# REVIEWS (IV)

*Comments*

(1) Consider any subset $\mathscr{B}$ of the power set $\mathscr{P}(X)$ of a set X which is closed under interections, unions, and complements. Equip $\mathscr{B}$ with the *addition* $A + B = (A \setminus B) \cup (B \setminus A)$ and *multiplication* $A.B = A \cap B$. Since $A + A = A$ and $A.A = A$ we see that $\mathscr{B}$ is a **Boolean algebra**, i.e. an algebra over the field $\mathbb{F}_2$ of 2 elements in which all elements are idempotents. A theorem of **Stone** [see p.168 of Kelley] assures us that all Boolean algebras are isomorphic to such algebras $\mathscr{B} \subseteq \mathscr{P}(X)$.

A **closure algebra** is a Boolean algebra equipped with an idempotent obeying Kuratowski's conditions with respect to the order defined by $A \leq B$ iff $A = A.B$. A theorem of **McKinsey-Tarski** [Annals (45), 1944] assures us that any closure algebra is isomorphic to a Boolean algebra $\mathscr{B} \subseteq \mathscr{P}(X)$ preserved by the closure operator of a topology on X.

(2) *"The free closure algebra generated by one element has 16 elements"* ! This "theorem" occurs on page 180 of **Birkhoff**'s "Lattice Theory", and is attributed as being in Kuratowski's thesis [Fund. Math. 3 (1922), 182-231].

It implies that, in any topological space X, and for any $A \subseteq X$, the algebra $\mathscr{B}(A) \subseteq \mathscr{P}(X)$ over $\mathbb{F}_2$ obtained by applying the processes of closure, complementation, and intersection, has at most 16 elements [the additional sets being obviously none other than $\emptyset$ and X] and so is at most 4 - dimensional !

Now $\mathscr{B}(A)$ contains in particular A's **boundary** $bd(A) = cl(A) \cap cl(A')$, which does not in general lie in S(A). Regarding $bd: \mathscr{P}(X) \rightarrow \mathscr{P}(X)$, which is not in general $\mathbb{F}_2$-linear, an interesting fact is that one has $bd \circ bd \circ bd = bd \circ bd$ always [see (13), p.56, of Kuratowski's "Topology"]. Thus the semigroup generated by bd has only two elements and one has $(bd)^t(A) = 0$ iff this equation holds with $t \leq 2$.

However $\partial \circ \partial = 0^{®}$ [*the registered trademark of 20th century mathematics* !] certainly does not have the analogue $bd \circ bd = 0$ and $bd\, bd$ is quite distinct from $\partial \circ \partial$ even for a geometrical simplex: it yields its boundary rather than 0. For $\mathbb{Q} \subseteq \mathbb{R}$ one has $bd \circ bd(\mathbb{Q}) = 0$ while $bd(\mathbb{Q}) = \mathbb{R}$.

$bd \circ bd(\mathbb{Q} \cap [0,1]) = \{0,1\} \neq 0$

(3) **Unfortunately the above "theorem" from Birkhoff's book is false !** In fact in his thesis [op. cit. p.197] Kuratowski gave *an example of a subset* A *of a space of ordinals for which* $\mathscr{B}$(A) *is infinite* ! [Later on McKinsey-Tarski also made use of this same example !] It is surprising that Birkhoff, who gives these as his sources, made above mistake ! [Even if one adds to the set of $\mathbb{R}$ above given, which yielded 14 sets, the interval [4,5], one sees already that $\mathscr{B}$(A) has at least 17 elements.]

Kuratowski's result was understood better by **Hammer** [see Kuratowski's book p.43 for ref.] who showed [in manner indicated above] that *if* i *is an order reversing involution of a poset* $\mathscr{P}$ *and* p *is an order preserving* *and expanding idempotent of* $\mathcal{P}$, *then the semigroup* $S_p$ *generated by* i *and* p *has at most* 14 *elements*. [There are also two refs. to a **Chapman** who apparently studied the semigroup S(X) further.]

(4) There might be some homology, defined in terms of [the closure operator of] the topology $\mathcal{T}$ of X, such that $H_0(X)$ is the free abelian group generated by the **components** [as against the **path components** ] of X, and which need not obey the "homotopy axiom", but which coincides on polyhedra with the usual homology ?

Since bd(A) = 0 iff A is open and closed, it seems that bd still might be involved in the definition of such a homology ? Also the homology of bd: ker(bd∘bd) → ker(bd∘bd) : i.e. the homology obtained by cutting down $\mathcal{P}$(X) to the mod 2 subspace ker(bd∘bd), might be pertinent ?
↪ ?? bd was not $\mathbb{F}_2$-linear !

Another problem: can Hammer's result be augmented by an interesting characterization of all [or say all order-preserving] idempotents p: $\mathcal{P}(X) \to \mathcal{P}(X)$, p∘p = p, for which the semigroup $S_p$ is finite? Also, do such questions tie up with some (?) known finiteness theorem re Boolean algebras of some (?) **Cohen**, to which **Bourgain** once alluded ?

Another interesting problem might be to explore the Kuratowski semigroup of a space [i.e. the definition X ↦ S(X)] from the categorical viewpoint, e.g. it seems that this simple topological invariant provides obstructions to embeddability of an X in Y?

(5) It is convenient to employ the *interval* notation for any *totally ordered* set X, so e.g (.,x] will denote {y ∈ X : y ≤ x}. [Some care is necessary however to distinguish an interval (a,b) from the ordered pair (a,b), and also to distinguish intervals in different sets: e.g. the interval (0,1) of ordinals is empty, but that of real numbers is something else again.] The **order topology** of X is the one generated by all intervals of the type (.,x) or (y,.).

**Theorem**. *For any totally ordered set* X *the intervals* [x,y] *are compact*.

*Proof*. Let $\mathcal{E}$ be a family of open sets of [x,y] which covers it, and let c be the supremum of all z such that some finite subfamily of $\mathcal{E}$ covers [x,z]. Now choose a member U of $\mathcal{E}$ containing c, and a z in I slightly to the left of c such that [z,c] is in U. Add U to a finite subfamily covering [x,z] to get another finite subfamily covering [x,c]. Unless c = y this finite subfamily will contradict the maximality of c. *q.e.d.*

Note that for $\mathbb{R}$, which is an example of a totally ordered space, the above result implies the classical **Heine-Borel Theorem**, i.e that a subset of real numbers is compact iff it is closed and bounded.

Recall also that the standard covers argument shows that *any compact and Hausdorff* X *is normal*, so, for any ordered set, all intervals of type [x,y] are normal.

(6) **Examples involving ordinals.** In the following all intervals are of ordinals, and equipped with the order topology, with ω, resp. Ω, being

Observe p idempotent ⟺ semigroup of p has just one element. So:



[...]igroup generated by the order reversing involution i and any order preserving obviously semigroup generated by p alone is finite. Is this sufficient when p is expanding?

the first infinite, resp. uncountable, ordinal. [For the definitions of other highlighted terms see Kelley.]

(i) No sequence in $[0,\Omega)$ $\subset [0,\Omega]$ converges to its boundary point $\Omega$.

(ii) Both $[0,\Omega]$ and $[0,\Omega)$ are normal but their product is not.

(iii) Though the rectangle $[0,\Omega] \times [0,\omega]$ is normal, the subspace obtained from it by deleting its corner $(\Omega,\omega)$ is not.

(iv) The Stone-Cech compactification of $[0,\Omega)$ coincides with its **one-point compactification** which of course is $[0,\Omega]$.

(v) The space $[0,\Omega)$ is **uniformizable**, and there is a unique **uniformity** compatible with its topology, and with respect to this it is not **complete**.

For more see pp. 29-30, 59-60, 76, 131-2, 163-5, 167, 172, 204 and Appendix of Kelley.

(7) **Frechet's convergence axioms.** For a topological space X the set $\mathcal{E}$ of all pairs $(S,x)$, where $S: D \to X$ is a net, $x \in X$, and $\lim_d S(d) = x$, has the following properties:

(i) If $(S,x)$ is such that $S(d) = x$ for all d, then $(S,x) \in \mathcal{E}$.
(ii) If $(S,x) \in \mathcal{E}$ and T is a subnet of S then $(T,x) \in \mathcal{E}$.
(iii) $(S,x) \notin \mathcal{E}$ implies $(T,x) \notin \mathcal{E}$ for all subnets T of some subnet of S.
(iv) If $S: D \times E \to X$ and $T: D \to E$ are such that $(S(d,.),T(d)) \in \mathcal{E}$ for all d, and $(T,x) \in \mathcal{E}$ for some $x \in \mathcal{E}$, then there is a function $f: D \to E$ such that $(R,x) \in \mathcal{E}$ where $R: D \to X$ is given by $R(d) = S(d,f(d))$.

Kelley shows that conversely a set $\mathcal{E}$ of pairs $(S,x)$, with S a net in X and $x \in X$, which satisfies (i)-(iv), determines a unique topology on X such that $\lim S = x$ iff $(S,x) \in \mathcal{E}$.

[As a matter of fact Kelley's (iv) is more complicated : his iterated limit theorem involves a function $S(d,e)$ with $d \in D$ and $e \in E_d$, a directed set *depending* on d. Replacing each $E_d$ by $E = \bigcup_d E_d$, and extending S to $D \times E$ by imaging new points $(d,e)$ to the limits $\lim_e S(d,e)$, it seems that our version, i.e. axiom (iv), is equally good.]

To see this he defines $p(A)$ to consist of all $x \in X$ such that $(S,x) \in \mathcal{E}$ for some net S in A. This self-map p of $\mathcal{P}(X)$ is shown to satisfy Kuratowski's condition [with $p \circ p = p$ following from (iv)]. So there is a unique topology such that $p(A) = cl(A)$, etc., etc.

(8) Urysohn's theorem characterizes second countable metric spaces. To see this note that Hausdorfness is obvious and that it is also true that *a metric space is normal*:

This follows by noting that, for any $A \subseteq X$, the function $x \mapsto d(x,A) =$

inf{d(x,y): y ∈ A}, *distance from* x *to* A, is continuous, because | d(x,A) − d(y,A)| ≤ d(x,y) by the triangle inequality. So closure of any A consists of points at zero distance from it, and two disjoint closed sets can be contained in the two disjoint open sets consisting of all points of the space nearer to one of them in comparison with the other.

*The Hilbert cube is thus a universal second countable metric space*, in the sense that any second countable metric space embeds in it.

Abstract characterizations of non second countable metric spaces were discovered later by **Smirnov**, **Nagata**, et al.

(9) With the product topology, the **space** $\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$ **of all subsets of** $\mathbb{N}$ [and more generally $X^{\mathbb{N}}$ where X is any topological space] has the property that any countable power of this space is homeomorphic to itself.

We can think of the elements f of $\mathcal{P}(\mathbb{N})$ as all sequences $f:\mathbb{N} \to \{0,1\}$. Associating to each f the corresponding *binary decimal* we get a *continuous surjection* of this space onto the unit interval. Using above homeomorphism, and a countable product of this surjection, we get a continuous surjection of $\mathcal{P}(\mathbb{N})$ onto any countable power of [0,1].

On the other hand if we think of the elements f of $2^{\mathbb{N}}$ as all sequences $f: \mathbb{N} \to \{0,2\}$, and associate to each f the corresponding *ternary decimal*, we get an *embedding* of $\mathcal{P}(\mathbb{N})$ in [0,1], the image being the well known **Cantor set**, which is obtained from [0,1] by successively excluding the open middle third intervals.

The aforementioned continuous surjection of $\mathcal{P}(\mathbb{N})$ onto any countable power of [0,1] can now be extended, by using the arc connectedness of the cube, to these excluded intervals, thus obtaining a **Peano curve**, i.e. a continuous surjection of [0,1] onto any countable power of [0,1].

In fact [see pp.164-65 of Kelley] *any compact and arc connected metric space is a continuous image of* [0,1].

## WHITEHEAD GROUPS

Talk of 16.4.93 by **F.T.Farrell** [based on joint work with **L.E.Jones**]:

(1) In his famous work on combinatorial homotopy, **J.H.C.Whitehead** defined, for each group Γ, an abelian group Wh(Γ) by

$$Wh(\Gamma) = \lim \Gamma\backslash GL_n(\mathbb{Z}\Gamma)/[GL_n(\mathbb{Z}\Gamma),GL_n(\mathbb{Z}\Gamma)].$$

It measures the stable obstruction to reducing a matrix over $\mathbb{Z}\Gamma$ to a diagonal one having ±(gp.elts) only.

(2) The algebraical task of computing these groups was well-begun by Whitehead's student **Higman**, and essentially by continuously developing Higman's ideas, much is known. For example, **Bass** showed that $Wh(\mathbb{Z}/p\mathbb{Z})$ =

free abelian group of rank $(p-3)/2$, while **Bass-Heller-Swan** showed that for any free abelian group it is zero, and later **Stallings** showed it is zero for free groups. The conjecture whether $Wh(\Gamma) = 0$ for all torsion free groups $\Gamma$ still remains. Again, Bass showed that for $\Gamma$ finite, $Wh(\Gamma)$ is finitely generated, but on the other hand **Murthy** has shown that, for many ordinary infinite groups, e.g. for $\Gamma = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$, it is not finitely generated.

