Roots of Functions

K. S. Sarkaria

(June 18, 2007, a preliminary version)

§1. Introduction. This paper is a sequel to [11] which had dealt only with roots of translations. It turns out that the geometric method used there – I had used this method previously in linear algebra, see [9] and [10] – extends in a very nice way to all functions k, and for any injective k yields a convenient classification of its qth roots. The continuous results of that paper also generalize, giving all homeomorphisms of a locally path connected Xthat are qth roots of a homeomorphism k without periodic points such that X covers X/k. We also look at increasing roots of increasing integer functions.

§2. Roots using pictures. By a qth *root* of $k : X \to X$ we mean a function $f : X \to X$ such that $f^q = k$. The basic idea that we shall use to get at these roots is simply to visualize the given function k by means of the directed graph whose vertices are all the points of X, and whose arrows are all ordered pairs $(x, k(x))$. Since 'graph of a function' has a different and entrenched meaning, we'll call this directed graph the **picture** of k, and denote it by Pic(k). Note that it is rather special: there is just one arrow issuing out of each vertex x. The arrow may loop back to x, this happens iff x is a *fixed point* of k; more generally, a finite sequence of arrows may form a closed **loop**, this happens iff the function has a *periodic point*, i.e., an x such that some positive iterate of k maps x to x. We partition the set X into the subsets C consisting of vertices belonging to the same **component** (which will also be denoted C) of the picture. Alternatively, C is a maximal subset of X such that any two points are mapped by suitable iterates of k to the same point.

(2.1) *A component of a picture is either an endless* **tree***, or else, has one and only one loop, which must be its* **sink**, that is to say, the components of any picture are typically as depicted below.

Figure 1

This follows because an arrow not belonging to a loop can conceivably end at one of its vertices, but it can never issue out of one, for then that vertex would have at least two arrows issuing from it.

(2.2) *A root of k maps arrows of Pic(k) to arrows.*

This is the same as saying that a qth root f of k *commutes* with k, that is $fk = kf$, which is true because both sides equal f^{q+1}.

(2.3) It follows from (2.2) that the root f images a component into a component. Starting with any component C_0 of $\mathrm{Pic}(k)$, let C_1 denote the component into which f maps it, then let C_2 be the component into which f maps C_1, etc. From $f^q = k$ it follows that after d steps, with d some divisor of q, we must have $C_d = C_0$. Thus the set of all components partitions off into cyclically ordered finite subsets, which we call the **orbits** of f; each has a cardinality that divides q, and its cyclic order indicates how its components 'rotate' under f.

(2.4) **Theorem.** *If the cardinality of an orbit is a proper divisor d of q, then each component C of this orbit must contains a loop.*

Let $x \in C$. Then $f^d(x)$ is also in the component C. So we can find u and v such that $k^u(x) = k^v(f^d(x))$, i.e., $f^{uq}(x) = f^{vq+d}(x)$. Here uq and $vq + d$ are distinct multiples of d, because only the former is divisible by q. So it follows that some $y = f^r(x)$, where r is a multiple of d – so $y \in C$ – is a periodic point of f. It must also be a periodic point of k because $f^t(y) = y$ implies $f^{tq}(y) = y$ and so $k^t(y) = y$.

(2.5) So, if $\mathrm{Pic}(k)$ has no loops, i.e., if its **first Betti number** $b_1(k)$ is zero, then each orbit of a qth root must have cardinality exactly q. If, moreover, the **zeroth Betti number** $b_0(k)$ of $\mathrm{Pic}(k)$– recall that this counts its components – is finite, then a qth root of k can exist at all only if q divides $b_0(k)$. We note that this homological statement is a far-reaching generalization of that I.M.0. 1987 problem which had initiated [1].

§3. Roots of injective functions. In this section we give a complete description—see (3.3) to (3.5)—and enumeration—see (3.7)—of the qth roots of an injective k, using the fact that such a k has a very simple picture.

(3.1) *If $k : X \to X$ is injective, then any component of* $\mathrm{Pic}(k)$ *must be a loop, or a* **doubly infinite string** $\cdots \bullet \longrightarrow \bullet \longrightarrow \bullet \cdots$ *or else, a* **singly infinite string** $\bullet \longrightarrow \bullet \longrightarrow \bullet \cdots$

This follows because now each vertex has at most one arrow entering it, as well as a unique arrow leaving it.

Note also that the injective function is bijective on the union of its loops and doubly infinite strings, and the points of X which are not in $\mathrm{Im}(k)$ are the initial vertices of its singly infinite strings.

(3.2) We begin by noting that since its qth power k is injective, the root f and all its other powers are also injective. Since these injective maps image arrows to arrows, it follows using (3.1) that *components belonging to the same orbit of f must be isomorphic to each other.*[1] So only three cases can occur.

[1]On the other hand, easy examples show that an orbit of a root of a non-injective k can contain non-isomorphic components.

(3.3) **CASE 1.** *The orbit has cardinality d and all its components are loops with r arrows* (recall that d is a divisor of q).

Suppose $x \in C_0$ is u arrows behind $f^d(x) \in C_0$. Since $f^q(x) = k(x)$ is situated one arrow after x in this r-arrow loop, we must have $uq/d = vr + 1$ for some integer v, so the following condition is necessary.

(3.3.1) *q/d and r must be relatively prime to each other.*

Conversely, let d be such a divisor of q. Then we can find integers u and v such that $uq/d = vr + 1$. Since all such u's are obtainable from one of them by adding multiples of r to it, we can assume our u to be positive. Let $C_0, C_1, \ldots, C_{d-1}, C_d = C_0$, be any cyclically ordered cardinality d set of r-arrow loops of $\mathrm{Pic}(k)$. Choose any $x_0 \in C_0, x_1 \in C_1, \ldots x_{d-1} \in C_{d-1}$, and let f be the arrow preserving bijection on the union of these loops which images x_0 to x_1, ... , x_{d-2} to x_{d-1}, and x_{d-1} to the vertex of C_0 which is u arrows ahead of x_0 in the loop C_0. Then $f^q = k$ on the union of these loops.

(3.4) **CASE 2.** *A cardinality q orbit with components doubly infinite strings.*
Take any $x_0 \in C_0$, then f is determined on orbit once we know $f(x_0) = x_1 \in C_1, \ldots f(x_{q-2}) = x_{q-1} \in C_{q-1}$. Conversely, if $C_0, C_1, \ldots, C_{q-1}, C_q = C_0$ is any cyclically ordered cardinality q set of doubly infinite strings of $\mathrm{Pic}(k)$, with an $x_i \in C_i$ chosen in each string, and f is the arrow preserving bijection on the union of these strings which takes x_0 to x_1, x_1 to x_2, ... , x_{q-2} to x_{q-1}, and x_{q-1} to $k(x_0) \in C_0$, then $f^q = k$ on the union of these strings.

(3.5) **CASE 3.** *A cardinality q orbit with components singly infinite strings.*
The one-to-one f cannot be onto on the union of these strings, because then $f^q = k$ would be onto. However, if some point is in Im f, then so are all the subsequent vertices of that string. Therefore, the initial point of at least one of these strings is not in Im f. We will use C_0 to denote such a string and call its initial point x_0. Since f images $f^{q-1}x_0 \in C_{q-1}$ to the next point $k(x_0)$ of C_0, it follows that $f^{q-1}x_0$ must be the initial point x_{q-1} of C_{q-1}: otherwise, the previous point of C_{q-1} will have nowhere to go to under the arrow preserving f. So f images $f^{q-2}x_0 \in C_{q-2}$ to the initial point of C_{q-1}, which in turn implies that $f^{q-2}x_0$ must be the initial point x_{q-2} of C_{q-2}: otherwise, the previous point of C_{q-2} will have nowhere to go to under the arrow preserving f. Continuing thus, we see that the first $q-1$ applications of f must image x_0 to the initial points $x_1 \in C_1, \ \ldots \ , x_{q-1} \in C_{q-1}$ of the other strings. Hence x_0 is, in fact, within the union of these strings, the *unique* point which is not in Im f, and the root f is determined once we know this distinguished point.

Conversely, if $C_0, C_1, \ldots, C_{q-1}$ is any totally ordered cardinality q set of singly infinite string components of $\mathrm{Pic}(k)$, with initial points $x_i \in C_i$, and f is the arrow preserving bijection on the union of these strings which takes x_0 to x_1, x_1 to x_2, ... , x_{q-2} to x_{q-1}, and x_{q-1} to $k(x_0) \in C_0$, then $f^q = k$ on the union of these strings.

