Riemann, Jordan, Pontryagin, S&E (1988), etc.

The tiling[1] defines an Arf invariant 0 'square root' $\vartheta : H^1(M^2; \mathbb{F}_2) \to \mathbb{F}_2$ of the mod 2 cup product, and any such refinement is realized thus. $\square$ We turn now to the genesis of ϑ from theta functions.

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The initial summary below to refresh memory was from Bellman's thin book (1961) especially its last pages 64-72. The names in it are all well-known, often confusingly for the same thing, though just as often it was in fact some other now lesser known mathematician who got there first. But my object was not its history, just the result involving ϑ that Riemann left behind for us in 1866, so what follows next is what I was able to grasp, after struggling for many weeks, and with this limited aim only, from Riemann's papers VI, XI and XXXI, this numbering being that in the 1892 edition of his Werke.

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Multidimensional theta functions :- The Legendre ambiguity of any rational integral $\int R(x,y)dx$ under polynomial constraint $P(x,y) = 0$, Abel had inverted to $2p$-periodic functions, which Jacobi saw were obtainable from the p-fold Fourier series $\theta(z,T) = \sum_n e^{2\pi \mathbf{i} n.z - n.Tn}$, where z and n are integral and complex p-vectors and T a $p \times p$ symmetric matrix with eigenvalues of $Re(T)$ positive for absolute and uniform convergence on compact sets; a *quasi* $2p$*-periodicty* is clear, viz., $\theta(z+e_k, T) = \theta(z,T)$ and $\theta(z + \mathbf{i}Te_k) = e^{-2\pi \mathbf{i} e_k.z + e_k.Te_k}\theta(z,T)$, where e_k is kth unit vector; but modular *functional equation* $\theta(z,T) = \frac{\pi^{p/2}}{\sqrt{\det(T)}}\theta(T^{-1}z, T^{-1})$ needs a p-dimensional Poisson formula for averaging functions over $\mathbb{Z}^p$ and a p-fold Gaussian integral; for genus $p = 1$ using this functional equation Riemann analytically extends Euler's *zeta function* in his VII, and like results are known for some number theoretical matrices T. $\square$

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[1] Not in above stand-in for a lost figure.

Riemann in §17 of VI shows that, for each $p \times p$ symmetric $a_{i,j}$ with real part negative definite, there is upto a constant factor a unique analytic θ, finite valued on all complex $(v_1, ..., v_p)$, which is unchanged if we add $\pi\mathbf{i}$ to any variable, but gets multiplied by $e^{2v_\mu + a_{\mu,\mu}}$ if we add $(a_{1,\mu}, \ldots, a_{p,\mu})$; integral combinations of these $2p$ independent translations he calls the associated *moduli of* θ and often $(v_1, \ldots, v_p) \equiv (w_1, \ldots, w_p)$ will denote variables related by them; also from its expansion – as above with $T = -a, \pi\mathbf{i}z = v$ – over integers $(n_1, \ldots, n_p)$ he notes in §23 that it is *even* $\theta(v_1, ..., v_p) = \theta(-v_1, \ldots, -v_p)$. □

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In XXXI he notes $(v_1, \ldots, v_p) \equiv (-v_1, \ldots, -v_p)$ holds only on all half integral combinations $(\frac{1}{2}\varepsilon_1'\pi\mathbf{i} + \frac{1}{2}\varepsilon_1 a_{1,1} + \cdots + \frac{1}{2}\varepsilon_p a_{p,1}, \ldots, \frac{1}{2}\varepsilon_p'\pi\mathbf{i} + \frac{1}{2}\varepsilon_1 a_{1,p} + \cdots + \frac{1}{2}\varepsilon_p a_{p,p})$ and that *mod two* exactly $2^{p-1}(2^p - 1)$ of the 2^{2p} choices $\varepsilon_1\varepsilon_1' + \cdots + \varepsilon_p\varepsilon_p'$ are nonzero:- there are $2^p - 1$ not all zero choices for the ε_i' and if $\varepsilon_i' = 1$ any of the 2^{p-1} choices for $\varepsilon_j, j \neq i$ occurs with $\varepsilon_i = 0$ or 1, respectively, depending on whether the sum over the other indices is 1 or 0. □ Riemann's papers are easier on the eye than the standard reference below.

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Mumford's *Tata Lectures on Theta* I, II (1982) curiously makes heavy weather of many things proved so quickly by Riemann. Mere manipulations, badly typeset to boot, make stretches real eye-sores. However its well-written Chapter 2 puts Riemann's VI, XI and XXXI in a modern context. Chapter 3 or II, which uses an heavier commutative algebra jargon, is turgider.

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Riemann surfaces in VI are graphs of polynomial equations, in Chapter 2 any connected and closed X of genus g. That g independent holomorphic 1-forms exist is in VI, that no more can is due to Roch (a student who died four months after him). VI too uses through any P_0 a homology basis of g pairs A_i, B_i of loops intersecting each other–but no other loop–once more. Let $\Omega_{ij} = \int_{B_j} d\omega_i$, the ith holomorphic 1-form ω_i dual to the loops A, evaluated on the jth conjugate loop B_j, then $\Omega_{ij} = \Omega_{ji}$ with imaginary part positive:- use wedge product of a holomorphic 1-form with another is zero, while with its conjugate is imaginary with non-positive coefficient. □ All this too is in VI except $g = p$ and dual integrals or 1-forms are $\pi\mathbf{i}$ times with periods $\pi\mathbf{i}\Omega_{ij} = a_{i,j}$. So any X dissected as above comes with the $\theta(z_1, \ldots z_g, \Omega)$, the $2g$-torus $\mathbb{J}$ obtained by dividing $\mathbb{C}^g$ out by the moduli of this theta, with X complex analytically embedded in $\mathbb{J}$ using $P \mapsto [\int_{P_0}^P \omega_1, \ldots, \int_{P_0}^P \omega_g]$.

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It is shown in VI and XI that just like $\frac{\prod_{i=1}^d (z - P_i)}{\prod_{j=1}^d (z - Q_j)}$ is meromorphic on $\widehat{\mathbb{C}}$ with zeros P_i and poles Q_j–their number with multiplicity must be the same—*unless*

$d < 2g$, for any disjoint d-multisets $\{P_i\}$ and $\{Q_j\}$ of X a meromorphic function on X having precisely these zeros and poles is $\frac{\prod_{i=1}^{d} \theta(e_1+\int_{P_i}^{z} \omega_1,\ldots,e_g+\int_{P_i}^{z} \omega_g)}{\prod_{j=1}^{d} \theta(e_1+\int_{Q_j}^{z} \omega_1,\ldots,e_g+\int_{Q_j}^{z} \omega_g)}$ *for some zero $(e_1, \ldots, e_g) \in C^g$ of θ* :- That $\theta(\int_x^y \omega) \equiv 0\ \forall y$ can happen for some x was noted in VI, and XI shows this can't happen for all x. Using this, for the finitely many given points, there does exist a $\theta(e) = 0$ such that neither the numerator nor the denominator of the expression is identically zero. So it is meromorphic and has indicated zeros and poles. Also, if the paths of integration are chosen with due care, it is $2g$-periodic, i.e., it is the pull-back of a meromorphic function on $\mathbb{J}$ uniquely extending its restriction on X. That this function has no other zeros and poles on X follows by checking that each factor of this alternating product does have other zeros, but always the same $2g - 2$ points, so they cancel out. □

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The above theorem was found by Abel, who avoided complex integrals. By Riemann's time, Cauchy-Liouville theory was in place, to which he added an existential insight about Laplace's equation. This Dirichlet Principle underpins his discovery of the Riemann Mapping Theorem in I, and is used for Riemann-Roch as well. As frequently happens with a new idea, it was flawed – his rival Weierstrass gave counterexamples to this principle as stated by him – but as happens just as often, if there is a cartesian naturality in the idea, many paths around these flaws slowly emerged, leading on to the discoveries of ... Lefschetz, Hodge, Hirzebruch, Atiyah, Grothendieck ...

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We recall FTA says all d-multisets, or *divisors* of positive degree d, of any X make a closed $2d$-manifold $\mathrm{Sym}^d(X)$. □ [A new NB from 21/06/21 post Mr π; should speak Riemannese of VI à la Abel addition in Goursat II; Lahore unlike filmi run was before Rome, led to Milkha book, no 1J 16A but daily Delhi drivel, older by ਪਾਸ਼ even less; yes ਦਵਾਈ ਵੀ ਅਤੇ ਕੜਾਈ ਵੀ on vaccine certificates is not Punjabi; on 27th K told he's working on Equivelar.]

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An exciting higher dimensional possibility :- Our method generalizes at least to any n-cube (or its dual polytope) inscribed in the unit $S^{n-1} \subset \mathbb{R}^n$. It tiles the n-ball of infinite radius $\mathbb{R}^n$ and has *sum of the solid n-angles at all vertices* already $vol(S^{n-1})$. Anyway, much as before, we bend the facets of this n-cube equally inwards, so at any time they lie on spheres hitting a concentric S_c^{n-1} of radius $c \geq 1$ normally, to continuously decrease the sum of the solid angles at all vertices of this now curved n-cube from that value at $c = \infty$ to zero at $c = 1$. Of particular interest are the c's for which this sum is a *rational,* especially, *an integral prime p divisor of $vol(S^{n-1})$*. Then it generates by glide half-rotations in the geometry of the ball B_c^n of radius c, *a p-fold curved tiling of this open n-ball branched only at vertices,* with quotient, *a closed hyperbolic n-manifold.* A construction of such manifolds so much more vivid than of Hyperbolic manifolds that it should be easy to spot among these many that are *almost parallelizable,* thus by-passing the étale route via Deligne-Sullivan (1975) to satisfy this extra condition. We recall that it was using the existence of such hyperbolic manifolds that Sullivan (1977) saw [2] by a relativistic version of a Pontryagin reminiscent toral argument of Kirby that, *outside dimension four, any topological n-manifold has a unique Lipschitz conjecture.* □

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Goursat closes §101 thus: "The study of these integrals is made very easy by the aid of plane surfaces composed of several sheets, called *Riemann surfaces.* We shall not have occasion to consider them here. We shall only give, on account of its thoroughly elementary character, the demonstration of a fundamental theorem, discovered by Abel." The *'very easy'* shows at his level Riemann's VI was thoroughly absorbed by then, and his §102 on this *'thoroughly elementary'* addition theorem of Abelian integrals exemplifies the easy now (yet rigorous, but never as in rigor mortis) style of this masterly text. Because of the regression brought about by our subservience to set-theoretic language, I don't think (the original) Riemann is well understood now.

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Lest we forget, the *space of all* (polynomial) *equations* $\odot$ over reals with its *stratifications* by degree, number and multiplicity of (real) roots–that led us to use Thom's usage *swallowtail* profusely–is the natural object.[3] Focussing on a degree d, using an ∞ to compactify, and (reluctantly) two-dimensional numbers for extra room, e.g., FTA, mere means towards an end: *a better understanding of this cartesian thing!* This reminder made, we note each equation $\odot$ in the top most stratum of the compactified degree d complex swallowtail $\heartsuit$ has its *Riemann surface* $S(\odot)$: the graph of the algebraic function y of x defined by this degree d homogenous equation $F(x, y) = 0$ sharing no roots with its derivative with respect to x. *Abel's addition theorem gives us something about $S(\odot)$ that is constant as $\odot$ dances around in this space.*

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About 5 years after conjecturing it, I circulated a tantalizing "proof" of the higher Heawood inequality in Shifting & embeddability (1988), but withdrew it

[2] Section (2.10) of Gromov's *Partial differential relations* (1986) begins with these isometric but far from smooth Nash immersions of almost parallelizable manifolds, and ends with Cauchy rigid embeddings of minimally triangulated simplicial manifolds; this last connection was also independently seen by Kalai (1987).

[3] This ball was set rolling by ਬੀਜ ਗਣਿਤ ਦਾ ਮੂਲ ਮਸਲਾ (2017) wherein is shown: *the equations we used to solve in school using the quadratic formula form a Möbius strip!* This pretty result was my gift to my wife on 18/04/23, so it's now Minni's theorem; by then I'd also made a shahmukhi transliteration of sorts of its statement and proof, using a qaida which had been her gift to me a week before on 11/04/23.

after acceptance from *Topology.* It had made quite an impact on Gil Kalai, who claiming a bit more, had then used a pretty argument to “show” the g-conjecture for simplicial spheres.[4] Of which he informed me excitedly. Alas! my return email bore my misgivings. Soon after, when with my family I was for a few days near San Francisco, I met him for the first time. This interaction continued by mail, as we proceeded to IHES for a month, then an academic year in MPI, including a month away at Djursholm, where I met Gil again. Just to break the ice, I told him a short proof of Tverberg's theorem from Linear embeddability (1991)—notes on many things, old and new, that I had been typing: my paper with Brehm showing, contrary to what Grünbaum had thought, how much more restrictive this was than piecewise linear embeddability of simplicial complexes is also from about then. I had no idea he would like it so much! It was soon a separate short paper, that appeared in his journal, and much later in 2008, in his own words—but adorned with a photo of mine!—on his popular blog. The more serious job of reviving the alluring argument of S&E was never far from my and I suspect his thoughts, for example, I talked on my last day in MLI on shifting (many variants were being tried) and a bit of this is in a Bonn talk too. In 1994 Gil invited me for a very pleasant month in Jerusalem, where we two had a good go at it again, but that ‘just a bit more’ remained elusive. Soon after Gil included this long conjectured inequality in his ICM talk. I've often enough mused about it in these writings, and told you that in 2018 it was finally proved for piecewise linear embeddings—so possibly it is false for some wild embeddings—by Karim Adiprasito :- as such this medal-worthy paper takes a different tack, but has in it some neat generic linear algebra arguments, including one he later found was known to Kronecker (1890), which suggest that the fish can also be reeled in–maybe even before the next ICM rolls around!–in a more cartesian manner using shifting. $\square$

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Liouville showed (1) a doubly-periodic (meromorphic) function with no poles is constant, others (2) have more than one pole in any tile with (3) same number of zeros, also (4) sum of zeros $\equiv$ sum of poles, and (5) two such function with same periods satisfy a polynomial equation:- For (2), (3) use $\frac{1}{2\pi\mathbf{i}}\oint f(z)dz = 0$, $\frac{1}{2\pi\mathbf{i}}\oint \frac{f'(z)}{f(z)}dz = 0$ on boundary (make symmetric detours around poles on it if any) of tile, for the contributions of opposite edges cancel, so invoking *Cauchy* the residues of $f(z)$, $\frac{f'(z)}{f(z)}$ in tile sum to zero. Then (1): if $f(z)$ has no poles, nor has $f(z)-f(z_0)$ but has zeros. For (4) note $\frac{1}{2\pi\mathbf{i}}\oint z\frac{f'(z)}{f(z)}dz$ is integral combination

[4]Conversely, in Gil Kalai's *The diameters of graphs of convex polytopes and f-vector theory* (1991), there is a reformulation – for my exposition see Reviews I, pages 19-22 – of the by then proved g-conjecture for simplicial polytopes in the language of commutative shifting, to prove *the Heawood inequality for subcomplexes K of simplicial polytopes!* Over the complex numbers forms with polynomial coefficients sit in smooth Thom forms on the simplicial complex, so building a Lefschetz-Kähler-Hodge theory in this setting was indicated from the outset as a promising path of attack for all simplicial spheres. However complexification also washes out much, for example swallowtails, so most cartesian and much more vivid would be to establish directly this real reformulation of the g-conjecture using deleted joins and equivariant shifting, without ever resorting to complex numbers.

of the periods by which z differs on opposite edges. For (5) that polynomial $P(f(z), g(z))$ in the functions has same periodicities, and for degree big its coefficients can be so chosen that it has no poles: because these can be only those of $f(z)$ or $g(z)$, and if homogeneous linear equations outnumber unknowns there is a nonzero solution. $\Box$

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Generating all doubly-periodic (meromorphic) functions using biquadratics from *Jacobi*'s elliptic functions is then what *Briot et Bouquet* do; later texts, e.g., *Goursat* and *Ahlfors*, prefer cubics and *Weierstrass*'s $\wp(z)$. Save (4) the results above hold also relativistically, i.e., for the multi-periodic automorphic functions of *Poincaré* on an open disk of radius $c < \infty$:- same proofs (for we avoided using the non-relativistic result[5] that a bounded entire function is constant). $\Box$ We'll deal mainly with those tied to its half-turn tilings.

