SIMPLICIAL CONSTRUCTIONS

From the book by May:

1. The work of Eilenberg-Zilber/Maclane, Milnor, Kan, Moore, Cartan et al. exploits the following *categorical generalization* [in the sense of 6 below] *of simplicial complexes* [with vertices ordered] which is superior to that provided by [e.g "CW"] "cell complexes":

A [**complex** or] **simplicial object** [i.e. set, group, etc.] is a contravariant functor K from the **category** Δ **of monotone maps** $\{0,1, \ldots ,n\} \to \{0,1, \ldots ,m\}$ [to the category of sets, groups, etc.]; and a natural transformation $f: K \to L$ between two such functors is called a **simplicial map**.

2. The nth object, $n \geq 0$, of K is denoted K_n. The **face maps** arise from the monotone injections $n-1 \to n$: omitting i, $0 \leq i \leq n$, gives $\partial_i : K_n \to K_{n-1}$: while the **degeneracy maps** arise from the monotone surjections $n+1 \to n$: repeating i,, $0 \leq i \leq n$, gives $s_i : K_n \to K_{n+1}$.

[Originally, Eilenberg-Zilber only demanded degeneracy maps, i.e. they worked with the smaller category of monotone injections $\{0,1, \ldots ,n\} \to \{0,1, \ldots ,m\}$.]

Since any map is a composition of injections and surjections, it follows that a simplicial map is characterized by the fact that its **components** $f_n : K_n \to L_n$ commute with the face and degeneracy maps.

The members of K_n are called the **n-simplices** of the complex K. If L and K are complexes with $L_n \subseteq K_n$ for all n, then L is called a **subcomplex** of K. Note that a complex is determined or **generated** by the the set of all its **non-degenerate** simplices, i.e. those not in the image of any degeneracy map. Note also that the subcomplex generated by a chosen **base point**, or 0-simplex, has one n-simplex for each $n \geq 0$.

3. A simplicial object, followed by a covariant functor, gives another simplicial object: e.g. given a simplicial set K we have the simplicial group C(K) with $C_n(K)$ = free abelian group over K_n.

The **(co)homology** H(K) of K is H(C(K)), i.e. the (co)homology of C(K) under the (dual of) alternating sum of the face maps .

4. Likewise, a **cosimplicial object**, i.e. a covariant functor from Δ, followed by a contravariant functor, also yields a simplicial object.

Consider e.g. the cosimplicial object which associates to each n the **geometrical n-simplex** $|n|$ = space of all continuous functions $\{0,1, \ldots$

,n) $\to \mathbb{R}^+$ with sum 1, and to each order preserving map n $\to$ m the linear map $|n| \to |m|$ prescribed by it. Composing it with the functor which associates to each space the set of all continuous maps from it to a fixed space X, we get the **singular complex** S(X) of X. So $S_n(X)$ = all continuous maps from $|n|$ to X.

5. The **linear complex** L(T) of a simplicial complex T has L_n = all linear maps from closed n-simplex [0,1, .. ,n] to T = all sequences of length n+1 supported on simplices of T. [This simple construction has been largely ignored: e.g. after defining it in 1.4, May never returns to it again: the preferred construction is the following.]

6. The **ordered complex** K(T), of a simplicial complex T with vert(T) ordered, has K_n = all increasing sequences of length n supported on simplices of T.

If T = closed simplex on {0,1, .. ,n} then K(T) will be written Δ_n and called the **standard n-simplex**. So $(\Delta_t)_n$ = all monotone maps {0,1, .. ,n} $\to$ {0,1, .. ,t}. Note that any order preserving map {0,1, .. ,t} $\to$ {0,1, .. ,s} determines a simplicial map $\Delta_t \to \Delta_s$ between the corresponding standard simplices.

A simple but important point is that each n*-simplex of any simplicial object* K *determines, and is determined by, a simplicial map* $\Delta_n \to$ K.

For example it is this [see p.14 onwards of May] which identifies the definition of homotopy groups, given in 10 below, with the more obvious definition, using the homotopy of simplicial maps of 12.

7. The sequence of all but the kth face, i.e. the **kth Schlegel diagram** $(\theta_0, \dots, \theta_{k-1}, \theta_{k+1}, \dots, \theta_n) = (\partial_0\sigma, \dots, \partial_{k-1}\sigma, \partial_{k+1}\sigma, \dots, \partial_n\sigma)$ of a simplex $\sigma \in K_n$, $n \geq 2$, obeys the **compatibility conditions**

$$\partial_i\theta_j = \partial_{j-1}\theta_i \quad \forall\ i < j \text{ other than } k.$$

K is called a **Kan complex** if, conversely, any sequence $(\theta_0, \dots, \theta_{k-1}, \theta_{k+1}, \dots, \theta_n)$ from K_{n-1}, $n \geq 2$, which obeys these compatibility conditions, is the the kth Schlegel diagram of some $\sigma \in K_n$.

8. **Proposition.** *Singular complexes* S(X) *are Kan complexes.*

The above, which is obvious, is the basis on which Kan generalized classical homotopy theory to his combinatorial homotopy theory [see 14].

[If the simplicial complex T is *full*, then L(T) and K(T) are also Kan complexes. But this is unimportant because these complexes are *rigid*: a simplex is determined by its sequence of faces: and so their homotopy groups — see 10 — are trivial.]

[Note also that, in a singular complex, a sequence of n+1 (n-1)-simplices $(\theta_0, \ldots, \theta_n)$, satisfying the necessary "compatibility conditions", is seldom a **simplicial boundary**, i.e. the sequence $(\partial_0\sigma, \ldots, \partial_n\sigma)$ of *all* faces of an n-simplex σ: in fact the given sequence determines a continuous map of the (n-1)-sphere into K, and this map would have to be trivial for such a σ to exist.]

9. **Proposition.** *Simplicial groups are Kan complexes.* [(17.1)]

The above, which is not quite obvious, is due to Moore, and enabled him to give some unexpected applications [see 16] of the combinatorial homotopy theory of Kan complexes.

10. Two n-simplices, $n \geq 1$, which have the same sequence of faces are called **homotopic simplices** if they are the two last faces of an (n+1)-simplex, whose other faces are obtained from the corresponding faces of either of the two simplices by applying the last degeneracy operator.

Proposition. *For Kan complexes homotopy of simplices is an equivalence relation.*

The same is true for "homotopy rel subcomplex" etc. The various **homotopy groups** are defined as these equivalence classes: e.g. $\pi_n(K,1)$, $n \geq 1$, are the homotopy classes of n-simplices with all faces in the subcomplex generated by the base point 1. The 0th group $\pi_0(K,1)$ is defined as the equivalence classes of 0-simplices lying in the same **path component**, i.e. those which can be connected by a train of 1-simplices.

11. The **product of complexes** K and L is the complex $F \times B$ for which $(F \times B)_n = F_n \times B_n$ with $\partial_i(x,y) = (\partial_i x, \partial_i y)$ and $s_i(x,y) = (s_i x, s_i y)$.

[Applied to example 6 this gives the standard triangulation — see Eilenberg-**Steenrod** — of the product of two simplices: quite possibly it was this which motivated the definition of simplicial objects ?]

There is also a more general notion of **twisted cartesian product** [= combinatorial **fibre bundle**] $F \times_\tau B$, with prescribed **twisting function** τ mapping each B_n to G_{n-1}, G being a simplicial group with a prescribed **left action** on the fibre F, ... [see (18.3) for this definition, § 19 first para and (20.5) to see equivalence with another definition.]

12. The **function complex** L^K [= Hom(K,L)] has $(L^K)_n$ = "n-parameter families of simplicial maps from K to L" = simplicial maps $K \times \Delta_n \to L$, while the face and degeneracy maps use the maps between Δ_n and $\Delta_{n-1}, \Delta_{n+1}$ [see (6.4)]. 1-simplices of L^K are also called **homotopies of simplicial maps**.

[*Shuffle permutations* — see pp.17-18 — enter into a more explicit description of the n-simplices of the function complex; they enter again on p.133 in definition of a homotopy inverse of the *Alexander–Whitney map* used to define cup products.]

Homotopy of two n-simplices [see 11] corresponds to the homotopy rel boundary subcomplex of the associated simplicial maps from Δ_n.

13. A Kan complex is called a **minimal complex** iff two simplices having the same kth Schlegel diagram have the same kth face. This is equivalent to demanding that homotopic simplices be same.

Proposition. *Every Kan complex has as strong deformation retract a minimal complex which is unique upto isomorphism.* [(9.5),(9.8)]

This construction of Eilenberg-Zilber proceeds inductively on dimension, choosing one representative, degenerate whenever possible, from each homotopy class of simplices.

Are minimal complexes related to the crystallizations of **Caviochilli** et al. [and maybe even to self-dual simplicial complexes] ?

14. **Combinatorial homotopy theory** *mimics classical homotopy theory of CW complexes:*

First, generalizing definition of 7 from complexes to simplicial maps, one obtains the analogue of the *Serre fibration*, now called a **Kan fibration** [likewise generalizing definition of 13 gives a **minimal fibration**: each Kan fibration has an essentially unique minimal fibration to which it is equivalent]. Standard things like *covering homotopy property*, *homotopy extension property*, *exact sequence of fibration*, etc., extend easily to this setting, a bonus being the following canonical *Postnikov factorisation* of a Kan complex [and an analogous one of a Kan fibration].

Proposition. *Given any Kan complex* K, *let* $K^{(n)}$ *be the quotient obtained by identifying simplices which* [considered as maps from a standard simplex] *coincide in dimensions* $\leq$ n. *The quotient maps* $K^{(n+1)} \to K^{(n)}$ *are Kan fibrations whose fiber has only one non-trivial homotopy group, the nth, which coincides with the nth homotopy group of* K. [(8.7)]

Such Kan complexes with only one homotopy group are called **Eilenberg-Maclane complexes**: in case they are also minimal they are called $K(\pi,n)$'s [see also 16 below].

15. **Realization** is a covariant functor from simplicial sets to topological spaces, $|K|$ being defined as explained before by using obvious identifications on $\bigcup_n (K_n \times |n|)$.

Proposition. $|K|$ *is a CW complex with one cell for each non-degenerate simplex.* $|K \times L|$ *is homeomorphic to* $|K| \times |L|$ *whenever it is a CW complex. Upto homotopy the singular functor is the* **adjoint** *of this realization functor.* [(14.1),(14.3),§16]

The above is due to Milnor and Kan. Leaving aside the exact meaning of the last phrase, it implies [(16.6),(16.7)]

(a) $\pi_n(X)$, of any CW complex X, coincides with $\pi_n(|S(X)|)$, and

(b) $\pi_n(K)$, of any Kan complex K, coincides with $\pi_n(S|K|)$. [*We can thus use this to define* $\pi_n(K)$ *even when* K *is not Kan.*]

Also $|K(T)| = |T|$ but $|L(T)|$ has more cells: but maybe $|L(T)|$ has always the same homotopy type as $|T|$?

16. As mentioned before it is the combinatorial homotopy theory of simplicial groups which threw up some really new stuff:

Proposition. *The homotopy groups of any simplicial group* G *coincide with the homology groups of the chain complex* N(G) *obtained by using the last face map on the kernel of all the other face maps. Also,* G *is minimal iff this chain complex is trivial, i.e. has homology* N(G). [§17: (17.3), (17.5)]

The above is due to Moore, who also obtained, for the abelian case, the following *Dold-Thom theorem* showing that homology groups constitute a special case of homotopy groups:

Proposition. *The homotopy groups of any abelian simplicial group* G *coincide with its homology groups under the alternating sum of the face maps* [with above N(G) being now a quotient of this chain complex] *Thus the homology groups* [see 3]*of any simplicial set* K *coincides with the homotopy groups of* C(K). [(22.1),(22.3),(22.6)]

Then Dold and Kan showed that the earlier work of Eilenberg-Maclane fits into this setting as follows:

Proposition. *There is an explicit and simple functor* Γ *from chain complexes to simplicial abelian groups* [defined by fattenning up each dimension to account for degeneracies, etc.] *which is adjoint to the above functor* N. [pp.95-96 for Γ, (22.4)]

This implies that the minimal Eilenberg-Maclane complexes $K(\pi,n)$*'s arise simply by applying* Γ *to the chain complex having zero everywhere except a* π *at level* n ! [(23.7)]

17. **Loop group** G(K) of a complex K is a simplicial group having [for the "reduced case" when K has only one vertex] $G_n(K) =$ (free group generated by all (n+1)-simplices of K)/(normal subgroup generated by the images of the first degeneracy map), etc. [(26.3)]

Proposition. *The principal fiber bundle* $G(K) \times_\tau K$*, with the obvious twisting function* $\tau: K_{n+1} \to G_n(K)$*, has a contractible realisation. The homotopy groups of Kan complex* K *are the homotopy groups, of one less* *dimension, of* G(K). *Also the homology groups of* K *are the homotopy groups, of one less dimension, of the abelianization* A(K) *of* G(K). [p.121]

Loop monoid $G^+(K)$. This variant is made just like G(K) except one uses free monoids instead of groups: *for reduced Kan complexes one however still gets* G(K). [(27.8,27.9)]. This may be close to deleted joins ?