(3) The following is the culmination of work of **Smale**, **Stallings**, **Barden**, **Mazur**, **Kirby-Siebenmann** etc.

**h-COBORDISM THEOREM.** *For any closed manifold* M *of dimension* 5 *or more, there is a bijection* $W \leftrightarrow \tau W$, *between the set of all h-cobordisms with base* M, *and the Whitehead group* $Wh(\pi_1 M)$ *of the fundamental group of* M

Here by " h-cobordism with base M " is meant a manifold-with-boundary W with $\partial W$ a disjoint union of M and N [the "top of the cobordism"] which is a homotopy cylinder [i.e. deformation retracts to both top and bottom]. The above $\tau W \in Wh(\pi_1 M)$, called the **torsion** of the h-cobordism, is zero iff the h-cobordism is the genuine cylinder $M \times I$.

(4) Theorem [Farrell and Jones]. *If* $\Gamma$ *is a discrete subgroup of* $O(n,1)$, *then* $Wh(\Gamma) = 0$.

This they prove geometrically by showing that h-cobordisms having as base the following kind of manifold must be all trivial.

A riemannian manifold $M^n$ is called **hyperbolic** if its sectional curvature is identically -1. It is known that these are quotients of **hyperbolic n-space** $\mathbb{H}$ by a discrete subgroup $\Gamma$ of $O(n,1)$. One can use the **Poincare model** of $\mathbb{H}$ : unit disk in euclidean n-space with geodesics diameters or circular arcs cutting boundary perpendicularly.

Earlier **Hsiang** and Farrell had proved same result with M **flat**.

(5) One ingredient in the proof is a refinement of the next result which applies to the induced h-cobordism having as base an appropriate bundle SM over M.

Ferry's theorem. *For each* M *there exists an* $\varepsilon > 0$, *such that any h-cobordism having* M *as base and having all* **tracks** *of size bounded by* $\varepsilon$, *is trivial.*

Here by "a track" of W one means the loop in M obtained by projecting the deformation of a point of W into the end M.

Besides this refinement the triviality of h-cobordisms on SM uses the hyperbolic structure, and the **geodesic flow** in these bundles, to "shrink" tracks far enough to apply this theorem.

A product formula, for the torsion of the h-cobordism over SM in terms of that over M, now shows that the h-cobordisms over M are also trivial.

Analogous tools give calculations for $Wh(\Gamma)$ when $\Gamma$ is a discrete subgroup of any Lie group and some interesting general conjectures.

## WU'S "A THEORY OF IMBEDDING ... "

**(A) PREFACE.** The space $X_*$ of all injective mappings from $\mathbb{Z}_2$ to a space X, i.e. the space of all ordered pairs $(x_1,x_2)$ of distinct points of a space X, will be equipped with the free involution $(x_1,x_2) \longleftrightarrow (x_2.x_1)$.

This $\mathbb{Z}_2$-space is important for embedding theory because **X embeds in Y only if there is a continuous $\mathbb{Z}_2$-map $X_* \to Y_*$.** In fact note that each embedding $f: X \to Y$ induces an involution preserving *embedding* $f_* : X_* \to Y_*$, viz. the one defined by $f_*(x_1,x_2) = (f(x_1),f(x_2))$, however the additional fact that the $\mathbb{Z}_2$-map $X_* \to Y_*$ is one-one will be ignored.

Any continuous $\mathbb{Z}_2$-map $X_* \to Y_*$ pulls back the equivariant **Smith class** $o^i(Y_*) \in H^i_s(Y_*)$ of the free $\mathbb{Z}_2$-space $Y_*$ to the Smith class $o^i(X_*) \in H^i_s(X_*)$ of $X_*$. So we have the embeddability criterion: **if X embeds in Y, and the ith Smith class of $Y_*$ is zero, then that of $X_*$ must also be zero. For example, if X embeds, in $\mathbb{R}^m$ then $o^m(X_*) = 0$.** This last follows from the fact that $(\mathbb{R}^m)_*$ has the $\mathbb{Z}_2$-homotopy type of the antipodal (m-1)-sphere.

For a simplicial complex K we will denote by $K_*$ the cell complex consisting of all $\sigma \times \theta$, where $\sigma$ and $\theta$ are disjoint simplices of K, and equip it with the involution $\sigma \times \theta \longleftrightarrow \theta \times \sigma$.

The equivariant cohomology class of $K_*$ which counts the isolated and separated double points of a general position piecewise linear map f of an n-complex K in 2n-space, is in fact *independent* of f. The vanishing of this **obstruction class** is obviously necessary for the piecewise linear embeddability of an n-complex K in 2n-space. We will see that this embeddability criterion of **Van Kampen** is included in the above embeddability criterion because **this obstruction to p.l. embeddability of an n-complex K in 2n-space coincides with $o^{2n}(|K|_*)$.** The key point in this proof will be that $K_*$ is a $\mathbb{Z}_2$-deformation retract of $|K|_*$.

Completing an argument given by Van Kampen, we will show conversely that **if $n \geq 3$, and $o^{2n}(X_*) = 0$, then the n-polyhedron X embeds piecewise linearly in 2n-space.** This result shows in particular that piecewise linear n-manifolds embed piecewise linearly in 2n-space: a corollary proved directly by Van Kampen. In fact the key additional idea used by Wu and Shapiro to complete Van Kampen's argument was the one which was used by **Whitney** to obtain the *smooth* analogue of this corollary, viz. that a smooth n-manifold embeds smoothly in 2n-space. [Far reaching improvements of these constructions were given later by **Haefliger**.]

It will be shown that the mod 2 **cohomology operations** of any polyhedron X can be defined in terms of the mod 2 classes $o(X_*)$, in particular we'll see that an embeddability criterion of **Thom** is included in our criterion by showing that **if** $o^m(X_*) = 0$ **mod 2, then the dual mod 2 operations** $\underline{Sq}^i : H^r(X) \to H^{r+i}(X)$ **of X vanish for** $2i+r \geq m$.

It will be shown likewise that the mod 2 **characteristic classes** of any closed manifold X can be defined in terms of the mod 2 classes $o(X_*)$, in particular we'll see that an embeddability criterion of **Stiefel** and **Whitney** is included in the above criterion by showing that **if** $o^m(X_*) = 0$ **mod 2, and X is an n-manifold, then the dual mod 2 characteristic classes** $\underline{sw}_i(X)$ **of X vanish in dimensions** $i \geq m - n$.

Besides considering the space $X_*$ of injective functions from $\mathbb{Z}_2$ to X, we'll also introduce some analogous spaces of functions from $\mathbb{Z}_p$ or $S^1$ to X, and indicate how their equivariant characteristic classes should determine the remaining cohomology operations, resp. characteristic classes, of the polyhedron, resp. manifold X.

Also we'll consider analogous results concerning obstructions to immersions and isotopies.

**(B) CHAPTER ONE.** An obvious invariant of an embedding $Y \subset X$ is the homotopy type [or even the topological type] of the **complement** $X \setminus Y$.

[As Wu mentions in the preface, non-embeddability arguments based on complements have been given by **Hopf**, **Hantzsche**, **Thom**, and **Peterson**, with the latter two considering ring structure and cohomology operations. For example, by using **Alexander's duality theorem**, Hopf showed that $\mathbb{R}P^n$ does not embed in $\mathbb{R}^{n+1}$.]

An embedding $Y \subset X$ of polyhedra is called **tame** if (X,Y) is homeomorphic to some pair (K,L) of simplicial complexes, and any such (K,L) is then called a [topological] **triangulation** of (X,Y). Note that, by additional subdivisions if need be, we can always assume that L is **full** in K, i.e. is such that any simplex of K which has all its proper faces in L is itself in L.

**Theorem 1a.** *If* L *is full in* K*, then there is a deformation retraction of its complement* $K \setminus L$ *onto* $[K \setminus L]$ *the largest simplicial complex contained in it.*

*Proof.* The fullness guarantees that any point p of the topological complement K, which is not in the simplicial complement, is an interior point of a *unique* **line segment** [x,y] all of whose interior points are of this type, and which has x in L, and y in [K\L]. Compressing each [p,y] to [y] one gets the required deformation retraction. *q.e.d.*

Likewise, for L full in K, there is a deformation retraction of the open [simplicial] **neighbourhood** K \ [K \ L] of L onto [K \ [K \ L]] = L. So its homotopy type is also an [albeit not very interesting !] topological invariant of (K,L).

We will see below that the homotopy type of the **deleted neighbourhood** K \ [K \ L] \ L [which of course is not the same as the homotopy type of [K \ [K \ L] \ L] = Ø !] is also a topological invariant of (K,L).

However for the proof it is convenient, and for a later result necessary, to work with a somewhat smaller neighbourhood, which Wu defines by means of a "preliminary subdivision" as follows.

Consider the continuous surjection r: K ⟶ [0,1] which maps L to 0, [K \ L] to 1, and which is linear on each of the segments mentioned in the proof of Theorem 1. We define the closed **tubular neighbourhood** of L in K by $N_t(L,K) = r^{-1}[0,t]$, the **tube** of L in K by $T_t(L,K) = r^{-1}[t]$, and the **tubular complement** of L in K by $E_t(L,K) = r^{-1}[t,1]$, where $t \in (0,1)$. Since the combinatorial type of these [geometric] *cell complexes* is unaffected by the choice $t \in (0,1)$, we will frequently just write N(L,K), T(L,K), and E(L,K).

Obviously the homotopy types of N(L,K), T(L,K), and E(L,K) coincide, respectively, with those of the neighbourhood, deleted neighbourhood, and complement of L in K.

**Theorem 1b.** *The homotopy type of the deleted neighbourhood, of a full subcomplex* L *of a simplicial complex* K, *is a topological invariant of the pair* (K,L).

The following is an easy generalization of the argument given in **Seifert-Threlfall** [pp. 125-128 of english translation] for the special case L = {v} [when the tube T(v,K) happens to be homeomorphic to the link of the vertex v].

*Proof.* Let (K',L') and (K,L) be any two full [topological] triangulations of the tame polyhedral pair (X,Y).

Choose in succession numbers $t_1$, $t_1'$, $t_2$, $t_2'$, $t_3$, $t_3'$ in (0,1) such that the tubular neighbourhoods $N_i = N_{t_i}(L,K)$ and $N_i' = N_{t_i'}(L',K')$, of Y in X, are nested in each other as follows:

$$N_1 \supseteq N_1' \supseteq N_2 \supseteq N_2' \supseteq N_3 \supseteq N_3'.$$

Consider any point p of the bounding tube $T_2$ of $N_2$. As we linearly shrink $N_1'$ to $N_2'$, p traces a path $p_t$ ending [after time $t_1' - t_2'$] at a point q of the bounding tube $T_2'$ of $N_2'$.

Likewise, as we linearly expand $N_3$ to $N_2$, any point q of the bounding tube $T_2'$ of $N_2'$ traces a path $q_t$ ending [after time $t_2 - t_3$] at a point s of the bounding tube $T_2$ of $N_2$.

A juxtaposition $p_t \cdot q_t$ of two such paths is nested within $N_1$ and $N_3$. We now linearly shrink this annulus to the bounding tube $T_2$ of $N_2$. The resultant projections of such juxtapositions on $T_2$ show that the identity map of $T_2$ is homotopic to the map $\psi \circ \phi$: $T_2 \rightarrow T_2$, where $\phi$: $T_2 \rightarrow T_2'$ is given by $p \longmapsto q$, and $\psi$: $T_2' \rightarrow T_2$ is given by $q \longmapsto s$.

Likewise, using the fact that a juxtaposition $q_t \cdot p_t$ of such paths is nested within $N_1'$ and $N_3'$, it follows that $\phi \circ \psi$ is homotopic to the identity map of $T_2'$. *q.e.d.*

*The homotopy types of the closure and boundary of the open simplicial neighbourhood of a full subcomplex* L *of* K [i.e. of the closed star and link of L in K] *are not topological invariants of* (K,L):

For example, a 3-vertex circle K is the closure of the simplicial neighbourhood of a closed edge L, while the boundary of this neighbourhood is the vertex not in L; and if we subdivide K by using a fourth vertex outside L, both homotopy types change.

But, for the smaller or tubular neighbourhoods defined above, one does have the following pleasant fact.

**Theorem 1.** *If* L *is full in* K *then the homotopy type of the closure algebra generated in* K *by* L *and* intN(L,K) [the open tubular neighbourhood of L in K] *is a topological invariant of the pair* (K,L).

This generalization is proved by Wu by making some straightforward modifications in the arguments of Theorem 1b.

[As Wu mentions in the preface, non-embeddability arguments based on homotopy invariance of the pair (tubular neighbourhood, tube) have been given by **Whitney**, **Pontrjagin**, **Thom**, **Massey** and **Atiyah**, with the last two considering ring structure and K-theory also.]

Also it is easy to generalize all these *Seifert-Threlfall type results* to any pair of cell complexes (K,L), with L "full" in K in the sense that a cell lying in neither [K \ L] nor L should be the join of two faces, one in [K \ L] and the other in L.