(3.6) The picture of any one-to-one k is clearly determined, up to an isomorphism of directed graphs, by the **invariants** c_r, $s_{\mp\infty}$, and $s_{+\infty}$ which count, respectively, the number of its r-arrow loops, doubly infinite strings, and singly infinite strings. From a knowledge of these cardinal numbers, we can work out exactly how many qth roots k has, and what these roots look like.

(3.7) **ENUMERATION.** *A one-to-one map k has a qth root iff its invariants c_r, $s_{\mp\infty}$, and $s_{+\infty}$ are divisible by (q,r), q and q respectively. If, moreover, $s_{\mp\infty} > 0$ or $\sum_r c_r$ or $s_{+\infty}$ is infinite, then k has infinitely many qth roots; otherwise, the number of its qth roots is finite and given by*

$$\frac{s_{+\infty}!}{(s_{+\infty}/q)!} \prod_{r,R,d} \frac{R_d!}{(R_d/d)!d^{(R_d/d)}}\; r^{R_d}.$$

Here, for each $r \geq 1$, R runs over all partitions of the cardinality c_r set of r-arrow loops into parts R_d indexed and divisible by divisors d of q such that q/d is relatively prime to r. Such d's are multiples of the least among them, viz. (q,r), so such a partition R exists iff c_r is divisible by (q,r); the nature and number of these R's depends on r, c_r and q only.

From Case 2, a qth root partitions the $s_{\mp\infty}$ doubly infinite strings into cyclically ordered cardinality q subsets – so q divides $s_{\mp\infty}$ – each contributing to the root one doubly infinite string interweaving (in the prescribed cyclic order) through all the vertices of all the q strings of each such subset. Since the number of such interweaving strings is infinite, there are infinitely many roots if $s_{\mp\infty}$ is a positive multiple of q.

From Case 3, a qth root partitions the $s_{+\infty}$ singly infinite string into totally ordered cardinality q subsets – so q divides $s_{+\infty}$ – each contributing to the root *the* singly infinite string that starts at the initial vertex of the first string and interweaves (in the induced cyclic order) through all the vertices of all the q strings. The first factor of the displayed formula arises because a cardinality uv set partitions into totally ordered cardinality u sets in $(uv)!/v!$ ways.

From Case 1, a qth root partitions, for each $r \geq 1$, the c_r r-arrow loops into cyclically ordered cardinality d subsets, where d is a divisor of q such that q/d is relatively prime to r, with each of these subsets contributing to the root one of the rd-arrow loop interweaving (in the prescribed cyclic order) through all the vertices of all its d loops. The last factor of the displayed formula arises because there are r^d such interweaving loops. The remaining factors arise because, if we forget cyclic orders and combine subsets of the same size d we get a partition R with parts R_d indexed and divisible by such divisors d, and a cardinality uv set partitions into cyclically ordered cardinality u sets in $(uv)!/v!u^v$ ways.

(3.8) Note that we have also worked out **the possible pictures of a root:** $s_{\mp\infty}(f) = s_{\mp\infty}(k)/q$, $s_{+\infty}(f) = s_{+\infty}(k)/q$, and $c_t(f) = \sum_r (r/t)R_{t/r}$, summation over all divisors r of t, and each R a partition of a cardinality $c_r(k)$ set into parts R_d indexed and divisible by divisors d of q such that $(d/q, r) = 1$.

4. Homeomorphisms without periodic points. Before we go further, it will be convenient to *reformulate* the above results in another language.

We begin by noting that any $k : X \to X$ induces the identity map on the quotient set X/k of all components of $\mathrm{Pic}(k)$, while a qth root $f : X \to X$ of k induces a qth root $\phi : X/k \to X/k$ of this identity map.

Thus what we did above was that, for injections k, we worked out exactly which qth roots $\phi : X/k \to X/k$ of the identity **lift** to qth roots $f : X \to X$ of k, and how, and to how many.

For example, for a bijection k with no periodic points we saw that ϕ lifts, and then in infinitely many ways, if and only if all orbits of ϕ have cardinality q, and such ϕ's exist if and only if the cardinality of X/k is divisible by q, in particular if X/k is infinite.

We'll now work out a 'continuous' analogue—that is, now X shall carry a non-discrete topology, and we'll be interested only in maps well-behaved with respect to this topology—of this example.

(4.1) *A homeomorphism $k : X \to X$ with no periodic points can have a homeomorphism as a qth root only if the quotient topological space X/k admits a* **free $\mathbb{Z}/q$-action**.

This is immediate, because the map $\phi : X/k \to X/k$ induced by such a root f is also a homeomorphism, and as already remarked above, all its orbits[2] must have cardinality q.

(4.2) Obvious as it is, the above observation (4.1) is quite useful, because many interesting results about free $\mathbb{Z}/q$-actions are known.

For starters, a product of circles admits the free $\mathbb{Z}/q$-action that rotates a nonempty subset of factors by $2\pi/q$. Likewise, by rotating all coordinates of $\mathbb{C}^n$ by $2\pi/q$ one obtains a free $\mathbb{Z}/q$-action on the $(2n-1)$-dimensional sphere formed by points at distance 1 from the origin.

As against this, *for $q > 2$, an even dimensional sphere does not admit a free $\mathbb{Z}/q$-action:* ϕ^2 preserves the sphere's orientation, so its Lefschetz number (the alternating trace sum of the maps induced in homology) is 2, which contradicts the fact—see, e.g., Spanier [15], p. 195—that ϕ^2 has no fixed point.

Somewhat less elementary is a 1941 result of P. A. Smith [13]: *for q a prime power, a Euclidean space can not admit a free $\mathbb{Z}/q$-action.*[3]

For more on topological group actions see, for example, Bredon [2].

(4.3) *Suppose one knows all the free $\mathbb{Z}_q$-actions ϕ on X/k, can one then work out all the qth roots f of k?*

The answer to this natural question is 'no' in general. Indeed, a good general correspondence between ϕ's and f's is ruled out because the topology of X/k may not be at all close to that of X. For example, a rotation $k : S^1 \to S^1$ of the circle by an irrational multiple of 2π is a homeomorphism without periodic points whose quotient S^1/k inherits the indiscrete topology.

[2]We note that the usual definition $\{\phi^m(C) : m \in \mathbb{Z}\}$ of an 'orbit of ϕ' coincides with that of an 'orbit of f' given in (2.3).

[3]It was [13] which ushered in equivariant cohomology, a tool that also suffices to establish the intermediary – as in Smith [14] and Eilenberg [5] – generalized Borsuk-Ulam theorem. The linear versions of these Borsuk-Ulam type of results are also interesting, and can be proved directly without the prime power restriction: see [8].

(4.4) However we'll see below that there is a very satisfactory answer to the above question (4.3) for the case when X is a connected **manifold**, and the quotient map $p : X \to X/k$ is a **covering map**.[4]

The assumed connectedness of X now packs a considerable punch, it implies that the base manifold $B := X/k$ *is connected, but not simply connected.* So, for example, now X/k cannot be an n-sphere with $n \geq 2$, even though in (4.2) we saw that, for n odd, these admit free $\mathbb{Z}/q$-actions for any q.

We choose any $b_0 \in B$ and a point $x_0 \in X$ above it, $p(x_0) = b_0$. The **fundamental groups** (of homotopy classes of loops) of the two manifolds at these chosen base points will be denoted $\pi_1(B)$ and $\pi_1(X)$ respectively. We recall that the covering map $p : X \to B$ induces a group monomorphism $p_* : \pi_1(X) \to \pi_1(B)$.

Given a free $\mathbb{Z}_q$-action ϕ on B, choose a path λ from b_0 to $\phi(b_0)$. Concatenating the q paths λ, $\phi(\lambda)$, ..., $\phi^{q-1}(\lambda)$ one obtains a loop at b_0 which threads through the orbit of this point. The unique path of the covering manifold, starting at its base point x_0, and lying above this loop, must end at some point $k^n(x_0)$ of the fiber $p^{-1}(b_0)$. We shall denote this integer n by $n(\phi, \lambda)$.

(4.5) **Theorem.** *For a free $\mathbb{Z}_q$-action $\phi : B \to B$ to lift to a homeomorphism of the connected manifold X which is a qth root of k* (with k satisfying the conditions above) *it is necessary and sufficient that the induced group isomorphism ϕ_* of $\pi_1(B)$ maps the subgroup $p_*(\pi_1(X))$ onto itself, and that $n(\phi, \lambda) = 1$ mod q; and when these conditions hold, there is only one such qth root.*

Since $\phi p = pf$ implies $\phi_* p_* = p_* f_*$ the condition $\phi_*(p_*(\pi_1(X)) = p_*(\pi_1(X))$ is necessary. Conversely, if this condition holds, there is a unique homeomorphism $f : X \to X$ satisfying $\phi p = pf$, and with $f(x_0)$ a preassigned point of X above $\phi(b_0)$. We recall—cf. Greenberg [3], p. 22—the definition of f. Given any $x \in X$, and a path μ from x_0 to x, $\phi(p(\mu))$ is a path with initial point $\phi(b_0)$. The final point of the lifted path with initial point $f(x_0)$ does not depend, thanks to the condition, on the path μ from x_0 to x. This final point is $f(x)$. Note that f commutes with k, $f^q = k^{n(f)}$ for some integer $n(f)$, and that the lifted homeomorphism with value $k^m(f(x_0))$ at x_0 is $k^m f$.