Loop complex L(K). This variant is defined by mimicking the *Serre fibration*: first one defines a **path complex** P(K) whose $P_n(K)$ = all (n+1)-simplices of K, and dropping the first face and degeneracy maps. [So this is precisely what figures in the odd columns of the **Tsygan** double complex !] The complex [not simplicial group] L(K) is the fibre of the fibration P(K) → K given by the first face map. [(23.3)]

Classifying complex W(G). This is adjoint of the functor G(K). W(G) has $W_n(G)$ = cartesian product of the groups $G_0, \dots ,G_n$, etc. The realization of W(G) is contractible, and W(G) comes with an action of G: so W(G)/G serves to classify bundles having group G. [§21: p.87, etc.]

Suspension E(K). This is adjoint of the functor L(K). One has $E_n(K)$ = all ordered pairs (i,x) where x is an (n-i)-simplex of K, etc. The realization of E(K) is homeomorphic to the reduced suspension of the realization of K. The adjointness corresponds to the topological fact that homotopy classes of maps from X to the loop space of Y are in 1-1 correspondence with homotopy classes of maps from the suspension of X into Y. [(27.10),(27.6)]

[The definitions above are due to various people: Eilenberg and Maclane, Milnor, Kan, Moore, etc.]

18. The remaining §§ 28-32 of May are aimed at establishing the main properties of the *Serre spectral sequence*. To calculate the second term he proves a theorem (31.7) of **Brown** giving an equivalence between $A(K \times_\tau L)$ and $A(K) \otimes_\tau A(L)$, i.e. between twisted cartesian products and twisted tensor products [this generalizes the untwisted case (29.3) which was the theorem of **Eilenberg-Zilber**]. The proof requires the method of **acyclic models** (§28) which furnishes the required homotopies, though a couple of explicit maps (29.7) of "Alexander-Whitney" and "Eilenberg-Maclane" also occur. The above equivalence also furnishes a construction of **cup product**, **Pontrjagin product** [for a G-complex], and **cap product** which occur in § 30.

CARTAN 1956/57

These papers of Cartan are also on "simplicial topology" à la **Eilenberg**: exposé 1 was a talk which gave, besides a few new results, an extremely nice account of **Kan's** simplicial homotopy theory [it was based on announcements/preprints of Kan, and lectures of **Moore** and **Milnor**], and its sequels [= exposés 3 and 4] established more new results:

1. Simplicial categories. Given any category $\mathscr{C}$ there is the associated category $\mathscr{C}^s$ of contravariant functors from Δ.

Alternatively, each object K of $\mathscr{C}^s$ consists of a sequence K_n, $n \geq 0$, of objects of $\mathscr{C}$, together with n+1 face maps d_i from K_n to K_{n-1}, and a like number of degeneracy maps s_i from K_n to K_{n+1}, obeying the usual commutation laws. And, a morphism $f:K \to L$ in $\mathscr{C}^s$ consists of a sequence of maps $K_n \to L_n$ commuting with the d_i's and s_i's.

2. Homotopy groups. Cartan recalls Kan's definition of the extension condition, but refers to Moore's lectures for the definition of homotopy groups of Kan complexes, and to Moore's Exposé 18 of 1954/55 [p.18-04 and §4] for the following.

Proposition. *Simplicial groups* G *are always Kan, and their homotopy groups* $\pi_n(G)$ *coincide with the homology groups* $H_n(N(G))$ *of the chain complex* $(N(G),d_0)$, *where* N(G) = *kernel of all face maps other than the first,* d_0.

Using this he checks that these $\pi_n(G)$ are always abelian for all $n \geq 1$. Then he turns to the case when the simplicial group G is abelian.

Proposition. *For a simplicial abelian group* G, $(N(G),d_0)$ *is a sub chain complex of* (G,d), *where* d = *alternating sum of the face maps, and the inclusion* $N(G) \to G$ *induces an isomorphism in homology.*

For this Cartan considers the **filtration** of sub chain complexes,

$$G = F^0 \supseteq F^1 \supseteq \dots ,$$

where, for $p \leq n$, $(F^p)_n$ = kernel of all face maps other than $d_0, \dots, d_{n-p}$, and for $p > n$, $(F^p)_n$ = kernel of all face maps other than d_0. He then checks that each inclusion $F^{p+1} \to F^p$ induces an isomorphism in homology.

[This proposition and/or its proof might lead to a conceptual understanding of **Kalai'theorem**: *exterior shifting preserves homology* ?]

The **homology group** $H_n(K)$ **of simplicial set** K is by definition the homology of the free simplicial abelian group C(K) generated by K. So we get from the above,

$$H_n(K) \cong \pi_n(C(K)).$$

[Dividing C(K) out by the subgroup generated by the subcomplex of K generated by a *base point* one gets the *reduced version*.]

The **homotopy group** $\pi_n(X)$ **of a space** X is defined to be that of its singular complex S(X). The above result of Moore was motivated by the earlier one of **Dold-Thom** which we should learn more about: *the homology groups of a space identify with the homotopy groups of its "symmetric product"*.

3. Realizations. For *any* simplicial set K, Milnor considered $|K|$, the quotient of the disjoint union

$$\cup \{ |x| : x \in K_n, n \geq 0\}$$

of *affine simplices* $|x|$, under the obvious identifications, and showed:

It is reasonable, for any K, *to define* $\pi_n(K)$ *to be the homotopy group* $\pi_n(|K|)$ *of this realization* $|K|$.

More precisely, he showed that, when K is Kan, the above definition coincides with Kan's because of the following.

Proposition. *For any* K, resp. *any* K *which is Kan, the monomorphism,*

$$K \to S(|K|),\ x \in K_n \mapsto (|x| : \Delta_n \to |K|),$$

induces an isomorphism of homolopy, resp. *homotopy, groups.*

In case $x \in K_n$ is non-degenerate, the aforementioned *singular simplices* $|x|$ are homeomorphisms restricted to the interior of Δ_n: this is the key point in checking that $|K|$ *is a CW complex, with one cell for each non-degenerate simplex of* K.

Furthermore, in case the K_n's consist of all non-decreasing sequences of vertices supported on the simplices of an *ordered simplicial complex*, then $|K|$ coincides with this simplicial complex.

[So, contrary to a remark in previous review, such a K is *not Kan*: otherwise its trivial homotopy would coincide with that of $|K|$. Likewise the "linear complexes" are also *not Kan*.]

[In Eilenberg-Steenrod, p.68, it was checked that *if* K *and* L *arise as above from two ordered simplicial complexes, then* $|K \times L|$ *coincides with* *the cartesian product of the two simplicial complexes*: this useful "product rule"— which incidentally can be generalized a lot — might well have been the small beginning of Eilenberg's great work ?]

Proposition. *The realization* $|G|$ *of a countable simplicial group* G *is a topological group.*

This follows because the simplicial map $G \times G \to G$ consisting of the group multiplications $G_n \times G_n \to G_n$, induces a continuous map $|G \times G| \to |G|$ of realizations, and the two simplicial projections define a continuous bijection $|G \times G| \to |G| \times |G|$, whose inverse is continuous at least when G is countable. [Likewise the "generalized product rule" also requires some hypothesis to ensure continuity of the inverse.]

4. **Fibrations.** These are maps $E \xrightarrow{p} B$ satisfying Kan's condition: this generalizes the definition of a Kan complex which is the case B = {pt}. More generally, the fibres $F = p^{-1}(b)$ of a fibration are always Kan. However, in general E or B need not be Kan, but it so happens that *the total space* E *is Kan iff the base space* B *is Kan*.

$X \to Y$ *is a Serre fibration iff the singular* $S(X) \to S(Y)$ *is a Kan fibration* :this easy but important observation might well have been the starting point of Kan's work ?

Also, if $E \to B$ is a Kan fibration, then its realization $|E| \to |B|$, though not quite a Serre fibration [see **May**: p.65], is still good enough to have the usual *exact homotopy sequence of a fibration*: thus [using definition of 3] a fibration $E \to B$ has an exact homotopy sequence even when E and B are not Kan.

Proposition. *If* G *is a simplicial group with a prescribed "free action"* $G \times E \to E$, *then the quotient map* $p: E \to E/G$ *satisfies Kan's condition.*

The special case E = G is Moore's result that simplicial groups are Kan, and the proof of the above generalization uses this special case.

All fibrations [see exposé 4, Prop.1], so in particular above **principal bundles**, always admit *a pseudosection* ρ *which commutes with all degeneracy maps, and also with all face maps except* d_0.

Using such a pseudosection ρ one has a bijection of E with $G \times B$. Under this bijection, the face and degeneracy maps of E identify with the usual face and degeneracy maps of a product, except d_0, which instead of being $(x,y) \mapsto (d_0x, d_0y)$ is $(x,y) \mapsto (d_0x \cdot \tau y, d_0y)$, where the **twisting function** $\tau: B_n \to G_{n-1}$ is determined by, and determines, ρ. The net upshot: *principal fibre bundles coincide with twisted cartesian products* $G \times_\tau B$.

5. **Loop groups.** This construction of Kan associates to each simplicial set K a free simplicial group G(K) with $G_n(K)$ = *free group on the*

(n+1)-*simplices of* K *except those lying in the image of* s_0. Its d_i's and s_i's are [on elements x determined by simplices x] the d_{i+1}'s and s_{i+1}'s of K, except that d_0x is $(d_1x)(d_0x)^{-1}$.

Proposition. *The formula* $\pi_n(G(K)) = \pi_{n+1}(K)$ *holds for any* K *having only one vertex.*

This important result of Kan [who also has a version which dispenses with the last condition] prescribes *a combinatorial way of calculating homotopy groups of any simplicial set* K.

To prove the above it is checked that the principal fibre bundle $G(K) \times_\tau K$, where the twisting function is the quotient $K_n \to G_{n-1}(K)$, is contractible, and then one uses the exact homotopy sequence of this fibration.

6. **Abelianized loop group.** Denote G(K)/[G(K),G(K)] by A(K): so $A_n(K)$ is $C_{n+1}(K)$ divided out by the image of s_0. However it turns out that this division is unimportant homologically and *homology of* A(K) *equals homology of* K *in one higher dimension*.

7. **Noncommutative chains.** Kan's definition of G(K) was preceded by one of Milnor, which simply associated, to each K, the simplicial group F(K) with $F_n(K)$ = *free group on the* n-*simplices of* K [or its reduced version], the d_i's and s_i's being, on elements determined by simplices of K, simply the d_i's and s_i's of K. Thus F(K) is simply the noncommutative version of C(K), which is its abelianization.

Proposition. $\pi_n(F(K))$ *is isomorphic to the* (n+1)*th homotopy group of the suspension of* $|K|$.

The suspension theorem of **Freudenthal** shows that the last named group is often [not always] the nth homotopy group of K: thus Milnor's result sort of shows that homotopy groups are "noncommutative analogues" of homology groups.

Milnor's construction was subsumed in Kan's by defining a simplicial analogue of suspension, and showing that F(K) was the loop group of the the suspension of K.

8. Kan got some abstract *Hurwicz-type theorems*, e.g. the following.

Proposition. *Let* F *be any free simplicial group and* A *its abelianization. If the homotopy groups of* F *are trivial below dimension* n, $n \geq 1$, *then* $\pi_n(F) \to \pi_n(A)$ *is an isomorphism.*

9. **Universal principal simplicial G-bundles.** For any simplicial group G this is the principal simplicial G-bundle W(G) for which

$$W_n(G) = G_n \times G_{n-1} \times \ldots \times G_0,$$

the face maps d_i are given by

$$d_i(x_n, x_{n-1}, \ldots, x_0) = (d_i x_n, d_{i-1} x_{n-1}, \ldots, d_0 x_{n-i} \cdot x_{n-i-1}, \ldots, x_0),$$

and the degeneracy maps s_i are defined by

$$s_i(x_n, x_{n-1}, \ldots, x_0) = (s_i x_n, s_{i-1} x_{n-1}, \ldots, s_0 x_{n-i}, 1_{n-i}, x_{n-i-1}, \ldots, x_0).$$

The free action of G_n on $W_n(G)$ is on its first factor, thus the nth object of W(G)/G is $G_{n-1} \times \ldots \times G_0$, and mapping any of its elements $(x_{n-1}, \ldots, x_0)$ to $(1_n, x_{n-1}, \ldots, x_0)$ gives the *canonical pseudosection* of this bundle. The key property of W(G) is that its realization is contractible.

The above definition is essentially Eilenberg-Maclane's, but Cartan refers to the 1954/55 exposés 12 and 13 of Moore [explicit formulae above are on p.13-06], and gives a resumé on pp.07-08 of his exposé 4.

Cartan's exposés 3 and 4 give the delicate proofs [e.g. regarding the "Kan-ness" of Hom(K,L)] which are necessary to develop, one by one, the *simplicial analogues of some well known facts of homotopy theory* [e.g that an isomorphism class of principal simplicial G-bundles over B', pulls back, under a homotopy class of simplicial maps B → B', to an isomorphism class of principal simplicial G-bumdles over B] and thus establish the following *classification theorem of principal simplicial G-bundles*.

Proposition. *A principal simplicial* G-*bundle on* B, *together with a pseudosection, determines, and is determined by, a simplicial map from* B *into* W(G)/G. *This sets up a bijection between isomorphism classes of principal simplicial* G-*bundles on* B, *and homotopy classes of simplicial maps* B → W(G)/G.