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A *saral parcha* on Jordan's solution of generic polynomial equations, and its analogue with compact tiles, entails shifting the hobson miscellany for what we need. For another, an equivariant shifting cartesian proof of the higher Heawood inequality, we'll do the same for the generic linear algebra in papers from S&E (1988) to Adiprasito (2018).[6]

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There was an equation for a complex *horizon* of n cyclically ordered distinct complex numbers z_j with $u_j = \frac{z_{j+1}}{z_j}$, viz., $\sum_m E_{2m+1}(\frac{c^2-1}{c^2+1} \cdot \frac{u_j+1}{u_j-1}) = 0$, where $E_t(\ )$ is the elementary degree t symmetric function of n quantities, which we developed in the hobson miscellany. From the condition for the *seminal case,* z_j in order on the unit circle, that the sum of the n angles between the circular arcs $z_j z_{j+1}$ normal to the concentric circle of radius c, be an integer times 2π. This is true for $c = 1$ when angles are zero, and if n is even for $c = \infty$. Are one or both of these solutions tied to a Riemann theta function of n variables? On which hinges Jordan's theorem as to how the z_j can be written in terms of their elementary symmetric functions, made explicit by Thomae and Umemura. So we'll first look at the question just posed. The other horizons $c > 1$ for the seminal case give compact possibly branched half-turn tilings of the concentric disk of this radius. Using the relativistic theta functions that enabled Poincaré in 1881 to make automorphic functions, there ought to be similar solutions of

[5] This strengthening of (1) for $c = \infty$ was aired by Cauchy before Liouville. How extremely non-relativistic this stronger property is was "shown" – a dodgy Dirichlet principle took half a century more before it was ach so! – by Riemann: there is a holomorphic bijection between the open disk and any proper open simply connected subset of $\mathbb{C}$. This holomorphic flexibility of the finite open disk is in play also in our method of solving any equation as it moves from the seminal to the fully generic case.

[6] On back-burner for now is a third ਸਰਲ ਪਰਚਾ on cartesian birth from relativistic motion of any Lipschitz manifold, with a cartesian version of Sullivan's proof that outside dimension four all manifolds have a unique Lipschitz structure.

generic polynomial equations for these. The horizons $0 < c < 1$ give tilings of the disk complement, so something mod 2 is also afoot.

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A power series $\sum_{-\infty}^{+\infty} a_n u^n$ valid in an annulus around the origin unwraps under $u = e^{\frac{2\pi \mathbf{i}}{\omega} z}$ to a trigonometric series of a function with period ω analytic on an infinite strip. □ So complex integration helped, but for more on multi-periodic functions, for example their existence via thetas, we need to mess around like *Abel* say for starters with *Fourier*'s series.

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In S&E (1988) we used Kalai's shifting of generic forms, made equivariant, and van Kampen's obstruction to embeddability of a *simplicial complex* K^n *in* $\mathbb{C}^n$, as in Wu. We cited Sullivan's rational homotopical computations with generic forms, but now obstruction is mod two. So maybe—and this seems the way of choice if the higher Heawood inequality is true even for all topological embeddings—*just shift the deleted join in some big characteristic two field?* As is suggested by APP (2021) which also traces back to Kronecker that generic argument–like that hoped-for disjointness preserving equivariant shifting in the big field–in Adiprasito (2018). This and matroidal AHK (2015) mimic Kähler package, so *maybe shifting holomorphic forms near embedded complex is more conceptual?* This package is Poincaré duality, as honed by Lefschetz, with Hodge relations on the bilinear form. For complex dimension one, note this is nothing but symmetry and positivity of the period matrix Ω that goes into the definition of Riemann's theta function. So all this too goes back to his VI + XI + XXXI, inclusive of mod two Arf aspect, which we saw, in the last installment of this miscellany, was the algebraical key turned by Pontryagin and Rokhlin to probe from within what manifolds really are. □

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This 1890 paper of Kronecker, see Werke Band 3-2, was on the classification of pairs so a pencil of (f.d.) bilinear forms, that is, a projective analogue to that of linear maps in Jordan's 1870 book, which ties to that of abelian groups, see e.g., Herstein. An update by Zavadskij (2007) on Kronecker's problem helped, also musing on Zoltek and positively curved (1986) led to the winding number argument of Milnor for reducing a pair of bilinear forms in Greub's book, then how a question of Dinesh much later had led to some infinite dimensional (2006) (im)possibilities. Wedderburn's book on matrices which follows another note in Band 3-2 is perhaps best for Kronecker's reduction. That equivariant cartesian shifting is like reduction of one symmetric bilinear form, generically containing Kuratowski complexes (1991), thanks to a perturbation lemma like the one used by AKZ (2021) : *if* $\beta(\ker\alpha) \cap \mathrm{im}\alpha = 0$ *then any generic linear combination of linear maps* α *and* β *has a bigger rank.* □ In this paper Kronecker's canonical forms are in fact used to prove this over any infinite field, algebraic analogues of the Heawood inequality conjectured for multilinear forms, and some proved for binary and ternary forms.

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Note hypothesis of this lemma may not hold if α and β are interchanged, but Adiprasito's iteration uses it with a Poincaré duality, which stems from piecewise linearity: then, see Akin (1975), a bigger simplicial complex L triangulates $\widehat{\mathbb{C}^n}$ and it is this simplicial sphere which is examined. $\Box$ Our equivariant shifting will otoh use just one symmetric bilinear form over a field of characteristic two, so big that Heawood inequality false for K^n implies generically a $(2n+1)$-dimensional pseudomanifold, so a Kuratowski subcomplex, etc.

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S&E (1988) has §1 Introduction, §2 Shifting, §3 Embeddability, then thanks to Björner for sending his paper [4] with Kalai, and to Adler for a conversation, and ends with references [1]-[30], of which six are to my own work including Kuratowski Complexes, also written at Barrick Street.

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§2 begins by noting that oriented simplicial coboundary of K stays of order 2 if omission of any vertex v goes with multiplication by $\omega(v) \in \mathbb{F}$ instead of 1 of the field, and has cohomology isomorphic to $H^*(K)$ if all $\omega(v) \neq 0$. $\Box$ Which is dubbed ellipticity of the 1-form ω on the free vector space V over all vertices, because of an analogy with a symbol sequence of the de Rham d, when however K is a closed simplex and this sequence is exact.

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So the convex hull of the vertices of any simplex $\sigma \neq \emptyset$ lies on the hyperplane sum of all coordinates 1, also deem $\emptyset$ this sum function. Any oriented σ as the constant $|\sigma|$-form which is ± 1 on its vertices, but 0 if any different, makes the oriented cochain complex $(C^*(K), \delta)$ a quotient subalgebra $(\Omega^{*+1}(K), d)$ of the DGA of all smooth forms on V. On it d is wedge product with the 1-form which maps vertices to 1, instead we can use any elliptic ω. $\Box$ We note this is not cup product, p-cochain $\wedge$ q-cochain is a $(p+q+1)$-cochain, and recall a twisted d_ω was Witten's way [7] into Atiyah-Singer index theory.

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I'd mused here on a small canonical quotient subalgebra of this DGA that gives not only the cup product but, à la Sullivan, the homotopy type of K over

[7] Indeed *shifting belongs to the discrete aspect of Morse theory* : any partial order of vertices which totally orders all simplices extrapolates simplexwise linearly to a real map ξ with only nondegenerate critical points at some vertices v, namely those with $a(v, \xi) \neq 0$ in the notation of Banchoff (1967); so any total order of the vertices of K and the induced partial order of the vertices of K_* can be construed as Morse functions on $|K|$ and $|K_*|$; but such a total order is precisely the tool we need to do lexicographic sieving etc.

char($\mathbb{F}$), but don't remember much :- that of all smooth forms on V zero outside a tubular neighbourhood of K does the job, but one can even from the DGA of all forms on V with polynomial coefficients, find small examples by imposing other conditions depending on K. $\square$

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Our $d_\omega = \omega\wedge$ is d conjugated by the algebra isomorphism of $\Omega^\star(K)$ generated by perturbing the vertices v in $V \setminus \{0\}$ to $\omega(v)v$ on their own axes. The kernel and image of d_ω are preserved by many linear isomorphisms, notably, the graded linear basis $B(K)$ left after lexicographically sieving, from the power set of any basis $\omega, \omega_2, \omega_3, \dots$ of V, all sets which as forms of $\Omega^*(K)$ linearly depend on the preceding, is a simplicial complex with the same cohomology:- If oriented set σ is linear combination of preceding, $\wedge$ with $\theta \setminus \sigma$ shows so is bigger set θ; also we have the linear bijection $L : \Omega^*(K) \to \Omega^*(B)$ commuting with d_ω's mapping $\sigma \in B$ to $\sum_{\theta<\sigma} c_\theta \theta + \sigma$ where $\omega \wedge \theta \in B, \omega \wedge \sigma = \omega \wedge \sum_{\theta \leq \sigma} c_\theta \theta$.

In above lower triangular recipe it is clear when a form is considered in the old quotient $\Omega^*(K)$ of all constant forms Ω^* on V, and when in its new quotient $\Omega^*(B)$. Being just its first vertex ω is not elliptic for B but gives a Hodge basis: $St_B\omega$ is a basis of $\mathrm{im}(d_\omega)$ on which $Lk_B\omega$ maps bijectively under d_ω, so rest β give us a basis of $\ker(d_\omega)/\mathrm{im}(d_\omega)$; also if $\omega_j \in \beta \notin \overline{St}_B\omega$ then $\omega \cup (\beta \setminus \omega_j) \in B$:- for if an l.c. in $\Omega^*(K)$ of preceding part of $St_B\omega$ inner product with ω contradicts $\beta \setminus \omega_j \in B$. That is, $Ud_\omega = dU$ holds for the upper triangular linear bijection U of $\Omega^*(B)$, which keeps simplices of subcomplex $[B \setminus \omega]$ fixed, and maps any $\sigma \in St_B\omega$ to $d(\sigma \setminus \omega) = \sigma + \sum_j \omega_j \cup (\sigma \setminus \omega)$. $\square$

For example, for any $\mathbb{F}$, replacing a basis of vertices $v_1, v_2, \dots, v_N$ of K with the new basis $v_1 + \cdots + v_N, v_2, \dots, v_N$ of V gives such a 'near-cone' B, also for this elliptic $\omega = \omega_1$ no initial conjugation is needed.

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If $\mathbb{F}$ is big we can choose ω_j to make near-cone B still simpler, the point being that any field automorphism π of $\mathbb{F}$ extends to a differential graded algebra automorphism of $\Omega^*(K)$ linear over its fixed subfield which keeps v_j and so all of K fixed, but can move and even permute linear combinations. For example, if we choose ω_j on the moment curve $v_1t + v_2t^2 + \cdots + v_Nt^N$ for values $t_1, t_2, \dots, t_N \in \mathbb{F}$ permutable in all possible ways by field automorphisms, then B is closed with respect to replacement by *any* smaller vertex:-

If $\omega_{j_1} < \cdots < \omega_{j_r}$ is dominated by $\omega_{k_1} < \cdots < \omega_{k_r}$ and $\omega_{j_1} \wedge \cdots \wedge \omega_{j_r}$ is a linear combination of preceding forms an automorphism π of $\Omega^*(K)$ strictly increasing from $\{\omega_{j_1}, \dots, \omega_{j_r}\}$ and its complement onto $\{\omega_{k_1}, \dots, \omega_{k_r}\}$ and its complement shows that $\omega_{k_1} \wedge \cdots \wedge \omega_{k_r}$ is also such a combination. $\square$ Also, since replacement by ω_1 is always on, this 'shifted' conclusion is true even if $t_1 = 1$ and $t_2, \dots, t_N$ are permutable in all ways by field automorphisms.

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The obstruction to $K^n \hookrightarrow \mathbb{C}^n$ lives in the cochain complex of its deleted join $K * \overline{K}$ equipped with conjugation, which has the following lexicographically first $\mathbb{Z}/2$-bases beneath the octahedral sphere $\Delta * \overline{\Delta}$ with simplices $\alpha \cup \overline{\beta}$ for *all* disjoint pairs $(\alpha, \beta), |\alpha| = p, |\beta| = q$ of subsets of vertices of K:- We perturb a canonical basis of vertices $v_1, v_2, \ldots, v_N$ of V to $\omega_1, \omega_2, \ldots, \omega_N$, and of conjugate vertices of $\overline{V}$ to its conjugate $\overline{\omega}_1, \overline{\omega}_2, \ldots, \overline{\omega}_N$. Then, *all increasing sequences of ω's with p bare and q overlined,* as forms on $V \oplus \overline{V}$, descend to a basis of $\Omega^{p,q}(\Delta * \overline{\Delta})$, so a spanning set of its quotient $\Omega^{p,q}(K * \overline{K})$ which is nonzero only if $p, q \leq n+1$. And, increasing sequences of v's of the same type, i.e., overlined at same spots, give direct sum decompositions of $\Omega^{p,q}(K * \overline{K})$, e.g., $\Omega^{2,3}(K * \overline{K})$ may have a nonzero summand $\Omega^{11010}(K * \overline{K})$, to which length five ω-words with the first two and the fourth letter overlined shall descend as a spanning set. So, sieving out in each type forms depending linearly on the preceding, gives a new $\mathbb{Z}/2$-basis $B(K * \overline{K})$ of $\Omega(K * \overline{K})$. $\square$ Likewise $\mathbb{Z}/2$-bases for the cochain complex of any $\mathbb{Z}/2$-subcomplex of the octahedral sphere.

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The argument we used before to show closure under inclusion doesn't work now, e.g., $B(K * \overline{K})$ may well contain $\omega_3 \wedge \overline{\omega}_4 \wedge \overline{\omega}_6 \wedge \omega_7$ of type 0110 but not $\omega_3 \wedge \omega_7$ when the latter depends on preceding forms amongst them $\omega_3 \wedge \omega_5$, for wedge of this form with $\overline{\omega}_4 \wedge \overline{\omega}_6$ has another type 0101; indeed $B(K * \overline{K})$ is seldom a simplicial complex:- when it is one then $B(K * \overline{K}) \subseteq B * \overline{B}$ because $B = B(K)$ and $B(\overline{K}) = \overline{B}$ are in it; but a generic $B * \overline{B}$ is usually smaller in size than $K * \overline{K}$, for example K a square has as its B a triangle with a fourth edge $\omega_1\omega_4$ attached to its least vertex. $\square$

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However for $\mathbb{F}$ big, $\omega_1, \ldots, \omega_N$ a generic basis of V as above, and $\overline{\omega}_1, \ldots \overline{\omega}_N$ the conjugate basis of $\overline{V}$, an action of $\mathrm{Aut}(\mathbb{F})$ shows that, any ω-word dominated by a word of the same (length and) type in $B(K * \overline{K})$ is also in it:- for the permutation π taking the increasing sequence of letters–bare or overlined ω's–in the dominated word termwise on the dominating, and letters not in the first in an increasing way on those not in the second, can be effected by a type preserving algebra automorphism π over a subfield of $\Omega(K * \overline{K})$ which keeps all of $K * \overline{K}$ eerily[8] fixed and commutes with conjugation. $\square$ But replacing an ω by any smaller is now different, can change type, and is not on.

We emphasize our $\mathbb{Z}/2$-typed fine grading of $\Omega(K * \overline{K})$ depends heavily on the basis, i.e., the order of vertices $v_1, \ldots, v_N$, which stay put under this action of $\mathrm{Aut}(\mathbb{F})$ on the coefficients of linear combinations of the typed v-words. The quotient subset of ω-words of its type in a summand is not preserved, but under above shuffles π those lexicographically preceding a dominated word are mapped into those preceding the given word.

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[8]For how uncannily useful such totally discontinuous Galois symmetries can be to grasp things quite continuous, Sullivan's MIT Notes (1970) are still best.

That $K^n \hookrightarrow \mathbb{C}^n$ implies Heawood's inequality $f_n(K) < (n+2)f_{n-1}(K)$, where f_i means the number of i-simplices of, is strongly suggested by the fact that, if K does not satisfy this inequality then its generic B contains the n-skeleton P of the simplex $\{\omega_1, \ldots, \omega_{2n+3}\}$:- if all n-simplices of B have smallest vertex in $\{\omega_1, \ldots, \omega_{n+2}\}$ deleting it counts a subset of $(n-1)$-simplices of B not containing ω_1, none counted more than $n+2$ times, so $f_n(B) < (n+2)f_{n-1}(B)$; so there must be an n-simplex in B with first vertex at least ω_{n+3}, but all n-simplices of the above $(2n+2)$-simplex are dominated by it. $\square$ We recall P^n is non-embeddable in $\mathbb{C}^n$:- which Flores showed by checking that $P * \overline{P}$ is an S^{2n+1} with antipodal action, which makes it, since there does not exist a continuous $\mathbb{Z}/2$-map $S^{2n+1} \to S^{2n}$, an obstruction to embeddability because $K^n \hookrightarrow \mathbb{C}^n$ implies there exists a $\mathbb{Z}/2$-map $K * \overline{K} \to S^{2n}$. $\square$

Were generic $B * \overline{B}$ smaller in size than $K * \overline{K}$ because in fact $B * \overline{B} \subseteq B(K * \overline{K})$ is always true–this however is moot–then à fortiori we would get an obstruction $P * \overline{P} \subseteq B(K * \overline{K})$ from the negated Heawood inequality (NHI), so we'll seek a direct counting argument generalizing above.

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Facets opposite smallest vertices of simplices of $B(K * \overline{K})$ are also in it:- for, replacing smallest vertex by ω_1 or $\overline{\omega}_1$ gives a dominated word of the same type, but if facet were a linear combination of preceding words of its type, $\wedge$ with ω_1 or $\overline{\omega}_1$ would show this word is not in $B(K * \overline{K})$. $\square$ So, if $f_\tau(K * \overline{K}) \geq (2n+3-\ell(\tau))f_{\tau_1}(K * \overline{K})$, where f_τ is the number of simplices of type τ of length $\ell(\tau)$, and τ_1 the type obtained by erasing τ's first 0 or 1, then all type τ simplices of $P * \overline{P}$ are in $B(K * \overline{K})$:- the argument above using the NHI, i.e., the case $\tau = 00\ldots0$ of length $n+1$, works in general. $\square$

So for any K^n, with an ordering of its N vertices, we have now in hand some numerical inequalities—one of them the NHI—which ensure the presence of the obstructing $\mathbb{Z}/2$-sphere $P * \overline{P}$ in the generic $B(K * \overline{K})$.

