n-q} \xrightarrow{\subseteq} V_{n,q} \longrightarrow V_{n,q-1}.$$

Again, to establish the requiste degenerescence, extra effort is needed on just one generator: this consists of comparing with the case q = 2 of tangent sphere bundle of (n-1)-sphere. Then [over $\mathbb{Z}$] this differential is multiplication by 0 or 2: more generally, for the tangent sphere bundle of *any* M, this differential is multiplication by the **Euler characteristic** of M.

(C) SIMMPLICIAL FIBRATIONS. In case the projection π is a simplicial epimorphism, we refine the dimensional grading $C(E) = \sum_m C^m(E)$, of cochains of E to a finer grading

$$C(E) = \sum_{(p,q)} C^{p,q}(E),$$

where $C^{p,q}(E)$ denotes the subspace of cochains spanned by all (p+q)-dimensional simplices $\sigma \in E$ having **filtration** $\mathrm{filt}(\sigma) = \dim(f\sigma)$ equal to p.

A double complex. The coboundary δ of C(E), which obeys $\delta\circ\delta = 0$, can be written as a sum $\delta_0 + \delta_1$, where δ_0 and δ_1 are homogenous with respect to this fine grading, and are of bidegrees (0,1) and (1,0), and obey

$$\delta_0\circ\delta_0 = \delta_1\circ\delta_1 = \delta_0\circ\delta_1 + \delta_1\circ\delta_0 = 0.$$

A decreasing filtration. Let $C_p(E) = \sum_{s\geq p} C^{s,q}(E)$ be the subspace of C(E) spanned by simplices of filtration $\geq$ p. We note that $\delta(C_p(E)) \subseteq C_p(E)$, and that

$$C(E) = C_0(E) \supseteq C_1(E) \supseteq \dots \supseteq C_N(E) = 0,$$

where N > dim(B).

The required spectral sequence is that of this decreasing filtration:

I.e. one first sets

$$E_0^{p,q} = C_p^{p+q}(E)/C_{p+1}^{p+q}$$

with d_0: $E_0 \to E_0$ being induced by the coboundaries δ: $C_p(E) \to C_p(E)$ of the filtered complex. Note that d_0 is homogenous of bidegree (0,1) [and coincides, under the identification $E_0^{p,q} = C^{p,q}(E)$ given by the canonical basis of E, with the δ_0 considered above.]

Then, for each $r \geq 0$, let $Z_r^{p,q} \subseteq C_p^{p+q}(E)/C_{p+1}^{p+q}$ consist of all cosets containing an x whose coboundary $\delta(x)$ has filtration $\geq$ p+r: thus this is a sequence decreasing from E_0 to subspace of cosets represented by cycles. [The definition of d_r: $E_r \to E_r$ uses this $x \mapsto \delta x$.]

And, for each $r \geq 0$, let $B_r^{p,q} \subseteq C_p^{p+q}(E)/C_{p+1}^{p+q}$ consist of all cosets containing the coboundary δy of a y having filtration > p-r: thus this is a sequence increasing from 0 to the subspace of E_0 consisting of cosets represented by coboundaries.

So if we define

$$E_r^{p,q} = Z_r^{p,q} / B_r^{p,q} ,$$

then it is clear that these spaces "decrease" and finally become E_∞ = H(E).

Of course to check all the facts of (A) much more work is required [e.g. one needs to define d_r's as indicated above to precise the "decrease" of the E_r's into $H(E_i) = H(E_{i+1})$].

(D) SMOOTH FIBRATIONS. When F, E, and B, are smooth manifolds with a smooth map of maximum rank, and we want to work over $\mathbb{F} = \mathbb{R}$, then the **de Rham complex** (Λ(E),d) of E can be used.

We say that a **differential form** $\omega \in \Lambda^r(E)$ is of **filtration** $\geq$ p iff it vanishes at any point whenever more than r-p of the **tangent vectors** are tangent to the fiber.

The formula for d shows that the subspace $\Lambda_p(E)$ of all forms of filtration $\geq$ p is preserved by d. Thus once again we have found a **decreasing filtration**

$$\Lambda(E) = \Lambda_0(E) \supseteq \Lambda_1(E) \supseteq \dots \supseteq \Lambda_N(E) = 0,$$

where $N > \dim(B)$.

Its spectral sequence turns out to fulfill the requirements of (A) [e.g. the required products in E_r are simply those induced by the **exterior product** of differential forms].

Comments

(1) **Analogues of Stiefel manifolds with q bigger than n.** General position sequences in $\mathbb{R}^n$ of lengths $\equiv n$ constitute GL(n), which has the compact retract O(n), and, more generally, if lengths are $\equiv q \leq n$, we have the compact retract $V_{n,q}$. *It is equally interesting* (cf. **Tverberg-Sierksma**) *to consider* $q > n$ *: are the cohomologies of such spaces known ?*

(2) **Spectral sequence of a foliated manifold.** This is defined just as in (D) and was introduced by **Sarkaria** in 1974. Since now no nice B or F is available there is much that is different from the case of fibrations. However a number of facts, e.g. finiteness, duality, etc. emerged under suitable conditions. However the complete story got understood much later by **Connes-Skandalis**. An exposition of part of their work is in the book of **Moore-Schochet**.

(3) **Simplicial Hodge theory and duality.** The *increasing filtration*

$$0 \subseteq C^1(E) \subseteq C^2(E) \ldots\ldots,\quad C^p(E) = \Sigma_{s \leq p} C^{s,q},$$

of the double complex of (C) is preserved by the dual *boundary* ∂ of C(E). [The duality of δ and ∂ being fixed by the canonical basis of C(E) provided by the simplices of E.]

If one is not interested in products, this spectral sequence, whose differentials d^r point the "other way", and which converges to the *homology* of E, is just as good. However, if products can be sacrificed, one can dispense with both filtrations!

The double complex determines another nice spectral sequence whose terms are equipped with dual pairs d_r, d^r of differentials:

Let $Z_*(E)$ and $Z^*(E)$ be the subspaces of C(E) consisting respectively of all cycles and cocycles. We know that additively **(co)homology**

$$H(E) \cong Z_*(E) \cap Z^*(E).$$

For each $\psi \in C(E)$, let $a^*(\psi)$ and $a_*(\psi)$ denote the smallest and biggest filtrations of the simplices of ψ. Consider the subspace $Z_*^r(E)$ of C(E) consisting of all ψ such that $a_*(\psi) - a_*(\partial\psi) \geq r$, and likewise the subspace $Z_r^*(E)$ of $C(E)$ consisting of all ψ such that $a^*(\psi) - a^*(\delta\psi) \geq r$. Note that both these sequences of subspaces start, for $r = 0$, from $C(E)$, and decrease, for r sufficiently [$\geq$ dimB] big, to $Z_*(E)$ and $Z^*(E)$, respectively. With the above equation as our cue, we define the rth term by

$$E_r = E^r = Z_*^r(E) \cap Z_r^*(E).$$

Note that ∂ preserves $Z_*^r(E)$: its biggest component, which is a differential, will be denoted d^r. Likewise the smallest component of the restriction of δ to $Z_r^*(E)$ gives us the differential d_r. On the intersection $E_r = E^r$ they are both available, and are dual to each other. Furthermore one has

$$(\ker d^r) \cap (\ker d_r) = E_{r+1} = E^{r+1}.$$