As usual the pull-back of the cohomology of W(G)/G gives the *characteristic classes* of the principal G-bundle over B.

THEORIES COHOMOLOGIQUES

Cartan [in *Invent.* 35, 1976] develops further Swan's note re Thom's definition of a *de Rham cohomology for simplicial complexes*.

1. Some standard algebraic terms:

Module = an abelian group M, equipped with a "compatible" scalar multiplication of some commutative ring R with unity = **vector space** if R is a field.

Graded module = module M in which a direct sum decomposition $M = \sum_n M^n$ into submodules indexed by the non-negative integers n is specified.

Differential graded module = graded module equipped with linear maps δ: $M^n \longrightarrow M^{n+1}$ such that $\delta \circ \delta = 0$. Its **cohomology** $H(M) = (\ker\delta)/(\mathrm{im}\delta)$ is a graded module with $H^n(M) = (\ker \delta\colon M^n \to M^{n+1})/(\mathrm{im}\ \delta\colon M^{n-1} \to M^n)$.

Algebra = module M equipped with a "compatible" product.

Graded algebra = an algebra which is a graded module with product imaging each $M^a \times M^b$ into M^{a+b}. Note that M^0 is an *algebra*, while M^n, for $n \geq 1$, are only modules.

Differential graded algebra = graded algebra M which is a graded differential module with the differential δ obeying the usual signed product rule. Its cohomology H(M) is a graded algebra, so $H^0(M)$ is an algebra.

2. Each of the above notions has of course a *simplicial version*, e.g. we have **simplicial differential graded algebras** A:

By this we mean that, for each $p \geq 0$, we have a DGA [= differential graded algebra] $A_p = \sum_n A_p^n$, and the face and degeneracy maps $d_i\colon A_p \to A_{p-1}$ and $s_i\colon A_p \to A_{p+1}$ are DGA morphisms. Setting $A = \sum A^n$, where $A^n = \sum_p A_p^n$, we will also consider the simplicial DGA A as an *ordinary* [= non-simplicial] DGA. Note that [the components A_p^n of] the A^n 's constitute **simplicial modules**, with A^0 being even a **simplicial algebra**. Note also that the cohomology H(A) is a **simplicial graded algebra**, so $H^0(A)$ is a simplicial algebra.

3. Given a simplicial set K, and a simplicial object A of the above kind, we put A(K) = **set of all simplicial maps from K to A**, with algebraic structure induced from the ordinary structure of A.

E.g., if K is a simplicial set K and A is a simplicial DGA, then the ordinary DGA A(K) = all simplicial maps from K to the simplicial set A, with the DGA structure induced by that of the DGA A. Thus $A(K) = \sum_n A^n(K)$, where $A^n(K)$ = all simplicial maps from K to the simplicial set A^n, with the module structure being the one induced from the module A^n.

2. **Homologically trivial simplicial DGM.** By this is meant not only that the cohomology of the cochain complex

$$A^0 \xrightarrow{\delta} \cdots \xrightarrow{\delta} A^n \xrightarrow{\delta} A^{n+1} \xrightarrow{\delta} \cdots,$$

of simplicial modules is zero in all positive dimensions, but also that the zeroth cohomology $H^0(A) = Z^0(A)$ is **simplicially trivial**, i.e. all the face and degeneracy maps of this simplicial module are isomorphisms: using this we will identify it with the **module** R(A) constituting its 0th object.

When A is a simplicial DGA then of course $H^0(A)$ is a simplicial algebra and R(A) is an algebra.

Homotopically trivial simplicial DGM A. By this is meant that all the *homotopy groups*, of each of the simplicial modules A^n, $n \geq 0$, are zero.

3. Theorem A. *In case the simplicial differential graded module* A *is homologically and homotopically trivial, the contravariant functor which associates to each simplicial set* K *the graded module* H(A(K)), *coincides with the usual cohomology of* K *with coefficients in the module* R(A).

Cartan establishes this by a beautiful application of *simplicial homotopy theory* :

Proof. We have $H^n(A(K)) = Z^n(A(K))/B^n(A(K))$, where $Z^n(A(K))$ = module of n-cocycles of A(K) = simplicial maps from K to $Z^n(A)$, the simplicial module of all n-cocycles of A, and $B^n(A(K))$ = module of n-coboundaries of A(K) = simplicial maps from K to $Z^n(A)$ whose image is in $B^n(A) = \delta(A^{n-1})$, i.e. those which can be lifted over the *fibration* $\delta : A^{n-1} \to Z^n(A)$.

Now, using the *covering homotopy property* of δ, it follows that any homotopically trivial simplicial map from K to $Z^n(A)$ can be thus lifted. Conversely, since A^{n-1} is homotopically trivial, any such liftable simplicial map from K to $Z^n(A)$ is homotopically trivial. Thus we have seen that

$$H^n(A(K)) = [K, Z^n(A)],$$

i.e. that the nth cohomology of A(K) identifies with the *simplicial homotopy classes of simplicial maps* from K to $Z^n(A)$.

We now use the fact that the usual *nth cohomology of* K *with coefficients in* R(A), identifies naturally with the set of simplicial homotopy classes of simplicial maps from K to an *Eilenberg-Maclane complex*, whose sole homotopy group is the nth, and equals R(A). So to complete the proof it suffices to check

$$\pi_i(Z^n(A)) = 0 \text{ if } i \neq n, \text{ and } \pi_n(Z^n(A)) = R(A).$$

For this note, because of the homological triviality of A, that the fiber of the fibration $\delta: A^{n-1} \longrightarrow Z^n(A)$ is $Z^{n-1}(A)$. So the required result follows by an induction on n by using the *exact homotopy sequence* of this fibration. *q.e.d.*

4. Ordered cochains. We have the simplicial differential graded module C for which C_p = all ordered cochains of *standard p-simplex* Δ_p with coefficients in ring R. Thus associated ordinary differential graded module = direct sum over $p \geq 0$ of all these cochain complexes. Note that R(C) = R.

Proposition. *The simplicial* DGM C *is homologically and homotopically trivial.*

So Theorem A applies to C: in fact the usual cohomology of K with coefficients in R is usually defined as cohomology of C(K). Also note that when K corresponds to an ordered simplicial complex, then C(K) is the ordered cochain complex of K.

Products in C. There is the obvious one: *juxtaposition*: but this is not well behaved with respect to the grading by dimensions: it is however well behaved with respect to the grading by *degrees* [= cardinalities]: and the differential obeys a product rule with respect to this: however it seems [proof ?] that the induced product in cohomology H(C(K)) is trivial.

A less obvious product is that introduced by *Alexander*, *Kolomogrov*, *Whitney* et al. : this is well behaved with respect to dimension grading and obeys the product rule too: so, with this C becomes a simplicial differential graded algebra: the induced *cup product* in cohomology is non-trivial, and H(C(K)) is [unlike DGA C(K)] also *graded-commutative*.

5. Smooth forms. This is the simplicial DGA Ω for which Ω_p = *de Rham*'s DGA of smooth forms on the standard p-simplex Δ_p [i.e. equipped with the *exterior product* and differential is *exterior differentiation*]. Note that the ordinary DGA Ω is graded-commutative and that $R(\Omega) = \mathbb{R}$. For this Ω one has $\Omega_p^n = 0$ for n bigger than p [so Theorem B below applies].

Proposition. *The simplicial* DGA Ω *is homologically and homotopically trivial.*

Thus to each simplicial set K [e.g. the one corresponding to an ordered simplicial complex, or the singular complex of a space] the cohomology of **Thom's** graded commutative DGA $\Omega(K)$ yields the ordinary real cohomology of K. In fact [using Theorems B and C below] the two cohomology algebras are also same.

Rational forms. This variant of the above, due to **Quillen**, **Sullivan**, et al., uses Ω_p = all forms over Δ_p which, in barycentric coordinates, have rational polynomial coefficients. Everything said above still works.

6. Integration. For the case of forms, **Stokes'** formula provides a *homomorphism* [non-multiplicative] between the de Rham complex and the simplicial cochain complex which induces the isomorphism of Theorem A.

Even this integration map has an interesting algebraical generalization.

Theorem B. *Let* A *be a homologically trivial simplicial differential graded module for which* $A^n_p = 0$ *whenever* $n > p$. *Then there exists one and only one morphism from* A *into the simplicial differential graded module* C *of ordered cochains of standard simplices with coefficients in* R(A), *which is the identity on* $Z^0(A)$.

Proof. By induction on n. Assume a unique extension upto level n-1, and into n-cocycles has been done. But this map $Z^n(A) \longrightarrow Z^n(C)$ must have a unique extension to a map $A^n \longrightarrow C^n$, because $A^n_n = (Z^n(A))_n$ because of $A^{n+1}_n = 0$. By homological triviality $A^n/Z^n(A) \cong Z^{n+1}(A)$, so we have a unique extension upto level n and into (n+1)-cocycles. *q.e.d.*

Remark. This "integration" induces a homomorphism $H(A(K)) \rightarrow H(C(K))$ for any simplicial set. If A is also homotopically trivial the left side is isomorphic to the right by the previous theorem. Presumably now this *homomorphism in cohomology is an isomorphism* [maybe it follows from uniqueness above ?] but Cartan forgets to say it explicitly !

7. **Theorem C.** *Let* A *and* B *be homologically and homotopically trivial simplicial differential graded algebras. Then any simplicial differential graded* module *homomorphism* $A \rightarrow B$, *which is an algebra homomorphism of* R(A) *into* R(B), *induces, for all simplicial sets* K, *a graded* algebra *homomorphism* $H(A(K)) \rightarrow H(B(K))$.

Proof. Continuing the simplicial homotopy theory of §3, what we need to do is to check that the following diagram, where the vertical arrows are defined by the given homomorphism, is commutative upto simplicial homotopy.

$$\begin{array}{ccc} Z^a(A) \times Z^b(A) & \longrightarrow & Z^{a+b}(A) \\ \downarrow & & \downarrow \\ Z^a(B) \times Z^b(B) & \longrightarrow & Z^{a+b}(B) \end{array}$$

For this it suffices to check that the diagrams obtained by applying any π_i to above diagram commute. But the two cartesian products on the left are themselves Eilenberg-Maclane complexes, their only nonzero homotopy group being in dimensions a+b, this being $R(A) \times R(A)$ and $R(B) \times R(B)$. So the required commutativity follows from the hypothesis that the given morphism is an algebra morphism of R(A) into R(B). *q*.e.d.

We remark that, apparently, the very last "follows" of the above argument is not true in general: this because Cartan inserts another condition in the statement — that the modules $Z^n(A)$ and $Z^n(B)$ be *flat*

[?] over the ring R — for this very step of the proof. However flatness is true if R = field or $\mathbb{Z}$ or free abelian groups, etc., so this gap is not too important.

Comments

(1) How exactly does justaposition give rise to the Alexander and the higher products ?

(2) Consider for each $p \geq 0$ the exterior algebra Λ_p = over the p+1 vertices {0,1, .. ,p} of Δ_p equipped with the differential $\delta(\omega)$ = (sum of vertices) $\wedge\ \omega$. The simplicial DGM Λ [the product does not make it a DGA] gives, for an ordered simplicial complex K, $\Lambda(K)$ = *oriented cochain complex* of K.

(3) How does $\Lambda(K)$ relate to Thom's $\Omega(K)$? In particular how does above exterior product of Λ relate to that of Ω ?

(4) Formulate **Kalai**'s *shifting process* functorially in the above setting: starting from any simplicial algebra A : one would have to work out the changed grading etc.?

(5) Integral cohomology ring can not be computed from a graded-commutative DGA. [Why?] However **Grothendieck** [see §§ 6,7 of Cartan's paper] gives such a simplicial graded-commutative which does yield the integral cohomology module, and also some additional data is set down whose knowledge prescribes the cup product structure. Similar results were found also by **Miller**.

(6) When K is a smooth triangulation of a smooth manifold M, then it seems $\Omega(K)$ coincides with the classical $\Omega(M)$ of **E.Cartan** and de Rham.

(7) Instead of thinking of $\Lambda(K)$ as all simplicial maps from K to Λ we can also think of it as all simplicial maps from the *chain group* of K to Λ. To consider *homology theories* we should, instead of $\Lambda(K)$ look at the tensor product of this chain group with Λ.

Eilenberg-Zilber

Following is a brief section-by-section account of this 1950 *Annals* classic:

(1) Discarding the requirement that simplices be uniquely determined by their faces, the authors generalize the notion of an ordered simplicial simplicial complex to a **semi-simplicial complex**, i.e. only *face maps* used. They point out that all of (co)homology, e.g. *Alexander formula* and *cup i-products* make sense for this generalization.

(2) Also *local coefficients* cohomology fits in very nicely: now to each vertex is associated a group, and to each edge is associated a group isomorphism, these being subject to the *cocycle condition*: the appropriate (co)chains now pair simplices with elements of group based

on their *leading vertex*: the 0th face operator gets **twisted**: otherwise the coboundary formula is the usual one.

(3) Now they turn to the chief motivation of their study: the singular simplices of a space. When a *base point* is specified they introduce the sub semi-simplicial complexes for which singular simplicial faces of dimension below an n are all on base point

(4) Next they define the notion of *homotopic simplices* for singular simplices having same faces. They define the notion of a **minimal semi-simplicial subcomplex** of the singular complex. They show how to get one, and also that, if the homotopy groups vanish till some n, that it lies in the subcomplex of (3).