So we can use the following to apply these results to the pth **diagonal embedding** $\Delta$: $|K| \rightarrow |K|^p$ of K, which associates to each point of K the corresponding *constant map* $\{1,2,...,p\} \rightarrow |K|$.

**Theorem 2.** *Let $K^p_*$ denote the sub cell complex of $K^p$, the p-fold product of a simplicial complex K, which consists of all cells of the type $\Sigma = \sigma_1 \times \ldots \times \sigma_p$ with $\sigma_1 \cap \ldots \cap \sigma_p = \emptyset$. Then the disjoint union of $\Delta(K)$ and $K^p_*$, together with all joins $\Delta(\sigma)\cdot\Sigma$, where $\Sigma \in K^p_*$ is such that $\sigma\cup\sigma_i \in K$ for all i, is a cell subdivision of $K^p$.*

We indicate below an argument [see also c.(6)] which shows that it is in fact the joins version of the above result of Wu which is more natural.

*Proof.* To be more precise, $K^p$ is the product ${}^1K \times .. \times {}^pK$ of p disjoint copies of K; so, using the notation ${}^j\theta \in {}^jK$ for the jth copy of $\theta \in K$, each member of this *cell complex* is of the type ${}^1\sigma_1 \times \ldots \times {}^p\sigma_p$, where the $\sigma_i$'s are nonempty simplices of K.

We will consider also the join $\mathbb{K}^p = {}^1K\cdot \ldots \cdot{}^pK$, a *simplicial complex* each of whose members is of the type ${}^1\sigma_1 \cup \ldots \cup {}^p\sigma_p$ [this disjoint union is also written ${}^1\sigma_1\cdot \ldots \cdot{}^p\sigma_p$, or even $\sigma_1\cdot \ldots \cdot\sigma_p$ if no confusion is possible] where now each $\sigma_i$ is any [possibly empty] simplex of K.

The space $|\mathbb{K}^p|$ is the disjoint union of all closed (p-1)-dimensional geometrical simplices with vertices $({}^1x_1, \ldots, {}^px_p)$, where again ${}^jy \in |{}^jK|$ denotes the jth copy of a point $y \in |K|$. We will identify $|K^p|$ with the subspace of $|\mathbb{K}^p|$ consisting of the centroids of these simplices.

The subcomplex of $\mathbb{K}^p$ consisting of all simplices ${}^1\sigma_1 \cup \ldots\cup {}^p\sigma_p$ with $\sigma_1 \cap \ldots \cap \sigma_p = \emptyset$ will be denoted $\mathbb{K}^p_*$. Note that the intersection of $|\mathbb{K}^p_*|$ with $|K^p|$ equals $|K^p_*|$, where $K^p_*$ is the aforementioned sub cell complex of $K^p$ consisting of all cells ${}^1\sigma_1 \times \ldots\times {}^p\sigma_p$ with $\sigma_1 \cap \ldots\cap \sigma_p = \emptyset$.

We now note that there is a unique way [take $\sigma = \sigma_1 \cap \ldots \cap \sigma_p$] of writing any simplex $\sigma_1\cdot \ldots \cdot\sigma_p$ of $\mathbb{K}^p$ as the join of a [possibly empty] simplex $\sigma\cdot \ldots \cdot\sigma$ having all factors "same", and a [possibly empty] simplex $\theta_1\cdot \ldots \cdot\theta_p$, $\theta_i = \sigma_i \setminus \sigma$ of $\mathbb{K}^p_*$.

We assert that there is a simplicial subdivision $W(\mathbb{K}^p)$ of $\mathbb{K}^p$ which, restricted to each simplex $\sigma_1\cdot \ldots \cdot\sigma_p$, is the join of the face $\theta_1\cdot \ldots \cdot\theta_p$, with a subdivision of the complementary face $\sigma\cdot \ldots \cdot\sigma$, and which is such that the *closure of each simplex of the type $\sigma\cdot \ldots \cdot\sigma$ gets* *retriangulated as the join* $\Delta(\bar{\sigma})\cdot(\bar{\sigma})^p_*$.

The main point in the verification of the above assertion is that $(\bar{\sigma})^p_*$ is a simplicial sphere of the right dimension. This follows at once by using the **multiplicative property**,

$$(K\cdot L)^p_* \cong K^p_*\cdot L^p_* ,$$

of this construction, and the fact that the simplicial complex $\{v\}^p_*$ consists of all proper subsets of the cardinality p set $\{{}^1v, \ldots ,{}^pv\}$.

The intersection with $|K^p|$, of the aforementioned simplicial subdivision $|W(K^p)|$ of $|K^p|$, gives a cell complex $W(K^p)$, which is the required cell $W(K^p)$ subdivision of $K^p$. *q.e.d.*

*So* $K^p_*$ *is a deformation retract of the of the space of all non constant functions* $\{1, \ldots ,p\} \to |K|$.

[**Shapiro**'s direct proof of this corollary was erroneous. Also note that for $p \geq 3$, **the pth product configuration space** of K, i.e. the subspace of $|K|^p$ consisting of all *one-one functions* $\{1, \ldots ,p\} \to |K|$, does not have the same homotopy type as the sub cell complex of $K^p$ determined by the condition that the factors $\sigma_i$ of the cells $\sigma_1 \times \ldots \times \sigma_p$ be *pairwise disjoint*. For example, if K is a closed 1-simplex and p = 3, then there is no such cell, but certainly $|K|$ has 3-tuples of distinct points.]

The **symmetric group** of all permutations $\pi$ of $\{1, \ldots ,p\}$, and so in particular the **cyclic subgroup** $\mathbb{Z}_p$ generated by the rotation $\pi = (p, 1, 2, \ldots p-1)$, acts on $|K|^p$ by $(x_1, \ldots ,x_p) \longmapsto (x_{\pi(1)}, \ldots ,x_{\pi(p)})$. Likewise there are group actions on $K^p_*$, etc. It is important to observe that *the aforementioned deformation retraction commutes with these group actions*.

From now on, for the sake of simplicity, Wu confines himself to the case when p is **prime**: so this cyclic action is **free** in the complement of the diagonal, and the quotient of the above **Wu triangulation** of $K^p$ gives an equally nice triangulation of $K^p/\mathbb{Z}_p$.

Having checked that *all* homotopy invariants of the complement, tube, etc., of a diagonal embedding $K \to K^p$, are p.l. [even topological] invariants of K, the chapter ends with the following result of **Lee** concerning *enumerative* invariants for the case p = 2.

**Theorem 3.** *Let* $V_2$ *be the subspace of all sequences* $c_{ij,k} \in \mathbb{R}$ *having the* *property that*

$$\Sigma_{i,j,k}\ c_{ij,k}\cdot|\{(\sigma,\theta):\ \sigma\in K,\ \theta\in K,\ |\sigma|=i,\ |\theta|=j,\ |\sigma\cap\theta|=k\}|,$$

*is invariant under subdivision for all simplicial complexes* K. *Then* $V_2$ *is* 3-*dimensional and has an integral basis which, applied to any* K, *yields the Euler characteristics of* $K^2$, K *and* $K^2_*$.

*Proof.* Since it seems more natural we'll in fact first establish that there is a joins version of the above basis.

Under the set theoretic surjection $W(K^2)\rightarrow K^2$, defined by $\alpha\cdot\beta\cdot\gamma\ \mapsto\ ((\sigma=\alpha\cup\beta,\ \theta=\alpha\cup\gamma))$, the pre-image of any $(\sigma,\theta)$ with $|\sigma|=i$, $|\theta|=j$ and $|\sigma\cap\theta|=k$, consists of precisely $2^s.\binom{k}{s}$ simplices of cardinality i+j-k+s, for each $0\le s\le k$.

This follows because this pre-image consists precisely of all simplices $\alpha\cdot\beta\cdot\gamma=(\alpha\cdot(\sigma\backslash\alpha)\cdot(\theta\backslash\alpha))\cdot(\emptyset\cdot\lambda\cdot\mu)$, where $\alpha=\sigma\cap\theta$, and $\lambda$ and $\mu$ are any two disjoint faces of $\alpha$, and so the required number coincides with the number of cardinality s simplices $\lambda\cdot\mu$ of $(\bar{\alpha})^2_*$, a k-fold join of 2 points.

So the number of cardinality t simplices in $W(K^2)$, and its subcomplexes $\Delta(K)$ and $K^2_*$, is given by

$$f_t(W(K^2))=\Sigma_{i+j-k+s=t}\ 2^s.\binom{k}{s}.f_{ij,k}(K),$$

$$f_t(\Delta(K))=f_{tt,t}(K),\ \text{and}\ f_t(K^2_*)=\Sigma_{i+j=t}\ f_{ij,0}(K),$$

respectively, where $f_{ij,k}=|\{(\sigma,\theta):\ \sigma\in K,\ \theta\in K,\ |\sigma|=i,\ |\theta|=j,\ |\sigma\cap\theta|=k\}|$. The second and third formulae follow from the first because a cardinality t simplex of $W(K^2)$ is of the type $\alpha\cdot\emptyset\cdot\emptyset$ iff i=j=k=t, and s=0, and it is of the type $\emptyset\cdot\beta\cdot\gamma$ iff k=s=0 and i+j=t

Since the Euler characteristic of $K^2$ coincides with the alternating sum of the face numbers of its subdivision $W(K^2)$, it follows that the integral element of $V_2$ given by

$$c_{ij,k}=\Sigma_s\ (-1)^{i+j-k+s}.2^s.\binom{k}{s},$$

*calculates* $\chi(K^2)$, i.e. $\chi(K^2)=\Sigma\ c_{ij,k}.f_{ij,k}(K)$ for any K. Likewise the integral element given by $c_{ii,i}=(-1)^i$ and = 0 otherwise, calculates

$\chi(K)$, and the integral element given by $c_{ij,0} = \sum (-1)^{i+j}$ and $= 0$ otherwise, calculates $\chi(K^2_*)$. It is obvious that these three elements of $V_2$ are linearly independent, so $\dim(V_2) \geq 3$.

We only sketch Lee's method for checking $\dim(V_2) \leq 3$, because it is [probably unnecessarily ?] laborious:

He assumes inductively that it is true that the *truncations* of the above elements, determined by $i,j < n$, do span the space $V_{2,n-1}$ of truncations $\chi_{n-1}$ of elements $\chi \in V_2$. Next he applies any $\chi \in V_2$ to all degree n complexes of the type $K = \bar{\sigma} \cup \bar{\theta}$, and also to subdivisions obtained from them by deriving one edge. The above inductive hypothesis, plus the invariance under subdivision of $\chi$, is then used to grind out the inductive step.

To pass to the products version of this basis simply note that cells of $K^2$ correspond to simplices of $\mathbf{K}^2$ having both factors nonempty, and having dimension one more than the cells, so $-\chi(K^2) = \chi(\mathbf{K}^2) - 2.\chi(K)$ and $-\chi(K^2_*) = \chi(\mathbf{K}^2_*) - 2.\chi(K)$. *q.e.d.*

[For p = 1, **Mayer** had previously considered the analogous space of linear combinatorial invariants, and shown that it is one-dimensional and spanned by the Euler characteristic: the above method of Lee does give a very simple proof of Mayer's theorem, but obviously ought to be simplified further to consider the cases $p \geq 3$.]

Another nice integral element of $V_2$ is that which calculates the Euler characteristic of the tube $\mathbf{K}^2_o = T(W(\mathbf{K}^2),\Delta(K))$ [which coincides with its products version $K^2_o = T(W(K^2),\Delta(K))$]. This can be easily calculated by noting that each cell of this tube corresponds to a simplex, of one dimension more, of $W(\mathbf{K}^2)$, which is neither in $\mathbf{K}^2_*$ nor in $\Delta(K)$. Thus $-\chi(K^2_o) = \chi(\mathbf{K}^2) - \chi(\mathbf{K}^2_*) - \chi(K)$.

**(C) CHAPTER TWO.** Given an action of a group G on a simplicial complex E, there is the induced action of its group ring $\mathbb{Z}G$ on its cochain complex $(C^*(E),\delta)$, and so each $\rho \in \mathbb{Z}G$ gives rise to the canonical short exact sequence of cochain complexes,

$$0 \longrightarrow \ker(\rho) \longrightarrow C^*(E) \xrightarrow{\ \rho\ } \mathrm{im}(\rho) \longrightarrow 0,$$

and thus an associated long exact cohomology sequence.

**Theorem 4.** *If* E *is any* G*-complex, and* $t \in G$ *is any group element of finite order* p *which acts freely on* E, *then*

$$\mathrm{im}(1 - t) = \ker(1 + t + \dots + t^{p-1}) \text{ and}$$

$$\mathrm{im}(1 + t + \dots + t^{p-1}) = \ker(1 - t),$$

*in each* $C^i(E)$, $i \geq 0$.

*Proof.* The inclusions $\subseteq$ are obvious. For the reverse inclusions $\supseteq$ we will use the fact that the orbit of each *nonempty* simplex under t has p distinct members. Thus, if we choose an ordering $(\sigma, t\sigma, \dots, t^{p-1}\sigma)$ of each such orbit, then there exists one and only one cochain having any specified length p *sequences of values* on these orbits.

