Shifting and Embeddability

by

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§1. Introduction

It is easy to see that a graph, i.e. a (finite) one-dimensional simplicial complex, embeds in $\mathbb{R}^2$ only if the number of its edges is less than 3 times the number of its vertices. This elementary fact is the foundation for many results of planar graph theory. For example, it shows that a planar graph must have a vertex incident to 5 or less edges, and therefore that 6 colors can be assigned to its vertices in such a way that no 2 adjacent vertices have the same color. By means of a somewhat more involved argument Heawood [7] was able to reduce the number of colors to 5. The celebrated conjecture that infact 4 colors suffice remained open for a long time till it was finally settled by Appel and Haken [2].

Barring some low codimension cases, no such results are known for higher dimensional embedded simplicial complexes $K^n \subset \mathbb{R}^m$, $n \leq m \leq 2n$. This is not surprising because even the basic problem (see e.g. Grünbaum [6], pp. 152-153) regarding the existence of a linear inequality $f_n(K) < C_1.f_{n-1}(K) + C_2.f_{n-2}(K) + \ldots$ for all $K^n \subset \mathbb{R}^m$, $n \leq m \leq 2n$, has remained open for a long time. The object of this note is to give an affirmative solution of this existence problem, and thereby show that it is possible to develop a higher dimensional analogue of planar graph theory for all embedded simplicial complexes $K^n \subset \mathbb{R}^m$, $n \leq m \leq 2n$. The following theorem also establishes some conjectures of [21].

THEOREM (7.2). <u>There exists a constant $C = C(n,m)$ such that a simplicial complex embeds in $\mathbb{R}^m$, $n \leq m \leq 2n$, only if the number of its n-simplices is less than C times the number of its (n-1)-simplices.</u>

There are two main ingredients involved in our proof:

First, we use the method of shifting initiated by Kalai in his thesis [8]. In §2 we will give a self-contained account of this method and develop it further to consider the case of shiftings which preserve a given coloring and group actions preserving this coloring. This work was inspired by the proof of Theorem (3.1) of Björner and Kalai [4]. (They mention that a different proof was given first in Kalai [9]. Note that in particular, for $t = 1$, §2 gives a simple proof of this theorem: it follows at once from the isomorphism $C(K) \cong C(B_K)$ of (4.4).) We have preferred to develop shifting theory in the dual, and apparently more natural , setting of forms on K. This shows e.g. that shifting is one more manifestation of the fruitful idea of Sullivan [27] that problems pertaining to the forms of K can often be profitably studied by replacing K by a suitable algebraic <u>model</u> obtained by considering a subset (e.g. a basis or a set of generators, etc.) of forms which is generic with respect to some conditions.

Second, we use the obstructions to embeddability discovered by van Kampen [28]. In §3 we will recall the Smith theoretic interpretation, due to Wu [30], of these characteristic classes, and then use shifting to examine their non triviality. We note that our proof is quite constructive and gives reasonably small numbers $C(n,m)$ which can obviously be further lowered without much effort. However the problem of finding their best possible values is one of many interesting problems (see no. 8) requiring more new ideas, and will, hopefully, be pursued elsewhere.

§2. Shifting

1. Generalized cohomologies

Let K be a simplicial complex whose set of vertices, $\mathcal{K} = \{v_1, \dots, v_N\}$, is a basis of the vector space V over the field F. We recall that an F valued function on the oriented i-simplices of K, whose values change sign with changes of orientation, is said to be an i-dimensional cochain of K with coefficients in the field F. All such cochains form a vector space $C^i(F)$; we set $C(K) = \sum_{i \geq -1} C^i(K)$. For each $\omega \in V^*$ one has a generalized simplicial coboundary operator $\delta_\omega : C(K) \longrightarrow C(K)$ defined by

$$(1.1)\quad (\delta_\omega c)[v_1, \dots, v_{j+1}] = \sum_{1 \leq k \leq j+1} (-1)^{k-1} \omega(v_k) . c[v_1, \dots, \hat{v}_k, \dots, v_{j+1}].$$