(5) A chain homotopy [approx. acyclic models] is constructed.

(6) Using above it is shown that minimal complexes have the same (co)homology. Another application is that if lower homotopy vanishes the subcomplexes of (3) carry all (co)homology.

(7) Now, for space connected, the minimal semi-simplicial complex is shown to be unique upto simplicial isomorphism.

(8) At this point they note that in fact all order preserving maps acts on singular complexes, and they define **complete semi-simplicial complexes**.

(9) Parallel to this they now define **minimal complete semi-simplicial complexes** and construct these iteratively also.

(10) The last section proves, by an explicit algorithm in which the now well known commutation rules re face and degeneracy maps are freely used, that **normalization**, i.e. the process of dividing out by the degenerate simplices does not effect (co)homology, with local coefficients, of a complete semi-simplicial complex.

Comments

(1) The analogy of minimal complexes to shifting is uncanny: from an X we go to S(X) within which we find this M(X) whose realization still has the homotopy type of X ...

(2) Re (8) one may ask if, analogously, had E-Z been studying the example of the *linear complex* of a simplicial complex, and seen that *all* maps [n] $\rightarrow$ [m] act on it, would they have from this introduced the notion of a *cyclic object* ?

(3) The algorithm of (10) should bear on **Bier**'s theorem.

ACYCLIC MODELS

The original papers of **Eilenberg-Maclane/Zilber**, in *Amer.J.* of 1953, are much smoother to read than **Spanier** [who gives a weaker version] or **May**.

(A) A covariant functor $K: \mathscr{C} \to \mathscr{A}$, where the category $\mathscr{A}$ is abelian, is said to be **representable** by a subset M of Obj($\mathscr{C}$), if there is some natural transformation which *lifts* it to the **associated** functor $K_M: \mathscr{C} \to \mathscr{A}$ having

$$K_M(a) = \langle(\mu,x):\mu \in \mathrm{Hom}(m,a), x \in K(m), m \in M\rangle,\ K_M(a \xrightarrow{\alpha} b) = (\alpha\circ\mu, x),$$

over the natural transformation,

$$K_M(a) \to K(a) = \{(\mu,x) \mapsto K(\mu)(x)\},\ a \in \mathrm{Obj}(\mathscr{C}).$$

Any such lifting $K(a) \to K_M(a)$, $a \in \mathrm{Obj}(\mathscr{C})$, is said to be a **representation** of K by the **model** objects $m \in M$.

(B) **Theorem 1.** *Let* K *and* L *be two functors from* $\mathscr{C}$ *to the category* $\mathscr{A}$ *of chain complexes, such that* K *is representable by a set of model objects of* $\mathscr{C}$ *whose homology under* L *is trivial. Then there is a natural transformation* $f:K(a) \to L(a)$, $a \in \mathrm{Obj}(\mathscr{C})$, *which is unique upto chain homotopy.*

Proof. Note that each natural transformation $f: K(a) \to L(a)$ has the associated natural transformation $f_M : K_M(a) \to L(a)$ obtained by composing it with the functor $K_M(a) \to K(a)$, and that conversely, by composing any natural transformation $K_M(a) \to L(a)$ with a representation $K(a) \to K_M(a)$ of K over the models M, one obtains a natural transformation $K(a) \to L(a)$.

Similar remarks apply to **partially defined** natural transformations $f: \mathscr{C} \to \mathscr{A}$, i.e. natural homomorphisms $f: K_p(a) \to L_p(a)$, $a \in \mathrm{Obj}(\mathscr{C})$, commuting with the boundaries ∂, for all , defined for dimensions p less than some n.

Given such a partially defined f, we choose, for each model object m, and $x \in K_n(m)$, a $z \in L_n(m)$ which bounds the cycle $(f\circ\partial)(x) \in L_{n-1}(m)$. Using this the associated partially defined natural transformation extends to the nth dimension by letting $f_M: K_{M,n}(a) \to L_n(a)$ be the map $(\mu,x) \mapsto L(\mu)(z)$.

The above argument, which used the vanishing of the (n-1)*th homology* of the models under L, serves as the nth step of our inductive construction of the required natural transformation $f: K(a) \to L(a)$.

A similar upward induction on dimension, whose nth step now uses the vanishing of the *nth homology* of the models under L, suffices to construct a chain homotopy $H: K_n(a) \to L_{n+1}(a)$, $\partial\circ H + H\partial = f - h$, between any two given natural transformations $f,g: K_n(a) \to L_n(a)$. *q.e.d.*

Eilenberg-Maclane also give *variants of this result*, for partially defined natural transformations and chain homotopies between them, which are obvious once one keeps track of which homology's vanishing is required at which step in the above proof.

(C) **Cubical singular homology.** The main application of the above theorem given in this paper is that **Serre**'s cubical singular homology coincides with the usual singular homology.

To define this one has **singular n-cubes** of a space X, i.e. continuous functions $\sigma(x_1, .. ,x_n)$, of n real variables satisfying $0 \le x_i \le 1$, taking their values in X. A singular n-cube has 2n (n-1)-dimensional faces: for each i, $1 \le i \le n$, there is the **front ith face** $\sigma(x_1, .. ,x_{i-1},0,x_i, .. ,x_n)$, and the **back ith face** $\sigma(x_1, .. ,x_{i-1},1,x_i, .. ,x_n)$. The **cubical singular complex** $\square(X)$ of a space is the free abelian group on its singular n-cubes which is equipped with the boundary ∂ given by taking the alternating sum over i of the differences (front ith face - back ith face).

The homology of the cubical singular complex $(\square(X),\partial)$ *is* not *the usual cohomology of a point.*

In fact for X = {pt.}, $\square(X)$ has one singular n-cube, for each $n \ge 0$, and above ∂ vanishes, giving homology $\cong \mathbb{Z}$ in all dimensions.

A singular n-cube σ is **degenerate** if there is some i, $1 \le i \le n$, and a singular (n-1)-cube θ such that $\sigma(x_1, .. ,x_n) = \theta(x_1, .. ,x_{i-1},x_{i+1}, .. ,x_n)$. These determine a subcomplex of $\square(X)$. The quotient complex $\blacksquare(X)$ is called the **normalized cubical singular complex** of X.

Theorem 2. *The homology of the normalized cubical singular complex* $(\blacksquare(X),\partial)$ *coincides with the usual singular cohomology of* X.

This follows from the acyclic models theorem by using the non-degenerate singular cubes as the models. Their acyclicity under $\blacksquare$ is easily checked [after an augmentation]. The usual **singular simplices** are of course continuous functions $\sigma(x_0, .. ,x_n)$, of n+1 non-negative real varibles having sum 1, having values in X. The **simplicial singular complex** $\Delta(X)$ is the free abelian group on these equipped with the boundary ∂ which is the alternating sum of the faces, there being just one ith face, viz. $\sigma(x_0, .. ,x_{i-1},0,x_i, .. ,x_n)$. The singular simplices serve as the acyclic models of $\Delta(X)$.

For the simplicial case too there is a smaller **normalized simplicial singular complex** [by dividing out by the degenerate simplices σ which arise from a simplex θ of one lower dimension as $\sigma(x_0, .. ,x_i+x_{i+1}, .. ,x_n)$] which too gives the same homology. But, unlike the cubical case, now normalization is only optional.

(D) We now go to another application, given in a paper which immediately follows the above paper in the same journal.

Eilenberg-Zilber Theorem. *If K and L are any two simplicial abelian groups, then their cartesian product K × L and tensor product K ⊗ L, considered as functors into the category of chain complexes, are homotopy equivalent.*

It is interesting that in this note [which is before **Kan**'s work] E-Z quite clearly think of a complete semi-simplicial complex as a contravariant *functor*, and having defined the q-simplices of K × L as all ordered pairs (σ,θ), where σ is a q-simplices of K, and θ a q-simplex of L, make any monotone α act on these by $\alpha(\sigma,\theta) = (\alpha(\sigma),\alpha(\theta))$. The product K × L becomes a chain complex by equipping it with boundary ∂ equal tò the alternating sum of the faces.

On the other hand K ⊗ L is the graded abelian group whose qth summand is freely generated by the synbols $\sigma \otimes \theta$, where σ is an r-simplex of K, and θ is an s-simplex of L, with r and s such that $q = r+s$. And the boundary ∂ is defined by $\partial(\sigma \otimes \theta) = \partial(\sigma) \otimes \theta + (-1)^r \sigma \otimes \partial(\theta)$.

To prove the above result they use $\mathcal{C}$ = category whose objects are ordered pairs (K,L) of simplicial complexes. The two products are then checked to have as acyclic models all pairs of [ordered complexes supported on pairs of] standard simplices.

Explicit equivalences are given, in one direction by the recipe of **Alexander-Kolomogrov-Whitney**, and in the other by that of Eilenberg-Maclane [involving shuffle transformations]: for the latter see their papers on $K(\pi,n)$'s and May.

The Eilenberg-Zilber Theorem gives very quick definitions of the products in (co)homology:

E.g. in conjunction with the **diagonal map** K → K × K one at once gets the **cup product** in cohomology H(K) ⊗ H(K) ⟶ H(K) [originally defined by using the explicit cochain map of A-K-W.] And, in conjunction with a given [simplicial] *group action* G × F → F, it at once gives the **Pontrjagin product** H(G) ⊗ H(F) → H(F) in homology. [Thus, for the case F = G one gets, in cohomology H(G), besides the cup product,the *comultiplication* which makes it into a **Hopf algebra**.] For more see e.g. May who also treats the **cap product**.

Comments

(1) *It should be very interesting to work out an analogue of the Eilenberg-Zilber theorem with joins instead of products."*

Of course this would entail finding the right definitions, and here the **Wu** triangulations of joins should help. A result of above type would explain, without messing around with explicit (co)chain formulae, how the cup, and all other cup-i products, arise all together [as "**Hirsch** components"] of the juxtaposition product [explaining why Pontrjagin classes are "Hirsch components" of the **van Kampen**-Wu obstruction classes].

(2) **Brown** [see May, or original § in *Annals* of 1959] established a **generalized Eilenberg-Zilber theorem** giving homotopy equivalence between a **twisted** cartesian product, and a twisted tensor product of two simplicial groups: this gives another proof re nature of the second term of the *Serre spectral sequence of a fibration* [which incidentally is where singular cubes first arose].

(3) In the above paper of Eilenberg-Zilber they also have a result re the *linear complexes* [of all sequences of vertices supported on the simplices] *of simplicial complexes*. [No total ordering of the vertices is involved]. The cartesian product of these is called the **simplicial product of simplicial complexes**: the E-Z theorem identifies the homology of simplicial products with that of the product space (same ex. is on p.359 of Spanier).

SERRE'S EXPOSE

From *Sem. Cartan* 1954/55, exposé 1:

(A) **Eilenberg's isomorphism.** *There is a natural isomorphism*

$$H^n(X;\pi) \cong [X, K(\pi,n)],$$

where X *is any* CW-*complex, and* $K(\pi,n)$ *denotes one whose sole non-trivial group is the* nth, *being isomorphic to* π.

Such a $K(\pi,n)$ is called an **Eilenberg-Maclane space**.

Proof. Note that $K(\pi,n)$ has trivial integral homology below dimension n, while its nth homology group is π. So its nth cohomology group with coefficients π is given by

$$H^n(K(\pi,n),\pi) = \mathrm{Hom}(\pi,\pi).$$

Let the **fundamental class** in $H^n(K(\pi,n),\pi)$ be the one corresponding to the identity map $\pi \longrightarrow \pi$. Each continuous map $f\colon X \longrightarrow K(\pi,n)$ pulls this back to a class of $H^n(X;\pi)$.

The map $[X,K(\pi,n)] \longrightarrow H^n(X;\pi)$ thus obtained was checked to be an isomorphism by means of an [**obstruction theoretic**] upward induction on the skeletons of the complexes. *q.e.d.*

Some examples of Eilenberg spaces:

$S^1 = K(\mathbb{Z},1)$, $K(\pi,n) \times K(\psi,n) \cong K(\pi \times \psi,n)$ e.g. $T^2 = K(\mathbb{Z}^2,1)$, $\mathbb{R}P^\infty = K(\mathbb{Z}_2,1)$, and more generally for any discrete group G one has $BG = K(G,1)$, $\mathbb{C}P^\infty = K(\mathbb{Z},2)$.

(2) The importance of Eilenberg-Maclane spaces is clear once one notes that **cohomology operations**, i.e. natural maps $H^n(X;\pi) \cong [X,K(\pi,n)] \mapsto [X,K(\psi,p)] \cong H^p(X;\psi)$, are [all of them?] determined by elements of $[K(\pi,n),K(\psi,p)] \cong H^p(K(\pi,n);\psi)$ [for more see Eilenberg-Maclane's $H(\pi,n)$ papers and Serre.]

Their (co)homology $H^p(K(\pi,n);\psi)$ and $H_p(K(\pi,n);\psi)$ are denoted $H^p(\pi,n;\psi)$ and $H_p(\pi,n;\psi)$ by Eilenberg-Maclane because of the following.

Theorem. *For each abelian* π *and* $n \geq 1$, *there is, upto homotopy type, a unique* $K(\pi,n)$.