If we use for $f(x_0)$ the final point of the path $\tilde{\lambda}$ from x_0 which lifts λ, then the prolongation $f(\tilde{\lambda})$ of this path lifts $\phi(\lambda)$. So the final point, of the path from x_0 lifting the concatenation of λ and $\phi(\lambda)$, is none other than $f^2(x_0)$. Continuing thus we see $n(\phi, \lambda) = n(f)$. One has $n(f) = 1$ if and only if f is a qth root of k, then $n(\phi, \lambda) = 1$. Conversely, if $n(\phi, \lambda) = 1 + qm$, then $k^{-m} f$ is the required and unique qth root of k which is a lift of ϕ.

(4.6) We note that the above proof showed in fact that if ϕ can be lifted, its lifts are qth roots of k^i, where i runs through the mod q residue class of $n(\phi, \lambda)$. We'll call this residue class the k**-index** of the action ϕ.

[4]In other words, the free action of $\mathbb{Z}$ on X, via the integral powers of the homeomorphism k, should be 'properly discontinuous'. The local condition 'manifold' has been adopted to reduce verbiage: the requisite covering space theory—see, e.g., Greenberg [7], pp. 17-27—works just as well for all 'locally path connected and semi-locally simply connected' spaces.

For example, if $k : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ is the translation of the second factor by 2π, then $p = (\mathrm{id}, \exp)$: $S^1 \times \mathbb{R} \to S^1 \times S^1$ is a covering map, and

(a) the free $\mathbb{Z}/2$-action which switches the two circles does not lift,

(b) the free $\mathbb{Z}/q$-action which rotates the first circle by $2\pi/q$ lifts, but since it has k-index 0 mod q, none of its lifts is a qth root of k, while

(c) the free $\mathbb{Z}/q$-action which rotates the second circle by $2\pi/q$ has k-index 1 mod q, so one and only one of its lifts is a qth root of k.

(4.7) **A generalization.** If the manifold X is not assumed connected in (4.4) then one gets an interesting mélange of the discrete and the continuous.

If homeomorphism $k : X \to X$ has qth root $f : X \to X$, then the induced bijection $k_\sharp : \pi_0(X) \to \pi_0(X)$ of the **set** $\pi_0(X)$ **of path components** has qth root $f_\sharp : \pi_0(X) \to \pi_0(X)$. So, using (3.4) and (3.3), we already know that the doubly infinite strings of $\mathrm{Pic}(k_\sharp)$ must partition off into cardinality q subsets, while for each $t \geq 1$, the t-arrow loops of $\mathrm{Pic}(k_\sharp)$ must partition off into some cardinality d subsets, with each d some divisor of q such that q/d is relatively prime to t (and each of these parts is cyclically permuted by $f_\sharp$).

Any $E \in \pi_0(X)$ projects under the covering map $p : X \to X/k$ to a path component $B := p(E)$ of X/k. Set $t = \infty$ if the path components $k^n(E), n \in \mathbb{Z}$, of X are all distinct,[5] otherwise let t be the least positive integer such that $k^t(E) = E$. So either $p^{-1}(B)$, the space covering B, has infinitely many path components, or else its path components are $E, k(E), \ldots, k^{t-1}(E)$.

Coming to $\mathrm{Pic}(\phi_\sharp)$, this must be a disjoint union of d-arrow *loops* $\{\phi^d(B) = B, \phi(B), \ldots, \phi^{d-1}(B)\}$ *of homeomorphic components of* X/k *having isomorphic coverings, with* d *a divisor of* q *such that* $(q/d, t) = 1$, where t denotes the number of path components in the total space of each of these isomorphic coverings. Here we have adopted the convention $(n, \infty) = 1$ iff $n = 1$, so for $t = \infty$ only $d = q$ is allowed, while for $t = 1$ any divisor d of q is eligible.

The isomorphism (f, ϕ) from the covering of B to the covering of $\phi(B)$ gives *group isomorphisms* $\phi_* : \pi_1(B) \to \pi_1(\phi(B))$ *that map subgroup* $p_*(\pi_1(p^{-1}(B)))$ *isomorphically onto* $p_*(\pi_1(p^{-1}(\phi(B))))$. Conversely, these group theoretic conditions ensure that each homeomorphism $\phi : B \to \phi(B)$ of path components of the base extends to an isomorphism of their coverings.

For t finite choose integers r and s such that $rq/d + st = 1$. Note that s is uniquely defined mod q/d, its residue class in $\mathbb{Z}/(q/d)$ being in fact the *inverse* of the residue class of t. The free $\mathbb{Z}/(q/d)$-action $\phi^d : B \to B$ lifts, in the connected covering space E of $B = E/k^t$, to the (q/d)th root $k^{-r}f^d : E \to E$ of k^{st}. (Let $k^{-r}f^d(E) = k^v(E)$, then vq/d—where q/d is prime to t—is divisible by t, so v must itself be divisible by t.) By (4.5)-(4.6) this implies that *the* k^t*-index of* ϕ^d *should be equal to* s *mod* q/d, and conversely, this condition ensures that ϕ^d lifts to a homeomorphism $\Phi : E \to E$ which is a (q/d)th root of k^{st}.

The conditions listed above are sufficient that ϕ *lifts, over the union of the path components* $\{\phi^d(B) = B, \phi(B), \ldots, \phi^{d-1}(B)\}$, *to a* q*th root of* k. Indeed, for t finite, the required qth roots of k can all be obtained from any given lift of ϕ over this union, by adjusting it by means of composition with suitable powers

[5] Now the covering is trivial, $p^{-1}(B) \cong B \times \mathbb{Z}$, and B can be simply connected.

of k over the d components, in such a way that the dth iterate of the adjusted lift coincides over B with the lift of ϕ^d over B which extends the homeomorphism $k^r\Phi : E \to k^r(E)$. For the case $t = \infty$, $d = q$, we adjust in the same manner to make the qth iterate coincide over B with the lift k of $\phi^q = id$.

(4.8) **Homeomorphisms of the line.** Specializing now to $X = \mathbb{R}$, we note that a homeomorphism of $\mathbb{R}$ is orientation reversing iff it is strictly decreasing, so it must have one and only one fixed point. On the other hand, any periodic point of an orientation preserving homeomorphism of $\mathbb{R}$ must be a fixed point of the same, because it is strictly increasing. So a *fixed point free homeomorphism k of $\mathbb{R}$ has no periodic points,* furthermore (4.4) holds, indeed we assert that the quotient map $p : \mathbb{R} \to \mathbb{R}/k$ *is a covering of the circle.*

(4.8.1) To see this note that a fixed point free strictly increasing homeomorphism is either *progressive* or *regressive,* i.e., one has either $k(x) > x$, or else $k(x) < x$, for all x. Accordingly, all its doubly infinite strings $\{k^n(x), n \in \mathbb{Z}\}$ are discretely embedded in $\mathbb{R}$ as strictly increasing, or else strictly decreasing, unbounded doubly infinite sequences. If we fix one string, any other string has one and only one member in each of the disjoint open intervals of $\mathbb{R}$ constituting its complement, and this open set is the union of all these strings. So the quotient space $\mathbb{R}/k$ is a compact 1-dimensional manifold, and the quotient map p is a covering map of this (topological) circle.

(4.8.2) We shall work out in this subsection all *the free $\mathbb{Z}/q$-actions ϕ on $\mathbb{R}/k \cong S^1$ and their k-indices.*

Let's say that the usual orientation of $\mathbb{R}$ corresponds under p to the *clockwise* orientation of S^1. Any free $\mathbb{Z}/q$-action ϕ has to preserves this orientation, otherwise it would have a fixed point. However, this does not imply that the q points $\{b, \phi(b), \ldots, \phi^{q-1}(b)\}$ of an orbit must occur in clockwise order, all it tells us is that if b is followed by $\phi^s(b)$, then after this should come $\phi^{2s}(b)$, and so on. Since the q points are distinct, $s \in \{0, 1, \ldots, q-1\}$ *is relatively prime to q, and is the same for all $b \in S^1$*, because it depends continuously on b.