A cochain c lies in $\ker(1 + t + \dots + t^{p-1})$ iff the sequences of its values $(c_0, c_1, \dots, c_{p-1})$ have sum zero. For each such zero sum sequence it is possible to choose [starting with any initial term $c'_0$] a [unique] sequence $(c'_0, c'_1, \dots, c'_{p-1})$ such that each $c'_i$ is $c_i$ more than the cyclically preceding $c'_{i-1}$. Clearly the corresponding cochain c' satisfies $(1 - t)(c') = c$.

On the other hand c lies in $\ker(1 - t)$ iff the sequences of its values $(c_0, c_1, \dots, c_{p-1})$ are constant. For each such constant sequence choose any sequence $(c'_0, c'_1, \dots, c'_{p-1})$ which sums to this constant value. Clearly the corresponding cochain c' satisfies $(1 + t + \dots + t^{p-1})(c') = c$. *q.e.d.*

Thus, if t acts freely on E, the cohomology sequences associated to the pair $(d = 1 - t,\ s = 1 + t + \dots + t^{p-1}) \in \mathbb{Z}G$ are inter-related, viz. these **Richardson-Smith sequences** run

$$\dots \to H^i_d(E) \to H^i(E) \to H^i_s(E) \to H^{i+1}_d(E) \to \dots,$$

$$\dots \to H^i_s(E) \to H^i(E) \to H^i_d(E) \to H^{i+1}_s(E) \to \dots,$$

where $i \geq 0$, and $H^k(E)$, $H^k_d(E)$, and $H^k_s(E)$ denote the kth cohomologies of $C^*(E)$, $C^*_d(E) = \ker(d)$ and $C^*_s(E) = \ker(s)$, respectively.

**Equivariant Kronecker duality.** The t's of cochains and chains are dual to each other, so the same is true for the s's and d's. We define, for any cochain-chain pair x,y which is killed by the d's, resp. s's,

$$\langle x,y\rangle_d = \langle s(a),c\rangle = \langle a,s(c)\rangle,$$

resp.

$$\langle x,y\rangle_s = \langle d(a),c\rangle = \langle a,d(c)\rangle,$$

where a,c is any cochain-chain pair such that $s(a) = x$ and $s(c) = y$,

resp. $d(a) = x$ and $d(c) = y$. It is easily checked that this gives a unimodular pairing [unlike $\langle x, y\rangle$ restricted to such x's and y's].
The equivariant Stokes formulae $\langle \delta x, y\rangle_d = \langle x, \partial y\rangle_d$ and $\langle \delta x, y\rangle_s = \langle x, \partial y\rangle_s = \langle x, y\rangle$

follow at once from the ordinary Stokes formula $\langle \delta x, y\rangle = \langle x, \partial y\rangle$.

Using these, Wu establishes that, over field coefficients, there is a perfect duality, between the above cohomological RS sequences, and the analogous homological RS sequences.

**Smith classes of** t . These are the classes $o^{2k}(E) \in H_d^{2k}(E)$, $o^{2k+1}(E) \in H_s^{2k+1}(E)$, where the zeroth class is represented by the cocycle which is 1 on each vertex, and the other classes are obtained successively from it, by alternately applying the connecting homomorphisms $H_d^i(E) \rightarrow H_s^{i+1}(E)$, $[sc] \mapsto [\delta c]$, and $H_s^i(E) \rightarrow H_d^{i+1}(E)$ $[dc] \mapsto [\delta c]$ of the above RS sequences.

**Topological invariance.** Wu states without proof that the RS sequences depend only on the equivariant homotopy type of $(X,t)$, $X = |K|$, and that in fact they identify with their *singular* versions which can be defined analogously whenever t is a free self-homeomorphism of order p of a topological space X.

**Examples.** These are powers of $\mathbb{R}^m$ minus the diagonal with cyclic action, so with quotients projective or lens spaces; and finite groups acting on spheres; and the antipodal involution in a tangent sphere bundle of a manifold.

**(D) CHAPTER THREE.** (To be continued.)

*Comments*

(1) Amongst the many interesting ***embedding techniques of general topology*** are those given by **Cantor** [using n-ary expansions, and leading to **Peano curves**], by **Urysohn** [using **Stone-Cech** families of functions, and leading to metrization], by **Menger-Noebeling** [using finite dimensionality and **Baire category theorem** for metric spaces], etc.

(2) **Pontrjagin's original definition of characteristic classes for manifolds was just like Van Kampen's definition of "characteristic classes" for polyhedra:** these were cohomology classes dual to some cycles residing on any general position self-intersection of the manifold in a suitable euclidean space. Thus, just like Van Kampen's embeddability criterion for polyhedra, the Pontrjagin or Stiefel-Whitney embeddability criteria for manifolds followed immediately from this original **extrinsic definition** of characteristic classes.

The progression of ideas "tubular neighbourhoods, normal bundles, tangent bundles, bundles ... " then led to an **intrinsic definition** of characteristic classes of manifolds. Analogously, for polyhedra, Van Kampen's definition was made intrinsic by Wu by using $X_*$ etc.

(3) Wu's tubular neighbourhood of a full subcomplex of a simplicial complex [though itself not a simplicial complex] is small and thus apparently more convenient for enumerative purposes than the original [simplicial] one of **Whitehead**, in which the "preliminary subdivision" consists in going without any ado to the second derived.

However it does seem to be more natural to adhere to the standard practice in p.l. topology of confining attention to only [full and] **piecewise linear triangulations** (K,L) of (X,Y), i.e. those which are p.l. homeomorphic to the polyhedral pair (X,Y):

The "Whitehead variant" of Seifert-Threlfall's [original] result is that **the p.l. type of the link of a vertex** v **is a p.l. invariant of** (K,v), and, more generally, the variant of Theorem 1 is that the p.l. type of the closure algebra generated by L and its open tubular neighbourhood in K is a p.l. invariant of the full pair (K,L). However note, as against the p.l. type of the tubular complement E(L,K), it is still only the homotopy type of [L \ K] which is a p.l. invariant.

For more on p.l. topology see **Whitehead**, **Zeeman**, **Stallings**, **Hudson**, **Rourke-Sanderson**, etc. For instance, for the case when X is a [p.l.] manifold, it is known that the tubular neighbourhood and tubular complement of any subpolyhedron are always manifolds-with-boundary which have the tube as their common bounding manifold.

(4) The above review shows that Van Kampen Theory needs only [mostly finite] simplicial complexes, and some concomitant special kinds of geometric cell complexes [which are still "simplicial", but in the *categorical* sense].

However, as in **Lefschetz**'s "Algebraic Topology", 1942 (AMS Colloq. Pub., v.27), the "complexes" used in Wu's book are the following very general ones which had been introduced by **Tucker**:

A poset P, equipped with a *dimension function* $P \to \mathbb{N}$, and an *incidence function* $P \times P \to \{1,0,-1\}$ supported on its covering relation $C \subset P \times P$, such that $\dim(\sigma) = \dim(\theta) + 1 \ \forall\ (\sigma,\theta) \in C$ and $\sum_{\phi} [\sigma:\phi][\phi:\theta] = 0 \ \forall\ (\sigma,\theta) \in P \times P$, is called an **abstract cell complex**.

For more on such early generalizations of simplicial complex see **Steenrod**'s "Reviews". These [somewhat ad hoc] definitions have now lost their original purpose because, by interpreting it categorically, **Eilenberg**, **Kan** et al. have shown that the domain of validity of the [more natural and elegant] simplicial method is very large.

(5) **Finer Wu subdivisions.** The homotopy type of *the pth join configuration space* of $X = |K|$ coincides with the subcomplex of $K^{\mathbf{p}}$ consisting of all simplices $\sigma_1 \cdot \ldots \cdot \sigma_p$ with $\sigma_i$'s pairwise disjoint.

This can be seen by using a further subdivision $\mathbf{W}(K^{\mathbf{p}})$ of the subdivision $W(K^{\mathbf{p}})$ of $K^{\mathbf{p}}$ which was suggested to us by **Bier** [p.46 of 13.2.92-24.5.92].

Recall that $W(K^{\mathbf{p}})$ consisted [of joins] of all sequences $\theta_\alpha, \theta_1, \ldots, \theta_p)$, with $\theta_\alpha \cup \theta_i \in K$, and $\cap_{1\le i\le p} \theta_i = \emptyset$

On the other hand $\mathbf{W}(K^{\mathbf{p}})$ consists [of joins] of all "sequences" $\{\theta_\alpha : \emptyset \neq \alpha \subseteq \{1, \ldots, p\}\}$, with $\cup_{\alpha\in C}\theta_\alpha \in K$ whenever C is totally ordered by $\subseteq$, and $\theta_\alpha$ disjoint from $\theta_\beta$ whenever $\alpha$ and $\beta$ are incomparable under $\subseteq$. [A proof that $\mathbf{W}(K^{\mathbf{p}})$ is indeed a subdivision of $K^{\mathbf{p}}$ is sketched in (6) below. This proof will show also that Wu's and Bier's subdivisions are but two of a whole class of nice subdivisions.]

Note that any permutation $\pi$ of $\{1, \ldots, p\}$ maps each nonempty set $\alpha$ to a nonempty set $\pi(\alpha)$, so there is a corresponding simplicial isomorphism $\pi$ of $\mathbf{W}(K^{\mathbf{p}})$, and **the fixed points of any** $\pi : \mathbf{W}(K^{\mathbf{p}}) \to \mathbf{W}(K^{\mathbf{p}})$ **form a subcomplex**, viz. the subcomplex determined by the condition that $\theta_\alpha = \emptyset$ whenever $\alpha$ is not fixed under $\pi$.

Thus the quotient of $\mathbf{W}(K^{\mathbf{p}})$, by any subgroup G of such permutations, will be a simplicial triangulation of $X^p/G$.

(6) **The multiplicative property seems to be basic in Van Kampen Theory** because, firstly, ***the joins versions of*** all its basic constructions, $K \longmapsto F(K)$, seem to obey this property:

$$F(K\cdot L) \cong F(K)\cdot F(L).$$

Secondly, ***recognition of multiplicativity simplifies proofs drastically:***

For example, to verify that Bier's simplicial complex $\mathbf{W}(K^{\mathbf{p}})$ is indeed a subdivision of $K^{\mathbf{p}}$, the main thing to note is that $F(K) = K^{\mathbf{p}}$ or $\mathbf{W}(K^{\mathbf{p}})$ are both multiplicative. Note further that $K \subseteq \Sigma$, the iterated join of the vertices of K, and that $F(K) \subseteq F(\Sigma)$. This reduces the verification to the case $K = \{v\}$, in which case it is easily checked that $\mathbf{W}(K^{\mathbf{p}})$ is the derived complex of $K^{\mathbf{p}}$, the closed simplex on the vertices $\{{}^1v, \ldots, {}^pv\}$.

[The above proof shows that any subdivision of $\{v\}^{\mathbf{p}}$ will lead to a Wu type subdivision, e.g. just deriving the top simplex of this corresponds to the original Wu triangulation of Theorem 2.]

Thirdly, and most importantly, we will see that this multiplicativity gives ***product formulae*** for Van Kampen classes, which imply [for the case of manifolds, via ***Thom complexes*** of their tangent bundles] the ***Whitney addition*** formulae, ***multiplicative sequences***, and other such things, of the theory of characteristic classes of manifolds.

(7) ***The multiplicative property also seems to drastically simplify Lee's proof that*** $\dim(V_2) \le 3$. For this the key point is to observe that any **characteristic** $\chi \in V_2$ satisfies

$$\chi(K\cdot L) = \chi(K).\chi(L),$$

and thus is determined by its value on a vertex {v}. But this can be subdivided no further, and {1,2} has 3 nonempty sets, so the value of $V_2$ on {v} is a 3-dimensional vector space.

Regarding *the free $\mathbb{Z}$-module consisting of the integral elements of* $V_2$, it seems that it is generated [not by the integral bases of $V_2$ given in Theorem 3 but] by the basis which computes the Euler characteristics of $\Delta(K)$, $K^2_o$, and $K^2_*$ .

**Characteristic space** $V_p$ **for** $p \geq 3$. Bier's subdivision $\mathbf{W}(K^p)$ suggests that a resonable definition would be to to consider all "sequences" $c_\lambda$ of real numbers, indexed by integral functions $\lambda$ on the set of all nonempty subsets $\alpha$ of {1, ... ,p}, such that

$$\Sigma_\lambda\ c_\lambda \cdot |\{(\sigma_1, \ldots ,\sigma_p) : \sigma_i \in K,\ |\cap_{i\in\alpha} \sigma_i| = \lambda(\alpha)\ \forall\ \alpha\}|,$$

is invariant under subdivision, for all simplicial complexes K.

Once again the multiplicativity of the elements of $V_p$ should quickly establish the obvious guess $\dim(V_p) = 2^p - 1$, and probably one can even display some integral basis of $V_p$ coming from the Euler characteristics of some minimal invariant subsets of $\mathbf{W}(K^p)$, and there might be interesting connections with results of **Brown** and **Quillen** concerning the Euler characteristics of groups and the poset of subgroups of the symmetric group on p letters ?