Proof. A CW cell complex which is a $K(\pi,n)$ can be constructed by using an inductive argument going back to **Whitehead**.

The uniqueness is a consequence of the isomorphism of (1): let X also have only the nth homotopy group non-trivial and equal to π: so

$[X,K(\pi,n)] \cong H^n(X;\pi) = \mathrm{Hom}(H_n(X),\pi) = \mathrm{Hom}(\pi,\pi)$: the map $X \longrightarrow K(\pi,n)$ corresponding to the identity map $\pi \longrightarrow \pi$ induces an isomorphism of homotopy groups, and thus is a homotopy equivalence. *q.e.d.*

The Eilenberg-Maclane groups $H^p(\pi,n;\psi)$ *generalize* **Hopf's** *notion of* **cohomology** $H^p(\pi;\psi)$ *of a group* π *with coefficients* ψ, *which corresponds to the case* $n = 1$: $H^p(\pi,1;\psi) = H^p(\pi;\psi)$. This follows because the quotient of a contractible space under a free action of a discrete group π is a $K(\pi,1)$.

In fact Serre's calculations of these more general groups starts from the case $n = 1$, and uses an induction on n based on the following:

There is a contractible fibre space with fiber $K(\pi,n-1)$ *and base* $K(\pi,n)$; namely the space of all paths of $K(\pi,n)$: thus $K(\pi,n-1)$ is the **loop space** of $K(\pi,n)$.

Note also that loop composition gives a group operation also in the homology groups $H_p(\pi,n;\psi)$ *which is associative and anticommutative.*

(3) **Postnikov towers** [also found independently by **Zilber**] give a way of constructing a space having a given sequence of homotopy groups π_1, π_2, π_3, ... One starts with an $X_1 = K(\pi_1,1)$ and builds over it a fibration having $K(\pi_2,2)$ as fibre: it turns out that these are classified by an **invariant** $k_3 \in H^3(X_1;\pi_2)$. And then, over the total space X_2 of this fibration, another fibration with $K(\pi_3,3)$ as fibre, determined by an invariant $k_4 \in H^4(X_2;\pi_2)$, and so on.

Theorem. *Each space has the homotopy type of a unique Postnikov tower, so its homotopy groups and Postnikov invariants characterize its homotopy type.*

Generalizing the idea of cohomology operations **Massey** and **Adem** considered some conditionally defined "secondary" operations which are classified by some special Postnikov towers.

Comments

(1) Exposé 20 of same year is also by Serre and deals with **homotopy operations** [in possibly many variables] of **G.Whitehead** etc., and **Hilton**'s theorem regarding homotopy groups of bouquets of spheres.

It is curious that the cohomology groups of Eilenberg-Maclane spaces, which are spaces with just one non-trivial homotopy group, are responsible for cohomology operations, while the homotopy groups of spheres, which are spaces with just one non-trivial cohomology group, are responsible for homotopy operations.

(2) Cohomology operations $H^p(X;\pi) \times H^q(X;\psi) \to H^s(X;\theta)$ of *two* variables [possibly all of them?] are determined by maps $K(\pi,p) \times K(\psi,q) \to K(\theta,s)$, which constitute $H^s(K(\pi,p) \times K(\psi,q);\theta)$, which can be calculated by using the Kunneth formula.

(3) Serre also shows in first § how, upto homotopy type, each map can be replaced by either an inclusion or a fibration: using **mapping cyclinders** or **path/loop spaces** respectively. Such constructions are basic for proofs of homotopy theory.

SET THEORY

Halmos's book, *Naive Set Theory*, which is written in the usual informal [but formalizable] language of mathematics, gives all that one usually needs from this body of facts:

(1) Letters [usually roman small, capital, or script] will denote **sets**, an undefined notion [like e.g. "point" in geometry] and $x \in A$ will be read as: x **belongs** to A, or, x is an **element** of A: and assumed to have the following property.

Axiom of extension. *Sets* A *and* B *are same iff* $x \in A \Leftrightarrow x \in B$

This sameness or **equality** of A and B will be denoted $A = B$, while **inclusion** $A \subseteq B$ [or that A is a **subset** of B] will mean that only $x \in A \Rightarrow x \in B$ holds. Thus above axiom says: $A = B$ *iff* $A \subseteq B$ *and* $B \subseteq A$, which of course is the common way of checking the coincidence of sets.

(2) An equally common way of defining new sets is the following.

Axiom of specification. *If* A *is a set then, for each* **condition** S(x), *there is a set* $B \subseteq A$ *which consists precisely of those elements* x *of* A *for which the condition* S(x) *is true.*

This set B is also denoted by $\{x \in A : S(x)\}$. Note that the axiom of extension shows that this set is uniquely determined by A and the given condition.

More precisely the above condition S(x) is a **sentence**, built up from the **atomic sentences** of the type $A = B$ and $x \in A$, by using the logical connectives **and**, **or**, **not**, $\rightarrow$, $\forall$, $\exists$, with at least one occurence of x **free**, i.e. not preceded by $\forall$ or $\exists$.

We use $A \neq B$ for not($A = B$), $x \notin A$ for not($x \in A$), etc.

Theorem. *For any* A, *one has* $y = \{x \in A: x \notin x\} \notin A$

Proof. If we had $y \in A$, then $y \notin y$ implies $y \in y$ while $y \in y$ implies $y \notin y$. *q.e.d.*

Thus the existence of a "universe", i.e. "a set containing all sets", leads to a contradiction, which is called **Russell's paradox**.

Remark. Such *unsets* [like a "universe": also see (20) and (25)] are called "classes" in the treatment of set theory given in **Kelley**'s appendix. However *class* and *collection* are more frequently used simply as synonyms for set.

Though in general $\{x: S(x)\}$ will have no meaning, we will consider it as equivalent to $\{x \in A: S(x)\}$ when the condition S(x) implies $x \in A$. In this case we'll say that the set $\{x: S(x)\}$ is **well-defined**.

(3) The next axiom ensures in particular that there is a set.

Axiom of pairing. *There exists a set* A *which has given sets* u *and* v *as elements.*

Of course such an A may have other elements also, but its subset $\{x \in A: x = u \text{ or } x = v\}$, has precisely u and v. This set, which is uniquely determined by u and v, is denoted $\{u,v\}$, and called an **unordered pair**.

If $v = u$, the pair $\{u,u\}$ is also written $\{u\}$ and called a **singleton**.

Take any set A [it exists by virtue of the above]. Its subset $\{x \in A: x \neq x\}$ is called the **empty set** $\emptyset$ [because it has no elements]: the uniqueness of $\emptyset$ follows from the axiom of extension.

(4) Given a *nonempty* set $\mathscr{C}$ of sets, the **intersection** of the member sets of $\mathscr{C}$ is defined by using the axiom of specification to cut down any one of them to the subset whose elements lie in all of them: by the axiom of extension this set is uniquely determined by $\mathscr{C}$: it is denoted by $\bigcap \mathscr{C}$, $\bigcap \{X: X \in \mathscr{C}\}$, $\bigcap_{X\in\mathscr{C}} X$ etc., with $A \cap B$ being the preferred notation when $\mathscr{C} = \{A,B\}$.

Axiom of unions. *Given any set* $\mathscr{C}$ *of sets there exists another which contains all their elements.*

Again, by using the axiom of specification, we can cut down to the subset containing just these elements, and, by the axiom of extension this **union** of the member sets of $\mathscr{C}$, is uniquely determined by $\mathscr{C}$.

The parallel notations for the union of the member sets of a collection $\mathscr{C}$ of sets are $\bigcup \mathscr{C}$, $\bigcup \{X: X \in \mathscr{C}\}$, $\bigcup_{X\in\mathscr{C}} X$, $A \cup B$, etc.

Since $\{a,b\} = \{a\} \cup \{b\}$, we generalize the notion of unordered pairs to **unordered triples**, etc., by defining $\{a,b,c\} = \{a\} \cup \{b\} \cup \{c\}$, etc. [there is no need to put extra parantheses on the right because of the easily checked "associativity" of unions].

(5) Before writing down the next axiom we note that we already have enough tools in hand to do the usual *algebra of sets*: distributivity of union [intersection] over intersections [unions], *de Morgan's laws* regarding **complements** $A - B = \{x \in A: x \notin B\}$ [also denoted B' in case A is understood], *Boolean addition* $A \Delta B = (A-B) \cup (B-A)$], etc.

Axiom of powers. *Given any set* X *there exists another which has all subsets of* X *as its elements.*

The **power set** $\mathscr{P}(X)$ will be the unique set whose elements are precisely all the subsets of X.

(6) **Ordered pairs** are defined, somewhat surprisingly, by $(a,b) = \{\{a\},\{a,b\}\}$, the justification being the following.

Theorem. (i) $(a,b) = (u,v)$ *if and only if* $a = u$ *and* $b = v$.

(ii) *The* **cartesian product** $A \times B = \{x: x = (u,v),\ u \in A,\ v \in B\}$ *is a well-defined set.*

(iii) *If* R *is a* **relation**, *i.e. a set whose elements are ordered pairs, then*

$$\mathrm{dom}(R) = \{a: \exists\ b \text{ s.t. } (a,b) \in R\} \text{ and } \mathrm{ran}(R) = \{b: \exists\ a \text{ s.t. } (a,b) \in R\}$$

are well-defined sets, and $R \subseteq \mathrm{dom}(R) \times \mathrm{ran}(R)$.

Proof. (i) needs a straightforward verification.

(ii) follows from the definition of ordered pair, which shows that the condition $x = (u,v)$, $u \in A$, $v \in B$ implies that x belongs to the set $\mathcal{P}(\mathcal{P}(A \cup B))$.

(iii) follows again from this definition because it implies that any a [resp. b] such that $(a,b) \in R$ for some b [resp. $(a,b) \in R$ for some a] belongs to the set $\bigcup\bigcup R$. *q.e.d.*

(7-10) Besides the usual facts pertaining to [compositions etc. of] relations, the following notions are important.

A **partition** $\mathcal{E}$ of a set X is a set of subsets whose pairwise intersections are empty, and whose union is X. Correspondingly we have the **equivalence relation** $R \subseteq X \times X$ consisting of all (u,v) such that u and v belong to the same member of $\mathcal{E}$.

Another important kind of relation, called a **function**, and denoted $f: A \to B$, or $x \mapsto f(x)$, $x \in A$, is a subset $f \subseteq A \times B$ with $\mathrm{dom}(f) = A$ and such that for each $a \in A$, there is a unique $b \in B$ such that $(a,b) \in f$. Sometimes this unique b is denoted b_a, and it is convenient to think of the function f as a **family** $\{b_a\}$ *indexed* by the members a of A. If $C \subseteq A$, then $f|C$ will denote the *restriction* of the function $f: A \to B$ to C.

Given any family $\{B_a\}$, $a \in A$, of sets, we can then look at the set of all families $\{b_a\}$, $a \in A$, with $b_a \in B_a$ for all a. This set is called the **cartesian product of the family** $\{B_a\}$ of sets, and is denoted by $\mathsf{X}_{a\in A}(B_a)$, or $\prod_{a\in A}(B_a)$, etc. When A is an ordered pair this notion identifies in an obvious way with the $A \times B$ of (6).

Associated to each function $f: A \to B$ there is an induced function $f: \mathcal{P}(A) \to \mathcal{P}(B)$ which associates to each subset of A its *image* in B under f, and also an induced function $f^{-1}: \mathcal{P}(B) \to \mathcal{P}(A)$ which associates to each subset of B its *preimage* in A under f. The latter commutes with unions, intersections, and relative complements, while the former commutes in general only with unions.

(11) The **successor** x^+ of a set x is defined by $x^+ = x \cup \{x\}$. The empty set $\emptyset$ is also called **zero**, its successor is called **one**, and one's successor is called **two**, and two's **three**, and so on. We adopt the usual

notation 0, 1, 2, 3, ..., for these **natural numbers**.

Axiom of infinity. *There exists a set* A *such that* $\emptyset \in$ A *and for which* $x \in A \rightarrow x^+ \in A$ *holds.*

The above axiom implies that there is a set ω consisting precisely of all natural numbers, and that ω is the smallest set of the kind mentioned in the axiom.

Families (b_a), $a \in n$, having as indexing set some natural number n, are called **finite sequences**, while families (b_a), $a \in \omega$, indexed by the set ω of all natural numbers are called **sequences**.

(12) The fact that ω is the smallest set obeying the axiom of infinity has the following immediate but important consequence.

Principle of mathematical induction. *If a subset* S *of* ω *satisfies* $0 \in S$, *and* $n \in S \rightarrow n^+ \in S$, *then* $S = \omega$.

Proofs using this fact are called *inductive proofs*.

For example to prove that ω *is a* **transitive set**, i.e. that $x \in y \in \omega$ implies $x \in \omega$, it obviously satisfies to show that all natural numbers are transitive sets, and this follows by noting that the set S of all transitive natural numbers obeys the above requirements and so coincides with ω.

It can be checked that the above hypotheses "$0 \in S$ and $n \in S \rightarrow n^+ \in S$" are equivalent to "$n \subseteq S \rightarrow n \in S$". [Anticipating the definition of order of natural numbers given in (13) this says "if all numbers less than n are in S, then so is S": it is this reformulation of induction which is generalized in (17) to all well ordered sets.]