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In general the other inequalities don't follow but, if we discard from a K^n satisfying the NHI all simplices not incident to any $(n-1)$-simplex then $f_j(K) \geq (2n+2-j)f_{j-1}(K) \, \forall j \leq n$:- $(j+1)f_j = \sum_\theta \mathrm{val}(\theta)$ over all $(j-1)$-simplices θ; so $\geq \sum_\theta n - j + \mathrm{val}(\sigma(\theta))$ for any choice of $(n-1)$-simplex $\sigma(\theta)$ containing θ; if we choose to maximise valence then $\geq (n-j+(n+1)(n+2))f_{j-1}$ because NHI says average valence of an $(n-1)$-simplex is $\geq (n+2)(n+1)$; but $n-j+(n+1)(n+2) \geq (j+1)(2n+2-j) \forall j \leq n$. $\square$ So indeed, $f_{a,b} \geq (2n+3-a-b)(f_{a-1,b} \text{ or } f_{a,b-1})$ where $f_{p,q} = \dim \Omega^{p,q}(K * \overline{K})$:- The average number of times a $(j-1)$-simplex of K occurs as the facet opposite the first (or second or ... last) vertex of its j-simplices is $\frac{f_j(K)}{f_{j-1}(K)} \geq 2n+2-j$; for a simplex of $K * \overline{K}$ having a un- and b over-lined vertices knocking out the first from former reduces the 'uns' to $a-1$ and from latter the 'overs' to $b-1$; and conversely for any such simplex the uns respectively the overs can be augmented in the beginning by on an average at least $2n+3-a$ respectively $2n+3-b$ choices of an un respectively over vertex; of which choices at most b respectively a are banned by disjointness of uns from overs, so $\geq 2n+3-a-b$ choices remain. $\square$ A stronger inequality with *or* replaced by $+$ is however implied by the type inequalities.

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We can streamline K more:- if $f_n > (n+2)f_{n-1}$ just knock out an n-simplex, if any $(n-1)$-simplex has valence less than $n+3$ erase its star, and demand that the complement of $(n-2)$-skeleton be connected, i.e., use a *minimal* K satisfying the NHI. $\square$ However it seems the *choice of vertex order* matters and should be involved, and a way is suggested by the interpretation above of $\frac{f_n(K)}{f_{n-1}(K)}$ as the average frequency of an $(n-1)$-simplex as the *first facet,* i.e., that obtained by deleting their first vertex, of n-simplices:- from the top simplices of $K * \overline{K}$–on which we assume the inequalities for all types τ with $a = b = n+1$–downwards, *we strip off all simplices that do not occur as first facets,* then down from these leaves along the first facet path on the concomitant *binary tree of types* the lower typed inequalities follow. $\square$ Leaving thus only these top typed inequalities to be dealt with; till then there is a chance that these numerical conditions are strictly stronger than the NHI, but even then note we are now dealing with topological non-embeddability, not just piecewise linear, and our approach is ways more direct than via the Kähler package.

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Heawood's inequality is by no means sufficient for embeddability in double dimensional space:- the latter is unaffected by replacing K by its derived, with simplices its $\subset$ chains, but for $n \geq 2$ deleting the topmost barycentre shows now $f_n < f_{n-1}$ and for derived graphs a factor 2 on right will do. Even our critically non-embeddable complexes, i.e., joins $\prod_i P^{n_i}$, satisfy HI: if an $(n-1)$-simplex lacks a vertex of P^{n_i} it has valence $2n_i+2$, so average valence $\frac{(n+1)f_n}{f_{n-1}} \leq 2n+2$, i.e., $f_n \leq \frac{2n+2}{n+1} f_{n-1} < (n+2)f_{n-1}$. $\square$

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A really pleasing weak generalization of the Pontryagin-Kuratowski theorem would be this:- just as in the deleted join of a non-planar graph $|P^0P^0|$ or $|P^1|$ give us 3-spheres, a 'refined' equivariant shifting should enable us, for $n > 2$, to characterize $K^n \hookrightarrow \mathbb{C}^n$ by the *generic absence* in $K * \overline{K}$ of any $\prod_i P^{n_i} * \overline{\prod_i P^{n_i}}$, which are all antipodal $(2n+1)$-spheres. $\square$

The 16-year old blind prodigy's graph theoretic result was <u>not</u> isolated from the characteristic classes which he and others in his train discovered: he switched from homology to cohomology to avoid singularities of obstructing cycles, but for a clear grasp of say Pontryagin numbers we should perhaps now switch back to homology and examine generic obstructing cycles.

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The particular characteristic class detected by the above obstructing $\mathbb{Z}/2$-spheres–note they are as many as the number of partitions of $n+1$–is that of van Kampen and vanishes for manifolds: it is the first step in the difficult problem of characterizing when K^n is an n-manifold. Anyway this first step was itself considerable: its vanishing is not only necessary but, outside dimension $n = 2$, sufficient for embeddability of K^n in $\mathbb{C}^n$. That 'refinement' we seek should not destroy any of these $\pi(n+1)$ obstructions, so genericity has to be controlled: so maybe shifting equivariantly the deleted join of the derived, using somehow its canonical vertex colouring by the $n+1$ weights of the barycentres of K^n to exercise this needed control? □

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Deleted joins, typing, equivariant shifting, characteristic classes, etc., extend naturally[9] from $\mathbb{Z}/2$ to any group G, for example, if deleted join of $\{pt\}$ is G, then–join formula–that of the simplex with N vertices is the ambient N-fold join $G \cdot G \cdot \quad \cdot G$—ambient octahedral sphere for $G = \mathbb{Z}/2$—which for $N \to \infty$ has as its quotient Milnor's universal space BG: so developing the combinatorial ideas above for $G = S^1 \subset \mathbb{C}^\times$ and its finite subgroups should already give us a much better grasp of Pontryagin numbers. □

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However it is $\mathbb{Z}/2$ that is the most subtle, because of that example on which Rokhlin stumbled in pushing Pontryagin's picture of $\pi_4(S^2)$ to the first group $\pi_5(S^3)$ on its stem: the insufficiency, for $n = 2$ only, of van Kampen vanishing for $K^n \hookrightarrow \mathbb{C}^n$, and the mod 2 obstruction, for $n = 4$ only, to creating a manifold M^n by a relativistic cartesian motion, are both tied to it. □

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Should we call $\overline{K}$ the negative rather than the conjugate of K? So far it has been the latter: say conjugation in the 45 degree line, with K itself the $+1$ or real and $\overline{K}$ the $+i$ or imaginary copy amongst an S^1 worth of copies forming in $S^1 \cdot S^1 \cdot \quad \cdot S^1 \subset \mathbb{C}^N$ the S^1-deleted join of K whose tubular nhbd's Dolbeault diamonds $E_r^{p,q}(U)$ converge to its cohomology over $\mathbb{C}$. □

However this time we won't use conjugation, i.e., reflections of S^1, and $\overline{K}$ is say the $-1 \in S^1$ or negative copy of K, with $\mathbb{Z}/2$ the subgroup $\{\pm 1\}$ of S^1: so the real part of the S^1-deleted join of K is $K * \overline{K}$ in the octahedral $(N-1)$-sphere $\{\pm 1\} \cdot \{\pm 1\} \cdot \quad \cdot \{\pm 1\} \subset \mathbb{R}^N$. □ Also the age-old 'big' field of real numbers only will be used to shift this real part.

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Work out generic $B(K * \overline{K})$ for $K = 3pts \cdot \overline{3pts}$ type graded by two different orders of V. Recall colored shifting and that it works for joins, but bicolored

[9] I've played with this basic idea in many papers starting with the third of my 1988-9 trilogy, and these should evolve into a beautiful theory.

colored shifting of deleted joins, e.g., $3pts * \overline{3pts}$, is a fatter complex see S&E. Otoh equivariant is same size though not complex, but for deleted joins of P^n's, e.g., for $3pts$, keeps them fixed. Under same sieving using partial order definition we should maybe expect $n+1$ colored K' to become a bigger complex, does it? What happens if we totally order vertices by concatenation, bearing color 0, then all colored 1, and so on? Of course just $B(K')$ now is ordinary shifted and since HI holds no Kuratowski complex can be expected in it. But we use now any such concatenated total order to define a finer type grading for $K' * \overline{K'}$ and equivariantly shift it. It is same size, is it now even an s.c.? Crazy confusing.

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Since type inequalities seem dependent on choice of vertex order, before we start looking for one which makes them true, we ought to be certain that their consequent inequalities $f_{a,b}(K * \overline{K}) \geq (2n+3-a-b)(f_{a-1,b}(K * \overline{K}) + f_{a,b-1}(K * \overline{K}))$, which certainly do not depend on vertex order, are true. For a K^n obeying the NHI and such that any simplex is incident to some $(n-1)$-simplex we checked them for cases $b = 0$. That is we know that the average valence of a cardinality $a-1$ simplex of K is $a(2n-3-a)$. Now note that any simplex with a un and b over-lined vertices has a facets with uns one less. To atone for the fact that one of these a facets itself has either itself a vertex that occurs over-lined in the second factor, or one of its incident cardinality a simplices of K has such a vertex, we need to subtract ab, thus giving us an inequality $af_{a,b} \geq a(2n+3-a-b)f_{a-1,b}$. Likewise we have the inequality $bf_{a,b} \geq b(2n+3-a-b)f_{a,b-1}$. hmm ... hmm ...

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The plan–p. 29–to scour 'papers from S&E (1988) to Adiprasito (2018)' was a non-starter: after reading a bit of S&E I put it aside too, and bad memory and all, decided that what was needed from the past shall come back slowly to me on its own! Soon after type knocked, I realized it was from the past, and taking time out, found, scanned and uploaded[10] this forgotten work: but it was two months later, on November 1, 2021, that I refreshed myself on what all is in the 67 pages of mli (1992) : wow (though I'm saying it myself)! My spur had been maybe I'll find the needed type inequalities there, so could save time: these however I didn't find. But it is a treasure trove of other ideas – especially in its §8 and §8bis, both quite long – and has pithy yet complete accounts of things due to others: like Kalai's pretty argument using (1) cohen-macauleyness of a shifted simplicial sphere and (2) the absence in it of $\sigma_t^t \sigma_{s-1}^{2s}, t+2s = m+1$ to prove the g-conjecture, etc. The attempt in §8 at HI used typing and forms only–just like now–and the concept later of a reduced deleted join seemed promising as a way to a generic generalized kuratowki theorem that doesn't mess around with the derived at all! There are lots of ?s though scattered. Which had led in §8bis, maybe to get around the point that type grading does not automatically tell us

[10] My scanner is slow too, a lot remains to be posted.

a spanning set is there in each type summand after moving to a new basis from vertices, to : a star algebra which seems defined with very good motivation, and (imnsho) just all those coboundaries show virtuosity! Its natural flow however leads us to a cyclic cohomology.[11]

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Type-shifting over $\mathbb{R}$ evokes a continuous finale using: embedding of $|K^n|$ in $2n$-space gives a continuous $\{\pm 1\}$-map from $|K * \overline{K}|$ to $(2n+1)$-space minus 0:- If f embeds $|K|$ in the sum of coordinates one hyperplane $\mathbb{A}^{2n} \subset \mathbb{R}^{2n+1}$ then $f \cdot \overline{f}$ embeds $|K \cdot \overline{K}|$ in $\mathbb{A}^{2n} \cdot \overline{\mathbb{A}^{2n}} \subset \mathbb{A}^{4n+1} \subset \mathbb{R}^{4n+2} = \mathbb{R}^{2n+1} \times \overline{\mathbb{R}^{2n+1}}$ with only the diagonal going to the $(2n+1)$-dimensional diagonal subspace; so its restriction composed with the projection on the orthogonal subspace gives us the required $\{\pm 1\}$-map $F : |K * \overline{K}| \to \mathbb{E}^{2n+1} \setminus 0$. $\square$

Now $|K|$ sits in the hyperplane sum of coefficients one of $V = \Omega^1(K)$, all real combinations of its vertices $v_1, \ldots, v_N$, which is also all real combinations of the nearby generic vertices $\omega_1, \ldots, \omega_N$ of the lexicographically first basis $|B(K)|$ of $\Omega(K)$ sitting in the analogous nearby hyperplane. Likewise $|K * \overline{K}|$ and the realization $|B(K * \overline{K})|$ of the typewise lexicographically first basis of $\Omega(K * \overline{K})$, obtained by sieving it under $\Omega(\Delta * \overline{\Delta})$, sit in nearby affine hyperplanes of $V \oplus \overline{V}$ minus the diagonal subspace.[12] This typewise bijection $K * \overline{K} \rightsquigarrow B(K * \overline{K})$ is very discontinuous for higher simplices, but it seems $|B(K * \overline{K})|$ pops up in an open set U of $V \oplus \overline{V}$ containing $|K * \overline{K}|$ to which the map F into $\mathbb{E}^{2n+1} \setminus 0$ extends equivariantly: otoh if (typed) NHI holds $B(K * \overline{K})$ contains an antipodal sphere of dimension $> 2n$. $\square$ For a 'pleasing pontryagin' or PP we need also that VKO is not killed and a variant of type-shifting may be necessary.

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Type-shifted $B(E)$ of any $\mathbb{Z}/2$-complex E with fixed subcomplex F, e.g., any Wu subdivision $W(K \cdot \overline{K})$:- If E is a subcomplex of $W(\Delta \cdot \overline{\Delta})$, i.e., $\Delta * \overline{\Delta}$ join a fixed copy $\dot{\Delta} \supseteq F$, the type τ of a simplex is the sequence from $\{+1, 0, -1\}$ which tells us whether its vertices in increasing order are positive, fixed, or negative; real[13] subspaces spanned by simplices of the same type give $\Omega(E) = \oplus_\tau \Omega_\tau(E)$; by continuity any perturbation of these vertex decomposable forms

[11] That talk–see p. 28–on my last day in MLI was from this paper–also 20 of its pages were circulated 'for the grudging admiration' of a handful of colleagues–but the month-long spell was broken a few hours later when someone carrying a suitcase for a friend walked away from a baggage belt in Bonn with that containing its diskette etcetra! By the time it found its way back, more than a month later, my mind had switched to other problems, as well as things non-mathematical, above all the murders-by-encounter of thousands in the Punjab to which we soon returned ... So I had gone to Jerusalem in 1994 too 'cold' but it started coming back and again I felt a conceptual understanding of shifting is the key–HI shall then fall out of its own–while Gil focussed more on that 'just a bit more'.

[12] This is true also for $B(K) * \overline{B(K)}$ but the conjecture which follows seems less likely for it; note also that any $K^n * \overline{K^n}$ can be made in $\mathbb{R}^{4n+3} \setminus 0$ and if $N \leq 2n+3$ even $\mathbb{R}^{2n+2} \setminus 0$, but what is the best function of N?

[13] Continuity can be used for completions $\mathbb{Q}_p$ of $\mathbb{Q}$ also, but the result is true even for a field of coefficients having a big enough degree over its prime subfield.

gives a graded basis; so *all* type τ perturbed-vertex decomposable forms give a spanning set of $\Omega_\tau(E)$; the lexicographically first bases $B_\tau(E)$ in these spanning sets are closed under domination if perturbation is generic over $\mathbb{Q}$. □

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All $(2n+1)$-simplices of types $\{\tau, \overline{\tau}\}$ define a mod 2 equivariant cohomology class of $K * \overline{K}$, alternating types give us van Kampen's characteristic class, the other classes are trivial:- put vertices on a monotone curve of $\mathbb{R}^{2n}$ to immerse K^n linearly in it and use: n-simplices with vertices from any $2n+2$ points on curve are on boundary of their octahedral hull, but for the two n-simplices with vertices alternating on curve, which intersect in it. □ Indeed $]\frac{m}{2}[$-simplices with vertices from any $N > m$ points on a monotone curve of $\mathbb{R}^m$ are on boundary of their cyclic polytopal hull, but for those cutting, i.e., having vertices alternating on curve with the vertices of some $[\frac{m}{2}]$-simplex; this gives the $]\frac{m}{2}[$-skeleton, so determines the simplicial structure[14] of this $(m-1)$-sphere:- for Dancis has shown this is so[15] even for all closed simplicial manifolds. □

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A simplicial complex with vertices ordered and closed under domination, i.e., a shifted complex B^n has nonzero van Kampen obstruction to embeddability in $\mathbb{R}^{2n}$ iff it contains one of the $\pi(n+1)$ joins $\prod_i P^{n_i-1}, \sum_i n_i = n+1$.