(8) Remarks re **Smith theory of free complexes** [= Wu's Chapter 2].

(i) The easiest and best way of presenting this theory would be to first work out the case of the **universal complex** $\mathbf{E} = \mathbb{Z}_p \cdot \mathbb{Z}_p \cdot \ldots$ of the group $\mathbb{Z}_p = \langle t \rangle$, and then restrict to any free $\mathbb{Z}_p$-subcomplex $E \subset \mathbf{E}$.

E.g. $H^k_d(\mathbf{E})$, the **group cohomology** of $\mathbb{Z}_p$, is easily seen to be $\mathbb{Z}$ in dimension zero, $\mathbb{Z}_p$ in all odd dimensions, and zero otherwise [see e.g. **Brown**, p.35]. Using the contractibility of $\mathbf{E}$ this computes $H^k_s(E)$ also.

[The RS sequences are thus closely related to the 2-step periodicity of the cohomology of finite cyclic groups. For a general G there may be no such apparatus for computing the G-characteristic classes.]

(ii) Maybe $1 - t \longleftrightarrow 1 + t + \ldots + t^{p-1}$ is only an instance of an **involution** $\rho \longleftrightarrow \bar{\rho}$ [?] defined throughout [the subring, of elements starting with 1, of] the group ring of the [finite] group G, and having the property that $im(\rho) = ker(\overline{\rho})$ and $im(\overline{\rho}) = ker(\rho)$ in $C^i(E)$, $i \geq 0$, for [E = G•G• ... and thus for] all free G-complexes E.

At least the **norm** s is defined for all finite G's, so **Tate**'s cohomology theory, which uses s, might be generalising some Smith theory to all G ?

(iii) If t is replaced by its **conjugate** $t^\alpha$ in $\mathbb{Z}_p$ [i.e. $\alpha$ is relatively prime to the order p of t] then s is unchanged. So $H^*_s(E)$ and $H^*_d(E)$, which are the cohomologies of ker(s) and im(s), and also the entire second RS sequence, remain the same. However $d = 1 - t$ gets multiplied by $1 + t + \ldots + t^{\alpha-1}$, to become $1 - t^\alpha$, so the morphism of the first RS sequence induced by d, as well as its connecting morphism, alter accordingly.

(iv) Smith classes of a $\mathbb{Z}_p$-complex are of order p.

Also, it seems that the reductions mod p, in the s or d cohomology, followed by the Bockstein of the d or s cohomology, coincides with the connecting homomorphisms of the RS sequences ?

The connecting homomorphisms of RS sequences also coincide with cup product with the class $o^1(E)$.

These **miscellaneous facts** from Wu's Ch.2 should become clear if viewed from the point of view of (i) as facets of the group cohomology of $\mathbb{Z}_p$.

(v) Some remarks are in order re the **quotient** E/t, especially since Wu spends a whole lot of time in bringing Smith theory down to it.

(a) Even if E is a free simplicial complex, this quotient is an abstract complex only. However the quotient E"/t of the second derived of such an E is a simplicial complex.

(b) For any $\rho$, there is a cochain complex $C^*(E/t;\rho(\mathbb{Z}))$, where $\rho(\mathbb{Z})$ is the subgroup of the p-fold sum $\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ defined by $\rho(.) = 0$, whose $\partial$ and whose isomorphism with $C^*_\rho(E;\mathbb{Z})$ both depend on a choice of orbital representatives.

(c) However, for the case $\rho = d = 1 - t$, the projection map $\pi: E \longrightarrow E/t$ induces a natural isomorphism $C^*(E/t;\mathbb{Z}) \longrightarrow C^*_d(E)$. Even for the case $\rho = s$, Wu gives a natural *homomorphism* $H^*_s(E;\mathbb{Z}) \longrightarrow H^*(E/t;\mathbb{Z}_p)$.

(d) There are Smith morphisms defined in $H^*(E)$ [involving reduction mod p if their length is odd] which are tied to the **Smith morphisms** [alternating compositions of connecting morphisms of RS sequences] of E by above maps.

From "On the combinatorics of the Euler characteristice" [= Ch.4 of "Finite Operator Calculus"]:

**1. There is one and only one measure on the poset of simplicial complexes taking any given values on the closed simplices.**

Here, by ***measure***, on the poset P of all [nonempty] simplicial complexes with vertices in a given set, is meant a map $\varkappa$ from P into some ring, which obeys

$$\varkappa(K \cup L) = \varkappa(K) + \varkappa(L) - \varkappa(K \cap L) \quad \forall\ K,L \in P.$$

From the point of view of generalizing to posets other than P, note that ***closed simplices*** S are characterized order-theoretically by the property that $S = A \cup B$ implies $A = S$ or $B = S$. So e.g. the ***smallest member*** $\mathbf{1}$ of P, viz. the simplicial complex containing just the empty simplex, is an example of a closed simplex.

*Proof.* This follows by using

$$\begin{aligned}\varkappa(A \cup B \cup C \cup \ldots \cup K \cup L) = {} & \varkappa(A) + \varkappa(B) + \varkappa(C) + \ldots + \varkappa(K) + \varkappa(L) \\ & - \varkappa(A \cap B) - \varkappa(A \cap C) - \ldots - \varkappa(K \cap L) \\ & + \varkappa(A \cap B \cap C) + \ldots \\ & \ldots \\ & \pm \varkappa(A \cap B \cap C \cap \ldots \cap K \cap L).\end{aligned}$$ *q.e.d.*

For example the ***reduced Euler characteristic*** X is the measure whose value on $\mathbf{1}$ is 1, and on all other closed simplices is zero, while the ***usual Euler characteristic*** is the measure which has value 0 on $\mathbf{1}$ , and the value 1 on all other closed simplices.

**2. If S is any closed simplex, and $\mu$ is the Mobius function of the poset of simplicial complexes, then**

$$\mu(\mathbf{1},S) = (-1)^{|S|}.$$

So the values of the Mobius function on its closed simplices determines the reduced Euler characteristic of any simplicial complex by

$$X(K) = \sum_{S \subseteq K} \mu(\mathbf{1},S).$$

*Proof.* For this recall that the ***Mobius function*** $\mu$: $P \times P \longrightarrow \mathbb{Z}$ of the poset P is zero outside $\leq$, 1 on =, and is defined elsewhere so as to satisfy $\sum_{x \leq z \leq y} \mu(x,y) = 0$. *q.e.d.*

Next one has the following generalization of the above, which shows that the correct order-theoretical interpretation of $(-1)^{|S|}$ comes from the reduced Euler characteristic of the poset of proper faces of S:

**For any** $K \in P$ **one has** $\mu(\mathbf{1},K) = -\chi(K_-)$, **where** $\chi(K_-)$ **is the reduced Euler characteristic** [of the simplicial complex of chains] **of the subposet** $K_- = \{L \in P: \mathbf{1} \neq L \subset K\}$.

Its proof requires more work, but once obtained paves the way for generalizations to other posets.

**3. Other posets.** Being purely order-theoretical now, the above results have interesting echoes in quite different P's:

(a) If P is the poset of natural numbers under divisibility, then the "usual Euler characteristic" of a natural number coincides with the number of distinct prime divisors of the natural number.

Likewise, for the poset P of partitions of a set under refinement, the Mobius function is known, so one can calculate X here also.

(b) If K is a **closed** q-**simplex**, i.e. the set of all subspaces of an n-dimensional vector space over the field $\mathbb{F}_q$, then the equation $\chi(K) = 0 = \sum_{S \subseteq K} \mu(\mathbf{1},S)$ coincides with an identity of **Euler** and **Cauchy** involving **Gaussian coefficients** $\begin{bmatrix} n \\ k \end{bmatrix}$, the number of k-dimensional subspaces of this vector space.

Note that now the poset P comprises all **q-complexes** K, i.e. sets of vector subspaces closed under $\subseteq$, and the special K's mentioned above were the "closed simplices" of this P. Regarding this P, Rota says the following:

" As $q \rightarrow 1$ (for **an imaginary field with 'one' element**) a q-complex becomes an ordinary simplicial complex, and a q-sphere becomes an ordinary homology sphere."

Here by q-sphere he means a closed q-simplex minus its top. [Such a fictional field of one element is dear to **Manin** also !]

(c) When P is the poset of faces K of a convex polytope, then $K_-$ is always spherical, which implies that one has $\mu(\mathbf{1},K) = \pm 1$, depending on the parity of the dimension of the face K.

*Comments*

(1) Rota considers the **Grothendieck group** obtained as the quotient, of the abelian group of all linear combinations of elements of P, by the subgroup generated by elements of the form $K + L - K \cup L - K \cap L$.

Since the coefficients of our linear combinations are from a ring, they can be multiplied in the obvious way, and it can be checked that this subgroup is an ideal, and so this Grothendieck group is in fact a *ring*.

Measures of P correspond to linear maps from the Grothendieck ring to the ring of coefficients. Also, in this ring, one has the identity

$$K_1 \cup \ldots \cup K_n = 1 - (1 - K_1)(1 - K_2) \cdots (1 - K_n),$$

for all $K_i \in P$. This explains why any such functional is determined by its values on the closed simplices of P.

(2) If one wants to look only at measures on P such that $\mu(K) = \mu(L)$ whenever the simplicial complexes K and L are isomorphic, then the right Grothendieck group is of *isomorphism classes* of simplicial complexes.

One has the still smaller Grothendieck group of *p.l. classes* of simplicial complexes, and measures descending to it are as follows.

**Mayer's Theorem.** *The only subdivision invariant measures on the poset of simplicial complexes are those which take a constant value on all closed simplices other than* **1**.

(3) There is another natural multiplication, that provided by the **join** $K{\cdot}L$ of simplicial complexes, which too descends to the above Grothendieck groups.

*The reduced Euler characteristic* $X$ *is the only subdivision invariant measure on* P *which is such that* $X(K{\cdot}L) = X(K).X(L)$ *for all simplicial complexes* K *and* L.

Finally, it seems that join multiplicativity and subdivision invariance, are of interest not only for *linear*, but also, à la **Lee**, for *polynomial* maps on the vector space spanned by P.

## QUILLEN'S COBORDISM PAPER

The [unoriented or complex] cobordism ring is the ring of coefficients of an extraordinary cohomology theory, and its structure [i.e. the theorem of **Thom** or **Milnor**] was shown by Quillen to follow from the properties of this geometric theory itself. [However, in the *complex* case, he used a homotopy-theoretically proved finiteness result. Also note for this case that all manifolds, maps, and vector bundles below will have an almost complex structure and so a preferred orientation.]

**(A) Cohomology theory** $U^*$. Since from the homotopy-theoretic point of view this entails no loss of generality [cf. Remark 1] we will work only with *smooth manifolds and smooth maps* in the following :

**(i)** $U^q(X)$ *will consist of all cobordism classes* [f] *of proper maps* $f: Z \to X$ *of dimension* $-q$:

Here **proper** means pull-backs of compact sets should be compact, **dimension** means $\dim(T_z Z) - \dim(T_{f(z)} X)$ for any $z \in Z$, and two maps $f_0$, $f_1$: $Z_0$, $Z_1 \to X$ are to be called **cobordant** if there is a map $F: W \to X \times [0,1]$, transversal to the two ends, whose restrictions to the inverse images of the two ends coincides with the given maps $f_0$ and $f_1$.

**(ii)** $U^q(X)$ *is equipped with addition* $[f] + [f'] = [f \sqcup f']$ *and multiplication* $[f]\cdot[f'] = \Delta^*[f \times f']$.

Here $f \sqcup f'\colon Z \sqcup Z' \to X$ is the **disjoint sum**, and $f \times f'\colon Z \times Z' \to X \times X$ the **cartesian product** of the maps $f\colon Z \to X$ and $f'\colon Z' \to X$, while $\Delta^*\colon U^q(X \times X) \to U^q(X)$ is the map induced by the **diagonal** $\Delta \to \Delta \times \Delta$ as follows.

[The **kth cup power** $[f] \mapsto [f]^k$ can be seen to coincide with the map $U^q(X) \to U^{kq}(X)$ defined by first mapping the cobordism class $[f]$ of any $f\colon Z \to X$ to the class in $U^{kq}(X^k)$ given by the k-fold product $f^k\colon Z^k \to X^k$ of f, and then using the map induced by the kth diagonal $X \to X^k$ as follows.]

**(iii)** *Each homotopy class of smooth maps* $\gamma : X \to Y$ *induces the functorial contravariant map* $\gamma^*\colon U^q(Y) \to U^q(X)$, $\gamma^*[f] = [g^*(f)]$.

Here $g^*(f)$ [also denoted by f' in (v) below] is the **pull-back,** under any member $g\colon X \to Y$ of the homotopy class $\gamma$ which is transversal to f, of the given map $f\colon Z \to Y$, i.e. the projection $X \times Z \to X$ restricted to $X \times_Y Z = \{(x,z) : g(x) = f(z)\}$.

The above $\gamma^*$ will also be written $g^*$ for any $g \in \gamma$, so as to have the usual $g \simeq h \Rightarrow g^* = h^*$.