Conversely, if a cardinality q subset $\{b_0, b_1, \ldots, b_{q-1}\}$ of the circle occurs on it in clockwise order $(b_0, b_s, b_{2s}, \ldots)$, where $(s, q) = 1$ and the suffixes are to be read mod q, then we can find all the ϕ's with $\phi^i(b_0) = b_i$ as follows. On the clockwise arc $[b_0, b_s]$, ϕ can be any increasing homeomorphism to arc $[b_1, b_{s+1}]$, then on this any increasing homeomorphism to arc $[b_2, b_{s+2}]$, till finally we arrive at clockwise arc $[b_{q-1}, b_{q-1+s}]$; on it ϕ must be equal to that homeomorphism to the original arc $[b_0, b_s]$ which ensures that ϕ^q becomes its identity map.

To compute the k-index (4.6) of this ϕ, we choose for the path λ of (4.4) the clockwise arc from b_0 to b_1. The points b_{is} subdivide it into t subintervals, where t is the smallest positive integer such that $ts = 1$ mod q. So the concatenation of paths $\lambda\phi(\lambda) \cdots \phi^q(\lambda)$ is the clockwise loop starting at b_0 and ending at it after going around the circle t times. It lifts in $\mathbb{R}$ to a path starting at an x_0 above b_0 and ending at $k^{\pm t}(x_0)$, the sign depending on whether the homeomorphism k is progressive or regressive. Thus $n(\phi, \lambda) = \pm t$ and so *the k-index of ϕ equals $\pm t$ mod q depending on whether k is progressive or regressive.*

(4.8.3) Using (4.5) which tells us that ϕ's with k-index 1 are in bijective correspondence qth roots of k we get the following.

For a progressive fixed point free k, the ϕ's that lift to roots have $t = 1$, i.e., their orbits must occurs in clockwise order; while, for a regressive k, the ϕ's that lift to roots have $t = q - 1$, i.e., their orbits occur in counterclockwise order.

For the apparently very special case of a k obtained by adding a nonzero constant—note that this *translation* is progressive or regressive depending on whether the constant is positive or negative—this result reduces to Proposition 6 of [3]. However, it is not hard to see that the fixed point free homeomorphisms of $\mathbb{R}$ belong to just two *conjugacy classes* – progressive and regressive – of the group of all orientation preserving homeomorphisms of $\mathbb{R}$. Likewise that, any free $\mathbb{Z}/q$ action ϕ of S^1 is conjugate, in the group of all orientation preserving homeomorphisms of the circle, to a *rotation* by $t(2\pi/q)$, for a unique positive integer t less than q and relatively prime to it.

(4.8.4) An *arbitrary orientation preserving homeomorphism* $k : \mathbb{R} \to \mathbb{R}$ restricts to strictly increasing fixed point free homeomorphism $k|I_j$ of each of the countably many disjoint open intervals I_j whose union is the complement of the closed set $\mathrm{Fix}(k)$ of fixed points. Since an open interval is homeomorphic to $\mathbb{R}$, we know from above all the qth roots of each $k|I_j$. Thus *the identity map of* $\mathrm{Fix}(k)$ *extends to homeomorphisms f of $\mathbb{R}$ that are qth roots of k:* just define each $f|I_j$ to be some qth root of $k|I_j$. Alternatively, we have just extended the identity map of $\mathrm{Fix}(k)$ by using roots on the complement supplied by this simple instance of (4.7) with $d = 1$ and $t = 1$: the complement of $\mathrm{Fix}(k)$ is a covering space over countably many circles which are preserved by ϕ, and the covering is a single copy of the line over each circle.

(4.8.5) As against this, *a strictly decreasing homeomorphism k of $\mathbb{R}$ has no qth root for q even,* as is obvious from orientation considerations. Now k has one and only one fixed point k_0, and the remaining periodic points have period 2, because they are also periodic points of k^2, which is orientation preserving on the two disjoint open intervals I_1 and I_2 whose union is $\mathbb{R} \setminus k_0$. So these come in pairs (x_1, x_2), $x_1 \in \mathrm{Fix}(k^2|I_1)$, $x_2 \in \mathrm{Fix}(k^2|I_2)$, such that $k(x_1) = x_2$ and $k(x_2) = x_1$. Note that, for any odd q, k is its own qth root on the closed set $\mathrm{Per}(k)$ of periodic points. Furthermore, *for any odd q, the restriction of k to* $\mathrm{Per}(k)$ *extends to homeomorphisms f of $\mathbb{R}$ that are qth roots of k.* One defines f on the complement to be any of the qth roots supplied by applying (4.7): under p this complement is a covering space over countably many circles which are preserved by ϕ, with two copies of the line over each circle – so d = 1, t = 2 – and $(q/d, t) = 1$ holds because it is the same as saying that q is odd.

§5. Monotonic roots. We have already dealt with an order theoretic problem above: the usual topology of $\mathbb{R}$ (or, more generally, of any interval of real numbers) coincides with its order topology, *an orientation preserving homeomorphism $k : \mathbb{R} \to \mathbb{R}$ is the same thing as an order preserving bijection of $\mathbb{R}$,* and we worked out in (4.8) others that are its qth roots.

(5.1) For $\mathbb{Z}$, the analogous problem is trivial: *an order preserving $k : \mathbb{Z} \to \mathbb{Z}$*

is a bijection iff it is a **translation**, that is, $k(x) = x + k(0)\ \forall x$, so k can have another such bijection f as a qth root iff $k(0)$ is divisible by q, and then this root is unique, being the translation $f(x) = x + k(0)/q$.

Thus, for $k(0) \neq 0$, though there are by (4.8) infinitely many increasing homeomorphisms of $\mathbb{R}$ whose qth iterate is the translation of $\mathbb{R}$ by $k(0)$, at most one of these roots maps $\mathbb{Z}$ to $\mathbb{Z}$. As against this, though there are by [11] or §3 infinitely many bijections of $\mathbb{Z}$ whose qth iterate is the translation k of $\mathbb{Z}$, at most one of these roots is increasing.

(5.2) This exemplifies two natural ways of attacking *the problem of finding the order preserving roots of a given fixed point free order preserving integer function k, say of the set $\mathbb{Z}_+$ of positive integers.*[6] In the first, one extends the given function to a homeomorphism of a real interval, and seeks, amongst the homeomorphic roots provided by (4.8), those that take integers to integers. For example, if k is a **dilatation**, i.e., if $k(x) = k(1)x\ \forall x \in \mathbb{Z}_+$ – the constant $k(1)$ will often be denoted K – we can search amongst the continuous roots of the dilatation of the positive reals $\mathbb{R}_+$ by the same multiple K.[7]

In the second – which is the one we'll pursue for the time being – we search in the set of all qth roots $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ of the injection k, which can be described fully by using §3. We note that $\mathrm{Pic}(k)$ is a disjoint union of singly infinite strings, with initial points 1 and the integers 'jumped' by k. So, since an open interval with integer end points contains one less integer than its length, *a qth root exists if and only if $1 + \sum_n (k(n+1) - k(n) - 1)$ is divisible by q.*

Moreover, we know that these roots f also have only singly infinite strings, and are in one-one correspondence with partitions of the strings of k into cardinality q totally ordered parts: each string of f 'interweaving', in this order, through all the points of the q strings. We'll call these qth **root strings** of k, and note that such a string is determined by any q consecutive points, because the next q consecutive points are their k images.

The subset of increasing roots f is determined by the following additional pictorial condition on the constituent root strings.

(5.3) *A fixed point free function f of the positive integers is strictly increasing iff* $\mathrm{Pic}(f)$ *consists of pairwise* **alternating** *increasing strings.*

From $f(1) > 1$ and f strictly increasing we get $f(n) > n$, so strings are increasing; also, if $n < m < f(n)$ then $f(n) < f(m) < f^2(n)$, so string of m alternates with that of n. Conversely, if $n < m$ are on the same string, then $f(n) < f(m)$ because string is increasing; otherwise $f^{i-1}(n) < m < f^i(n)$ for some $i \geq 1$, and the alternating condition again gives $f(n) < f(m)$.

(5.4) However, an arbitrary set of pairwise alternating increasing strings of positive integers cannot occur as a subset of strings of such an $f : \mathbb{Z}_+ \to \mathbb{Z}_+$, it

[6] An arbitrary integer set bounded below can be treated just like $\mathbb{Z}_+$; if set is bounded above, strings become decreasing; finally, a set unbounded above and below can be partitioned into a progressive (strings increasing) and a regressive (strings decreasing) part.