The following result [whose proof also uses induction] is the basis of all *inductive* [= recursive] *definitions*.

Recursion theorem. *Given a function* $f: X \rightarrow X$, *and* $a \in X$, *there exists a unique function* $u: \omega \longrightarrow X$ *obeying* $u(0) = a$ *and* $u(n^+) = f(u(n))$ *for all* $n \in \omega$.

Proof. We'll show in fact that u is the intersection of the [obviously nonempty] set $\mathscr{E}$ of all those subsets A of $\omega \times X$ which contain (0,a) and obey $(n,x) \in A \rightarrow (n^+, f(x)) \in A$.

For this note that u itself is in $\mathscr{E}$, so it suffices to prove that it is a function. For this we apply the principle of mathematical induction to the set S of all natural numbers n for which it is true that $(n,x) \in u$ for at most one x. *q.e.d.*

(13) In ω **addition** of m [to be denoted by $n \longmapsto n + m$] is defined to be the recursive function $u: \omega \longrightarrow \omega$ obeying $u(0) = m$ and $u(n^+) = (u(n))^+$.

Having defined this, we can define **multiplication** by m [to be denoted by $n \mapsto n.m$] as the recursive function $v: \omega \to \omega$ obeying $v(0) = 0$ and $v(n^+) = u(n) + m$.

Next, raising to mth **power** [to be denoted by $n \mapsto n^m$] is defined to be the recursive function $w: \omega \mapsto \omega$ obeying $w(0) = 1$ and $w(n^+) = w(n).m$.

The usual rules of *arithmetic in* ω can now be checked.

The **usual order** < of ω is the subset of $\omega \times \omega$ consisting of all (m,n) such that $m \in n$. Since natural numbers are transitive sets the corresponding weak relation ≤ is equivalent to $m \subseteq n$, and thus is a **partial order**, i.e. is a reflexive, antisymmetric, and transitive relation. In fact it is a **total order**, i.e. one also has $n \leq m$ or $m \leq n$ for all $m,n \in \omega$.

[For the last verification Halmos suggests using the following *Lemma* proved inductively in §12 of book "If $m \in n$ then we can not have $n \subseteq m$". He used this lemma to check $n^+ = m^+ \to n = m$ in ω: which is the sole property of ω, amongst those listed in §12 of book as the **Peano axioms** of ω, which is not immediate from the definition of ω.]

Of course ω is **equivalent**, i.e. related by a one-one onto function, to some proper subsets, but it can be checked inductively that no natural number has this strange property. **Finite sets** A are those which are equivalent to some natural number. This number can be seen to be uniquely determined by A, and is called the **number of elements** #(A) in the finite set A. **Infinite sets** are those which are not finite sets.

(14) The terminology for *partially ordered sets* A [i.e. ordered pairs $(A, (\leq) \subseteq A \times A)$] is mostly self-explanatory, but note that the [initial] **segment** s(a) determined by an $a \in A$, which consists of all $x < a$, should be distinguished from the corresponding **weak segment** $\underline{s}(a)$ which also has a in it, and again note that a **largest element** m is one for which $x \leq m$ is always true, while a **maximal element** m is one for which the $m < x$ is never true.

(15) By an induction on n it is easy to *prove* the **finite axiom of choice**, i.e. that the cartesian product $\times_{i \in n} A_i$ of any finite family $\{A_i\}$, $i \in n$, of sets is empty if [this is the trivial part, and is true even without finiteness] and only if [this is somewhat non-trivial] at least one of the sets A_i is empty.

Many interesting facts of mathematics however require the following generalization of the above result.

Axiom of choice. *The cartesian product of any nonempty family of nonempty sets is nonempty.*

Alternatively, any nonempty set $\mathcal{E}$ of nonempty sets admits a choice function, i.e. a function $f: \mathscr{C} \to \bigcup \mathscr{C}$ obeying $f(A) \in A$ for all $A \in \mathscr{C}$.

Theorem. *Every infinite set X has a subset equivalent to ω.*

Proof. Choose any choice function f for the collection $\mathscr{X}$ of all nonempty subsets of X.

Next, let $\mathscr{C}$ be the collection of all finite subsets of X, and define $g: \mathscr{C} \to \mathscr{C}$ by $g(A) = A \cup \{f(X - A)\}$.

By recursion we define the function $u: \omega \to \mathscr{C}$ obeying $u(0) = \emptyset$ and $u(n^+) = g(u(n))$. This function u can be seen to be one-one. *q.e.d.*

An easy corollary is that, like ω, every infinite set is equivalent to some proper subset. This shows that our definition of infinite set coincides with that of **Dedekind**: *a set is infinite iff it is equivalent to some proper subset.*

‹16› Many mathematical applications of the axiom of choice are made via the following intermediary.

Zorn's lemma. *If every totally ordered subset of the nonempty partially ordered set X has an upper bound in X, then X has a maximal element.*

Proof sketch. By considering $\mathscr{X}$, the set of all subsets of X which are subsets of weak initial segments of X, one can see that it would suffice to prove the above result when the partial order is *inclusion* on $\mathscr{X}$, a nonempty set of subsets of a nonempty set X for which $y \subseteq x \in \mathscr{X} \to y \in \mathscr{X}$ holds, and which is such that the union of any totally ordered subset of $\mathscr{X}$ is also in $\mathscr{X}$.

Define a function $g: \mathscr{X} \to \mathscr{X}$ by mapping each A to A unless by adjoining some element of X–A to A we can get a bigger set of $\mathscr{X}$. In all such cases we map A to $A \cup f(\bar{A})$, where f is a choice function for the set of all nonempty subsets of X, and $\bar{A}$ is the nonempty subset of X-A consisting of all elements whose adjunction to A gives a bigger subset of $\mathscr{X}$. What we need to show is that there is at least one A for which the first case, i.e. $g(A) = A$, occurs.

For this purpose consider all subsets of $\mathscr{X}$ which contain $\emptyset$, which are preserved by g, and which are such that the union of each totally ordered subset is also in it. One checks that the intersection of this collection, which also surely obeys all these conditions, is in fact totally ordered. Then the union A of this totally ordered set is checked to obey $g(A) = A$. *q.e.d.*

Conversely, a much easier argument shows that Zorn's lemma implies the axiom of choice.

‹17› A partially ordered set is called **well ordered** if *every* nonempty subset has a smallest element. Note that this is a much stronger requirement than demanding that it be totally ordered, i.e. that any subset having *two elements* has a smallest element.

However, one can check that *each totally ordered set has a* **cofinal** *well ordered subset*, i.e. one having an element dominating any given element of the bigger set.

Principle of transfinite induction. *If S is a subset of a well ordered set W obeying* $s(x) \subseteq S \Rightarrow x \in S$ *for all* $x \in W$, *then* $S = W$.

Proof. Otherwise the smallest element x of the nonempty set $W - S$ is outside S while its initial segment s(x) is contained in S. *q.e.d.*

This generalization of induction is valuable because, besides ω, there are well ordered sets galore.

Well ordering theorem. *Every set can be well ordered.*

Proof. Use Zorn's lemma on the set of all well ordered subsets of the given set, equipped with the partial order of being an initial segment. A maximal element must be a well ordering of the entire set, for otherwise we could lengthen this maximal element by sticking on a new element at the end. *q.e.d.*

(18) In the following generalization of the recursion theorem to all well ordered sets W, X_W denotes the set of all functions with domain any initial segment of W, and range in X.

Transfinite recursion theorem. *Given a function* $f: X_W \longrightarrow X$, *there exists a unique function* $u: W \longrightarrow X$ *obeying* $u(x) = f(u|s(x))$ *for all* $x \in W$.

Its proof uses a collection $\mathcal{E}$ of subsets of $W \times X$ defined analogously to that in the proof of the recursion theorem, whose intersection is verified, via transfinite induction, to be a function.

Two well ordered [or even partially ordered] sets are called **similar** if there is an order preserving bijection, or **similarity**, between them.

Comparability theorem. *If two well ordered sets are similar then there is a unique similarity between them, and if they are not similar then exactly one of them is similar to an initial segment of the other.*

Proof sketch. The key point is to check that a similarity f from W onto a subset of W must obey $a \leq f(a)$ for all $a \in W$. This gives the first part, and also shows that a well ordered set is never similar to any of its initial segments. To complete the proof of the second part a transfinite recursion is used to define a similarity of one of sets with an initial segment of the other. *q.e.d.*

(19) Following **von Neumann** we define an **ordinal number** α as a well ordered set in which each element x is equal to its segment s(x).

As per the above definition, the underlying set of an ordinal α is obviously *very* special. Note also that *the order of* α *is uniquely determined by its underlying set*: this follows because any partial order is determined by its initial segments, but now these coincide with the elements of the set.

Examples of ordinals. The natural numbers and ω are ordinals. Also note that if α is an ordinal, then so is α^+ with the obvious continuation of the well ordering. These successors of ω are called $\omega+1$, $\omega+2$, ...

To get more ordinals we use the following simple fact, which is known to be independent of the previous axioms.

Axiom of substitution. *Let* S(a,b) *be a sentence such that each* {b:S(a,b)}, a ∈ A, *is a well-defined set. Then* a ⟼ {b:S(a,b)} *is a function.*

[The argument "a ⟼ {b:S(a,b)} is a function because it takes values in the union of the singletons {{b:S(a,b)}}" is erroneous because such a "union" is defined only for families [= functions] {{b:S(a,b)}}, a ∈ A, and the existence of such a function begs the question asked.]

More examples of ordinals. Using above axiom we see that there is a function on ω such that n ⟼ ω+n = {b: b ∈ ω or b is the ith succesor of ω for some $0 \le i \le n$}. The the union of ω and the range of this function, when equipped with the obvious ordering, is an ordinal, which is denoted $\omega.2$. Its successors give $\omega.2 + 1$, $\omega.2 + 2$, ... Next, the union of $\omega.2$, and the range of the function n ⟼ $\omega.2 + n$, yields $\omega.3$, ... Having defined $\omega.n$, we can now make a similar use of the function n ⟼ $\omega.n$ to define ω^2, ... , etc., etc.

(20) Theorem. *The relation of being an initial segment well orders any set of ordinals.*

Proof. Note that each element ξ, or equivalently each initial segment $s(\xi)$, of an ordinal α, is also an ordinal under the restricted order, and that its least upper bound in α is ξ.

Using this an easy transfinite induction shows that one can have a similarity between two ordinals iff they *coincide*, and then the similarity has to be the identity map.

If two ordinals are not similar, then the comparability theorem gives a similarity from one of these well ordered sets onto an initial segment of the other, and same argument shows that the first ordinal coincides with this initial segment of the second.

So any set of ordinals is totally ordered. A small extra argument is needed to check that in fact it is well ordered. *q.e.d.*

Supremum of a set of ordinals. By this we mean their union equipped with the obvious ordering under which each of the given ordinals becomes an initial segment of this. It is easily verified that this indeed gives an ordinal, and is the smallest ordinal ≥ all the given ordinals.

It is false that "the set of all ordinals" exists. Otherwise, a contradiction, called the **Burali-Forti paradox**, follows: the supremum of such a set would be an ordinal $\geq$ all ordinals, which is silly, because its successor is still bigger.

Counting theorem. *Each well ordered set* W *is similar to a unique ordinal* ord(W).

The uniqueness is immediate from the above, the existence is not hard and uses axiom of substitution and transfinite induction.

(21) It is often useful [e.g. in the following definitions] to replace a family B_a, $a \in A$, by the family $\underline{B}_a$, $a \in A$ of *pairwise disjoint* sets, obtained by replacing each B_a by a **copy** $\underline{B}_a = B_a \times \{a\}$.

The **sum** $\alpha+\beta$ **of ordinals** is the ordinal corresponding to "W followed by U", where W and U are any disjoint well ordered sets such that ord(W) = α and ord(U) = β. When α and β are natural numbers this definition can be seen to coincide with that of (13).

The **product** $\alpha.\beta$ **of ordinals** is the ordinal corresponding to "W repeated U times", where again ord(W) = α and ord(U) = β. More precisely we take a pairwise disjoint family W_u, $u \in U$, of copies of the well ordered set W, form its union $\bigcup_{u\in U}(W_u)$, and equip this union with the order defined as follows: if $a \in W_i$ and $b \in W_j$ then $a < b$ means either $i < j$ or else $i = j$ and $a < b$ in $W_i = W_j$.

Since $\bigcup_{u\in U}(W_u) = W \times U$, this is same as saying $\alpha.\beta$ as the ordinal number of the well ordered set obtained by equipping the cartesian product $W \times U$ with the **reverse lexicographic order** $(a,i) < (b,j)$ iff $i < j$ or $i = j$ and $a < b$. When α and β are natural numbers this product can be seen to coincide with that of (13).

The arithmetic of ordinals comes with some surprises: one has e.g. $1+\omega \neq \omega+1$ [the left side is ω, while right side is ω^+], $2.\omega \neq \omega.2$ [the left side is ω, while the right side is the ordinal $\omega 2$ of (19)], and so $(1+1).\omega \neq \omega.(1+1)$. On the other hand many other laws of the arithmetic of ω have expected generalizations.

Warning. In ordinal arithmetic it is natural to define the **ordinal power** 2^ω as the ordinal obtained as the supremum of the ordinals 2, 2.2, 2.2.2, ... Note that its underlying set, being a countable union of finite sets is obviously countable. On the other hand the notation 2^ω is also frequently used for the [unordered] set of all functions from ω to 2, which, as we'll see later, is not countable.