Here $\wedge$ denotes omission. One has $\delta_\omega \delta_\omega = 0$. We will denote the corresponding cohomology, $\ker \delta_\omega / \operatorname{Im} \delta_\omega$, by $H_\omega(K) = \sum_{i \geq -1} H^i_\omega(K)$. The ordinary simplicial coboundary operator $\delta : C(K) \longrightarrow C(K)$ corresponds to the case $\omega = 1_K = v_1^* + \dots + v_N^*$. It gives rise to the ordinary (reduced) cohomology of K with coefficients in F, $\ker \delta / \operatorname{Im} \delta = H(K) = \sum_{i \geq -1} H^i(K)$.

2. Forms on K

We will denote by $A^i(V)$ the $\binom{N}{i}$-dimensional vector space of all skewsymmetric multilinear degree i forms $V \times \dots \times V$ (i times) $\longrightarrow F$. Note that $A^1(V) = V^*$, the dual of V, while $A^0(V) = F$ by convention. We recall that if $\omega \in A^i(V)$ and $\theta \in A^j(V)$, then their exterior product $\omega \wedge \theta \in A^{i+j}(V)$ is defined by

$$(2.1)\quad (\omega \wedge \theta)(v_1, \dots, v_{i+j}) = \sum_\pi (-1)^\pi \omega(v_{\pi(1)}, \dots, v_{\pi(i)}) . \theta(v_{\pi(i+1)}, \dots, v_{\pi(i+j)}).$$

Here π runs over all (i,j)-shuffle permutations of $\{1, \dots, i+j\}$ -- i.e. those which are order preserving on the subsets $\{1, \dots, i\}$ and $\{i+1, \dots, i+j\}$ -- and $(-1)^\pi = \pm 1$ depending on whether π is even or odd.

Under the exterior product, the direct sum $A(V) = \sum_{i \geq 0} A^i(V)$ becomes an associative graded algebra over F with identity $1 \in A^0(V) = F$. Furthermore this algebra is graded commutative, i.e. for any 2 homogenous elements ω and θ one has

$$(2.2)\qquad \omega \wedge \theta = (-1)^{\deg \omega . \deg \theta} (\theta \wedge \omega).$$

Using (2.1) we see that the forms vanishing on simplices of K constitute a graded ideal $I_K = \sum_{i \geq 2} I_K^i$ of A(V). The elements $[\theta]$, $\theta \in A(V)$, of the quotient algebra $A_K = A(V)/I_K$ will be called forms on K. Formula (2.2) shows that the operators δ_ω: $A_K \to A_K$, $\omega \in V^*$, defined by $\delta_\omega[\theta] = [\omega \wedge \theta]$ obey $\delta_\omega \delta_\omega = 0$. We will now identify the corresponding cohomologies $H_\omega(A_K) = \sum_{j \geq 0} H^j_\omega(A_K)$ with the ones defined above.

(2.3) Identification of forms and cochains. There is a canonical degree -1 vector space isomorphism $A_K \to C(K)$ which commutes with the coboundary operators δ_ω. From now on we will use this isomorphism to identify A^j_K with $C^{j-1}(K)$ and $H^j_\omega(A_K)$ with $H^{j-1}_\omega(K)$.

Proof. Each $\theta \in A^j(K)$ determines the cochain $\bar{\theta} \in C^{j-1}(K)$ given by $\bar{\theta}[v_1, \ldots, v_j] = \theta(v_1, \ldots, v_j)$. This linear map $\theta \mapsto \bar{\theta}$ is onto with kernel I^j_K. Thus there is an induced isomorphism $[\theta] \mapsto \bar{\theta}$ of $A^j_K = A^j(V)/I_K$ with $C^{j-1}(K)$. For any $\omega \in V^*$, (2.1) shows that $\delta_\omega : A(V) \to A(V)$ obeys $(\delta_\omega \theta)(v_1, \ldots, v_{j+1}) = \sum_{1 \leq k \leq j+1} (-1)^{k-1} \omega(v_k).\theta(v_1, \ldots, \hat{v}_k, \ldots, v_{j+1})$. Thus, by (1.1), it follows that $[\theta] \mapsto \bar{\theta}$ commutes with the operators δ_ω.