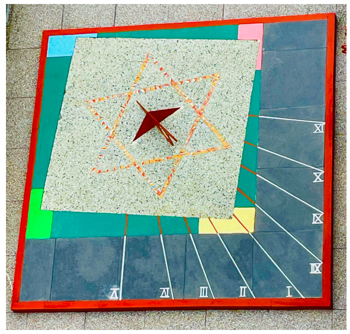
A shifted graph B^1 other than a subdivided edge uwv contains it only if it contains the unsubdivided edge uv:- for it is dominated by uw or wv unless w is smaller than both u and v, then replacing it in these two edges by the smallest vertex 1 we see $1u$ and $1v$ are in graph, which dominate $1w$, so it is an edge, which contradicts w has valence two unless $w = 1$ and graph is $u1v$. □ So, a nonplanar shifted graph contains one of the two graphs $K_5 = P^1$ and $K_{3,3} = P^0P^0$ without subdivision; alternatively this also follows, and generalizes to all n as sketched below, by examining $B * \overline{B}$ under VKO nonzero:-

Equivalently $B * \overline{B}$ has an equivariant $(2n+1)$-dimensional minimal mod 2 cycle—an even number of its simplices incident to any $2n$-simplex—connected by paths of $(2n+1)$-simplices sharing $2n$-simplices, which has an odd number of pairs of alternating $(2n+1)$-simplices; B being shifted any edge-path in its closure can be coned over a least vertex; this simple connectivity would allow us to choose a strictly smaller mod two cycle unless it is a pseudomanifold, that is, any $2n$-simplex is incident to either none or exactly two of its $(2n+1)$-simplices; which implies [16] that this subcomplex of $B * \overline{B}$ is one of the $\pi(n+1)$ antipodal $(2n+1)$-spheres $\prod_i P^{n_i} * \overline{\prod_i P^{n_i}}$. □ Also, all $\pi(n+1)$ complexes are needed, e.g., $\{12, 23, 13\}$ union $P^0P^0 = \{1,2,3\}\{4,5,6\}$, with this vertex order, is a shifted nonplanar B^1 not containing P^1.

[14] Cf. my omnibus paper, imho still the best entrée into this whole field.

[15] See LE, page 12 for a quick proof, and page 10 for Akin's theorem; mli, page 28 has a new proof $K \rightsquigarrow B(K)$ preserves cohen-macaulayness, which for $B(K)$ just says it's pure, and even the next result for a shifted B is stated on its page 9.

[16] See barrick one which underlines the special nature of deleted joins amongst all free $\mathbb{Z}/2$-simplicial complexes.



Our motif is that star of David, bounding its convex hull is the deleted join of $P^0 = \{1,2,3\}$, so join of 2 hexagons is that of $K_{3,3} = \{1,2,3\}\{4,5,6\}$: with this ordering of vertices $K_{3,3} * \overline{K_{3,3}}$ has 2×9 alternating tetrahedra, remaining 2×9 of the enclosed kind, e.g., $1\overline{24}5$ and conjugate, none of the third kind (with another vertex ordering this distribution can alter): it is not type-shifted and in trying to make it so we'll break its spherical 3-cycle: uncontrolled type-shifting is too drastic, it can make VKO zero.

In fact, only the irreducible antipodal sphere $P^n * \overline{P^n}$, atop the convex hull of the star of Flores—the union of two antipodal $(2n+2)$-simplices—is type-shifted : e.g., the 3-sphere $K_5 * \overline{K_5}$ has all five tetraheda of each of the six types, with deleted join of its polygons 123451 and 135241 conjugate pairs of solid tori, the latter containing all alternating tetrahedra.

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Since then Melikhov has also considered cell complexes whose deleted join is a $(2n+1)$-sphere, aiming maybe for, not just a generic, but a full classification of these obstructors in any $K * \overline{K}$: indeed Stiefel using barycentric derived had defined characteristic homology classes, but their dual classes were preferred by Whitney. Also: an $(n+1)$-manifold has $\pi(n+1)$ mod 2 characteristic numbers, and bounds iff they are all zero; characteristic classes are enumerated using cells of subspaces intersecting a fixed flag in the same way; oriented matroids are but arrangements of p.l. antipodal hyperspheres; allowing wild hyperspheres here may be tied to the subtleties unmasked by Pontryagin and Rokhlin's geometric method for $\pi_4(S^2)$ and $\pi_5(S^2)$.

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Post generic type-shifting a complex may no longer be closed under inclusion, and using its closure is artificial: is there a natural less drastic operation, for not all $\mathbb{Z}/2$-complexes, but just all deleted joins: a 'first' simplicial complex $P(K)$ on perturbed vertices such that $P(K) * \overline{P(K)}$ is a basis of $\Omega(K * \overline{K})$, for which the above $\pi(n+1)$ complexes are 'final' ?

For an n-pure K we could temper shifting thus: call vertices primary if they are first in some n-simplex, secondary if second in some n-simplex ... ; that the vertices of any 'final' have a non-decreasing $n+1$ colouring suggests we relist the primary vertices first, then secondary, and so on; then, in each dimension, sieve out simplices depending linearly on previous having the same colour sequence; over $\mathbb{R}$ all this is well-defined and extends to $\Omega(K * \overline{K})$ ...

This vertex colouring, obtained from any total order using purity, is hardly natural; as against this, ever since Poincaré used it to set up an up-down duality in manifolds, the barycentric derived, with its vertices coloured by dimensions, has played a key rôle in many contexts, including characteristic classes; here too the derived of $K \cdot \overline{K} \setminus$ diag seems useful. However any subdivision K' followed by fully generic shifting won't do: $B(K')$ is embeddable in $2n$-space for K 'final' but not the yin-yang[17] complex P^n:- e.g., if $B((K_{3,3})')$ were non-planar it would contain K_5 or $K_{3,3}$; the first subcomplex is ruled out because its Betti number $b_1(K_5) = 6$ is already bigger than $b_1(K_{3,3}) = 4$, which is preserved by both subdivision and shifting; the second because it itself is not shifted, and b_1 is bigger for its clusure under domination.□

Maybe sought 'partial shifting' $P(K')$ of the derived should not only preserve the Betti numbers of its deleted join but make it colour shifted ? That is, closed for simplices with lesser vertices of same colours; if so then output for any 'final' complex will contain the $n+1$ fold join of $P^0 = \{3\text{points}\}$, e.g.., $P((K_5)')$ with 5 perturbed vertices and 10 barycenters $a_1, \ldots, a_5, b_1, \ldots, b_{10}$ of K_5 being connected will contain a_5b_1, a_1b_{10} and so edges a_ib_1 and a_1b_j under them; then all a_ib_2 but we still have only 18 edges because a_1b_1 and a_1b_2 occured twice; this deficit of 2 is made up by the next edge a_2b_3 and then, skipping other a_2b_j for it decreases b_3 of the deleted join, the final edge a_3b_3; so it contains $K_{3,3} = \{a_1, a_2, a_3\} \cdot \{b_1, b_2, b_3\}$? But alas, this ain't so:- this graph has only 114 disjoint pairs of edges–list the 20 edges, count for each subsequent edges disjoint from it, and add–while $(K_5)'$ has 150. □

The constraint of a colouring, on top of all disjoint pairs remaining linearly independent was too much: it renders $(K_5)'$ rigid; but a $P((K_5)')$ without it contains K_5 on its first 5 vertices plus stuff featuring the last 10; the 'need' of a colouring, artificial or not, was a red herring:- we just forgot that an n-complex, whose deleted join is a $(2n+1)$-pseudomanifold, is automatically one of the $\pi(n+1)$ 'final' complexes. □

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{All simplicial complexes L on the generic perturbed vertices such that $L * \overline{L}$ is a bigraded basis of $\Omega(K * \overline{K})$} is a finite nonempty set because it contains the perturbation of K, the job is to define $P(K)$ as the 'least L' of this set and show that it has a subcomplex with deleted join a $(2n+1)$-pseudomanifold if and only if van Kampen obstruction of K is nonzero :-

We again deem simplices as increasing words in the perturbed vertices–their order that of the basis (of their generated vector space V) of vertices of which they are perturbations–but coefficients so generic Galois automorphism over $\mathbb{Q}$ can permute this order in any which way–or as exterior monomials in this order

[17] Meaning a subset of vertices is a simplex iff its complement is not: this usage for $\mathbb{R}P^2_6$ is in Grothendieck's *Récoltes et Sémailles,* $\mathbb{C}P^2_9$ is another nice–usage of Schild (1991)–example, but the full classification of yin-yang complexes seems to be still open.

of their duals; so simplicial complexes as increasing words of increasing words closed under subwords occuring before or after, like e.g., the subwords 'alm' and 'alms' of 'almost'; then $B(K)$ defined before can be seen to be lexicographically least in {all simplicial complexes L on the generic perturbed vertices such that L is a graded basis of $\Omega(K)$}, a bigger set because this constraint was milder; and it is a consequence of its leastness and the genericity of coefficients that $B(K)$ was closed under domination. $\square$

We cannot expect this fully shifted property to endure for the least L of a smaller set, that is, under an additional constraint, but it is reasonable to hope that often leastness and genericity make things simpler:- for example, if the given K is n-pure then so is its perturbation, and defining least just as above, but for the set of only all n-pure L which give us a graded basis of $\Omega(K)$, we shall get in general a simplicial complex different from $B(K)$–it is as mentioned before the same iff K is Cohen-Macaulay–but it will still be closed under a 'weak domination' in n-simplices by the same galois argument. $\square$ What precisely 'weak domination' here and below means we'll analyze later.

So the cartesian choice for $P(K)$ is the lexicographically least simplicial complex L on the generic perturbed vertices such that $L * \overline{L}$ is a bigraded basis of $\Omega(K * \overline{K})$; also we assume here $(2n+1)$-purity of deleted joins which is stronger than n-purity of complexes; now its leastness and genericity imply that all top dimensional simplices of $P(K) * \overline{P(K)}$ of the same type are closed under 'weak domination'; this implies that if NHI holds, i.e., $f_n(K) \geq (n+2) f_{n-1}(K)$, then it contains $P^n * \overline{P^n}$; and also it seems that if VKO of K is nonzero, then $P(K) * \overline{P(K)}$ contains a $(2n+1)$-pseudomanifold, i.e., the deleted join of a join of some yin-yang complexes $P^j, j \leq n$. $\square$

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This game should extend to Cartan's cup-i products, Steenrod's reduced powers, etc.; their combinatorics is beautiful but intricate in K; but upstairs in the relevant product of K, rather relevant join of K–this segue from affine to linear to bring out using Galois action the simplification in–a generic least basis, which suffices up here for we want only to preserve homology (the group action tied to the operation, e.g., of $\mathbb{Z}/2$ above, went into defining the relevant join) working always in the good old and big field $\mathbb{R}$ of Eudoxus: seems other yin-yang complexes like $\mathbb{R}P^2_6, \mathbb{C}P^2_9, \ldots$ will appear generically when other characteristic numbers like of Stiefel or Pontryagin are nonzero?

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There is a constrained sieving which gives the least L:- For an n-pure K, the perturbed n-simplices and their new faces, i.e., not faces of any lexicographically previous n-simplex, form a graded basis of $\Omega(K)$: so there *exists* a sieving of all n-simplices, i.e., increasing words of length $n+1$ in these vertices, each admission with all new subwords, giving the lexicographically least word of these words, that is, all the top simplices of the n-pure L. $\square$ Intuitively K slides down jerkily but within all n-pure bases to the least generic slot L.

The first n-simplex is in L:- *any* n-simplex σ in the perturbed vertices is a linear combination with all coefficients nonzero in the perturbed n-simplices of K, but a linear dependency in its faces wedge a form gives $\sigma = 0$. □

After this for $\sigma \in L$ we usually need more than: there is no linear dependency between its new faces and of all those already admitted:- This is needed for the n-pure simplicial complex being erected to be linearly independent in $\Omega(K)$, but if this alone is our admission criterion, we'll usually fall short of a basis, e.g., if K has non-trivial homology in any dimension less than n. This last because by the same argument as before n-simplices, so using purity *all* simplices, will be closed under domination, i.e., we'll get a shifted n-pure complex. These have trivial homology in dimensions less than n, indeed links of nonempty simplices also have homology concentrated only in their top dimensions. What we'll erect this way is in fact the subcomplex of $B(K)$ generated by its n-simplices which, as mentioned before, is strictly smaller, unless the same homological condition holds for all links of the given n-pure K.□ For example, P^n is cohen-macaulay, but the other kuratowski n-complexes are not.

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The original vertices stay put under any field automorphism of $\mathbb{R}$ applied to the generic coefficients of their perturbations, while any permutation π of the perturbed vertices extends by this galois action to an algebra automorphism of $\Omega(K)$ over $\mathbb{Q}$. Though no longer an algebra basis, the perturbation of the n-pure K is our primary example of an n-pure simplicial complex P on the perturbed vertices which is a graded linear basis of $\Omega(K)$ over the reals, but then, for any such P–these can be combinatorially very different from K–any $\pi(P)$ is another but only isomorphic example. □

The ordering of the original vertices passes to their perturbations and singles out L as the P whose n-simplices as increasing words are lexicographically least: for $\pi \neq \mathrm{id}$ the isomorph $\pi(L)$ of L is usually distinct, so bigger. □ Focussing now on shuffles [18] i.e., permutations π order preserving on some subset and its complement, we saw that the unconstrained least basis B is closed with respect to downward shuffles of simplices. What about L?

In the unconstrained case B is closed under a downward shuffle π of an n-simplex σ for otherwise $(B\backslash\sigma)\cup\pi(\sigma)$ would be a smaller basis. In the constrained case the extra condition on π is that after this replacement the total number of simplices generated by the n-simplices should remain the same.

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As for the question above, a downward shuffle $\theta = \pi(\sigma)$ of an n-simplex of $\sigma \in L$ is not in L iff replacing σ by θ decreases the total number of faces of top simplices:- We queue n-simplices in the perturbed vertices in dictionary order and admit an n-simplex iff (a) it is not a linear combination of those admitted; so (i) the very first is admitted, and (ii) by taking wedges it follows also that the

[18]They triangulate a product of simplices and define a wedge product without fractions.

simplicial complex generated by those admitted is linearly independent in $\Omega(K)$; and (b) after sieving out linearly dependent simplices due to its admission we should still have a spanning set of $\Omega(K)$ furnished by the faces of all admitted or still standing in queue; this gives L, etc. $\Box$ On the other hand if we use only (a) without constraint (b) this gives:- the shifted simplicial complex M generated by the least generic basis of $\Omega^n(K)$, which in general is only a graded linearly independent subset, not a basis of $\Omega(K)$. $\Box$

Since the van Kampen obstruction (VKO) to the embeddability of our K^n in $2n$-space $\mathfrak{o}(K)$ resides in the $(2n+1)$-dimensional simplices of the deleted join $K * \overline{K}$ we can assume its $(2n+1)$-purity which implies n-purity of K. $\Box$ We have then the lexicographically least $\mathcal{L}$ on the perturbed generic vertices having a $(2n+1)$-pure $\mathcal{L} * \overline{\mathcal{L}}$ bigraded basis of $\Omega(K * \overline{K})$, further it seems $\mathfrak{o}(K) \neq 0$ iff $\mathfrak{o}(\mathcal{L}) \neq 0$. $\Box$ Also we have the simpler complex $\mathcal{M}$ generated by the least set of n-simplices on the perturbed vertices such that all ordered disjoint pairs form a basis of $\Omega^{n,n}(K * \overline{K})$, now $\mathcal{M} * \overline{\mathcal{M}}$ is only a linearly independent set of $\Omega(K * \overline{K})$, but even $\mathfrak{o}(K) \neq 0$ iff $\mathfrak{o}(\mathcal{M}) \neq 0$ seems true. $\Box$

It seems if an n-simplex is in $\mathcal{M}$ then so is any dominated by it provided replacement by it does not decrease the number of disjoint pairs of n-simplices while for $\mathcal{L}$ an even weaker domination involving the number of total number of simplices in the deleted join is needed. $\Box$

A sieving for the lexicographically least $\mathcal{L}$ on the perturbed vertices having a $(2n+1)$-pure $\mathcal{L} * \overline{\mathcal{L}}$ bigraded basis of $\Omega(K * \overline{K})$ when $K * \overline{K}$ is $(2n+1)$-pure:- at each step (a) choose the first n-simplex which increases the dimension in $\Omega^{n,n}(K * \overline{K})$ of all disjoint pairs of admiited n-simplices, and (b) that these with those still unsieved due to this criterion, still generate–like the initial queue of all n-simplices in the perturbed vertices–a graded spanning set of $\Omega(K * \overline{K})$; note (i) first two disjoint n-simplices are in $\mathcal{L}$, (ii) wedge product again linear independence of its faces, but $\mathcal{L}$ is closed under a weaker domination: replacing by the dominated n-simplex should not reduce the number of simplices in the generated deleted join. $\Box$

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Replacing an n-simplex θ^n in *a Kuratowski n-complex on some vertices by a* ϕ^n not in *may not decrease the number of disjoint pairs*:- An in n-simplex uses r vertices of each factor σ^{2r}_{r-1} and any disjoint in n-simplex all but one of the remaining $r+1$, so θ^n has $\prod(r+1)$ disjoint in n-simplices. An n-simplex ϕ^n not in uses more from some factor, say σ^{2r}_{r-1}. If it uses even more than $r+1$ there is no disjoint in n-simplex. Now assume it uses $r+1$ of just this factor and $s-1$ of a second factor σ^{2s}_{s-1}. Then any in disjoint n-simplex of complex uses the r left in first factor and any s of the $s+2$ left in the second. So number of disjoint in n-simplices is (unless θ^n is disjoint to ϕ^n, then reduce by 1) given by replacing $(r+1)(s+1)$ in above by $\binom{s+2}{s}$: so it is now lesser, equal, or bigger depending on whether $s < 2r, s = 2r$ or $s > 2r$. $\blacksquare$

We note that in above ϕ^n may have just one more vertex from a number of factors, compensated by one or more less from as many or a smaller number of other factors. Anyway, since $s < 2r$ does not hold for $r = 1, s = 2$, we can't expect $\sigma_0^2 \cdot \sigma_1^4$ to be $\mathbb{Z}/2$-*shifted,* i.e., closed with respect to weak domination–a dominated simplex is also in unless replacement by it reduces the number of disjoint pairs–with respect to *all* vertex orders. On the other hand $\sigma_{r-1}^{2r} \cdot \sigma_{r-1}^{2r}$ is $\mathbb{Z}/2$-shifted for all vertex orders but is not $\{1\}$-*shifted,* that is shifted, for any. More generally, pushing above calculation: the t-fold join of σ_{r-1}^{2r} is $\mathbb{Z}/2$-shifted for any t with respect to all vertex orders.