(22) Equipping two given sets with any well orderings, and using the comparability theorem, we see that they are à fortiori comparable in the weaker sense that one of them must be equivalent to a subset of the other. This "relation" [in the "set of all sets"] is obviously reflexive and transitive in the usual sense, furthermore it is antisymmetric in the sense of the following.

Schroeder-Bernstein Theorem. *If* A *is equivalent to a subset of* B, *and* B *to a subset of* A, *then* A *and* B *are equivalent to each other.*

Proof. We are given an injection f: A $\to$ B, and another g: B $\to$ A, we want to define a bijection A $\leftrightarrow$ B.

Let $A_A \subseteq A$ be the subset of elements with "Adam" in A. ["Father" being an element of B whose g-image is the element, "grandfather" being its father, i.e. an element of A whose f-image is this new elewment, ...] Likewise let $A_B \subseteq A$ be the disjoint subset of elements with "Adam" in B. What remains, "the non-christians", form A_∞. Similarly partition B into three subsets.

The restrictions of f and g, respectively, are bijections $A_A \longleftrightarrow B_A$ and $A_B \longleftrightarrow B_B$. And the restriction of either f or g gives a bijection $A_\infty \leftrightarrow B_\infty$. *q.e.d.*

(23) *Re countable sets*, i.e. those which are finite or else equivalent to ω, one checks that: countable unions of countable sets are countable, so cartesian product of two countable sets is countable, etc., etc.

However the most interesting fact is that not all sets are countable, in fact one has the following.

Cantor's theorem. *A set is never equivalent to the set of all its subsets.*

Proof. If there were a bijection f: X $\to \mathcal{P}$(X), then in particular we would have some a $\in$ X such that f(a) = {x $\in$ X: x is not an element of f(x)}. For such an a both possibilities "a is in f(a)" and "a is not in f(a)" lead to contradictions. *q.e.d.*

(24) *Cardinal arithmetic*, i.e. relations between cardinal numbers, can be studied without knowing what cardinal numbers a are, or what one means when one says card(A) = a, i.e. that cardinality of A is a. [These definitions are in (25) below.] We simply need to know [as follows easily from the definition in (25)] that card(A) $\le$ card(B) iff A is equivalent to a subset of B, and that card(A) = card(B) iff A is equivalent to B.

So in terms of cardinal numbers the Schroeder-Bernstein theorem says a $\le$ b and b $\le$ a iff a = b.

Sum a+b of cardinals is given by a+b = card(A $\cup$ B), where A and B are disjoint sets such that a = card(A) and b = card(B).

Product a.b of cardinals is cardinality of "A added to itself B times" i.e. card(A × B).

Powers a^b **of cardinals** is cardinality of "A multiplied with itself B times" i.e. $card(A^B)$.

There are again some surprises: for infinite cardinals one gets a+a = a and a.a = a, and one has a+b = max(a,b) when right side is an infinite cardinal.

(25) An ordinal number a is called a **cardinal number** iff it is not equivalent to any smaller ordinal number. A set A is said to have **cardinality** a, and we write a = card(A), iff A is equivalent to the cardinal number a. [In other words out of all ordinal numbers arising from the various well orderings of A the smallest one is called the cardinal number of A.]

When A is a finite set then card(A) = ord(A) = #(A).

Ordinal arithmetic and cardinal arithmetic are two *different* generalizations of the arithmetic of natural numbers, i.e. even though though the arena of ordinal arithmetic is larger than that of cardinal arithmetic, the latter is definitely not a restriction of the former. For example, ω and 2 are cardinals whose ordinal sum ω+2 is not a cardinal while the cardinal sum, also unfortunately denoted ω+2, equals the cardinal ω. Likewise, the ordinal power 2^ω of these cardinals is not a cardinal, and should not be confused with the cardinal power, which too is unfortunately denoted by 2^ω.

It is false that "the set of all cardinals" exists. Otherwise, a contradiction, called the **Cantor paradox**, follows: the supremum of such a set would be an ordinal ≥ all cardinals, and the least of all such ordinals will be a cardinal ≥ all cardinals, which is silly, because the cardinality of its power set is still bigger.

There is an established notation for the members of the well ordered "unset" of all infinite cardinals. The least one of these [i.e. ω] is called **aleph-zero**. The next one [i.e. the least non-countable ordinal Ω] is called **aleph-one**. The **continuum hypothesis** is that $\Omega = 2^\omega$. More generally, the definition of the "β th aleph" [β any ordinal] is the expected one, and the "generalized continuum hypothesis" states that the "(β+1)th aleph has the cardinality of the power set of the βth aleph".

Comments

(1) It seems that well ordering is the key concept required to generalize the notion of *shifting* to infinite simplicial sets.

(2) There is possibility too of transfinite generalizations of theorems like the **Kneser** conjecture.

From the book by **Kelley**:

(1) By a **topology** $\mathcal{T}$ is meant a set of sets closed with respect to arbitrary unions and finite intersections.

One says then that $\mathcal{T}$ is a *topology on the set* $X = \bigcup \mathcal{T}$, and the pair $(X,\mathcal{T})$ (or even X, if $\mathcal{T}$ is understood) is called a *topological space*. The members of $\mathcal{T}$ are called the **open sets** of this space.

Thus a subset $\mathcal{T}$ of the *poset* $\mathcal{P}(X)$, of all subsets of X under $\subseteq$, is a topology on X, iff it is closed with respect to *supremums* of arbitrary subsets, and *infimums* of finite subsets. Note also that the *least element* $\emptyset$ and the *biggest element* X of $\mathcal{P}(X)$ are contained in any topology $\mathcal{T}$ on X.

(2) *The poset (under* $\subseteq$ *) of all topologies on* X has as least element the **indiscrete topology** $\mathcal{T} = \{\emptyset, X\}$, and as biggest element the **discrete topology** $\mathcal{P}(X)$. Furthermore each subset of this poset has an infimum, given by the intersection of the given topologies, as well as a supremum, given by the intersection of all topologies which contain the given topologies.

In this context note that the union of two topologies — e.g. of $\emptyset$, {a}, {a,b,c}} and {$\emptyset$, {b}, {a,b,c}} — need not be a topology.

Given any $\mathcal{S} \subseteq \mathcal{P}(X)$, the topology $\mathcal{T}$ *generated* by $\mathcal{S}$, or the topology having $\mathcal{S}$ as a **subbase**, is the intersection of the topologies containg $\mathcal{S}$, and is easily verified to consist of all unions from the set $\mathcal{B}$ of all finite intersections from $\mathcal{S}$. A set $\mathcal{B} \subseteq \mathcal{P}(X)$ of this type is said to be a **base** for $\mathcal{T}$.

(3) We equip the power set $\mathcal{P}(X)$ [of a nonempty X] with the [free] order reversing *involution* $i: \mathcal{P}(X) \to \mathcal{P}(X)$, $i \circ i = 1$, which associates to each A its complement A'. Also, any $p: \mathcal{P}(X) \to \mathcal{P}(X)$ such that $p \circ p = p$, will be called an *idempotent* or *projection operator* of $\mathcal{P}(X)$. Note that the composition $p \circ q$ of two commuting idempotents p and q is also an idempotent.

For a topological space X, the complements U' of its open sets U are called its **closed sets**, and its *closure operator* $cl: \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by cl(A) = intersection of all closed sets containing A.

If $p: \mathcal{P}(X) \to \mathcal{P}(X)$ is the closure operator of a topology on X, then it is easily seen that $A \subseteq p(A)$, , $p \circ p = p$, $p(\emptyset) = \emptyset$, and that p is *order-preserving*, i.e. $A \subseteq B \Rightarrow p(A) \subseteq p(B)$: in fact p even *preserves supremums of finite subsets*, i.e. $p(A \cup B) = p(A) \cup p(B)\ \forall A, B \in \mathcal{P}(X)$.

Kuratowski's theorem $\mathcal{T} \leftrightarrow cl_{\mathcal{T}}$ *is a bijective correspondence between the set of all topologies* $\mathcal{T}$ *on* X, *and the set of all non-decreasing projection operators* p *of the poset* $\mathcal{P}(X)$, *which map* $\emptyset$ *to* $\emptyset$ *and preserve supremums of finite subsets.*

Proof To check that $F \in \mathcal{P}(X)$ is a *fixed point* of the non-decreasing map $p: \mathcal{P}(X) \to \mathcal{P}(X)$ it suffices to verify $p(F) \subseteq F$. Note further that the set $\mathcal{F} = \{F \in \mathcal{P}(X): F = p(F)\}$ of these fixed points contains $\emptyset$ and X, and is obviously closed with respect to finite unions.

Since p is order-preserving it follows that, for any $\mathcal{A} \subseteq \mathcal{F}$, $p(\bigcap\mathcal{A}) \subseteq p(A) = A \ \forall A \in \mathcal{F}$, thus yielding $p(\bigcap\mathcal{A}) \subseteq \bigcap\mathcal{A}$. So $\mathcal{F}$ is also closed with respect to arbitrary intersections. Hence $\mathcal{T} = \{U \in \mathcal{P}(X): U' \in \mathcal{F}\}$ is a topology on X.

To see $p(A) = cl(A) = \bigcap \{F \in \mathcal{F}: F \supseteq B\}$, note that one of the F's in $\mathcal{F}$ which contains A is p(A), and all other such F's are bigger. *q.e.d.*

Clearly the *conjugate* $i \circ p \circ i$ of any idempotent p of $\mathcal{P}(X)$ is also an idempotent, e.g. for a topological space X, its **interior operator** int: $\mathcal{P}(X) \to \mathcal{P}(X)$, which associates to each A the union of all open sets contained in A, is the conjugate of its closure operator. Easy examples show that int and cl need *not* commute with each other.

The *semigroup* S_p (under $\circ$) generated by an idempotent p of $\mathcal{P}(X)$, and the involution i, consists of all self-maps of $\mathcal{P}(X)$ obtainable by an alternating compositions $..\circ i \circ p \circ i \circ ..$ of these two maps. Clearly the semigroup of p coincides with that of its conjugate. The **Kuratowski semigroup** of a topological space X is defined by $S(X) = S_{cl} = S_{int}$.

A priori it is not even clear that this topological invariant is a *finite* semigroup, so the following, which too was essentially in Kuratowski's thesis, is very striking.

Theorem *For any topological space X the self-map* $cl \circ int$ *of* $\mathcal{P}(X)$, *and its conjugate* $int \circ cl$, *are idempotents of* $\mathcal{P}(X)$. *Furthermore, the Kuratowski semigroup* S(X) *of any topological space X has at most* 14 *elements*.

Proof The first part follows by using the fact that while the order preserving idempotent cl is *expanding*, $A \subseteq cl(A)$, the order preserving idempotent int is *contracting*, $A \supseteq int(A)$.

Thus our semigroup S_p, p = cl, has, besides p and $i \circ p \circ i$, the idempotents $p \circ i \circ p \circ i$ and $i \circ p \circ i \circ p$. The equation $p \circ i \circ p \circ i \circ p \circ i \circ p \circ i = p \circ i \circ p \circ i$ shows that the eighth term of

$$i, p \circ i, i \circ p \circ i, \ldots$$

is a repetition, so this sequence has at most seven distinct elements. Likewise, $i \circ p \circ i \circ p \circ i \circ p \circ i \circ p = i \circ p \circ i \circ p$ implies $p \circ i \circ p \circ i \circ p \circ i \circ p = p \circ i \circ p$, and shows that the seventh term of

$$p, i \circ p, p \circ i \circ p, \ldots$$

is a repetition, and so this sequence has at most six distinct terms.

et f, g be two order preserving idempotents of a poset with f shrinking and g expanding. using the fact that f is shrinking twice $f \circ g \circ f \circ g \le g \circ f \circ g \le g \circ g = g$. Applying f to get $f \circ g \circ f \circ g \le f \circ g$. Likewise use g expanding twice to get $f \circ g \circ f \circ g \ge f \circ g \circ f \ge f \circ f = f$ pply g to it to get $f \circ g \circ f \circ g \ge f \circ g$. So $f \circ g \circ f \circ g = f \circ g$. Similarly $g \circ f \circ g \circ f = g \circ f$. qed

Adding 1 = i∘i we see that S_p has at most 14 elements. *q.e.d.*

For the **real line** $\mathbb{R}$, i.e. the real numbers with the topology having the set of all open intervals as a base, card($S(\mathbb{R})$) = 14 because the action of $S(\mathbb{R})$ on $A = (0,1) \cup (\mathbb{Q} \cap (1,2)) \cup \{3\}$ gives 14 distinct sets.

(4) A poset $(D,\geq)$ is called a **directed set** if for all $a,b \in D$ we can find c such that $c \geq a$ and $c \geq b$.

Generalizing the notion of a **sequence** in X, i.e. a function from the well ordered set $\mathbb{N}$ of non-negative integers to X, one defines a **net** in X to be a function S from any directed set D to X.

If X is equipped with a topology, then a **convergent net** S of X is one for which there is an $x \in X$ with the property that for any open set U containing x, the net S is *eventually*, i.e. for all $b \geq$ some $a(U) \in D$, in U.