[7] Note that $\exp : \mathbb{R} \to \mathbb{R}_+$ converts translations of $\mathbb{R}$ into dilatations of $\mathbb{R}_+$, but non-identity translations of $\mathbb{Z}$ are quite different from non-identity dilatations of $\mathbb{Z}_+$: the former have only finitely many doubly infinite strings, the latter infinitely many singly infinite strings.

must be **expanding** in $\mathbb{Z}_+$, i.e., if $a \to a'$ and $b \to b'$ are any two arrows with $a < b$, then $b - a \leq b' - a'$. This follows because f has to map the $b - a + 1$ integers between a and b into distinct integers between a' and b'.

In other words, the **gaps** $a_{i+1} - a_i$ of a string, $\cdots < a_i < a_{i+1} < a_{i+2} < \cdots$, as well as the gaps $b_i - a_i$ (lag) and $a_{i+1} - b_i$ (advance) between it and another string, $\cdots < a_i < b_i < a_{i+1} < b_{i+1} < \cdots$, should be non-decreasing with i.

Figure 3

These conditions suggest a mechanical procedure for constructing increasing roots, say, an increasing solution of $f^2(n) = 4n \ \forall n \geq 1$. For the procedure to begin at all we require an increasing root string through 1 with non-decreasing gaps. For the example being considered this requirement is met by the first string $1, 2, 4, \ldots$ shown in Figure 3 (the other increasing root string $1, 3, 4, \ldots$ from 1 does not satisfy the gap condition). The second string is to begin with the least number not already used, that is 3, and will be determined once we decide its next number. The alternating condition tells us that f should map 3 to a number between 4 and 8. We have chosen $f(3)$ to be the least such number, 5, noting that then the gaps of the new string, the amounts by which it lags the previous string, and, the amounts by which it is in advance of the previous string, are all non-decreasing as desired. Likewise, each subsequent root string of Figure 3 – note that each is determined by, and so we may only write, its first two numbers – starts from the least number not used in the previous strings, and its chosen f-image satisfies these conditions.

It remains to prove that once begun, the procedure shall never encounter an 'obstruction' to such a choice. This we do now, in more generality, to analogously construct increasing qth roots of many increasing and 'concave up' k's.

(5.5) **Theorem.** *A strictly increasing function $k : \mathbb{Z}_+ \to \mathbb{Z}_+$ with $k(1) > 1$ and non-decreasing slope $k(n+1) - k(n)$ has an increasing qth root iff it has an increasing qth root string with non-decreasing gaps starting from 1.*

Since k is fixed point free and injective, an increasing root f is fixed point free and strictly increasing, and from (5.2)-(5.4) we know that the component of $\mathrm{Pic}(f)$ containing 1 must be a string of the stated kind.

Conversely, *starting with the given string from 1, we'll construct an increasing qth root f, root string by root string, in ascending order of initial points.*

For the inductive step, let x_1 be the smallest number not in the already constructed finite and expanding set of pairwise alternating increasing qth root strings of k (if no such x_1 we are done). Put $a_1 = x_1 - 1$, and of the numbers already used and bigger than x_1, let b_1 be the least, and let $a_1, a_2, \ldots, a_q, a_{q+1} = k(a_1), \ldots$ and $b_1, b_2, \ldots, b_q, b_{q+1} = k(b_1), \ldots$ be the already constructed strings from a_1, b_1 onwards (at the first step both are portions of the given string).

Since no number from (a_1, b_1) has been used so far, no number in (a_2, b_2) can be a non-initial point of an already constructed string, but neither can it be an initial point, because it is bigger than x_1. Proceeding thus, we see that no integer has been used so far from any of the disjoint open intervals of non-decreasing length, $(a_n, b_n), n \geq 1$.

The new root string $x_1, x_2, \ldots, x_q, x_{q+1} = k(x_1), \ldots$ shall be in this available space between the two alternating strings. We have fixed $x_1 \in (a_1, b_1)$ – so we know $x_{tq+1} = k^t(x_1) \in (k^t(a_1), k^t(b_1)) = (a_{tq+1}, b_{tq+1})$ – it remains only to choose $x_2 \in (a_2, b_2), \ldots, x_q \in (a_q, b_q)$.

Any choice gives a root string, for the q numers $x_1, x_2, \ldots, x_q$ are indeed initial points of strings of k. To see this note that a k-preimage of x_1 would be a smaller number not already used which is impossible, for the same reason a k-preimage of an $x_2 \in (a_2, b_2)$ must be bigger than x_1 but this is impossible because the k image $x_{q+1} \in (a_{q+1}, b_{q+1})$ of x_1 is bigger than x_2, etc.

The choice $x_i = a_i + 1$ for all $2 \leq i \leq q$ will ensure that the segment length $|a_n x_n|$ – i.e., the amount by which the string 'x' is in advance of the string 'a' – is non-decreasing, but the segment length $|x_n b_n|$ – i.e., the amount by which the string 'x' lags the string 'b' – may not be non-decreasing, so we shall make the following amended choice.

The interval $(x_1 = a_1 + 1, b_1)$ is no longer than $(k(x_1), k(b_1)) = (x_{q+1}, b_{q+1})$, but it may happen that from some i onwards in $[2, q]$, one has $b_i - a_i - 1 > |x_{q+1} b_{q+1}|$. For all such integers i, and only for these, we replace $a_i + 1$ by the bigger number $x_i \in (a_i, b_i)$ such that $|x_i b_i| = |x_{q+1} b_{q+1}|$.

With this amendment, one has both $|a_1 x_1| \leq \cdots \leq |a_q x_q| \leq |a_{q+1} x_{q+1}|$ and $|x_1 b_1| \leq \cdots \leq |x_q b_q| \leq |x_{q+1} b_{q+1}|$. Since k has non-decreasing slope these inequalities imply – because $k(a_1) = a_{q+1}$, etc. – the next pair of q inequalities $|a_{q+1} x_{q+1}| \leq \cdots \leq |a_{2q} x_{2q}| \leq |a_{2q+1} x_{2q+1}|$ and $|x_{q+1} b_{q+1}| \leq \cdots \leq |x_{2q} b_{2q}| \leq |x_{2q+1} b_{2q+1}|$, and so on. Thus the new root string 'x' is in advance of the string 'a' by a non-decreasing amount $|a_n x_n|$, and at the same time it is lagging the string 'b' by a non-decreasing amount $|x_n b_n|$.

Since $|x_n x_{n+1}| = |x_n b_n| + |b_n a_{n+1}| + |a_{n+1} b_{n+1}|$, and all three quantities are non-decreasing – the middle is the amount by which 'b' *lags* 'a' – it follows that the gaps of 'x' are non-decreasing. Likewise, writing $|\alpha_n x_n| = |\alpha_n a_n| + \rightarrowtail |a_n x_n|$ we verify that 'x' is in advance of any extant string 'α' by a non-decreasing amount, and writing $|x_n \beta_n| = |x_n b_n| + |b_n \beta_n|$ we see that 'x' lags any extant string 'β' by a non-decreasing amount. In other words, the addition of the new string has given us a bigger expanding set of pairwise alternating increasing qth root strings of k, and the proof of the theorem is complete.

Note the *finite nature of the condition*, all we require is the answer to this query: are there integers $1 < a_2 < \cdots < a_q < k(1)$ with $a_2 - 1 \leq \cdots \leq a_q - a_{q-1} \leq k(1) - a_q \leq k(a_2) - k(1)$? If the answer is 'no', then k does not have an increasing qth root. If the answer is 'yes', then – because k has a non-decreasing slope – we do have an increasing qth root string of k with non-decreasing gaps, $1, a_2, \ldots, a_q, a_{q+1} = k(1), a_{q+2} = k(a_2), \ldots$, and the theorem tells us that k has an increasing qth root having this as one of its strings.

(5.6) **Remarks and refinements** (about above theorem and its proof).

(5.6.1) For $k(n) = C + n$, an increasing qth root string $1, a_2, \ldots$ has $a_{q+2} - a_{q+1} = (C + a_2) - (C + 1) = a_2 - 1$, so has non-decreasing gaps only if all are equal to this amount t and $C = tq$ is a multiple of q. Further, starting with $1, t+1, 2t+1 \ldots$, the inductive construction of (5.5) stops after t strings, and gives, in conformity with (5.1), the unique increasing qth root $f(n) = t + n$.

Also note that, *a function* k *as in* (5.5) *is a translation, iff it has constant slope* 1, *iff it has finitely many strings.* For, if i is so big that points a_i, of one of the m strings of such a k, are bigger than the m initial points, then (a_i, a_{i+1}) has length m, so slope is eventually 1, and therefore, because k has non-decreasing slope ≥ 1, always 1. It follows that, *the inductive construction of* (5.5) *requires infinitely many steps for all other k's,* there is always, at each step, an x_1 waiting and available, for starting the new root string from.