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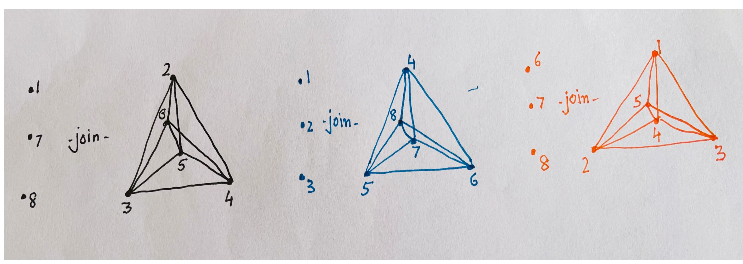
Vertex orders, in which all n-simplices, as an increasing word of increasing words, is lexicographically least, are as follows, *for a Kuratowski n-complex*:- Our cue, for $K_{3,3} = \sigma_0^2 \cdot \sigma_0^2$ such an order is $\{1, 5, 6\} \cdot \{2, 3, 4\}$: both factors being same any vertex can be first, if it is in the first factor to use the least words with it the next three must be all of the second, and the last two the remaining vertices of the first factor. Similarly for $\sigma_{s-1}^{2s} \cdot \sigma_{r-1}^{2r}$ except if $s < r$ the first s vertices must be any of the first factor, then all $2r + 1$ of the second in any order, and lastly the remaining $s + 1$ of the first factor. More generally, write any Kuratowski complex so that the dimensions of the factors are non-decreasing, and assume known a required vertex sequence for the complex obtained by omitting the first factor σ_{s-1}^{2s}, then put before and after this sequence, respectively, any s and the remaining $s + 1$ vertices of this factor. ■



Alas! *the Kuratowki 2-complex $\sigma_0^2 \cdot \sigma_1^4$ is* not $\mathbb{Z}/2$-*shifted with this vertex order,* e.g., the replacement of the triangle 568 in complex by the non-disjoint dominated triangle 148 having two vertices in the first factor keeps the number of disjoint pairs of triangles same. *So we'll dump this notion of leastness,* such a replacement destroys the obstructing antipodal 5-sphere $K * \overline{K}$ which any linear $\mathbb{Z}/2$-sieving worth its name should not do. Shown next is a vertex order which minimizes the number of first vertices of triangles, it does not work either; but the vertex order in orange, i.e., *minimizing the number of* last *vertices of the triangles of $\sigma_0^2 \cdot \sigma_1^4$ makes it $\mathbb{Z}/2$-shifted!*

More generally, *any Kuratowski n-complex is $\mathbb{Z}/2$-shifted with respect to any vertex order giving precedence to its higher dimensional factors*:- Again, let θ^n be in complex, and ϕ^n not in it which is disjoint to at least one n-simplex of complex. If $\phi^n < \theta^n$ with respect to such a vertex order, for each σ_{s-1}^{2s} in which ϕ^n has t less vertices than s, we can count off t other factors σ_{r-1}^{2r}, all with $r \geq s$, in each of which ϕ^n has $r + 1$ vertices. The result follows now from $(s+1)^{t+1} > (\frac{s}{t+1} + 1)(\frac{s}{t} + 1) \cdots (s + 1) = \binom{s+t+1}{s}$. ■

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The above suggests we seek some *nice* vertex orders, that are well-defined for *any* pure n-dimensional simplicial complex K, and which, when it is Kuratowski, perforce give precedence to its higher dimensional factors.

A first tentative is this:- with vertices (totally) ordered *last vertex* of any n-simplex makes sense, so the number of vertices that occur last in some n-simplex; for any vertex order which minimizes this number we consider the number of vertices that occur *second last* in some n-simplex; then for any vertex order which also minimizes this second number we consider … should the orders left after this iterated minimization be deemed nice?

A more constructive recipe emerges if we muse over this question for *Example:* $\sigma^{2r}_{r-1} \cdot \sigma^{2s}_{s-1}, r+s = n+1, r \geq s$:- The number of n-simplices incident to a vertex in the second factor is $\binom{2r+1}{r}\binom{2s}{s-1}$, which divided by $\binom{2r}{r-1}\binom{2s+1}{s}$, the number incident to a vertex in the first factor, gives us $\frac{(2r+1)(s+1)}{(r+1)(2s+1)}$, which is bigger than one if and only if $r > s$. This suggests that if $r > s$ *the highest vertex* ought to be chosen from the second factor; if $r = s$ it can be from either but switching factors if need be we'll make it the second.

Then we count the number of n-simplices, not containing the already chosen highest vertex, and incident to a given vertex: if this vertex is in the second factor this number is $\binom{2r+1}{r}\binom{2s-1}{s-1}$ and if in the first factor then $\binom{2r}{r-1}\binom{2s}{s}$ which is smaller. So the second highest vertex ought also be chosen from the second factor. The same criterion–*the biggest number of incident n-simplices containing* none *of the already chosen higher vertices*–dictates that the $s+2$ highest vertices be all chosen from the second factor because $\binom{2r+1}{r}\binom{2s-t}{s-1} \div \binom{2r}{r-1}\binom{2s-t+1}{s} = \frac{(2r+1)\div r}{(2s-t+1)\div s} > 1 \forall\ 1 \leq t \leq s+1$, in fact for all $t \geq 1$.

But, with only $s-1$ vertices left unchosen in this factor, we need to switch to counting number of n-simplices, *containing exactly* one *of the chosen,* and incident to a given vertex: if this vertex is in the second factor this number is $(s+2)\binom{2r+1}{r}$, and if in the first factor only $(s+2)\binom{2r}{r-1}$, so the choice of the next highest vertex must also be from the second factor. In the order being constructed this will be the highest vertex which does not occur last in any n-simplex, but occurs as the second last of some n-simplex.

Also, for this particular example, the only vertex of this type, because now we have only $s-2$ vertices of this factor in hand. So we switch now to counting number of n-simplices, *containing exactly* two *of the already chosen ones,* and incident to a given vertex, etc. We see that the $2s+1$ highest vertices ought all be chosen from the second factor. The remaining $2r+1$ vertices, all of the first factor, can be labelled in any which way. ■

The recipe given in the above example defines: *vertex orders for any pure simplicial n-complex, which, if it is Kuratowki, automatically display its factors with dimension non-increasing*:- The criterion for the highest vertex shows it must be in a lowest dimensional factor σ^{2s}_{s-1}. From this and using the same inequalities–now σ^{2r}_{r-1} is any of the other factors–the critera laid down for the second highest vertex, third highest vertex, etc., show that the $2s+1$ highest vertices must be all from σ^{2s}_{s-1}. Deleting this factor this reduces us to the same result for a Kuratowski complex of lesser dimension equipped with a vertex order constructed using the same recipe. ■

So should we call orders cooked by this recipe nice? [19] Provisionally we shall, but it is best to keep options open: enough for the moment that there is a simple definition answering our requirements, but as this analysis proceeds, it is quite on the cards that we might add desiderata.

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More generally, for K any *Pontryagin n-complex,* i.e., a join of some yin-yang complexes, *nice vertex orders automatically display its nice factors with number of top simplices non-increasing,* e.g., a factor $\mathbb{R}P^2_6$ will come after a factor σ^6_2 because it has less 2-simplices, likewise $\mathbb{C}P^2_9$ after σ^{10}_4.

Is its deleted join K_* a pseudomanifold iff K is Pontryagin? 'If' is easy and 'only if' was proved, for $\dim(K_*) = 2n+1$, in barrick one. With the hope of removing this restriction we'll now analyze that argument **:-**

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(1.2) *Notation.* Also we'll call K *yin-yang or nice* iff for any partition of its vertices into two nonempty parts, one and only one part is in it. So, a closed n-simplex is not nice for $n > 0$ – indeed *a nice complex factors as a join iff it is a cone over a nice complex other than* $\{\text{pt}\}$*,* which is vacuously nice – but being the $(n+1)$-fold join of the closed 0-simplex its deleted join is the octahedral n-sphere. Conversely, if the deleted join K_* of an n-complex K is an n-pseudomanifold, K cannot be strictly bigger than a closed n-simplex because then $\dim(K_*) > n$, nor smaller because the octahedral n-sphere does not properly contain another n-pseudomanifold. So 'only if' *is true for* $\dim(K_*) = n$. ■ So from here on in our analysis $\dim(K_*) = m$ *where* $n < m \leq 2n+1$*;* also dimensionally homogenous will be usually replaced by the synonym pure.

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If K_* is an m-pseudomanifold with least vertices is K nice? The m-purity of K_* does *not* imply n-purity of K for $m < 2n+1$, but, since $n < m$, for any inclusion maximal *top simplex* $\omega \in K$, there is an m-simplex $(\omega, \varphi) \in K_*$ with $\varphi \in K$ *nonempty,* and, incident to any $(\omega, \varphi \setminus v)$ one other m-simplex (ω, φ') of this m-pseudomanifold. So K *has at least $m+2$ vertices.* Conversely, if K has $m+2$ vertices, the complementary set of vertices ω^c can't be in K, but all its facets $\varphi, \varphi', \ldots$ are in K and $[K \setminus \omega] = \partial(\omega^c)$. Is this the same as saying that K is nice? Anyway given a partition $\{\alpha_1, \alpha_2\}$ of all $m+2$ vertices and any m-simplex $(\sigma_1, \sigma_2) \in K_*$ by replacing the latter by an adjacent m-simplex using the $(m+2)$th vertex available we can step-by-step improve the imbalance

[19] For such questions it should help to think of *an order as a Morse function ξ,* i.e., build K step by step by adding the sets K_v of all simplices with v as their biggest vertex; so the Euler characteristic—the additive invariant e which is ± 1 on open cells depending on whether their dimension is even or odd—satisfies $e(K) = \Sigma_v e(K_v)$. When K is a manifold $e(K_v) = 0$ or ± 1 with K_v an open cell in the latter case, and v is called a critical point of index r if this cell has dimension r; see Banchoff (1967), who writes $e(K_v)$ as $a(v, \xi)$, for more.

with respect to the partition to reach an m-simplex of K_* such that one of these disjoint simplices of K contains the smaller part.

(2.1) *The case* $m = 2n+1$ in the paper is special because now the purity of K_* implies [20] the purity of K. Also K_* *pseudomanifold implies any* $\sigma^{n-1} \in K$ *has at least* 3 *and at most* $n+3$ *vertices* v *as its link.* Because any $2n$-simplex $(\sigma, \omega), v \in \omega$ of K_* has at most n vertices other than v in ω. Further [21] *if a* σ^{n-1} *has* $n+3$ *vertices in its link there is* **no** *vertex* w *of* K *outside its star!* Because then a $2n$-simplex $(\sigma, \varphi), w \in \varphi$ of K_* would have valence bigger than 2. That is $K \subseteq \tau_n^{2n+2}$, where $\tau^{2n+2} = S_K\sigma$. So $K = \tau_n^{2n+2}$ because no $(2n+1)$-pseudomanifold is properly contained in $(\tau_n^{2n+2})_*$ since its top simplices can be joined to each other via codimension one simplices (in fact as we'll presently see it is an antipodal simplicial sphere, one of many). ■ Most apt would be a like jlt proof that K_* any m-pseudomanifold, $n < m \leq 2n+1$, implies the finiteness of vert(K), before going to the join factorization of K.

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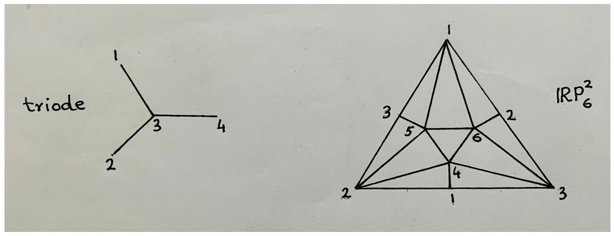
These join-irreducible *factors,* i.e., simplicial n-complexes with $m+2$ vertices, $n < m \leq 2n+1$, whose deleted join is an m-pseudomanifold, are legion, but are possibly all tied as follows to a very simple one, viz., the disjoint union of the boundary of an m-simplex and a point:- This $(m-1)$-complex $\partial(\sigma^m) \cup \{\text{pt}\}$ has $m+2$ vertices, is nice, and its deleted join is the octahedral m-sphere with a pair of its m-simplices derived. All yin-yang complexes K may be obtainable from these by *moves* in each of which we replace a top simplex ω by ω^c. The surgery on the deleted join induced by each move then shows that the m-pseudomanifold K_* *is in fact an antipodal m-sphere.* ■

For $m = 1$ there is no move to make on $\partial(12) \cup \{3\} = \{1, 2, 3\}$, i.e., σ_0^2, whose deleted join is a hexagon. For $m = 2$ besides $\partial(123) \cup \{4\}$ with deleted join the octahedral 2-sphere $(123)_*$ with triangle 123 derived at $\overline{4}$ and $\overline{123}$ at 4, a move gives also *triode* whose deleted join is the suspension of a hexagon. Next six non-isomorphic antipodal simplicial 3-spheres, including $(\sigma_1^4)_*$, the shortest sequence of moves from $\partial(1234) \cup \{5\}$ to σ_1^4 being to replace one after another the four triangles by the four complementary edges incident to 5. Similarly for $m = 2n+1$ we can move $\partial(\sigma^m) \cup \{\text{pt}\}$ to the unique least dimensional σ_n^{2n+2} by replacing one by one all $(m-1)$-simplices, then all $(m-2)$-simplices, ..., finally $(n+1)$-simplices by their complementary simplices.

[20]This is obvious; not so obvious is what is shown later using K_* pseudomanifold fully, viz., that *any* full subcomplex of K is pure, i.e, that K is a matroid; for $n < m < 2n+1$ a parallel argument on K_* itself, using oriented or $\mathbb{Z}/2$-matroids, may work.

[21]This dramatic point signalled K_* pseudomanifold is a very tight condition, but I was not able to close the deal as elegantly, and will need to check my NB to see if I had at least got vert(K) always finite just like that? Using a nice order a simpler reasoning avoiding matroids altogether may in fact fully join factorize K for all $n < m \leq 2n+1$.



If we insist that not only K_* but also that K be a pseudomanifold (is such a K perforce a manifold?) *then only a handful of examples is known*:- The first with K_* an antipodal 4-sphere can be obtained by replacing one by one all tetrahedra of $\partial(12345)\cup\{6\}$ by the five complementary edges joining 6 to the other vertices, and then replacing five triangles too by complementary triangles to make link of 6 a pentagon: as shown above this $K = \mathbb{R}P^2_6$. The next, with K_* an antipodal 7-sphere, can be made similarly from $\partial(12345678)\cup\{9\}$ so that the link of 9 is the unique neighbourly non-polytopal 8-vertex 3-sphere. Or, the machine can start from σ^8_3 and replace, guided by the pseudomanifold condition on K we want, 36 of its $\binom{9}{4} = 126$ tetrahedra by their complementary 4-simplices. Kühnel and Banchoff (1983) remains the best account of this $K = \mathbb{C}P^2_9$ including a simplicial analogue $\mathbb{C}P^2_9 \to S^4_6$ of the result that the complex projective plane divided by complex conjugation gives the 4-sphere. So we also get a simplicial analogue of the fact that S^7 is homeomorphic–but not diffeomorphic as Milnor showed using Pontryagin numbers–to the unit tangent bundle of S^4. The last three known all have 15 vertices–so K_* is an antipodal 13-sphere–and seem to be triangulations of the projective quaternionic plane. Despite this paucity of known examples imho [22] [23] are many more, maybe even infinitely many, but a Dynkin diagram type classification can be worked out.