Theorem *A subset* U *of a topological space* X *is open iff all nets of* X *which converge to points of* U *are eventually in* U.

Proof To check the non-trivial "if" let D_x be the set of all open sets containing $x \in U$, directed by the partial order $\subseteq$, and choose a net $S: D_x \to X$ for which $S(W) \in W$ with $S(W) \notin U$ if W is not a subset of U.

This net converges to x, since given any open set V containing x, S(W) is in V for all W in D such that $W \subseteq V$.

So, by hypothesis, this net should eventually be in U. So we can find some $W_x \in D$ such that $S(W_x) \in U$. This implies $W_x \subseteq U$. Being the union of all such open sets W_x, $x \in U$, the set U must be open. *q.e.d.*

The "if" part of the above result is false if we confine to sequences. However, for **first countable spaces**, i.e. those for which each each open set containing $x \in X$ contains one of a countable list $\mathcal{B}_x$ of such open sets, sequences do suffice.

If a net $S: D \to X$ converges to $x \in X$ it is useful to use the notation x = lim S [or $x = \lim_d S(d)$] even though there is some danger of confusion because x = lim S and y = lim S need not imply x = y. However, for **Hausdorff spaces**, i.e. those in which distinct points can be put in disjoint open sets, the limit of a convergent net is necessarily unique.

Even if a net is not convergent it can still have a **limit point** $x \in X$, i.e. a point with the property that for any open set U containing x, the net is *frequently*, i.e at least once after any $a \in D$, in U.

If x *is a limit point of a net* S *of* X *then one can show that there is a* **subnet** T *of* S *which converges to* x, provided by subnet we mean a net of X obtained by composing $S: D \to X$ with a map $N: E \to D$ such that N(e) is arbitrarily large when e is large enough.

However it is not always possibly to find some restriction $T: E \to X$ of S to a **cofinal** subset E of D, i.e. one containing an element $\geq$ any given element of D, which converges to x.

Theorem. *Let* $S: D \times E \to X$ *be a function from the product of directed sets* D *and* E *such that there is an* $x \in X$ *with* $x = \lim_d(\lim_e S(d,e))$. *Then one can find a function* $f: D \to E$ *such that* $x = \lim_d S(d,f(d))$.

This follows by the usual "diagonal method" for iterated limits of sequences. In above formulation the function f can not be fixed in advance and depends on S.

The *product partial order* makes the product of directed sets into a directed set. So e.g. set E^D of all functions f from D to E is directed. Obviously the conclusion of the last result is true for all sufficiently large functions f, so we have $x = \lim_{(d,f)}(\Delta_D \circ S)(d,f)$ where $\Delta_D : D \times E^D \to D \times E$ is defined by $(d,f) \mapsto (d,f(d))$. Likewise defining $\Delta_E : D^E \times E \to D \times E$ by $(g,e) \mapsto (g(e),e)$ one can establish the formula $\lim_e(\lim_d S(d,e)) = \lim_{(g,e)}(\Delta_E \circ S)(g,e)$. The advantage of this reformulation is that now the **diagonal maps** Δ_D and Δ_E are given in advance and are independent of S.

<5> A **filter** on X is a set $\mathcal{F}$ of nonempty sets of X which is closed with respect to supersets and finite intersections.

As far as convergence questions are concerned nets carry some "extra baggage": given any net S in X we will see that what is really important is the filter $\mathcal{F}_S$ consisting of all subsets A of X such that S is eventually in A.

Another example of filter: if X is equipped with a topology, then each $x \in X$ has its **neighbourhood filter** $\mathcal{N}_x$, which consists of all sets containing some open set containing x.

A **convergent filter** $\mathcal{F}$, of a topological space X, is one for which there is an $x \in X$ such that $\mathcal{F}$ contains $\mathcal{N}_x$, and then we write $x = \lim \mathcal{F}$ [even though $x = \lim \mathcal{F}$ and $y = \lim \mathcal{F}$ need not imply $x = y$ for non-Hausdorff spaces].

Note that $\lim S = x$ iff $\lim \mathcal{F}_S = x$. Conversely given a filter $\mathcal{F}$ on X let $D_{\mathcal{F}}$ be the directed set of all pairs (F,x), $F \in \mathcal{F}$, $x \in F$, equipped with $\subseteq$ of first factor, and define $S_{\mathcal{F}}: D_{\mathcal{F}} \to X$ by projecting to the second factor. Then it is easily seen that $\lim \mathcal{F} = x$ iff $\lim S_{\mathcal{F}} = x$.

The above devices help in translating results involving nets into results involving filters and conversely. For example a result proved above has the translation:

A subset U *of a topological space* X *is open iff* U *is contained in any filter of* X *which converges to a point of* U.

However, being devoid of the aforementioned "extra baggage", the language of filters is somewhat *simpler* than that of nets. For example, the notion of a bigger [with respect to $\subseteq$] filter is obviously simpler than the parallel notion of "subnet". Again, using Zorn's lemma, we see that each $\mathcal{F}$ is contained in some maximal or **ultrafilter** : defining the parallel of this useful notion for nets requires more effort.

Note that any subset $\mathcal{S}$ of a filter $\mathcal{F}$ satisfies the **finite intersection property**, i.e. the intersection of any of its finite subsets is nonempty. Conversely given any subset $\mathcal{S}$ of $\mathcal{P}(X)$ satisfying the f.i.p. the filter *generated* by $\mathcal{S}$, i.e. the intersection of all filters of X which contain $\mathcal{S}$, can be obtained as the family of all supersets of the family $\mathcal{B}$ obtained by taking all finite intersections in $\mathcal{S}$.

A **compact** topological space X is one for which any family of open sets having union X has a finite subfamily whose union is also X. Using the fact that it amounts to saying that any family of closed sets having the finite intersection property has a nonempty intersection one can check the following.

Theorem *A topological space* X *is compact iff all ultafilters of* X *are convergent.*

Equivalently X is compact iff each net of X has a convergent subnet.

Unless otherwise mentioned any subset Y of a space X is understood to have the **subspace topology**, i.e. the one whose open sets are the intersections with Y of the open sets of X. The classical examples [see c.(5)] of compact spaces are the closed and bounded subsets Y of the real line $\mathbb{R}$.

(6) A **continuous map** between topological spaces is one which pulls back each open set to an open set. The isomorphisms of the category of topological spaces and continuous maps are called **homeomorphisms**.

The **product topology**, on the cartesian product $X = \prod_{a\in A} X_a$ of topological spaces X_a, $a \in A$, is defined to be the smallest topology which makes each of the projections $\pi_a : X \rightarrow X_a$ continuous.

Thus this topology of $X = \prod_{a\in A} X_a$ is the one which has as a base all subsets of the type $\prod_{a\in A} U_a$, with U_a open in X_a, and such that $\{a: U_a \neq X_a\}$ is a *finite* subset of the indexing set A.

It is easily seen that the π_a's, besides being continuous, are also **open**

maps, i.e. they image each open set of X to an open set of X_a, and that a function into the product is continuous iff its composition with each of the projections is continuous.

A property shared of the coordinate spaces may or may not be shared by arbitrary products of such spaces. For example for Hausdorffness the answer is yes. On the other hand first countability is not enjoyed by arbitrary products of first countable spaces, in fact *a product of first countable spaces is first countable iff all but a countable number of the factors are* indiscrete *spaces* ! The most important result of this genre is however the following.

Tychonoff's Theorem. *Any product of compact spaces is compact.*

Proof. Choose any ultrafilter $\mathcal{F}$ of the product. Adding all supersets to its ath projection $\pi_a(\mathcal{F})$ we obtain an ultrafilter $\mathcal{F}_a$ of X_a. Since X_a is compact $\mathcal{F}_a$ converges to some point x_a, i.e. $\mathcal{F}_a$ contains all open sets U_a of X_a containing x_a. So $\mathcal{F}$ contains the pull backs $(\pi_a)^{-1}(U_a)$ of all such sets, and thus the neighbourhood filter of the point x of the product whose ath coordinate is x_a. *q.e.d.*

(7) A [partition or] **decomposition** X/R of a space X is usually provided with the biggest topology which keeps the quotient map π: X $\to$ X/R continuous.

It helps to think of the points of X/R as the **leafs** of X, e.g. in this language the open sets of X/R are precisely the **saturated** open sets of X, i.e those which contain complete leafs only, and we see that π is open iff the saturation of each open set of X is open.

On the other hand, the decomposition is **upper semi-continuous**, i.e. the quotient map is a **closed map**, i.e. images each closed set to a closed set, iff *leafs have arbitrarily small saturated neighbourhoods*, i.e. a saturated neighbourhood contained in any given neighbourhood.

The reason for above terminology is that if $F \subset \mathbb{R} \times \mathbb{R}$ is the the area under the graph of a non-negative function $f:\mathbb{R} \to \mathbb{R}$, then the decomposition of F into vertical segments is of the above type iff f(x) $= \lim_{y \to x}$ f(x) for all x.

The R in the above notation stands for the **equivalence relation** $R \subset X \times X$ corresponding to the partition. If this is a closed subset of $X \times X$ then X/R is Hausdorff. The converse is true also provided the quotient map is open.

(8) By an **embedding** e: X $\to$ Y is meant a continuous map which is open and one-one. [In general a continuous injection is not open, but it is so if X is compact and Y is Hausdorff.] Our embeddings will be constructed as follows:

Let F be a family of functions f: X $\to Y_f$ and let e: X $\to$ Y = $\prod_{f \in F} Y_f$ be defined by $(e(x))_f = f(x)$. Then the continuity of the f's ensures that e is continuous. Furthermore if for each closed set A of X and a point x of X not in it there is an f in F with f(x) not in cl(f(A)), i.e. if F **distinguishes points and closed sets**, then e is open. Lastly if for each distinct pair of points of X there is an f in F which images them to distinct points, i.e. if F **distinguishes points**, then e is also one-one.

So, to embed X into say a **cube**, i.e. a product of the interval I = [0,1] of real numbers, we need a family of continuous functions X → I which is sufficiently rich in the sense explained above. For this, we use the following celebrated result, where a **normal space** is one whose disjoint closed sets can be contained in disjoint open sets.

Urysohn's Lemma. *If* A *and* B *are disjoint closed sets of a normal space* X, *then there is a continuous function* f: X → [0,1] *such that* f(A) = 0 *and* f(B) = 1.

Proof. Because of normality we can choose an open set $U_{1/2}$ nested between U_0 = A and U_1 = B' whose closure is contained in the latter. Now find a similar $U_{1/4}$ nested between U_0 and $U_{1/2}$, and a $U_{3/4}$ nested between $U_{1/2}$ and U_1, and so on.

Define f: X → [0,1] by f(B) = 1, and $f(x) = \inf \{t: x \in U_t\}$ for $x \in B$. Clearly f(A) = 0.

To check the continuity of f it suffices to verify that the pull backs of all infinite open intervals of $\mathbb{R}$ are open, or equivalently, that all sets of the type $\{x: f(x) < r\}$ are open, and all of the type $\{x: f(x) \leq r\}$ are closed. This follows from

$$\{x: f(x) < r\} = \bigcup \{U_s: s < r\}, \text{ and}$$
$$\{x: f(x) \leq r\} = \bigcap \{U_s: s > r\} = \bigcap \{cl(U_s): s > r\}.$$

Here the first assertion only uses that $U_s \subseteq U_t$ whenever $s < t$. The first equality of the second assertion uses this as well as the fact that the dyadic rationals are dense. The last equality uses the fact that we have in fact $cl(U_s) \subseteq U_t$ whenever $s < t$. *q.e.d.*

Stone-Cech Compactification. It follows as an immediate corollary of Urysohn's Lemma, that if X is *Hausdorff and normal*, then the family F of all continuous functions X → I distinguishes points and closed sets. So, for all such spaces X, e: X → I^F is an embedding of X in the cube I^F. The closure of e(X) is the required compactification.

However, such an X need not embed in the **Hilbert cube** $I^{\mathbb{N}}$, because for this it is necessary to assume that, like this cube, X is **second countable**, i.e. that its topology has a countable base $\{U_1, U_2, \ldots\}$.

Embedding in Hilbert cube. Assuming that X is *Hausdorff and normal and has a countable base* $\{U_1, U_2, \ldots\}$, we can find within each U_i a smaller open set V_i whose closure is contained in U_i, and then, by Urysohn's Lemma, a countable family F of continuous functions $f_i : X \to I$ such that $f(V_i) = 0$ and $f(U_i') = 1$. Since F too can be seen to distinguish points and closed sets, it follows that $e: X \to I^{\mathbb{N}}$, where $(e(x))_i = f_i(x)$, is an embedding of X in the Hilbert cube.

Recall now that a set X which is equipped with a non-negative symmetric function $d: X \times X \to \mathbb{R}$, which is positive outside the diagonal, and obeys the triangle inequality $d(x,y) + d(y,z) \geq d(x,z)$, is called a *metric space*. Such an (X,d) can be equipped with the topology having the set of all balls $\{y: d(y,x) \leq r\}$ as a base. A topological space X is called **metrizable** if its topology is of this type.

Urysohn's Metrization Theorem *A Hausdorff, normal, and second countable space is metrizable.*

This follows at once from the above because it is easy to verify that the product topology of the Hilbert cube coincides with that arising from the metric $d(x,y) = \sum_i (1/2)^i . |x_i - y_i|$.