(5.6.2) *A non-identity dilatation* $k(n) = Kn$ *of the positive integers has an increasing qth root f iff* $K > q$. For, there is no strictly increasing sequence of integers such that $1 < a_2 < \cdots < a_q < q$, but if $K > q$, then $1, 2, \ldots, q, K, 2K, \ldots$ is an increasing qth root string with non-decreasing gaps, and result follows by Theorem (5.5), with its proof giving us the lexicographically least f which is an increasing qth root of the dilatation. For $q = 2$ and $K \geq 4$ this f coincides, but for an initial 0, with the sequence analyzed by Allouche et al. [1].

An important simplification occurs: *for dilatations the unamended choice* $x_i = a_i + 1$ *for all* $2 \leq i \leq q$ *is valid at any inductive step of* (5.5). That is, with this choice, the lengths $|x_n b_n|$ are non-decreasing. To see this observe that, if $|x_q b_q|$ were bigger than $|x_{q+1} b_{q+1}| = K|x_1 b_1| \geq K$, there is a multiple of K in (a_q, b_q), which, being the k-image of a number less than a_1, must have been already used in a previously constructed string, a contradiction.

The amendment is necessary in general. Suppose, for example, that we are constructing the lexicographically least cube root of $k(n) = 24 + 5n\ \forall n \geq 1$ having as first string, $1, 5, 9, 29, 49, 69, \ldots$ (which is, as required, increasing with non-decreasing gaps). Then the unamended choice of second string $2, 6, 10, 34, 54, 74, \ldots$ is invalid because $34 - 10 > 54 - 34$, the lexicographically least valid choice of a second string is $2, 6, 14, 34, 54, 94, \ldots$ (the first string being 'rigid' in the sense of (5.6.3) below, this is indeed the only valid choice).

However, this finesse is necessary only for the first some strings: *if, in the proof of* (5.5), *one has* $a_1 > k(1)$, *then the unamended choice* $x_i = a_i + 1, 2 \leq i \leq q$ *is valid.* Our hypothesis ensures that there is a biggest j such that $k(j) < x_q$. Since k is strictly increasing, j must be smaller than a_1 because $k(a_1) = a_{q+1}$ is bigger than x_q. Also, by the argument given in the paragraph before last, we know that (a_q, b_q) and so (x_q, b_q) has no point of $\mathrm{im}(k)$. So it follows that $k(j+1) \geq b_q$. Thus the slope of k at j is at least $|x_q b_q|$. On the other hand, the slope of k at x_1 can't exceed $|x_{q+1} b_{q+1}|$, because $x_{q+1} = k(x_1)$ and $b_{q+1} = k(b_1)$. Since k has non-decreasing slope, it follows that we cannot have $|x_q b_q| > |x_{q+1} b_{q+1}|$, i.e., no amendment is necessary.

(5.6.3) *The dilatation* $k(n) = (q+1)n$ *of the positive integers has a unique increasing qth root* f, e.g., the trebling map has a unique square root, viz., Propp's sequence, or A003605 of Sloane [12]).

An increasing qth root f has the string $1, 2, \ldots, q, q+1, 2(q+1), \ldots$ because this is the only increasing root string from 1. This string 'a' is **rigid**, by which we mean that each $a_{i+1} a_{i+2}$ has the same length as $a_i a_{i+1}$ or $a_{i+q} a_{i+1+q}$. In the first case, f maps any $n \in (a_i, a_{i+1})$ to the corresponding point of the equal interval (a_{i+1}, a_{i+2}), in the second to the unique point of (a_{i+1}, a_{i+2}) corresponding to the point $k(n)$ of the equal interval (a_{i+q}, a_{i+1+q}).

The same proof shows, for any $k : \mathbb{Z}_+ \to \mathbb{Z}_+$, that *an increasing qth root having a rigid string is uniquely determined by it and all its strings are rigid.*

Figure 3

For example, for the unique cube root of $n \rightarrowtail 4n$, to work out the string through 36 we observe – Figure 3 – that it is on the arrow $32 \to 48$ of the string through 1, and the next arrow $48 \to 64$ has the same length, but $64 \to 128$ is 4 times longer. This gives us the portion $\to 36 \to 52 \to 80 \to$ and so the entire root string through 36. It too is rigid.

Sometimes an extra condition ensures uniqueness, for example, *there is a unique increasing $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $f^3(n) = 7n$ and $f(1) = 3$,* because the string of f through 1 has to be $1, 3, 5, 7, 21, \ldots,$ and it is rigid.

(5.6.4) *Any dilatation $k(n) = Kn$ has an increasing qth root on $\mathbb{Z}_t$, the set of integers greater than or equal to $t =]\frac{q}{K-1}[$, and this root is unique if q is divisible by $K - 1$,* e.g., the doubling map has a unique square root on $\mathbb{Z}_2$, viz., Mallows' sequence or A007378 of Sloane [12].

All of the above holds with obvious changes if $\mathbb{Z}_+ = \mathbb{Z}_1$ is replaced by the order-isomorphic set $\mathbb{Z}_t$ (or any infinite set of integers bounded below). For the restriction $k : \mathbb{Z}_t \to \mathbb{Z}_t$, the qth root string $t, t+1, \ldots, t+q-1, Kt, \ldots$ from the least point t is increasing with non-decreasing gaps, so we can inductively construct an increasing qth root $f : \mathbb{Z}_t \to \mathbb{Z}_t$ having this as one of its strings. When $Kt = t + q$, i.e., when q is divisible by $K - 1$, this is the only such string, and it is rigid, so then k has a unique increasing qth root on $\mathbb{Z}_t$.

One may lose uniqueness on a smaller domain, but somewhat exceptionally, *on $\mathbb{Z}_2$, the trebling map has exactly* two *increasing square roots*: the possible (increasing with non-decreasing gaps) roots strings from 2 are $2, 3, 6, 9, \ldots,$ and $2, 4, 6, 12, \ldots,$ and both are rigid! On the order-isomorphic set $\mathbb{Z}_+$, this translates back to the statement that *there are precisely two increasing function f such that $f^2(n) = 3n + 2$ for all positive integers n.*

(5.6.5) For the inductive step in (5.5) we had only used *one* particular choice – it is the lexicographically least choice – of $q-1$ numbers, $x_2, \ldots, x_q$, such that the new qth root string 'x' $= x, x_2, \ldots, x_q, x_{q+1} = k(x), \ldots$ is in advance of the extant string 'a' by a non-decreasing amount, and at the same time, lags the extant string 'b' by a non-decreasing amount. If we allow *all* such choices – note that there are only finitely many – at each step, this construction clearly generates *all* the increasing roots of k. This shows that *the increasing qth roots of k are in one-one correspondence with the maximal paths of a directed loopless graph,* namely, the one whose nodes are the steps of the construction – each node will also represent the corresponding finite set of root strings – and the finitely many arrows issuing out of a node represent the aforementioned choices at this step. We shall denote this graph by $\mathrm{Stree}_q(k)$ and call it the qth **solution tree** of k. Since we avoid an 'empty node' with no root strings, this is really a forest with finitely many trees, rooted on initial nodes in one-one correspondence with the qth root strings of k from 1 which are increasing with non-decreasing gaps. On the other hand, *for a non-translation k as in* (5.5) *no (nonempty) node is final,* there is at least one arrow issuing out of it.

For example, $\mathrm{Stree}_3(7n)$ is the disjoint union of three trees with initial nodes the three cubic root strings of $n \rightarrowtail 7n$ from 1 which are increasing and have non-decreasing gaps, viz., $1, 2, 3, 7, \ldots$, $1, 2, 4, 7, \ldots$, and $1, 3, 5, 7, \ldots$. Using (5.6.3) we see that the third tree has no forks at all – i.e., that it is a singly infinite chain – because the third string is rigid. As against this, we'll see below that the first two trees have lots and lots of forks.

(5.6.6) *The number of arrows issuing from a node* of the solution tree, that is, the number of choices $x_2, x_3, \ldots, x_q$ (5.6.5) at this step, depends on the lengths $|a_1b_1| \leq |a_2b_2| \leq \cdots \leq |a_qb_q| \leq |a_{q+1}b_{q+1}|$ and the points $x_1 = x = a_1 + 1$ and $x_{q+1} = k(x)$ in the first and last segments, a_1b_1 and $a_{q+1}b_{q+1} = k(a_1)k(b_1)$. If we think of the last segment as a 'stick' above the first – see Figure 3 – with x_{q+1} vertically above x_1, then what we want *is the number of ways of putting the remaining sticks in order between these two sticks, in such a way that each* **covers** *the preceding*; the intersections of x_1x_{q+1} with these interpolated sticks give the corresponding choice $x_2, x_3, \ldots, x_q$.