**Remark 1.** Thanks to this homotopy invariance we can define $U^q(X)$ **of a simplicial complex** X by embedding it rectilinearly in some euclidean space, and then replacing it with the homotopy equivalent smooth manifold which occurs as its open tubular neighbourhood. [However note that the next property (iv) is for manifolds only.]

**Remark 2.** Though he strongly advocates the above geometric approach, Quillen's official definition of $U^*(X)$ is still homotopy theoretical via **Thom spectra** = {Thom spaces of the canonical vector bundles of the Grassmannians}. He did this partly to save time, since for spectral cohomologies it was well-known how to define the **relative cohomology** $U^*(X,A)$ **of pairs** and verify **the first six Eilenberg-Steenrod axioms**, but also because, in the *complex* case, he needed the fact, which he could not prove by purely geometric means, that $U^q(X)$ **is a finitely generated abelian group for any polyhedron** X.

**Remark 3.** That the geometric and homotopy theoretical definitions of $U^*(X)$ agree is a routine generalization of a celebrated theorem of Thom. In fact Thom's theorem is the case X = pt, because the **coefficient ring** $U^*(pt)$ [or just $U^*$] of our cohomology theory obviously coincides with Thom's ring of cobordism classes of smooth manifolds.

**(B) Thom isomorphisms in $U^*$.** Our cohomology theory [like ordinary cohomology or K-theory] happens to also have the following extra structure.

**(iv)** *Each proper map* $g: Z \to X$ *of manifolds of dimension* $-n$ *induces the functorial covariant map* $g_*: U^q(Z) \to U^{q+n}(X)$, $g_*[f] = [g \circ f]$ *which is such that for each* **fibre square**

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{h'} & Z \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{h} & X \end{array}$$

*one has* $h^* \circ g_* = (g')_* \circ (h')^*$. *Besides we have* $x \cdot (f_* z) = f_*((f^* x) \cdot z)$, *for all* $z \in U^q(Z)$ *and* $x \in U^{n+q}(X)$.

[For example the ordinary $f_*: H^n(X) \to H^0(pt) = \mathbb{Z}$ of an n-manifold X evaluates cohomology classes on the fundamental n-cycle of X.]

This structure suffices to ensure that, whenever $s: X \to V$ is the **zero section** of an n-dimensional **vector bundle** $\pi: V \to X$, then $s_*: U^*(X) \to U_c^{*+n}(V)$, defined again by $s_*[f] = [s \circ f]$, but now with values in the **compactly supported cobordism theory** $U_c^*$ of V, is an isomorphism. [To define this theory one puts the appropriate relation of cobordism on all proper maps $f: Z \to V$ whose images have compact closure, etc.]

In the late 1960's **motifs**, i.e. the [mostly conjectural] universal cohomology functors of **Grothendieck**, had begun to create a stir, which soon died down, but has now returned as a storm ! Influenced by Grothendieck's ideas, Quillen emphasized the important fact that $U^*$ is *universal* amongst cohomology functors possessing the above structure !!

**Proposition 1.** *Given any cohomology theory* $\mathcal{H}^*$ *satisfying* (i)-(iv), *there is a unique morphism* $U^* \to \mathcal{H}^*$ *which preserves this structure, and maps* $\mathbf{1} = [id] \in U^0(pt)$ *to a given element* $a \in \mathcal{H}^0(pt)$.

*Proof.* Since $f_* \circ (Z \to pt)^*[id] = [f]$ for any $[f] \in U^q(X)$, $f: Z \to X$, it follows that such a morphism of functors must map [f] to $f_* \circ (Z \to pt)^*(a) \in \mathcal{H}^q(X)$. The result follows because one can check that this element of $\mathcal{H}^q(X)$ is independent of the representative f of the cobordism class which is used in its definition. *q.e.d.*

**(C) Characteristic classes in $U^*$.** These can be defined, for *any* theory obeying (i)-(iv), in the standard way [cf. books of Hirzebruch or

Milnor-Stasheff]:

The **Euler class** $e(V) \in U^n(X)$ of an n-dimensional vector bundle $V \longrightarrow X$ with zero section s is defined by $e(V) = (s^* \circ s_*)(1)$.

The total **Chern class** of a sum $V = L_1 \oplus \ldots \oplus L_b$ of line bundles is defined by $c(V) = (1 + e(L_1)) \cdot \ldots \cdot (1 + e(L_b))$. Then the **splitting principle** is used to extend this definition to all V's as follows.

The **Leray-Hirsch structure theorem** for the cohomology $U^*(\mathbf{P}V)$ of the space of lines of V still holds because it is a consequence of (i)-(iv). We use $\pi$: $\mathbf{P}V \longrightarrow X$ to lift V to $\pi^*(V)$ which splits. Then using this theorem its $c(\pi^*(V))$ arises as $\pi^*$ image of a unique cohomology class in X, which is the required $c(V) \in U^*(X)$.

This gives a functorial map c: $K^*(X) \longrightarrow U^*(X)$ which obeys $c(V \oplus W) = c(V) \cdot c(W)$.

[Though we won't use it, we also have the **Chern character** ch: $K^*(X) \longrightarrow U^*(X) \otimes \mathbb{Q}$, a functorial *ring homomorphism*, which can also be defined by this splitting method, by summing the cohomology classes exp(e(L)) as L runs over the line bundle summands L of $\pi^*(V)$.]

More generally, there is a functorial map $c_t$ : $K^*(X) \longrightarrow U^*(X)[t_1, t_2, \ldots]$ obeying $c_t(V \oplus W) = c_t(V) \cdot c_t(W)$, defined in exactly the same way by associating to each line bundle summand L of $\pi^*(V)$ the factor $1 + t_1 e(L) + t_2 e(L)^2 + \ldots$ .

The following observation of **Novikov** heralded the explicit use of formal groups in topology.

**Proposition 2.** *There is a unique power series* $F(T_1, T_2)$ *in two variables, and with coefficients in* $U^*(pt)$, *such that for any two line bundles over any* X *one has*

$$e(L_1 \otimes L_2) = F(e(L_1), e(L_2)).$$

*Moreover* F *is a commutative* **formal group law**, *i.e. it obeys*

$$F(T_1, T_2) = F(T_2, T_1), \quad F(0, T) = 0 = F(T, 0), \quad \textit{and}$$

$$F(T_1, F(T_2, T_3)) = F(F(T_1, T_2), T_3).$$

*Proof.* Since complex projective space with their canonical line bundles are universal, it suffices to compute the $U^*$ of a product of such spaces using (i)-(iv). This turns out to be the truncated polynomial ring in the two Euler classes, etc., etc. *q.e.d.*

For ordinary cohomology one has $F(T_1, T_2) = T_1 + T_2$, while for complex K-theory it is $T_1 + T_2 - T_1T_2$. **Mischenko**, in an appendix to Novikov's paper, computed the *logarithm* of the formal group law of $U^*$, from which it follows easily [see Quillen's B.A.M.S. note] that the latter is the **universal formal group law** which had been studied before by **Lazard**. [However these arguments used the homotopy theoretically proved theorem of Milnor re the structure of the coefficient ring $U^*(pt)$.]

**(D) Operations in $U^*$.** The paper is based on a clever exploitation of a basic relationship between the following two operations:

**Novikov character** $s_t$: $U^*(X) \longrightarrow U^*(X)[t_1, t_2, \ldots]$ is the functorial ring homomorphism defined by $[f] \longmapsto f_*(c_t(\nu_f))$, where $\nu_f = f^*(TX) - TZ \in K(Z)$, denotes the virtual **normal bundle** of the proper map $f: Z \longrightarrow X$ of manifolds.

[For example if f is the constant map from an n-manifold Z, then the ordinary $s_t[f] \in \mathbb{Z}[t_1, t_2, \ldots]$, where $\deg(t_j) = j$, is homogenous of degree n, and its coefficients are the **Chern numbers** of the manifold Z.]

In the next definition Q is any manifold on which the cyclic group $\mathbb{Z}_k$ operates freely, and $B = Q/\mathbb{Z}_k$, e.g. we can even take $Q = \mathbb{Z}_k$ and B = pt.

**kth Steenrod power** $U^q(X) \longrightarrow U^{kq}(B \times X)$ is defined by first mapping any [f], where $f: Z \longrightarrow X$, to the element $[id \times f^k]_{eq}$ of the **equivariant cobordism** $U^{kq}_{eq}(Q \times X^k)$, and then using the map $(id \times \Delta)^*: U^{kq}_{eq}(Q \times X^k) \longrightarrow U^{kq}_{eq}(Q \times X) = U^{kq}(B \times X)$.

[Apparently for the case $Q = \mathbb{Z}_k$ one just gets the kth cup power ? However, when $Q = S^{2m+1}$, with the usual $\mathbb{Z}_k$ action, then one gets new stuff: note that for "$m = \infty$" the infinite-dimensional manifold $B = Q/\mathbb{Z}_k$ is a classifying space $B\mathbb{Z}_k$ of the group $\mathbb{Z}_k$.]

**(E)** The relationship between the above two operations is Proposition 3.17. It says [in *very rough* analogy with a result of **Wu** which might be its ordinary case] that "the Leray-Hirsch components of the kth Steenrod power are the Chern numbers of the manifold". Actually there are other terms involved, so it looks more like an **index formula**, and incorporates a non-trivial **integrality theorem**, i.e. implies that something à priori in $U^*(pt) \otimes \mathbb{Q}$ is in fact in $U^*(pt)$.

[Though Karoubi's exposition of the proof of this result is understandable, we don't *really* understand the meaning of this important index formula ... but we should return to it later, since the cyclic cohomology version of the aforementioned result of Wu will apparently shed new light on the topological invariance of rational Pontrjagin classes.]

The difficult [and apparently of not much interest for us] part of Quillen's paper is the deduction of the structure of $U^*(pt)$ from this index formula:

First, by using the integrality result, and some computations regarding the cobordism of lens spaces, Quillen deduces that $U^*(pt)$ coincides with the subring generated by the coefficients of $U^*$'s formal group law.

Then, by using a theorem of Lazard, he deduces from above Milnor's result that $U^*(pt)$ is a polynomial ring having one generator in each even dimension.

*Comments*

(1) Formal groups were defined by **Bochner** in the 40's to make some old calculations of **Lie** re "infinitesimal groups" more meaningful. The relationship *cohomology* $\longleftrightarrow$ *formal groups* came to the fore implicitly in **Hirzebruch**'s great book, and was made explicit shortly after by**Novikov** . After this came the above paper of **Quillen** , and its contemporay expositions by **Adams** and **Karoubi** .

(2) Note that the importance of another contemporary extraordinary cohomology, i.e. the $K^*$ of **Atiyah-Hirzebruch** , also stemmed from the fact that the structure theorem re its coefficient ring, i.e. **Bott periodicity**, was also a deep non-trivial fact.

However its universality ~~however~~ seems to make cobordism theory much more basic, e.g. Bott periodicity may follow from the structure theorem for $U^*(pt)$ ? Also this "motivic viewpoint" suggests that the first, i.e. the cobordism-dependent, proof of the **Atiyah-Singer theorem** was perhaps the "right" one after all ?

(3) For developements subsequent to (1) — e.g. a characterization theorem for formal groups arising from cohomologies, study of special cases like *elliptic cohomology*, and the relationship of formal groups to things like *binomial polynomials*, *functional equations*, and *umbral calculus* — see the 1991 paper of **Bukhstaber-Kholodov** and its references.

**CHAPTER THREE.** Though very simple, this is the heart of the book, since the basic criteria for embeddability, etc. are formulated here.

A continuous mapping $f: X \rightarrow Y$ between polyhedron is called an **embedding**, resp. a local embedding or **immersion**, if it is one-one, resp. locally one-one [i.e. each point has a neighbourhood on which f is one-one].

**Theorem.** *If* $|K|$ *embeds, resp. immerses, in* $\mathbb{R}^m$*, then, for each prime* p*, there is a continuous* $\mathbb{Z}_2$*-map from* $K^p_* \simeq W(K^p) \setminus \Delta K$*, resp.* $K^p_o \simeq N(WK^p, \Delta K) \setminus \Delta K$*, to a free* $\mathbb{Z}_p$*-sphere* $S^{pm-m-1}$.

*Proof.* Clearly any embedding, resp. immersion, $X \rightarrow Y$, induces an equivariant continuous map from the complement, resp. local complement, of the diagonal $\Delta X$ of the p-fold product $X^p$ of X, into that of the diagonal $\Delta Y$ of the p-fold product $Y^p$ of Y.

The result follows because a projection on the subspace orthogonal to the diagonal subspace $\Delta\mathbb{R}^m$, followed by a normalization, shows that the complement and local complement, of the diagonal of $(\mathbb{R}^m)^p$, both have the equivariant homotopy of a sphere $S^{pm-m-1}$, and the $\mathbb{Z}_p$-action is free because p is prime. *q.e.d.*

**Corollary.** *If* K *embeds, resp. immerses, in* $\mathbb{R}^m$*, then the Smith classes of* $K^p_*$*, resp. of* $K^p_o$ [which are images of those of $K^p_*$ under the map induced in equivariant cohomology by $K^p_o \simeq N(WK^p, \Delta K) \setminus \Delta K \subseteq W(K^p) \setminus \Delta K \simeq K^2_*$]*, must vanish in dimensions* $\geq$ pm-m.