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(2.2) If K_* is a $(2n+1)$-pseudomanifold then, *for all* $\sigma^{n-1} \in K, \theta^n \in K$ *and* $\lambda \in L_K\sigma \setminus \theta$, *the simplex* $(\sigma \setminus \theta) \cdot (L_K\sigma \setminus \theta \setminus \lambda)$ *lies in* K. [24]

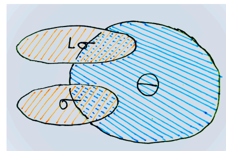
Indeed if $\sigma \cap \theta$ has t vertices, then $L\sigma \setminus \theta$ has at least 2 and at most $t+2$, with all proper faces in it, but **(2.5)** $L\sigma \setminus \theta$ *itself is not in* $Lk(\sigma \setminus \theta)$:-

[22] Otoh Datta has opined that any y-y pseudomanifold Y is perforce a manifold. If so duality plus the observation that the *Euler characterstic of any y-y complex is odd* show Y must be a non-spherical manifold with "few vertices", i.e. no more than $\frac{3}{2}\dim Y + 3$. As Brehm-Kühnel checked these admit a *Morse function with exactly three critical points.* So Y belongs to a handful of such manifolds allowed by the Hopf invariant one theorem of Adams, and studied in some detail by Eells-Kuiper (1961-2). So on second thoughts maybe there are only finitely many pure y-y complexes Y in which all codimension one simplices have the same valence t, the case $t = 2$ being that of pseudomanifolds?

[23] The deleted join $Y * \overline{Y}$ of a y-y complex occurs as the boundary of an $(m+1)$-polytope with $2(m+2)$ vertices iff Y is *tight* in this affine space: cycles contained in and not bounding in its intersection with a half space are non-trivial.

[24] I.e., *the link of* $\sigma \setminus \theta$ *contains* $\partial(L\sigma \setminus \theta)$. So K is a matroid. Then in (2.5) of paper this result is shown the best possible, i.e., *the link of* $\sigma \setminus \theta$ *does not contain* $L\sigma \setminus \theta$. Which enables us to identify the maximal subsets of vertices joinable to each other via the hollow simplices or circuits of K as the subsets spanning its factors σ^{2s}_{s-1}.

If $\underline{t=0}$, i.e., σ is disjoint from θ, this is the pseudomanifold property of the deleted join: $L\sigma \setminus \theta = \{\lambda_1, \lambda_2\}$, the 2 vertices in the link of σ not in θ but the edge $\lambda_1\lambda_2$ is not in it because K has no simplex with $n+2$ vertices.



If $\underline{t=1}$, i.e., $\sigma \cap \theta = \{v\}$, using purity of K_* let us replace θ^n by an $\xi^n \in K$ containing $\theta \setminus v$ and disjoint from σ.

If ξ has its new vertex w not in $L\sigma$ using K_* pseudomanifold we see that $L\sigma \setminus \theta = L\sigma \setminus \xi = \{\lambda_1, \lambda_2\}$, just 2 vertices. If $\lambda_1\lambda_2\sigma \setminus v$ were an n-simplex of K, for the same reason we would have a $u \notin \theta, u \neq \lambda_1$ such that $u\lambda_2\sigma \setminus v \in K$. Then a $w' \notin u\lambda_2\sigma \setminus v, w' \neq v$ such that $\theta' = w'.(\theta \setminus v)$ is in K. Also we can't have $w' = \lambda_1$ for then σ would have only λ_2 in its link outside the disjoint n-simplex θ'. So the 3 vertices v, λ_1 and u in the link of $\lambda_2\sigma \setminus v$ are distinct and outside the disjoint n-simplex θ' which is not possible.

If ξ has new vertex $w = \lambda_1$ in $L\sigma$ then $L\sigma \setminus \theta = \{\lambda_1, \lambda_2, \lambda_3\}$ has 3 vertices, with λ_2 resp. λ_3 also new vertices of n-simplices ξ' resp. ξ'' containing $\theta \setminus v$ and disjoint from σ, else the $2n$-simplex $(\lambda_3\sigma, \theta \setminus v)$ resp. $(\lambda_2\sigma, \theta \setminus v)$ of K_* has valence one. So $\theta \setminus v$ has besides v all three λ_i in its link, therefore any n-simplex β of K disjoint from θ contains all but one λ_i. So the triangle $L\sigma \setminus \theta$ is not even a simplex of K, but all its three edges must be, to provide the requisite 1-pseudomanifold link of the $(2n-1)$-simplex $(\alpha, \theta), \alpha = \sigma \setminus v$ of the $(2n+1)$-pseudomanifold K_*, viz., the hollow triangle $\partial(L\sigma \setminus \theta)$.

After a short break we'll see this proof extends to $t = 2, 3 \ldots$

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As this installment has been delayed too much, I'll post it now even amidst this analysis within … but probably will be making additions and changes, save in its final two pages, before moving on … also my aim is if possible self-contained polished expositions of some results later.

K S Sarkaria July 5, 2022

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The pure yin-yang manifold $\mathbb{R}P^2_6$ *is not a matroid* :- the full subcomplex on any 4 vertices is not pure: it has all 6 edges but only two of the 4 triangles, so has an edge not incident to a triangle. ■ Likewise $\mathbb{C}P^2_9$ and $\mathbb{H}P^2_{15}$ are not–indeed a matroid has non-trivial homology only in its top dimension–and maybe there is not much left to add to barrick one if we only want to find *all matroids with deleted join a pseudomanifold?* Probably this happens iff matroidal components are all yin-yang, however we'll keep going for the full conjecture, viz., K_* is a pseudomanifold iff K is Pontryagin. With this aim we are trying to make that old proof clearer and matroid theory free, though another and oriented matroid seems tied to the generalization, but let's see [25] ...

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If $\underline{t > 1}$ and $v \in \sigma \cap \theta$ we replace θ by an ξ containing $\theta \setminus v$ and disjoint from $v.(\sigma \setminus \theta)$, so $\sigma \cap \xi = \sigma \cap \theta \setminus v$ has $t-1$ vertices.

If ξ has its new vertex w not in $L\sigma$ then $L\sigma \setminus \theta = L\sigma \setminus \xi$ has inductively at most $t+1$ vertices with all proper faces in the link of $v.(\sigma \setminus \theta)$—so in the link of $\sigma \setminus \theta$—but $v.(\sigma \setminus \theta).(L\sigma \setminus \theta)$ is not in K. We claim that even $\gamma = (\sigma \setminus \theta).(L\sigma \setminus \theta)$ is not in K. Were γ an n-simplex of K we replace any $\lambda \in L\sigma \setminus \theta$ in it by a vertex $u \neq \lambda$ to get another n-simplex γ' still disjoint from θ. Then we replace v in θ by a vertex $w' \neq v$ to get another n-simplex θ' also like θ disjoint from γ'. If $w' \neq \lambda$, the $(n-1)$-simplex $\gamma \setminus \lambda$ disjoint from θ' has three distinct vertices v, λ and u in its link outside θ'. And, if $w' = \lambda$, the inductive hypothesis for $t-1$ is violated: $\sigma \cap \theta'$ has cardinality $t-1$ with $\sigma \setminus \theta' = v.(\sigma \setminus \theta)$ and $L\sigma \setminus \theta' = L\sigma \setminus \theta \setminus \lambda$ but their join is in K. In case γ has dimension less than n we argue likewise on an incident n-simplex $\tilde{\gamma}$ still disjoint from θ.

If no ξ has its new vertex w outside $L\sigma$ using inductive hypothesis on σ and ξ we know $L\sigma \setminus \xi = L\sigma \setminus \theta \setminus w$ has at least 2 but at most $t+1$ vertices with all proper subsets in the link of $\sigma \setminus \xi = v.(\sigma \setminus \theta)$. In fact any $\lambda \neq w$ in $L\sigma \setminus \theta$ is also the new vertex of some ξ', otherwise any $2n$-simplex $(\theta \setminus v, \beta)$ of K_* containing $(\theta \setminus v, v.(\sigma \setminus \theta).(L\sigma \setminus \theta \setminus \{w, \lambda\})$ has valence one; so any n-simplex β of K disjoint from θ contains all but one of the vertices of $L\sigma \setminus \theta$. In particular $L\sigma \setminus \theta$ is not a simplex of K, but any n-simplex disjoint from θ containing $\sigma \setminus \theta$ can be written $\alpha.(L\sigma \setminus \theta \setminus \lambda)$; and, the link of $(\alpha, \theta) \in K_*$ being a pseudomanifold of dimension two less than the cardinality of $L\sigma \setminus \theta$, all $\lambda \in L\sigma \setminus \theta$ must occur thus, i.e., all proper faces of $L\sigma \setminus \theta$ are in the link of $\sigma \setminus \theta$. ■

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A triangulable space comes stratified by intrinsic dimension, and admits full subcomplexes covering lower dimensional strata; so fullness is natural; and so for starters are complexes with all full subcomplexes dimensionally homogenous, i.e., matroids; so, as well, in the context of a group action, G-complexes with all full G-subcomplexes pure: and maybe for $G = \mathbb{Z}/2$ these are again only oriented matroids? However we'll stay away from these turbid waters yet; we'll go where analogous arguments take us starting from the more general hypothesis K_* an m-pseudomanifold, $n < m \leq 2n+1$ [26] ...

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So, resuming our analysis of barrick one, recall by (2.1) we are done if any σ^{n-1} has $2n+3$ vertices in its star; otherwise $S\sigma$ is a proper subset of all vertices; so, *it suffices to show the vertices of some factor are all in $S\sigma$*:- for, if $K = L \cdot N$,

[25]ici aussi time slows down ...

[26]et ici, two "lectures" on shadows and reflections ...

the easily verified (but very useful!) *join formula* $K_* = L_* \cdot N_*$ tells us that K_* is a highest possible dimensional pseudomanifold iff both L_* and N_* also are, so an induction on the dimension of K takes us home. ■ To show this we'll use (2.2+5) and the following, also implicit in **(2.7)** of paper.

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There is *another definition of join irreducible factors,* having at least two vertices, which but for a single word–hollow–is just like that of path components! *Two vertices are in the same factor if and only if we can go from one to the other such that successive vertices are in the same hollow simplex,* i.e., a simplex itself not in K, but all its proper faces are:- Let L, resp. N, be the full subcomplex of K on such an equivalence class of vertices, resp. its complement; then $K \subseteq L \cdot N$ is trivial, and $L \cdot N \subseteq K$ holds because if some $\alpha \cdot \beta$ with $\alpha \in L$ and $\beta \in N$ were not in K, it contains a hollow simplex of K with one vertex in the equivalence class and another outside it. ■

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This result shows up to permutations the complete join factorization of any simplicial complex K is unique and tied to its hollow simplices or *circuits.* The subcomplex of all proper faces of all circuits is usually strictly smaller even if K is circuit-connected, i.e., join-irreducible. Indeed even if K is a yin-yang complex, for example for $\mathbb{R}P^2_6$ this subcomplex is patently 1-dimensional. Circuits being the complements of their maximal simplices, what is true is that all yin-yang complexes, so also their joins, i.e. (conjecturally [27])*all Pontryagin complexes are determined by their circuits.* Hollow simplices also figured in a nice result of Dancis, viz., *a simplicial m-manifold is determined by its* $]\frac{m}{2}[$*-skeleton*:- because, from here on, Poincaré duality [28] tells us exactly which hollow simplices to fill to build back the entire complex. ■

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To wrap-up we recall how (2.2+5) *was used in the paper*:- From K_* a $(2n+1)$-pseudomanifold it is easily seen that *no* $L\sigma \setminus \lambda$ *is in* K; this and (2.2) applied to $\sigma^{n-1} = \omega^n \setminus u$ and θ^n shows that **(2.3)** K is a *matroid,* i.e., if we imagine $\mathrm{vert}(K)$ as all columns of a *matrix* such that a set of columns is linearly independent iff it is in K, the *basic property of linear independence*–given any two bases ω and θ any $u \in \omega \setminus \theta$ can be replaced by some $v \in \theta \setminus \omega$ to get another basis–holds. So hold as well for our K all *theorems* [29] formally deducible from it. Notably,

[27]this if we deem K to be *Pontryagin iff* $K * \overline{K}$ *is a pseudomanifold* : this definition extends naturally to groups $G \neq \mathbb{Z}_2$.

[28]otoh *an Alexander duality holds for a join* K *of yin-yang complexes*:- we can retract its complement in the sphere $|K * \overline{K}|$ onto its copy $|\overline{K}|$ because any point not in it lies on a unique half-open interval $(x, y]$ where $x \in |K|$ and $y \in |\overline{K}|$.

[29]Whitney's neat 25-page paper of 1935 is still best: its theorem 19 is transitivity of in same circuit, its §16 shows matrices have 'other theorems' by *a matroid* K^2 *whose 7 vertices cannot arise as columns of any matrix over any field with* $2 \neq 0$*,* its appendix characterizes those arising thus over $\{0, 1\}$, etc.

if u and v are in a minimal dependent set of columns, and v and w in another, then u and w are also in such a set of columns, i.e., the *transitivity* of the binary relation "u, v in same circuit" holds for K.[30] Likewise, that any independent set or simplex can be *augmented* by vertices from any basis or n-simplex to make another n-simplex, which is used in **(2.4)** to precise, all proper faces of $L\sigma \setminus \lambda$ are in K, which is immediate from (2.2), to: *a subset of vertices is a circuit iff it is an $L\sigma \setminus \lambda$.* Then, using transitivity of in same circuit, in **(2.6)** it is deduced from (2.2) and (2.5) that: if $v \in L\sigma$, its circuit-component, that is all vertices of the factor of K in which v lies, is contained in $S\sigma$; in fact the proof shows *the circuit-component of v is the intersection of all $S\sigma$ as we run over all $(n-1)$-simplices σ that have v in their link.* ■

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Apparently Whitney introduced matroids in 1935 while analysing arguments he used in a 1932 paper for a new planarity criterion for graphs[31] maybe to link it more tightly with that of Pontryagin and Kuratowski, and push at least some part to higher dimensions; and indeed his name like Pontryagin's is also tied to the theory of characteristic classes, which accomplished a lot, but dually, using obstructing cocyles; otoh, our aim is to revert to homology, but be content with Galois genericity, in which context we hope to see not only phantom riders like Pontryagin numbers, but pretty yin-yang carriers like $\mathbb{C}P_9^2$ themselves to emerge from the mist with cartesian naturality!

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So, *K_* a $(2n+1)$-pseudomanifold implies $K \cdot K$ is a matroid*:- indeed the join of *any*[32] number of disjoint copies ${}^gK, g \in G$ of K has all full subcomplexes pure *if and only if* K has this property. ■

Notation. Below we'll often use a capital letter Σ for the union (α, β) of the first and second copy of subsets α and β of vertices, $\overline{\Sigma} = (\beta, \alpha)$ for its antipode, the two-fold join $K \cdot K$ of K has all Σ such that $\alpha, \beta \in K$, and its deleted join K_* or $K * K \subset K \cdot K$ is the subcomplex given by $\Sigma \cap \overline{\Sigma} = \emptyset$. So associating to any Σ the complement of $\Sigma \cap \overline{\Sigma}$, i.e., the map $\Sigma \mapsto \Sigma \setminus \overline{\Sigma}$ or $(\alpha, \beta) \mapsto (\alpha \setminus \beta, \beta \setminus \alpha)$ equivariantly retracts the set $K \cdot K$ onto its subset $K * K$ but it is not monotone:

[30] More generally K_* any pseudomanifold should imply the binary relation "C_i, C_j intersect" on circuits is transitive, so any two circuits of any factor of K intersect, and indeed it is true that *any two circuits of a yin-yang complex intersect.*

[31] By 1931, after a thesis under Birkhoff on chromatic polynomials, Whitney had shown that *if a simplicial 2-sphere has no hollow triangles then it is hamiltonian* (by squeezing a lemma in this paper it is known now that this conclusion is still true if there are up to five hollow triangles, but may be false if there are six); which reduces proving the four colour theorem to simplicial 2-spheres obtained by gluing two simplicial closed disks with all vertices on their common boundary polygon; but note if we well colour the vertices of the two disks with three colours in general all nine ordered pairs of these are needed.

[32] If the fully deleted G-fold join of K is a pseudomanifold of the highest dimension, it seems (2.2+5) generalizes with similar conclusions, K a matroid, etc., and a similar classification, however our policy here is not to be tempted away by any low hanging fruit under the purview of the third part of our barrick trilogy.

it does not commute with inclusion; this combinatorial retraction underlies how uncannily just this subcomplex being a pseudomanifold of the highest possible dimension $2n+1$ influences the entire two-fold join!

Otoh, *when $K*K$ is a pseudomanifold of a lower dimension m, even the m-skeleton of $K\cdot K$ is seldom a matroid*:- for now $\dim(K)\le m$, so this would imply K a matroid, which we've seen is not usually so. ■ However we hope to show below that $K\cdot K$ is still a lot like a matroid.