Figure 3

Since its length is intermediate, we can always insert a_2b_2 so that $a_{q+1}b_{q+1}$ covers it and it covers a_1b_1, then a_3b_3 so that $a_{q+1}b_{q+1}$ covers it and it covers a_2b_2, etc. So at least one such interpolation exists, explaining again why (for a k with non-decreasing slope) no node of the solution tree is final.

If the sticks are all of the same length as either a_1b_1 or $a_{q+1}b_{q+1}$, then obviously only one interpolation is possible – cf. (5.6.3) – but the converse is *not* true. If the end a_{q+1} of the last stick is vertically above the end a_1 of the first stick, then too, irrespective of their lengths, there is only one way of interpolating the other sticks: with ends a_i above a_1. Note that this case happens iff k has slope 1 at a_1. If k has slope > 1, then $a_{i+1}b_{i+1}$ extends beyond a_1b_1 on both sides, a_{i+1} is not above a_1, nor b_{i+1} above b_1.

With this extra hypothesis the converse is also true, i.e., *for a k with slope bigger than* 1*, there is a unique arrow out of the node if and only if the segments* a_ib_i*,* $1 \leq i \leq q+1$*, are all of the same length as* a_1b_1 *or* $a_{q+1}b_{q+1}$*.*[8] The point being that there are at least two *positions,* for an a_tb_t of a strictly intermediate length, such that it covers the first, and is covered by the last stick. At the left-most position we must have a_t under a_{i+1} or b_t above b_1; in either case – because a_1 is not under a_{i+1} and length of a_tb_t is strictly intermediate – we can move one step to the right to a new position. But surely there are *at least as many arrows as the number of positions of any a_tb_t*: just interpolate – as in the second last paragraph – the a_ib_i's with $i \leq 2 < t$ between the first stick and a

[8] For use in (5.6.7) we note that, for a k with non-decreasing slope, this follows from the statement, "the segments a_ib_i, $2 \leq i \leq q+2$, are all of the same length as a_2b_2 or $a_{q+2}b_{q+2}$." For a k with constant slope, the two statements are equivalent.

chosen position of a_tb_t, and the remaining a_ib_i's between this position of a_tb_t and the last stick.

Note that the right-most positions of all the sticks gives an interpolation of sticks, it corresponds to the lexicographically least choice used in (5.5), now x_i is the least integer such that $1 = |a_1x| \leq |a_ix_i|$ and $|x_ib_i| \leq |k(x)k(b_1)|$. Likewise, the lexicographically biggest choice/arrow corresponds to the interpolation of sticks in which all of them are at their left-most positions, now x_i is the biggest integer such that $|xb_1| \leq |x_ib_i|$ and $|a_ix_i| \leq |k(a_1)k(x)|$.

We know (5.6.3) that there is a unique qth increasing root of k having a prescribed rigid root string through 1, using the criterio just established we'll now show that the situation is quite different if the first string is non-rigid.

(5.6.7) **Proposition.** *If a k as in* (5.5) *has a non-rigid increasing qth root string 'α' with non-decreasing gaps from* 1*, then it has uncountably many increasing qth roots f having 'α' as one of their root strings.*

We know that translations, i.e., k's with constant slope 1, have only rigid root strings, so the slope of the given k must be eventually bigger than 1. Also, the non-rigidity of 'α' means that for all j big enough, the $q+1$ successive gap lengths $\alpha_j\alpha_{j+1}, \dots \alpha_{j+q}\alpha_{j+q+1}$ have more than two values. So, at a node of the solution tree sufficiently many arrows after 'α', the point x_1 – from which the new string shall now be constructed – can be assumed to be in such an interval (α_j, α_{j+1}), with the slope of k bigger than 1 on it.

Now consider the subdivision of this interval (α_j, α_{j+1}) into subintervals (β_j, γ_j) by its points – one per string – that are in the already constructed strings. The slope being bigger than 1, all these subintervals become bigger if q is added to the indices: $\beta_{j+q}\gamma_{j+q} > \beta_j\gamma_q$. On the other hand, their sum $\alpha_j\alpha_{j+1}$ takes a strictly intermediate value if a suitable lesser number than q is added to the indices. It follows that *at least one of these subintervals must be such that the $q+1$ lengths $\beta_j\gamma_j, \dots \beta_{j+q}\gamma_{j+q}$ have more than two values.*

Using this, we'll deduce that *after finitely many arrows from this node we must arrive at a node from which there issue two or more arrows.* For this we need to examine how the next some steps of the inductive construction proceed. The new string starts from $x_1 = a_1 + 1$, the smallest point of *the subset N of (a_i, a_{i+1}) consisting of all points not already in the constructed strings of the given node,* and the remaining points of this string are bigger than a_{i+1}. It may well be that only one such string, i.e. a unique arrow out from the node, is possible. Then we note that, at the next node, the new string shall be built from the second smallest point x_{12} of N, with remaining points of this string again bigger than a_{i+1}. Et cetera. There shall come a stage when the point x_{it}, from which we are going to build the new string, is the first point of either the aforementioned (β_j, γ_j) or $(\beta_{j+1}, \gamma_{j+1})$. At this node, using (5.6.6), we are sure that it is possible to build at least two new strings starting from x_{1t}, i.e., that there are at least two arrows issuing out of this node.

Using this *we associate, to each infinite binary sequence, a distinct increasing qth root of k,* i.e., a distinct infinite path of the solution tree starting from 'α', as follows. The digits of the sequence indicate, in order, the direction which the path is to take from nodes with more than one out-arrows: it chooses the lexicographically least arrow if the digit is 0, and the lexicographically biggest if the digit is 1. Thus there are uncountably many distinct increasing qth roots with 'α' as one of their strings.

(5.6.8) **Slope sequences.** A qth root f of k is determined by $f(1)$ and its slopes $m_f(n) = f(n+1) - f(n),\ n \geq 1$. We note that, *if the slopes of k are bounded by K, then the slopes of f are also bounded by K:* the root string through any a_0 lags that through $b_0 = a_0 + 1$ by a non-decreasing amount, so $m(a_0) = |a_1 b_1| \leq |a_q b_q| = b_q - a_q = k(a_0+1) - k(a_0) = m_k(a_0) \leq K$. The non-decreasing slopes $m_k(n)$ coincide with the least bound K if n is large enough, but the slopes $m_f(n) = m(n)$ of a root are rarely non-decreasing.

Nevertheless, the next result shows that, for an interesting class $\mathcal{L}(k)$ of roots of such a k, the slope sequences $m_f(n)$ display *a simple universal asymptotic behaviour,* which depends only on this least bound K, and is quite independent of both $q \geq 2$, and the particular qth root $f \in \mathcal{L}(k)$ under consideration. Namely, we'll show this for the class of all **eventually least roots** of k, that is, we define $\mathcal{L}(k) = \bigcup_{q\geq 2} \mathcal{L}_q(k)$, where $\mathcal{L}_q(k)$ denotes all qth roots of k that correspond to maximal paths of $\mathrm{Stree}_q(k)$ – see (5.6.5) – in which the lexicographically least arrow is chosen from some node onwards. For example, for the qth root constructed in the proof of Theorem (5.5), this was so from the second node onwards, so it belongs to $\mathcal{L}_q(k)$. However, though $\mathcal{L}(k)$ is usually infinite, it is obviously countable: so, by (5.6.7), most roots of k are outside it.

Theorem. *If the slopes of a k as in* (5.5) *are bounded, with least upper bound $K \geq 2$, and $f \in \mathcal{L}_q(k)$, then, for all n sufficiently big, the slope $m_f(n) = m$ determines the subsequent K consecutive slopes $m_f(k(n)),\ m_f(k(n)+1),\ \ldots\ ,$ $m_f(k(n)+K-1)$ as follows: the first $m-1$ of these slopes are all equal to K, the next $K-m$ are all equal to 1, while the last equals m.*

More precisely, using the hypotheses and (5.6.2) we know that, for all n sufficiently big, one has (i) $m_k(n) = K$, and (ii) all new root strings from initial points bigger than $n+1$ are lexicographically least and unamended. We'll prove that *the above conclusion holds for all n such that* (i) *and* (ii) *hold.*

Therefore, once such an n is known, the entire slope sequence of f can be easily written down, if one knows the **initial segment** $m_f(1), \ldots, m_f(k(n)-1)$, by iteratively extending this initial segment by means of the following **substitutions** (the block of K's has length $m-1$, of 1's length $K-m$):

$$m \rightsquigarrow \underbrace{K \cdots K}\underbrace{1 \cdots 1}m.$$

The slope at n gives the next K terms after the initial segment should be, then the slope at $n+1$ gives the next K terms, and so on. The initial segment depends on the particular root $f \in \mathcal{L}_q$ of k being considered, its extension however involves only the least upper bound K of the slopes of k.