**Alternating cocycles.** It is important to note that the above obstructions to embeddability or immersibility, i.e. the classes $o^i \in H^i_{d/s}(K^p_*$ or $K^p_o)$, are defined *purely combinatorially*. The case p = 2 has been developed further as follows:

Depending on whether i is even or odd, consider the symmetric or skewsymmetric i-cochain $o^i$, which takes value 0 on any $\sigma \times \theta$, unless the vertices of $\sigma$ and $\theta$ **alternate** with respect to the total order, with the value being 1, if further the least vertex of $\sigma \cup \theta$ is in the first factor $\sigma$. Then it can be verified [it suffices to check the universal example of octahedral spheres] that $o^i$ is a cocycle which is in either $+o^i$ or $-o^i$, and even this sign can be worked out in terms of the congruence class of i mod 8.

**Examples.** Non-embeddability and non-immersibility of some complexes is now checked via above criteria, e.g. Wu reproves the **Van Kampen - Flores Theorem** [$\sigma_n^{2n+2}$ does not embed in $\mathbb{R}^{2n}$, etc.] by using the above alternating cocycles.

**Isotopy etc.** In p.l. topology a whole gamut of such definitions have now been analysed. E.g. two embeddings are called (resp. ambient) isotopic if they are related by a 1-parameter class of embeddings (resp. self-homeomorphisms of the ambient space). [Likewise one speaks of two locally isotopic local embeddings .. .] On the other hand two embeddings are called equivalent, or in **isoposition**, if they are related by a single self-homeomorphism, which sometimes is required to be orientation preserving, etc.

A choice of an orientation of $\mathbb{R}^m$ fixes a generator of $H^{m-1}((\mathbb{R}^m)^2 \setminus \Delta\mathbb{R}^m) \cong H^{m-1}(S^{m-1})$. Under an embedding, resp. immersion, of K in $\mathbb{R}^m$, this generator pulls back to a cohomology class in $H^{m-1}(K_*^2)$, resp. $H^{m-1}(K_o^2)$, which obviously does not change under isotopy [but does change sign under an orientation-reversing homeomorphism of $\mathbb{R}^m$]. So these classes can be sometimes used [as Wu shows via some examples] to check that two embeddings or two immersions are not isotopic, etc.

**CHAPTER FOUR.** This, the messiest chapter of this messy book, gave a "new" definition of **Steenrod squares**, so we'll return to it after having a look at an "old" definition first.

*Comments*

(1) It seems that the notions of embedding, immersion, etc., can be generalized so as to view the Smith classes, of any given invariant part of $\mathbf{W}(K^{\mathbf{p}})$, as suitable obstructions.

(2) Likewise it seems, e.g. by using **oriented matroids** other than the alternating one, that it will be able to make the definition of these classes combinatorially more explicit even for p ≥ 3.

(3) The most challenging problem of course is to understand the limiting [or motivic or universal] case "$\mathbb{Z}_p \to \mathbb{S}^1$" combinatorially, perhaps via **cyclic cohomology**, using the cyclic model of the group $S^1$.

## STEENROD-EPSTEIN

In these [pre-1962] lectures Steenrod gave a new construction [= Chapter VII] of cohomology operations which is based on some simple facts [= Chapter V] regarding equivariant cohomology.

[*In this book, the integral chain complex of a cell complex* K *is also denoted by* K, *rather than* $C_*(K)$; *however the authors prefer to use* K ⊗

L, *rather than just* $K \times L$, *to denote* $C_*(K) \otimes C_*(L) \cong C_*(K \times L)$.]

## A. Equivariant Cohomology.

Given a [possibly infinite] cell complex E and a module A, both with prescribed actions of a group G, all **equivariant cochains** $c \in C^*(E;A)$, i.e. those satisfying $c(g.\sigma) = g.c(\sigma)$ for all $g \in G$ and $\sigma \in E$, form a sub cochain complex $C_G^*(E;A)$, whose cohomology is denoted $h_G^*(E;A)$.

**Proposition 1.** *If the G-complex E is such that the faces of any cell preserved by a group element g are also preserved by g, then* $h_G^*(E;A)$ *is an invariant of the equivariant homotopy type of the G-space* $|E|$.

[Cf. first paragraph of the "Errata" of the book.]

*Proof sketch.* Under the given hypothesis, $h_G^*(E;A)$ coincides with its singular version $h_G^*(|E|;A)$. *q.e.d.*

A much more restrictive notion than the above is that of a **free action**, i.e. one in which the conjugates $g.\sigma$, $g \in G$, of any cell $\sigma \in E$, are pairwise disjoint as g runs over G.

**Proposition 2.** *For any group* G, *there exist free acyclic* G*-complexes* EG, *which are functorially* G*-homotopy equivalent to each other.*

So we can denote $h_G^*(EG;A)$ by $H^*(G;A)$, and call it the **cohomology of the group** G with coefficients in the G-module A.

*Proof.* Recall that an **(acyclic) carrier** S from E to F associates to each cell $\sigma$ of E an (acyclic) subcomplex $S(\sigma)$ of F in such a way that $\sigma \subseteq \theta$ implies $S(\sigma) \subseteq S(\theta)$.

On the other hand, an **equivariant (acyclic) carrier**, with respect to a given G- (resp. H-) action on E (resp. F) and a group homomorphism $\pi$: $G \to H$, will be one which also satisfies $\pi(g).S(\sigma) = S(g.\sigma)$ for all $g \in G$.

For example, each (equivariant) chain map $\phi$: $E \to F$ gives rise to a [not necessarily acyclic] (equivariant) carrier, viz. its **support** $\mathrm{supp}(\phi)$, which associates to each $\sigma \in E$ the subcomplex of F generated by the cells occuring [with nonzero coefficients] in the chain $\phi(\sigma)$.

If a chain map $\phi$: $e \to F$, from a subcomplex e of E, is **supported** by some known acyclic carrier S from E to F [in the sense that $\mathrm{supp}(\phi(\sigma)) \subseteq S(\sigma)$ $\forall\ \sigma \in e$] then it can be extended to a chain map $\phi$: $E \to F$ supported by S as follows:

One arranges the cells $\sigma$ of $E - e$ in order of increasing dimension, and $\phi$ is defined on each of these in turn so as to satisfy $\phi\partial = \partial\phi$, this being possible each time because of the acyclicity of $S(\sigma)$.

A similar argument [arrange the G-orbits, which are *pairwise disjoint* cells, in order of increasing dimension, ...] shows likewise that if an equivariant chain map $\phi$: e $\longrightarrow$ F, from an equivariant subcomplex e of a *free* G-complex E, is supported by some equivariant acyclic carrier S from E to F, then it can be extended to an equivariant chain map $\phi$: E $\longrightarrow$ F supported by S.

It follows easily from this that the required EG must be unique upto G-homotopy equivalence.

As far as the existence of EG goes, we can [following **Milnor**] take EG = G∘G∘ ... , where this **infinite join** of the point set G is to be provided with the (obviously free) diagonal G-action. This complex is acyclic because any cycle, which has to lie in finitely many factors, is bounded by its cone, which is available by using one more factor. *q.e.d.*

Continuing with general arguments of the above kind the authors partially check the fact that each G-module A determines a **cohomology theory** $K \longmapsto H^*_G(K;A) = h^*_G(EG \times K;A)$, i.e. an abelian functor satisfying the first six *Eilenberg-Steenrod axioms*, which will be called **G-equivariant cohomology** with coefficients in G-module A.

[As **Borel** pointed out it is useful to note that the diagonal action on the product of G-complexes is free as soon as *one* of the factors is free: e.g. the *homotopy axiom* for the above cohomology theory follows from Proposition 1 because EG × K is free. On the other hand note that the diagonal action of a join of G-complexes is free iff *all* of them are free.]

Furthermore, imitating the usual definition [i.e. cross product followed by the map induced in cohomology by the diagonal] they equip this cohomology with natural **cup products** $H^*_G(K;A) \otimes H^*_G(K;B) \longrightarrow H^*_G(K;A\otimes B)$, for any G-modules A and B.

Again, just as in the ordinary case G = 1, each short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of G-modules has an associated long exact **Bockstein sequence** in equivariant cohomology.

To "do sums" with this cohomology later we need its value on a point, so the authors compute some group cohomologies $H^*_G(pt;A) = H^*(G,A)$.

**Proposition 3.** **(a)** *Additively the cohomology of the cyclic prime order group* $\mathbb{Z}_p$ *with coefficients in the trivial* $\mathbb{Z}_p$*-module* $\mathbb{F}_p$ *is given by*

$$H^i(\mathbb{Z}_p;\mathbb{F}_p) \cong \mathbb{F}_p,\ \forall\ i \geq 0.$$

**(b)** *Furthermore we can choose generators* $w_i$, $i \geq 0$, *of these groups which behave as under with respect to cup product:*

$$\text{For } p = 2,\ w_i = (w_1)^i.$$

$$\text{For } p \text{ odd prime, } w_{2j} = (w_2)^j \text{ and } w_{2j+1} = (w_2)^j.w_1.$$

(c) *Also one has* $w_2 = \beta(w_1)$, *where* $\beta$ *is the connecting homomorphism of the Bockstein sequence of* $0 \to \mathbb{F}_p \to \mathbb{F}_{p^2} \to \mathbb{F}_p \to 0$.

(d) *The normalizer of the rotation subgroup* $\mathbb{Z}_p$ *of the symmetric group* $S_p$ *acts on* $H^i(\mathbb{Z}_p;\mathbb{F}_p)$ *via inner automorphisms* (resp. *inner automorphisms multiplied by the parity of the permutation*). *This action is trivial iff* i *is an even* (resp. *odd*) *multiple of* p-1, *or one less than such a multiple.*

[They obtain similar answers when p is any odd number, and surely, at least by now, the answers must be known even for any $p \in \mathbb{N}$ ?]

*Proof sketch.* The authors do the above computations combinatorially via coboundaries of an explicit $\mathbb{Z}_p$-equivariant subdivision of the unit sphere of eventually zero infinite sequences of complex numbers. *q.e.d.*

Finally they consider the **transfer** or **integration** [= summation over each coset] map between the cohomology of a group and that of a subgroup of *finite index*. [This map goes in a direction opposite to that of the obvious functorial map, and a composition of the two maps equals multiplying by this index, etc.] Using integration they check e.g. that each cohomology class of a finite group G has a finite order which divides the order of G.

**B. Cohomology Operations.** The interpretation of semi-simplicial cohomology groups as homotopy groups arose from the following.

**Proposition 4.** *Considering the module* A *as a chain complex nonzero only in dimension zero, one has a natural bijection from* $H^q(K;A)$ *to the set of chain homotopy classes of chain maps* $K \to A$ *of degree* -q.

[In fact we will also need the interpretations of "cochains", "cocycles", and "cohomologies" of K given in the proof below.]

*Proof.* By definition a *cochain* $u \in C^q(K;A)$ identifies with a *linear map* $u: K\ [= C_*(K)] \to A$ of degree -q. Furthermore $u \in Z^q(K;A)$, i.e. u is a *cocycle*, iff this map $u: K \to A$ vanishes on q-boundaries, i.e. iff it is a *chain map* of degree -q. Finally it can be checked that the difference $u - v$ of two such cocycles is in $B^q(K;A)$, i.e. u and v are *cohomologous*, iff there is a *chain homotopy* between these chain maps $u, v: K \to A$ of degree -q. *q.e.d.*

**Definition of external powers P.**

Let G be a subgroup of the group of all permutations of p letters, which will be assumed to act on the p-fold product $K^p$ of any complex by permuting the factors. We choose any EG and shall use the diagonal action of G on $EG \times K^p$.

Given a q-cochain $u\colon K \to A$ of K, and any $p \geq 2$, we denote by $\mathbf{P}u\colon EG \times K^p \to A^p$ the pq-cochain of $EG \times K^p$ obtained by composing its p-fold tensor product $u^p\colon K^p \to A^p$ with the map $EG \times K^p \to K^p$ obtained by taking the tensor product of the *augmentation* $\varepsilon\colon EG \to \mathbb{Z}$ and the identity map of $K^p$.

To ensure that $u^p\colon K^p \to A^p$ is a G-map we need to use, depending on whether $\dim(u) = q$ is even or odd, two *different actions* of G on the p-fold tensor product $A^p$: for q even we just permute the factors, while for q odd we also multiply by the parity of the permutation. These two G-modules will be denoted respectively by $A^p_+$ and $A^p_-$.