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If $K*K$ is an m-pseudomanifold, for $m=2n+1$ we saw that no nonempty $L\alpha\setminus\lambda, \alpha\in K$ is in K, and the minimal such, i.e., if $\dim(\alpha)=n-1$, gave us all the circuits of K; but, *for $m<2n+1$ a nonempty $L\alpha, \alpha\in K$ may be in K*:- for example $\partial(\sigma^m)\cup$ pt and $\mathbb{R}P^2_6$ have smallest nonempty $L\alpha$'s consisting of just two vertices, and that edge is also in these yin-yang complexes. ■

In the hope that the pairs version of the above for $m=2n+1$, viz., the smallest nonempty $L\Sigma\setminus\lambda, \Sigma\in K\cdot K$ are the circuits of $K\cdot K$, extends in some way – note $L\Sigma=(L\alpha,L\beta)$ if $\Sigma=(\alpha,\beta)\in K\cdot K$ – we now ask

Q. Does $K*K$ an m-pseudomanifold imply: *for any Σ^{m-1} in $K\cdot K$, the join of* $\Sigma\setminus\overline{\Sigma}$ *with any subset* $L\Sigma\setminus\overline{\Sigma}\setminus\lambda$ *of,* but <u>not</u> with $L\Sigma\setminus\overline{\Sigma}$, *is in it?* If $\Sigma^{m-1}\in K*K$ then yes:- now $\Sigma\setminus\overline{\Sigma}=\Sigma$, and the m-pseudomanifold condition tells us $L\Sigma\setminus\overline{\Sigma}$ has 2 vertices but the join of Σ with this edge is not in $K\cdot K$. Also, *for $m=2n+1$, this is just a reformulation of* (2.2+5):- say $\Sigma=(\sigma^{n-1},\theta^n)$, then $L\Sigma=(L\sigma,\emptyset)$, so $\Sigma\setminus\overline{\Sigma}=(\sigma\setminus\theta,\theta\setminus\sigma)$ and $L\Sigma\setminus\overline{\Sigma}=(L\sigma\setminus\theta,\emptyset)$. ■ So maybe a similar induction shows the answer is yes in general, but let's first look at some consequences it would have.

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$\forall\Omega^m\in K\cdot K, u\in\Omega\setminus\overline{\Omega}\ \exists v\in\overline{\Omega}\setminus\Omega$ s.t.$(\Omega\setminus u)\cup v\in K\cdot K$:- If no such v, first part of assumed Q for $\Sigma=\Omega\setminus u$ gives $(\Sigma\setminus\overline{\Sigma})\cdot(L\Sigma\setminus\overline{\Sigma}\setminus\lambda=L\Sigma\setminus\lambda)\in K\cdot K$, but second *seems* to imply $(\Sigma\setminus\overline{\Sigma})\cdot(L\Sigma\setminus\lambda)\notin K\cdot K\ \forall\ \Sigma^{m-1}\in K\cdot K$ ...

In *barrick one* for $m=2n+1$ we had first proved first part of Q, viz., (2.2), and more than above 'seems' was directly available which we saw in last item is untrue for smaller m, but this does not exclude 'seems' from second part of Q, which in that paper was (2.5) later; but maybe combined statement (2.2+5) contains 'K a matroid' in it?

Alas, *pairs reformulation of* (2.1)–any $\Sigma^{2n}\in K\cdot K$ has valence at least 3 and at most $n+3=\frac{m}{2}+\frac{5}{2}$, etc.—*is as such also false for $m<2n+1$*:- Yin-yang $K=\partial(\sigma^m)\cup\{\text{pt}\}$ has $K*K$ an m-sphere, but $\Sigma^{m-1}=(\alpha,\beta)\in K\cdot K$ has valence 2 if α or $\beta=\{\text{pt}\}$; if α and β are nonempty faces of σ^m with i and $m-i$ vertices then $L\alpha$ and $L\beta$ have $m+1-i$ and $i+1$ vertices, so $L\Sigma$ has $m+2$ vertices; and if α or β is empty then also $m+2$. Again, $K=\mathbb{R}P^2_6$ has $K*K$ a 4-sphere, but $\Sigma^3\in K\cdot K$ has valence 4 if α and β are edges, or valence 6 if one is a triangle and other a vertex of K. ■

Indeed Q *also has answer no*:- e.g., if $K=\mathbb{R}P^2_6$ and $\Sigma^3=(\alpha,\alpha)\in K\cdot K$ with α any edge of K, then $\Sigma\setminus\overline{\Sigma}$ is empty, but $L\Sigma\setminus\overline{\Sigma}=(\beta,\beta)$ where β is the edge of K having the two vertices in $L\alpha$ sure is in $K\cdot K$. ■

So the steps we took to do case $m = 2n+1$ don't generalise; more positively: *is there a simple general way we didn't take?* Not using a nice order, which lies more in the province of *barrick two,* i.e., genericity.

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Starting afresh, we are given a simplicial complex K such that $K * K$ is a pseudomanifold. Consider the complete join factorization of K. It follows by the join formula that the deleted join of each factor is a pseudomanifold. So our job is: *to show that a join-irreducible K with $K * K$ an m-pseudomanifold has $m + 2$ vertices and is a nice or yin-yang complex.*

We saw that join-irreducible is same as saying we can go from any vertex to any other such that consecutive vertices belong to a circuit. This suggests that we start with the m vertices of K in any $\Sigma^{m-1} \in K * K$ plus the two in its link. We want to show these are all the vertices of K. If there is an $(m+3)$rd vertex using circuit-connectivity we need to show that either $K * K$ is not m-pure or has an $(m-1)$-simplex with valence not equal to two.

Looking first at small m should clarify more, but only after a short break [33] here will we resume these cogitations.

sarkaria_2000@yahoo.com September 16, 2022

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September 22, 2022 : Z's birthday sketch done, so short break is now over, during which also mulled asides *Paper Boat* and *Religious Orientations* : both shall be mathematical only, but with some other end-notes. [34]

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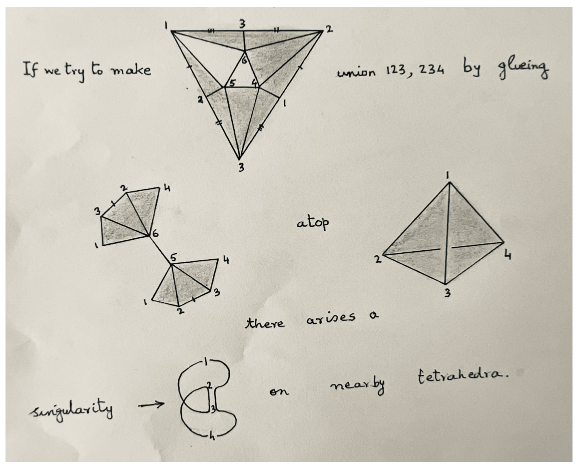
We are assuming $m > \dim(K)$; so, *if K has an $(m-1)$-simplex α, then it does have $m+2$ vertices*:- for $(\alpha, \emptyset) \in K * K$ has 2 vertices in its link. More generally, *the subcomplex complementing any maximal simplex α of K is a pseudomanifold $[K \setminus \alpha]$ of dimension $m - \dim(\alpha) - 1$.* Also, *if K is join-factorizable* [35] *so are all these pseudomanifolds $[K \setminus \alpha]$*:- maximal simplices of a join are joins of maximal simplices of its factors, with these complementing pseudomanifolds joins of their complements. ■ Conversely if K is join-irreducible our job is to show they are not just irreducible but spheres with the least number of vertices. Anyway we are now reduced to $n < m - 1 < 2n$, so we'll start with $(n, m) = (2, 4)$, and it seems the above constraints should suffice to show there are only finitely many K^n with $K * K$ an m-pseudomanifold ... [36]

$- : - : -$

[33] Hauptsächlich für etwas Skizzen und Papierschiffchen.
[34] Also saw a krait on beesmukhi today.
[35] with neither factor a point
[36] here also ਸੂਰਜ ਪਰਿਥੱਵੀ ਦੇ ਗਿਰਦ ਘੁਮਦਾ ਹੈ

There are only finitely many K^2 *with* $K*K$ *a* 4*-pseudomanifold: such a* K *has at most* 7 *vertices*:- If K has a maximal edge α its $[K \setminus \alpha]$ is a 2-pseudomanifold; if it has a singular vertex, or is non-singular with more than 5 vertices, it has–look at it à la Schlegel–a disjoint pair of triangles[37] which is a no-no because K_* is 4-dimensional; so $[K \setminus \alpha] = S^2_4$ *or* S^2_5.

Otherwise any maximal $\alpha \in K$ is a triangle, and being a 1-manifold $[K\setminus\alpha]$ is a disjoint union of polygons. Any ordered pair (σ, θ) of disjoint edges of $[K \setminus \alpha]$ gives a 3-simplex of $K * K$; since it has valence 2, both edges must be incident to only one triangle, and for these 2 triangles to be not disjoint their third vertices $v_\sigma, v_\theta \in \alpha$ must be the same. Unless $[K \setminus \alpha]$ *is a single polygon with at most* 4 *vertices*: given any two adjacent edges σ and σ', amongst edges θ disjoint from σ there is one also disjoint from σ'; so K^2 is the cone over a $v \in \alpha$ of its opposite edge and these polygons; which is not possible because $[K \setminus \beta]$ is not a union of polygons for any triangle $\beta \neq \alpha$. \blacksquare[38]



A yin-yang K^2

Examples. The cone of y-y $\partial\sigma^3 \cup \{\text{pt}\}$ gives a y-y K^3 with one maximal edge complemented by $\partial\sigma^3 = S^2_4$; otoh $\{\text{three points}\}$ joined to $\partial\sigma^2 \cup \{\text{pt}\}$ is a 7-vertex K^2 with all three maximal edges complemented by the join of the $\{\text{other two points}\}$ and $\partial\sigma^2$, i.e., an S^2_5; and $\{\text{pt}\} \cdot \{3 \text{ pts}\} \cdot \{3 \text{ pts }\}$ is a pure 7-vertex K^2 with all triangles complemented by 4-gons $\{2 \text{ pts}\} \cdot \{2 \text{ pts}\}$. Besides $\mathbb{R}P^2_6$ there are other (non-manifold) join-irreducible 6-vertex yin-yang K^2 with triangles complemented by 3-gons, and a maximal edge if any complemented by an S^2_4. At first flush it seems, for example, that the K^2 above with maximal edge 56, obtained from $\mathbb{R}P^2_6$ by replacing its incident triangles by their complements, embeds in $\mathbb{R}^3$. For, minus this edge, it is but $\partial(1234)$ union cones over 5 and 6 of the edge paths $12, 23, 34$ and $14, 32, 24$ from 1 to 4. However this edge forces us to erect these two cones on the same side of the tetrahedral boundary, and this

[37] but some triangles may not be disjoint from any other, e.g., the two unsubdivided triangles of the simplicial 2-spheres obtained by deriving one edge of σ^4_2 repeatedly.

[38] hier auch paper boat

cannot be done without a singularity because of the needed reversal of direction in the common middle edge of the two paths. ■

Likewise, using the hands-on method of van Kampen, this minimally non-embeddable nature of any Pontryagin complex was shown by Schild in his nice paper. Or, à la Flores, as we noted on p. 49 [39], *for any* $(m+2)$*-vertex yin-yang complex* K *the* m*-pseudomanifold* $K * K$ *is in fact an antipodal* m*-sphere,* so there is no embedding of K in $\mathbb{R}^{m-1}$, for it would give a $\mathbb{Z}/2$-map $K*K \to S^{m-1}$ contradicting Borsuk-Ulam. ■ We'll now elaborate on the 'surgery' which shows the spherical nature of these pseudomanifolds.

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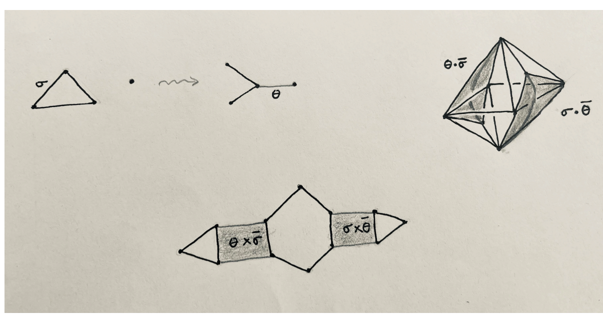
Note this 'surgery' $(\sigma, \partial\theta) \to (\partial\sigma, \theta)$ looks like usual Milnor surgery, indeed is the same if we interpret parentheses as products instead of joins. Let's see the difference by looking as well at the *deleted product,* the time $t = 1/2$ subspace of the deleted join, as we strip the four triangles of $\partial(\sigma^3) \cup \{\mathrm{pt}\}$ one by one and replace by complementary edges:- To start with this cell complex is a disjoint union of three 2-spheres: one antipodal and two that switch under the $\mathbb{Z}/2$-action. The first 'move' puts two antipodal handles to make a single antipodal sphere, then each move results in one by one attachment of three more pairs of handles, so: *the deleted product of the* 5*-vertex Kuratowski graph* σ_1^4 *is a surface of genus* 6 *equipped with a free* $\mathbb{Z}_2$*-action.* Otoh join-surgery uses disjoint 4-cells $\{\sigma \cdot \overline{\theta}, \theta \cdot \overline{\sigma}\}$, pasted atop extant 3-cells $\{\sigma \cdot \overline{\partial\theta}, \partial\theta \cdot \overline{\sigma}\}$, to replace them by the 3-cells $\{\partial\sigma \cdot \overline{\theta}, \theta \cdot \overline{\partial\sigma}\}$, on the other sides of the 2-spheres $\{\partial\sigma \cdot \overline{\partial\theta}, \partial\theta \cdot \overline{\partial\sigma}\}$: which does not modify the manifold.■ [40]



join vs product surgery

Likewise the deleted join K_* of any $K = \partial(\sigma^m) \cup \{\mathrm{pt}\}$ is obtained from the deleted join of σ^m, the octahedral m-sphere, by replacing $\{\sigma^m, \overline{\sigma^m}\}$ by the cones of their boundaries over $\{\overline{\mathrm{pt}}, \mathrm{pt}\}$. Otoh its deleted product $K_\#$ has three disjoint $(m-1)$-spheres: the two copies of $\partial(\sigma^m)$ whose vertices are the midpoints of the edges of these two cones, they switch under the $\mathbb{Z}/2$-action, and an antipodal

[39]Or even in a Reviewer's Note of 1993: luckily I found these scribbles in pencil, for the new AMS has distanced itself from the old wisdom that knowledge multiplies by sharing, MathSciNet is now exclusively for duly institutionalized members!

[40]See one-dimensional (1991) pp. 87-88 for more on deleted products and joins of K^1.

cellular $(m-1)$-sphere whose vertices are the remaining $t = 1/2$ midpoints of edges. To keep the set of vertices same *we disallow replacement of a maximal vertex by its complement,* so any first 'move' on K replaces an $(m-1)$-simplex of $\partial(\sigma^m)$ by an edge from the opposite vertex to $\{\text{pt}\}$. This puts two antipodal index 1 handles on $K_\#$ which modify it to a single antipodal $(m-1)$-sphere, the deleted product of the new y-y complex obtained after this move; and the same game is on again … ■ Apparently in the three decades I've been away from this part of our garden no one has paused to: *work out what all these* $\mathbb{Z}/2$*-manifolds* $Y_\#$ *can be,* as Y runs over this nice family of all yin-yang simplicial complexes, so we shall linger a bit more on them …

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Indeed, $K_\#$ *is a manifold if* K *is any Pontryagin complex,* i.e., join of y-y complexes:- By join formula we know K_* is a sphere, à fortiori a manifold, which implies star of any vertex $v \times \overline{w} \in K_\#$, i.e., the barycentre of an edge $v \cdot \overline{w} \in K_*$ with one end in K and the other in $\overline{K}$, is an open cell of dimension one less: for any simplex $\sigma \cdot \overline{\theta} \in K_*$ incident to $v \cdot \overline{w}$ cuts the $t = 1/2$ subspace of K_* which $K_\#$ covers, in the cell $\sigma \times \overline{\theta} \in K_*$ incident to $v \times \overline{w}$. ■ Our conjecture says these are all: $K_\#$ is a manifold *iff* K is Pontryagin.

Working out some aspects of these cute manifolds $K_\#$ is a pleasant job, for living as they are in spheres K_* of a dimension more, they are *orientable,* etc., and *the* $\mathbb{Z}/2$*-homotopy type of* $K_\#$ *is that of* $K_* \setminus K \cup \overline{K}$:- note this open set is the disjoint union of open segments, one through each point of $K_\#$, now shrink all these segments towards these midpoints. ■

So, e.g., $K_\#$ *is a surface of genus* 4 *for the Kuratowski graph* $K = \sigma_0^2 \cdot \sigma_0^2$:- by Alexander duality $S^3 \setminus K \cup \overline{K}$ has $b_1 = 8$. ■ And, *for any Kuratowski* n*-complex we know at least the Betti numbers of the* $2n$*-manifold* $K_\#$:- e.g., those of σ_2^6 are $b_0 = 1, b_1 = 0$ and $b_2 = 20$ because $\binom{7}{1} - \binom{7}{2} + \binom{7}{3} = 21$, so $(\sigma_2^6)_\#$ *is a simply connected* 4*-manifold with* $b_2 = 40$, etc. ■ Otoh, $(\mathbb{R}P_6^2)_\#$ *is the orientable* 3*-manifold with* π_1 *the Klein four group.* ■ Since this is pleasantly addictive but perhaps done better with a machine we'll leave this job here …

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Rebrowsing what Grothendieck circulated [41] in 1986 was sad, but reminded me why he stopped at just $\mathbb{R}P_6^2$:- the thing is he was in psychoanalysis by now; a christmas tree with stars of david and an icosahedral pendant models a web of yin-yang pairs of psyche; but he stresses this list can vary, and is unsure of the pseudomanifold constraint.