For the proof, let 'a' and 'b' denote the root strings of f through $a_0 = n$ and $b_0 = n+1$, and consider root strings with initial points between them in one of the disjoint intervals $(a_1, b_1), (a_2, b_2), \ldots, (a_q, b_q)$. The interval (a_1, b_1)

has length m, so $m-1$ of these root strings have their initial points in it. We'll think of the points of these strings as 'black dots' – see Figure 5, which depicts the case $q = 3, m = 4, K = 7$. By (ii) each of these strings is in advance of a previous string by 1 till (a_{q+1}, b_{q+1}) in which interval these gaps becomes K.

Figure 5

Since $a_q = f^q(a_0) = k(a_0) = k(n)$ and $b_q = f^q(b_0) = k(b_0) = k(n+1) = k(n) + m_k(n) = k(n) + K$ by (i), the interval (a_q, b_q) has length K and so precisely $K-1$ integer points, viz., $k(n)+1, \dots, k(n)+K-1$. Thus there are $(K-1)-(m-1) = K-m$ more strings of the kind being considered, we'll think of the points of these strings as 'white dots'. The initial points of these strings are in the intervals $(a_i, b_i), i \geq 2$, so by (ii) these strings are in advance of a previous string by 1 even in (a_{q+1}, b_{q+1}).

Thus, as we proceed from a_{q+1} into the length Km interval (a_{q+1}, b_{q+1}), we first encounter, after equal gaps of length K, the $m-1$ black dots; then in succession – i.e., with gaps of length 1 – the $K-m$ white dots; finally, there is a residual gap of length $m = Km - K(m-1) - (K-m)$ between the last white dot and b_{q+1}. These gaps coincide respectively with the K consecutive slopes $m_f(k(n)),\ m_f(k(n)+1),\ \dots, m_f(k(n)+K-1)$, so *q.e.d.*

EXAMPLES. It is often easy to find an n – or even the least such or n_f – for which (i) and (ii) hold, e.g., *the lexicographically least increasing qth root of the K-fold dilatation, $q < K$, has $n_f = 1$ with $m_f(1), \dots, m_f(K-1)$ given by $m_f(q-1) = K-q, m_f(K-1) = q+1$ and all other $m_f(i) = 1$.* Indeed, for this f even the first root string $1, 2, \dots, q, K, \dots$ is chosen lexicographically least, and by (5.6.2) all the subsequent least choices are unamended: so (ii) holds for $n = 1$, (i) is obvious. Also, the next $K-1-q$ root strings begin, $q+i, K+i, \dots$, which suffices to check that the first $K-1$ slopes are as stated.

A convenient notation. By above, the slope sequence – with its first 19 terms displayed – of the least increasing square root of the 4-fold dilatation is

$$\dot{2}\dot{1}\dot{3}(\dot{4}112)(1111)(4413)(4444)\cdots,$$

where the overdots and parentheses indicate how slopes yielded subsequent prolongations, $2 \rightsquigarrow (4112), 1 \rightsquigarrow (1111), 3 \rightsquigarrow (4413), 4 \rightsquigarrow (4444)$; now, putting an overdot on the fifth slope 1 and prolonging the display by its substitution (1111), we can go up to the first 23 terms, and so on, indefinitely. Likewise, the slope sequence of the least (in fact only) increasing cube root of this dilatation is

$$\dot{1}\dot{1}\dot{4}(\dot{1}111)(1111)(4444)(1111)\cdots,$$

while the slope sequences of the lexicographically least increasing square, cube, and quadruple roots of the 5-fold dilatation are, respectively, the following:

$$\dot{3}\dot{1}\dot{1}\dot{3}(\dot{5}5113)(11111)(11111)(55113)(55555)\cdots,$$

$$\dot{1}\dot{2}\dot{1}\dot{4}(\dot{1}1111)(51112)(11111)(55514)(11111)\cdots,$$

$$\dot{1}\dot{1}\dot{1}\dot{5}(\dot{1}1111)(11111)(11111)(55555)(11111)\cdots.$$

It is equally facile to fully work out (the slope sequences of) many other eventually least roots, e.g., generalizing the last example, *the lexicographically least increasing qth root f of the K-fold dilatation with a given first string* $1 = a_0, a_1, \ldots, a_{q-1}, a_q = K, \ldots$, where of course $1 < a_1 < \cdots < a_{q-1} < K$ and $a_1 - 1 \leq a_2 - a_1 \leq \ldots \leq K - a_{q-1}$, for the string is increasing with non-decreasing gaps. Again $n_f = 1$, and the first $K-1$ slopes $f(n+1) - f(n)$, equivalently the values $f(n)$ of the function for $1 \leq n \leq K$, can be computed quickly: one has $f(a_i) = a_{i+1}, f(a_i+1) = a_{i+1}+1, \ldots, f(a_{i+1}-1) = 2a_{i+1} - a_i - 1, f(a_{i+1}) = a_{i+2}$, for $0 \leq i < q-1$, while the remaining values are $f(a_{q-1}+1) = 2K, \ldots, f(a_{q-1} + a_1 - 1) = (a_1 - 1)K, f(a_{q-1} + a_1) = (a_1 - 1)K + 1, \ldots f(K-1) = (a_1 - 1)K + K - 1 - a_{q-1} - a_1, f(K) = a_1 K$. This initial segment of $K-1$ slopes can then be prolonged, just as before in spurts of length K, as per the same substitutions, to get the entire slope sequence.

For instance, the lexicographically least increasing square root f of the 6-fold dilatation with f(1) = 3 has first six values $3, 4, 6, 12, 13, 18$, so its first five slopes are $1, 2, 6, 1, 5$, and its slope sequence is

$$\dot{1}\dot{2}\dot{6}\dot{1}5(111111)(611112)(666666)(111111)\cdots.$$

Such computations may take long, even for a dilatation with a small K, because n_f can be big and the initial segment long, since on the solution tree we might come to the first stably least arrow of $f \in \mathcal{L}$ only after a very large number of nodes. However one can quickly verify that the 5-fold dilatation does have increasing square roots such that $f(1) = 2$ and $(3) = 7$ because the pair of square root strings $\{1, 2, 5, \ldots; 3, 7, 15, \ldots\}$ satisfies the gap conditions. And that, for the lexicographically least of all such roots, one has $n_f = 2$ – so the first overdot is on the second slope for this example – while the first 10 functional values are $2, 5, 7, 8, 10, 11, 15, 20, 21, 25$, which gives us the 9 term initial segment of slopes, and therefore as before the entire slope sequence:

$$3\dot{2}\dot{1}\dot{2}14514(51112)(11111)(51112)\cdots.$$

To be continued ... (Got pre-occupied in other matters at this point, so will finish typing up this paper, and will add the pictures, later on.)

References

[1] J.-P. Allouche, N. Rampersad, and J. Shallit, On integer sequences whose first iterates are linear, *Aequ. Math.* **69** (2005) 114-127.

[2] G. E. Bredon, *Introduction to Compact Transformation Groups,* Academic Press (1972).

[3] P. E. Conner and E. E. Floyd, On the construction of periodic maps without fixed points, *Proc. Amer. Math. Soc.* **10** (1959) 354-360.

[4] A. Cobham, Uniform tag sequences, *Mathematical Systems Theory* **6** (1972) 164-192.

[5] S. Eilenberg, On a theorem of P. A. Smith concerning fixed points for periodic transformations, *Duke Math. Jour.* **6** (1940) 428-437.

[6] —- , *Automata. Languages, and Machines: volume A,* Academic Press (1974).

[7] M. J. Greenberg, *Lectures on Algebraic Topology,* Benjamin (1967).

[8] K. S. Sarkaria, Tverberg partitions and Borsuk-Ulam theorems,

[9] —- , When is the locally nilpotent part a direct summand?, to appear.

[10] —- , On linear maps, to appear.

[11] —- , Roots of translations, *Aequ. Math.,* to appear.

[12] N. J. A. Sloane, *On-line Encyclopedia of Integer Sequences,* at website http://www.research.att.com

[13] P. A. Smith, A theorem on fixed points for periodic transformations, *Ann. of Math.* **35** (1934) 572-578.

[14] —- , A theorem on fixed points for periodic transformations, *Amer. J. Math.* **63** (1941) 1-8.

[15] E. H. Spanier, *Algebraic Topology,* McGraw-Hill (1966).

213, 16A,
Chandigarh 160015, INDIA.

E-mail: sarkaria_ 2000@yahoo.com
Website: http://www.geocities.com/sarkaria_2000/