With this precaution $\mathbf{P}u$ is equivariant, and so we have a [non-linear] function

$$\mathbf{P}\colon C^q(K;A) \to C^{pq}_G(EG \times K^p;A^p_\pm).$$

**Proposition 5.** *The map* $\mathbf{P}$ *images (cohomologous) cocycles to (equivariantly cohomologous) equivariant cocycles, and thus induces a* [non-linear] *map*

$$\mathbf{P}\colon H^q(K;A) \to h^{pq}_G(EG \times K^p;A^p_\pm).$$

*Proof.* The augmentation being a chain homotopy, it is clear that if the degree $-q$ map $u\colon K \to A$ is a chain map, then the degree $-pq$ equivariant map $\mathbf{P}u\colon EG \times K^p \to A^p_\pm$ is also a chain map. Likewise [using acyclic carriers] it is easy to check that if u and v are chain homotopic, then $\mathbf{P}u$ and $\mathbf{P}v$ are equivariantly chain homotopic. *q.e.d.*

Next the authors check that these maps $\mathbf{P}$ commute with the maps induced by any $K \to L$ and its p-fold cartesian product.

Also they show that if this external map $\mathbf{P}$ is composed with the map $h^{pq}_G(EG \times K^p;A^p_\pm) \to h^{pq}_G(K^p;A^p_\pm) \to H^{pq}(K^p;A^p)$, induced by the projection $EG \times K^p \to K^p$, then we obtain $H^q(K;A) \to H^{pq}(K^p;A^p)$, $[u] \mapsto [u] \times .. \times [u]$, i.e. the *p-fold cross product*.

[And so, composing further with the diagonal induced map, one would just obtain the *p-fold cup product* $H^q(K;A) \to H^{pq}(K;A^p)$, so we'll turn to what happens if we use the diagonal first.]

**Internal powers P.** These are the associated [non-linear] maps

$$P = d^* \circ \mathbf{P}:\ H^q(K;A) \longrightarrow H_G^{pq}(K;A_\pm^p),$$

where $d^*: h_G^{pq}(EG \times K^p; A_\pm^p) \to h_G^{pq}(EG \times K; A_\pm^p) = H_G^{pq}(K;A_\pm^p)$ is induced by the *diagonal map* $d: K \to K^p$.

[The authors avoided defining an internal P at the *cochain level* because there was no canonical choice of a *cellular* map $K \to K^p$ which induces $d^*$: however it seems it should be possible to repair this state of affairs by using the **Bier-Wu subdivision** of $K^p$.]

To say more about P it is necessary to *compute* the equivariant cohomology $H_G^{pq}(K;A_\pm^p)$, which they do for the following case.

**Case** $G$ = **rotation group** $\mathbb{Z}_p$, **p prime, and** $A = \mathbb{F}_p$. Now it can be checked that $A_+^p$ and $A_-^p$ both coincide with $\mathbb{F}_p$ with the trivial $\mathbb{Z}_p$-action.

The group actions of K and $\mathbb{F}_p$ being trivial, the required cohomology $H^*_{\mathbb{Z}_p}(K;\mathbb{F}_p) = h^*_{\mathbb{Z}_p}(E\mathbb{Z}_p \times K;\mathbb{F}_p)$ coincides with $H^*(B\mathbb{Z}_p \times K;\mathbb{F}_p)$, where $B\mathbb{Z}_p = E\mathbb{Z}_p/\mathbb{Z}_p$ has of course the same cohomology as the group $\mathbb{Z}_p$. So, by using Künneth's theorem, which applies since we have field coefficients,

$$H^*_{\mathbb{Z}_p}(K;\mathbb{F}_p) = H^*(\mathbb{Z}_p;\mathbb{F}_p) \otimes H^*(K;\mathbb{F}_p).$$

So we can define the Künneth **components** $P_k: H^q(K;\mathbb{F}_p) \to H^{pq-k}(K;\mathbb{F}_p)$ of the internal **cyclic powers** $P: H^q(K;A) \longrightarrow H^{pq}_{\mathbb{Z}_p}(K;\mathbb{F}_p)$ by

$$Pu = \Sigma_k\, w_k \times P_k u,$$

where the $w_k$ are as in Proposition 3.

For the case p = 2, we now define the **Steenrod squares** $Sq^i: H^q(K;\mathbb{F}_2) \to H^{q+i}(K;\mathbb{F}_2)$ by

$$Sq^i u = P_{q-i} u.$$

And, for p an odd prime, the **Steenrod reduced powers** $\mathcal{P}^i: H^q(K;\mathbb{F}_p) \to H^{q+2i(p-1)}(K;\mathbb{F}_p)$ are defined by

$$\mathcal{P}^i u = (a_{p,q})^{-1} . P_{(q-2i)(p-1)} u,$$

where the significance of the normalizing constant $a_{p,q}$ will be clear from (d) of the following.

**Proposition 6.** **(a)** *Cyclic powers* $P: H^q(K;A) \longrightarrow H^{pq}_{\mathbb{Z}_p}(K;\mathbb{F}_p)$ *are linear.*

**(b)** *Furthermore, their components* $P_k: H^q(K;\mathbb{F}_p) \to H^{pq-k}(K;\mathbb{F}_p)$ *are, for* q *even* (resp. *odd*), *zero unless* k *is an even* (resp. *odd*) *multiple of* p-1, *or one less than such a multiple.*

**(c)** *If* $p = 2$, *then we have the cross product rule*

$$P_k(u \times v) = \Sigma_{i+j=k} P_i u \times P_j v,$$

*while for an odd prime* p *one has*

$$P_{2k}(u \times v) = \pm \Sigma_{i+j=k} P_{2i} u \times P_{2j} v,$$

*where the sign equals the parity of* $\binom{p}{2}.\dim(u).\dim(v)$.

**(d)** *The* $P_k$*'s vanish also if* k *exceeds* $q(p-1)$, *and* $P_{q(p-1)}: H^q(K;\mathbb{F}_p) \to H^q(K;\mathbb{F}_p)$ *is multiplication by a nonzero constant* $a_{p,q} \in \mathbb{F}_p$.

*Proof sketch.* (a) One checks that $d^*$ vanishes on the image of the map

$$H^{pq}(E\mathbb{Z}_p \times K^p;\mathbb{F}_p) \longrightarrow h^{pq}_{\mathbb{Z}_p}(E\mathbb{Z}_p \times K^p;\mathbb{F}_p)$$

induced by integration. Then it is verified that, if u and v are q-cocycles, then $\mathbf{P}(u+v) - \mathbf{P}(u) - \mathbf{P}(v)$, which by definition is essentially $(u+v)^p - u^p - v^p$, lies in the image of this map. So $P = d^*\mathbf{P}$ is linear.

[Q. Find all subgroups of the symmetric group $S_p$ for which this works.]

(b) Consider the functorial maps from the internal powers of the normalizer of $\mathbb{Z}_p$ in $S_p$ to the internal cyclic powers. Now use the fact that actions via inner automorphisms are trivial for the normalizer's powers, while for the cyclic powers they have, by Proposition 3(d), trivial components only for the stated values of k.

(c) For any group G of permutations of p letters, if one applies to $\mathbf{P}u \times \mathbf{P}v$ the map induced by the diagonal group homomorphism $G \to G \times G$, then one gets $\mathbf{P}(u \times v)$ upto the above sign, because this is the change in orientation resulting from the shuffle $K^p \times L^p \leftrightarrow (K \times L)^p$.

For the case of the rotation subgroup one obtains the required rules for the components of the internal cyclic power because, for the case of an odd prime, we know from Proposition 3(b) that $w_a . w_b$ is zero when a and b are both odd.

(d) *But for the fact that the constant* $a_{p,q}$ *is nonzero, the rest will follow by repeatedly using the fact that powers are functorial in* K:

First note that homomorphisms induced in cohomology by the inclusion of its q-skeleton are monomorphism in dimensions $\le$ q, so, for a q-dimensional class u, $P_k u$ will be zero in K iff it is zero in its q-skeleton $K^q$.

Now find a map from $K^q$ to $S^q$, which pulls back the dual fundamental class of this oriented q-sphere to u. So, by functoriality again, it suffices to consider the case when K is an oriented $S^q$ and u is its dual fundamental class, the required $a_{p,q}$ *can be found by computing the homomorphism* $P_{q(p-1)}$ *of* $H^q(S^q;\mathbb{F}_p) = \mathbb{F}_p[u]$. This computation, which shows that it is nonzero, is sketched later.

As far as the vanishing assertion goes, it can be in doubt only for $P_{qp}$: $H^q(S^q;\mathbb{F}_p) \rightarrow H^0(S^q;\mathbb{F}_p)$, which must be zero since it commutes with homomorphisms induced by the inclusion $(pt) \subseteq S^q$.

*Computation of* $a_{p,q} = a_p$:

An application of the product rule to $K \times S^1$ gives $a_q = \pm\ a_{q-1} . a_1$, the sign being the parity of $\binom{p}{2} . \binom{q}{2}$.

So it only remains to compute $a_1$, i.e. the homomorphism $P_{p-1}$: $H^1(S^1;\mathbb{F}_p) \rightarrow H^1(S^1;\mathbb{F}_p)$ of an oriented circle. This [which incredibly is the hardest part of the whole proof !] is done combinatorially via coboundaries starting from the subdivision of the circle into two arcs. It turns out that $a_1 \in \mathbb{F}_p$ equals -1. *q.e.d.*

**Proposition 7.** **(a)** *The Steenrod squares* $Sq^i$: $H^q(K;\mathbb{F}_2) \rightarrow H^{q+i}(K;\mathbb{F}_2)$ *are natural transformations obeying* $Sq^0 = id$, $Sq^q u = u^2$, $Sq^i u = 0$ *for* $i > q$, *and* $Sq(x.y) = Sq(x).Sq(y)$, *where* $Sq = \sum_i Sq^i$.

**(b)** *For any odd prime* p, *the Steenrod reduced powers* $\mathcal{P}^i$: $H^q(K;\mathbb{F}_p) \rightarrow H^{q+2i(p-1)}(K;\mathbb{F}_p)$ *are natural transformations obeying* $\mathcal{P}^0 = id$, $\mathcal{P}^{q/2} u = u^p$, $\mathcal{P}^i u = 0$ *for* $i > q/2$, *and* $\mathcal{P}(x.y) = \mathcal{P}(x).\mathcal{P}(y)$, *where* $\mathcal{P} = \sum_i \mathcal{P}^i$.

*Proof.* Follows easily from Proposition 6, and the definitions of $Sq^i$

and $\mathcal{P}^i$. *q.e.d.*

The "axioms" listed above [the product rule is called **Cartan's formula**], not only imply the remaining "axioms" [Bockstein behaviour/**Adem's relations**], but also are enough to uniquely determine these cohomology operations: for this see Chapter VIII of the book. The rest of the book is based solely on these "axioms".

*Comments*

(1) In his 1947 *Annals* paper Steenrod had defined squares using *cup i-products*. The definition discussed above, which ties them up nicely with the *cohomology of the finite rotation groups*, seems to be a new version of that given in his 1953 *Commentari* paper.

Previously, in their 1936 *Annals* paper, **Richardson** and **Smith** had computed the *cyclically equivariant homology of pth powers of complexes*. The definition of the dual [or inverse] **Smith operations** $Sm^i$, $Sm \circ Sq$ = id, appeared implicitly in their calculation, as was pointed out later in **Wu**'s 1965 book.

Following **Milnor**, operations are also interpreted as the action of a known *Hopf algebra* [generated by symbols subject to relations suggested by Adem's formula, and equipped with the co-multiplication suggested by Cartan's formula] on cohomology. Analogously, following **Serre** and **Cartan**, they can be interpreted also as a homotopy-theoretic action of the *cohomology of an Eilenberg-Maclane space* on cohomology.

Amongst the striking *applications of operations* are the ones of **Thom** [embeddings, topological invariance of Stiefel-Whitney classes], and those of **Adams** [vector field and Hopf invariant problems] who used other operations also.

(2) *The definition of operations given in Wu's book is very close to that discussed here*. The only difference being that instead of associating to K the equivariant complex $EG \times K^p$ [G being say the cyclic permutation group on p letters] he works with $K^p$ [and its subcomplex $K^p_*$ and subdivision $W(K^p)$] *itself*. Again the operations are obtained by an "equivariant localization" of pth powers of cocycles to the diagonal.

(3) *One can replace products by joins in these definitions*. For example though $K \cdot \,..\, \cdot K \cdot G \cdot G \cdot \,...$, unlike its sub cell complex $K^p \times EG$, is not free, its G-action still satisfies the requirement of Proposition 1, and it has the same diagonal as the aforementioned sub cell complex.

This should enable us to use *join multiplicativity* to shorten some proofs [say the computation of the $a_{p,q}$'s ?] and should [using e.g. the fact that $EG = G \cdot G \cdot \,..$ is a deleted join] enable us to put Steenrod's and Wu's definitions in a single framework.

(4) *It would be interesting to generalize this combinatorial theory to*

*the infinite abelian group of circular rotations*. The methods of *cyclic cohomology* suggest that this is now possible, and it would be interesting to do it, because it might lead to a conceptual combinatorial definition of rational Pontrjagin classes, etc.

(5) *Also one should generalize this theory to some other finite, but non-abelian, permutation groups*. This should be possible, because starting with say **Cartan-Eilenberg**'s 1956 book [Chapter 12], a mass of information is available about group cohomology, the essential ingredient in the above method.

This should also relate to computations of cohomologies, and equivariant localizations of pth powers of cocycles, for *invariant parts of* $\mathbf{W}(K^{\mathbf{P}})$ other than the diagonal.