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October 30, 2022. Due to a sudden health emergency on this day I have to suspend mathematics for some time. Two weeks later things are looking up but it will still be a long haul. There were some important thoughts from

[41] That is reference [5] of linear vs piecewise-linear; see also $3\frac{1}{2}$, (50.41).

my notebook that I was about to enter here. By and by I shall at least try to put these down in this space. Also I'll be working on a note or a blurb if you will entitled Pontryagin Complexes which will set out for the convenience of the interested reader that despite all his weaknesses and to put it mildly a maddeningly different course the mathematician K S Sarkaria has changed the look of mathematics as a whole. You'll have to deal with it why he of all people was able following an eye for simplicity and beauty able to change the noisy market place – yes the clickety clack of 'results' is a part of this scene and I think I've had the pleasure of fitting in my share of puzzles too but this is only part of mathematics –into something as a whole beautiful and very simple ...

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December 25, 2022. Okay, so the blurb got done, remains to put here those thoughts that I was about to put down on October 29 when I fainted etc. I'll content myself with just one and final concerning what I was reading from Grothendieck's 1986 writings that I've referred to before. Note he was not at all old then, only 58, while I reading him was in my 77th year; he was done with mathematics, my best mathematics came after my 64th year!

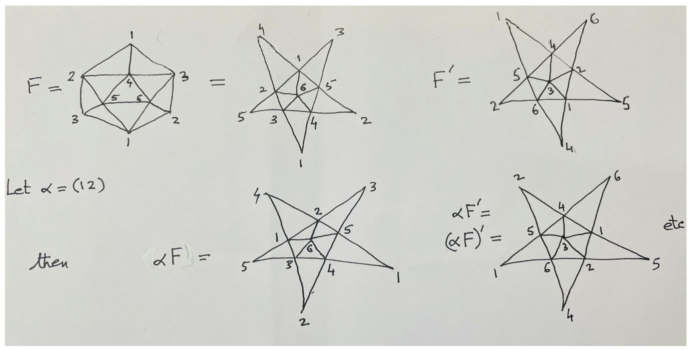
Anyway let me come to the item itself. What G calls an *icosahedral structure* F on a set S of six vertices is a choice of (necessarily ten) triangles such that each of the 15 edges is incident to precisely 2 of these triangles, that is it *is same as a* $\mathbb{R}P^2_6$. But (leaving aside the context–psychoanalysis–which gave rise to it) G's focus is on counting how many of these icosahedral structures are there. In fashion characteristic of all his writings he lays all this down as a long *Théorème 1* containing parts (a) to (f) and then proves it equally at length. However no competent mathematician should have any difficulty checking all the statements without looking at the proofs.

Basically we are just looking at the action of the group of all bijections of the six element set S on the set of all icosahedral structures on this same set. The result is any two of these structures are isomorphic under some permutation and *there are in all twelve distinct icosahedral structures* on S. These occur in pairs: associate to each F the icosahedral structure F' given by taking the complementary ten triangle. Denote by Σ the set of all *biicosahedral structures* $\{F, F'\}$ on the six vertex set S. Thus our action gives *an automorphism of the group of all bijections of S with those of Σ.*

Grothendieck doesn't say it but this action is *exotic* that is to say (after any identification of S and Σ) not an inner automorphism. On the other hand – Segal (1948) is one simple exposition – *the automorphisms of the group of all permutatins of any set of cardinality other than six are all inner.*This exoticity is clear from noting the action of a transposition α on Σ: it can be seen that it switches each biicosahedral structure with another one. This should be clear from the figures below, where incidentally unlike Grothendieck I've preferred [42] the most unforgettable way of drawing an $\mathbb{R}P^2_6$, viz., draw the usual

[42]He prefers the usual face-centred view of an icosahedron's half, the pentagram reflects a vertex-centred view of the same; a similar planar edge-centred representation with the closed

star of school, i.e. a pentagram, then derive the pentagon and identify five pairs of opposite boundary edges. That α becomes a product of three disjoint transpositions of Σ rules out an inner automorphism. ■



What about all simplices with vertices in S? The 5-simplex is fixed under all permutations. Deleting this we get S^4_6, the minimal trangulation of the 4-sphere. An altogether deeper question with the word 'exotic' is an outstanding problem of mathematics: *does S^4 have an exotic smooth structure?* The majority opinion is it has infinitely many though none has been found. Once mulling it over some I thought: *maybe S^4 has precisely two smooth structures?* It is a huge jump but I've this quixotic feeling that $Aut(S_6)/Inn(S_6) = \mathbb{Z}_2$ is somehow pointing towards this! *With this thought to chew on I will now close this miscellany* fully knowing many gaps and errors in it need to be addressed ... -

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star of edge in the interior of the polygon is not possible. However note all this is btw, we are dealing with topologically accurate triangulations only.

Does all shown in ζ paper VII hinge just on functional equation of that theta $2\psi+1$? Will theta–hobson link make this some Poincaré duality of tilings? Link ϑ–$\pi_i(S^2)$ is a cartesian–or haha kardashian?–hint of link with RH.

On April 20th sank in Riemann multidimensional theta only, adjective Siegel in Mumford-Umemura only screws chronology: solving biquadratics by four half turns was by Jacobi's time, by Riemann's all was ready, except that relativistic or automorphic lifting later by Poincaré, of meromorphic functions on surfaces of any genus, which were known since Abel. So that new application of monodromy left by Galois was Jordan's to mop up in 1870, it is (sum of) angles zero case in our hobson construction, so tilings modular not compact.

Maybe Riemann's theta θ vanishing theorem of XI is for seed case all angles 0 only? Or maybe more because he counts in XXXI all abelian functions of that genus as those on whose mod 2 homology class ϑ is nonzero? Such counting starting with VI uses Dirichlet principle, so laplacian, was polished by Roch his student, and pushed all the way to index formula by Atiyah; so no wonder he could again give a modern count this way. Of number of complex structures, relating them with spin cobordism. Maybe–using XI–one should say θ not identically zero eqvt to ϑ not zero mod two. Translated into tilings, this counting of ϑ should suffice for *first result in this continuation.* $\square$

Since above floundering but re-reading Johnson et al on relation of Atiyah's invariant to Arf's the nonsense in last para can perhaps now be weeded out and that first result correctly stated and shown.[43] Tantalisingly near is planarity criterion of Pontryagin and Kuratowski, the van Kampen invariant, its possible link to this of Arf-Atiyah, cf., Eccles Akhmetiev Melikhov, homotopy entrails of $\widehat{\mathbb{R}^4}$, all built maybe by relativistic periodic motions, showing exactly two distinct Lipschitz structures on 4-sphere?

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Theta embeddings of tori in complex projective spaces are in Lefschetz, 1927 and *TAMS,* and Lang's *Abelian varietes* has story from here to Weil conjectures. Counting toral structures is in internal or pontryagin analysis of manifolds, as well as relativistic building from motion only of all this matter. This all evoked those real analytic lie algebra toral structures with arbitrarily slow spectral non-degenerescence : we used infinite dimensionality of a non-final term, which is not available after complexification, but it is likely there are complex toral examples too : showing that beyond nice lattice tied presumably kähler structures even tori admit lots of complex wildness. About that flamboyant purely cartesian conjecture for the 4-sphere we made it implies that sliced knot computer search started by Freedman and now joined by Manolescu is going to end up with zilch: no exotic 4-sphere this way : but – as suggested by said cartesian broadly – the fact that $\pi_4(S^2)$ has two elements will give by motion matter identity a unique non-relativistic or exotic structure on it.

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[43] Should see this notebook and redraw first figure.

Vandermonde's determinant used by Belyi in second proof of his theorem that any surface is made by a rational polynomial branched over three points had evoked again connection with nice triangulations; at least for most genera our argument had in fact fourth color non-singular; then one can complexify to do over complex numbers; Weil's descent made slick by Grothendieck – a paper by Goldring – makes it over rationals, q.e.d? After Sjisling and Voight's "On calculating Belyi maps" now certain a nice triangulation with D trivalent gives Belyi's theorem, even via Andreev's circle packing. To join beehive and say $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with 'esquisse please', here's a *dessin d'enfant* [diagram]: all AB edges give one, for there is a prescribed cyclic order around each vertex, conversely from this data can make nice triangulation. $\square$ Old annotations on a 2001 Garrett page show only descent was in doubt then, now Kock with *field automorphisms of $\mathbb{C}$ preserving branched complex covering* helped, then nice overview by two above took care of rest, but joining a hive is not nice, a new one is: *nice triangulations are nicer than wise crow's dessins,* so cartesian asks: *can they all be made as above over $\mathbb{Q}$?* [Sept 13, 2021]

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From K a matrix, characteristic polynomial, Jordan canonical? Mused on Allan's musing about Heisenberg[44] but could only think of resemblance between join and $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ with group operation $(t_1+t_2, x_1+x_2, y_1+y_2)$ except in time also add half $x_1\cdot y_2 - y_1\cdot x_2$ for wee non-commutativity. Brodie's thesis recaps this group has up to unitary equivalence but one irreducible infinite dimensional representation, this gives quantum mechanics, and besides only one-dimensional representations that give classical mechanics. Possible close connection with lexicographic order also didn't give anything interesting, only usual ordering of weights, and of partitions like Young's. Thangavelu's book doing laplacian on it was there, and others, drier, on connection with theta functions, Weil representation, etc., that is in Mumford's Tata.

About the canonical chain homotopy of Weil maps : note Weil algebras and that of polynomial coefficient forms very close, also that $\iota_d + d\iota_X = L_X$ has rhs zero on constant forms, that alone feature in S&E. In that paper a Lie algebra was involved, so V had this structure, giving us the filtration of the thesis, from this infinitesimal formula was perhaps made a 2-chain homotopy with term E_2 of this spectral sequence being thus identified to Chern classes, the connection definition that is? But for moment this aside too ignored.

Preprints keep to chronology better, and despite–or maybe because of–their shortcomings are better too for someone else to get into the game. This apropos some other scans put, e.g., the two pages added to my zeta paper with Björner, but haven't got time yet to look at Loday again for that Gysin link to cyclic cohomology, nor read even induction argument.

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[44] After reading S&E when I was in MIT to give some other talk in Stanley's seminar.

Elliptic ω, $\mathbb{F} = C^{-1}(K) \to C^0(K)$ maps $1 = \emptyset$ to $v_1 + \cdots + v_N$, exterior shifting, when is $\overleftarrow{K} \cong \overset{\Leftarrow}{K}$? Square matrix after row permutation – pivoting, full pivoting involves columns too – can be reduced by Gauss elimination to a lower triangular L, the reducing process giving a U, so product of these two. We are doing something like this with chain isomorphisms now. Vandermonde, its square is discriminant – saw ANT made easy by Ash. Of course moment curve, but which – closed, as in discrete Fourier transform ? – order is there, and transcendental enough the discontinuous Galois symmetries. For equivariant set the stage in the quotient DGA determined by the biggest octahedral sphere with all conjugate pairs of vertices; one can again equivariantly stretch or squeeze axes to get twisting diagonal algebra automorphism, and now there is a moment curve and its conjugate, the order of the curve pair and conjugation **type** each basis element in the sense of mli – still Sept 2 2021 not scanned should do – and with genericity here we can expect type-shifted maybe.

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... applies to all groups G, and just the finite groups of motions of the unit circle of the plane should suffice for this theory of Chern Pontryagin classes. The various join formulas etc are clear. Beyond these finite groups the group S^1 itself should be roped in ... again it was here Pontryagin's work on $\pi_4(S^2)$ pushed into the next stem by his student Rokhlin, see last installment of this miscellany, which gave that remarkable 2-dimensional example showing van Kampen's iff cohomology obstruction doesn't work here–a natural homotopy theoretical necessary condition's sufficiency remains open. Should be maybe call $\overline{K}$ 'negative' rather than conjugate of K? We went for the latter–more precisely think of the conjugation that is reflection in the 45 degree line, so making K real part and $\overline{K}$ imaginary part, in a process of complexifying K to keep our set up close to the Hodge diamond and the Kahler package, however now working in a neighbourhood of this complexified K in complexification $\mathbb{C}^N$ of V, in this open context Poincaré duality used by Adiprasito etc doesn't work and things get different, as seen before Stein manifolds etc come to the fore. However thinking of the copy as its negative probably should be pursued as well, now we are restricting to orientation preserving groups of the unit circle

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On Steenrod's 50th death anniversary posted einstein and new List showing ਹੇਵੁਡ ਦੀ ਨਾਬਰਾਬਰੀ ahead of this ਨਿਕ ਸੁਕ; tentative titles for the other two saral parchas are ਯੌਰਦਾਂ ਦਾ ਤਰੀਕਾ and ਸੱਲੀਵਨ ਦਾ ਨਤੀਜਾ.

September 28, 2021, 9:12-9:15 a.m. I was totally surprised in 213 by our son, daughter and granddaughter !!! Minni was in on Manpreet's visit, but when she went to the airport, Mallika and Azeeza had taken her too by complete surprise there !! Together again after a long hiatus due to the virus, our days became happily busy, for example, November 7, 2021 was historic : conquest of bucket bridge by all five of us, *Z* leading, a piece of cake enfin ! Come December it was time for Manpreet to go back, but by then our son-in-law Prabhjot was with us, however by end January it was back to usual.

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November 19, 2021 was great leader's u-turn on "reforms" intended to hand farming here too to oligarchs to "develop," a policy so well-liked by Tweedledum and Tweedledee in all "democracies," that world media barely took notice of the year and a half long protest, and more than a thousand deaths on the road, that it had taken to bring about this temporary reversal.

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On December 18, 2021 two men were lynched, one almost in the *Darbar Sahib* itself, for alleged sacrilege during ਸੋ ਦਰੁ ਰਹਰਾਸਿ, by some alleged followers of the sweet and gentle teachings of Guru Nanak. Despite abundant video evidence, the police failed to identify the assailants: in fact no murder charges were filed! The two victims also remain unidentified, but seemed to be poor and probably mentally unhinged vagrants. Condemnation of the alleged sacrilege was loud, incessant and popular, but of the two murders almost zero: no politician wanted to risk his or her chances in the state elections.

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Usually it is a couple of sitcom episodes, but with the children here, I saw some movies too. *It's a Wonderful Life* (1946) suggested by our daughter-in-law Ravleen was nice, and *Chal Mera Putt* (2019) funny, but the others were jerkily directed: even with subtitles on, I had to ask the kids often to stop and tell me what happened! Inclusive of *Don't Look Up* and *Being the Ricardos* (both 2021) which were hot favourites for the 2022 Oscars.

The former has a Pentagon sized doomsday asteroid that the generals are all set to blow up and save the world, but the oligarch owning not only Dash Cellular but also POTUS wants to mine its precious metals first! He mimics the soft-spoken nerd, once at the top, whose lumbering word processing program comes built-in my laptop (and who owns the most farm land in the USA). For this reason, despite being packed with Tweedledum stars, this movie did not get as many from its newspapers. For the same reason, Tweedledee gave it a good review, even though the muscular oligarch now at the top, and actually given a free rein to mine space, seems more their kind of guy?

These tie-ups and rivalries between oligarchs, who imho now run almost all countries, are x-factors in the twists and turns of the neverending Great Game, but the way things are going, the odds are an asteroid won't be necessary, the insatiable greed of some 'dashboard' will suffice.

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Azeeza loves making and listening to stories, so had brought a lot of picture books; besides Minni borrowed many more, and has still some we used to read with Mallika and Manpreet back in Barrick Street! I loved *Oscar's Book* even more than Azeeza, especially her peals of laughter at how silly this grouch was, who couldn't stand people, yet was nothing without people to ignore! However, she is too small for the original *Alice's Adventures in Wonderland* and *Through the Looking Glass,* which I'd started reading on the side, but finished after she left. For how this inspired me to a discovery about the helicity of hair of right handed humans–it is the opposite to that of the actress in the other film from 2021 above–and much more, see "Actually it was Red's turn ..." which is all but complete since March and I'll post soon.

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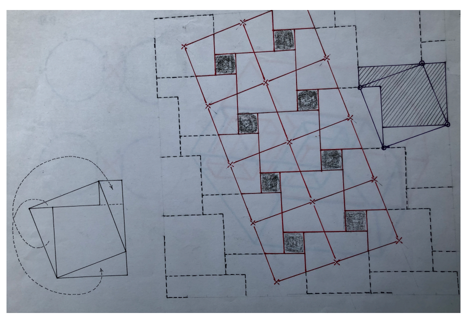
The logo of a coming event[45] is but "Grecian origami" of 213, 16A ! This led to my asking if the $3^n - 1 = \sum_{j\geq 1} \binom{n}{j} 2^j$ varying pieces of an n-cube minus a rotating small central n-cube, obtained by cutting along its hyperplanes, can be re-glued to make a third n-cube: the answer seems no for $n \geq 3$, but is there a nice grecian solution of $x^n + y^n = z^n$ for all n? [46]

  $\square$

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ਜਿਸੁ ਹਥਿ ਜੋਰੁ ਕਰਿ ਵੇਖੈ ਸੋਇ ॥
ਨਾਨਕ ਉਤਮੁ ਨੀਚੁ ਨਾ ਕੋਇ ॥੩੩॥

[45]Marred now by what I've often bemoaned, see e.g. (50.43) of $3\frac{1}{2}$

[46]A simple, but not so nice, grecian solution of any $x^n + y^n = z^n$ is also in the linked paper; and a nice but lost–for a margin was too small!–proof of Fermat tells us no solution is possible for $n \geq 3$ if we insist all pieces be n-cubes of the same size.