Let $\{v^* : v \in \kappa\}$ be the basis of V^* dual to the basis κ of V. For any $\alpha \subseteq \kappa$, $\alpha^* \in A(V)$ will denote the exterior product of the 1-forms v^*, $v \in \alpha$; note that α^* is defined only upto sign. (A similar notation will be used in no. 3 below for any basis β of V.) The following is immediate from the definition of A_K.

(2.4) Canonical basis of C(K). If $\alpha \subseteq \kappa$, then $[\alpha^*] \in A_K$ is nonzero iff $\alpha \in K$, and all such elements $[\alpha^*]$, $\alpha \in K$, constitute a basis of $A_K = C(K)$.

An $\omega \in V^*$ will be called elliptic with respect to K if $\omega(v)$ is nonzero for all vertices v of K.

(2.5) Diagonal isomorphisms D. For any elliptic $\omega \in V^*$ let D: $C(K) \to C(K)$ be the algebra isomorphism which multiplies each canonical basis element $[\alpha^*]$, $\alpha \in K$, by the product of the numbers $\omega(v)$, $v \in \alpha$. Then $D\delta = \delta_\omega D$.

Proof. It is easily checked that D is induced by an algebra isomorphism $\bar{D}$: $A(V) \to A(V)$ (-- viz. that which multiplies each α^*, $\alpha \subseteq \kappa$, with the product of the numbers $\omega(v)$, $v \in \alpha$ --) obeying $\bar{D}(I_K) = I_K$ and $\bar{D}(1_K) = \omega$. So $D\delta[\theta] = D[1_K \wedge \theta] =$

$[\bar{D} 1_K \wedge \bar{D}\theta] = [\omega \wedge \bar{D}\theta] = \delta_\omega[\bar{D}\theta] = \delta_\omega D[\theta]$ for all $\theta \in A(V)$.

3. Lexicographic bases of C(K)

Note. We will work with a fixed partition, or coloring, $\kappa = \kappa_1 \cup \ldots \cup \kappa_t$, of the vertices of K. Thus we have a direct sum decomposition $V = \sum_{1 \le i \le t} V_i$ where V_i denotes the subspace spanned by κ_i.

Let β be any basis of V which is compatible with this direct sum decomposition, i.e. $\beta = \beta_1 \cup \ldots \cup \beta_t$ where β_i is a basis of V_i. Each $\sigma \subseteq \beta$ has a unique partition $\sigma = \sigma_1 \cup \ldots \cup \sigma_t$, $\sigma_i \subseteq \beta_i$. We equip each β_i with a total order, and denote by $\le$ the resulting partial order on β. We now extend this partial order to a lexicographic partial order $\le_L$ on the subsets of β: $\theta \le_L \sigma$ iff $\theta_i \le_L \sigma_i$, $1 \le i \le t$, where $\theta_i \le_L \sigma_i$ iff $\theta_i \subseteq \sigma_i$ or $\min(\theta_i \,\Delta\, \sigma_i) \in \theta_i$. We will say that $[\theta^*]$ (see no.2 for notation) precedes $[\sigma^*]$ iff $\theta <_L \sigma$.

(3.1) Lexicographic bases B of C(K). The subset $\mathcal{B} = \{[\sigma^*] : \sigma \in B\}$ of $\mathcal{S} = \{[\sigma^*] : \sigma \subseteq \beta\}$, obtained by deleting all forms which are linear combinations of preceding forms, spans the vector space C(K). Furthermore B is a simplicial complex, and any basis $\bar{\mathcal{B}}$ of C(K) contained in $\mathcal{B}$ must contain $\mathcal{B}_i^* = \{[\sigma^*] : \sigma \in B,\ \sigma \subseteq \beta_i\}$, $1 \le i \le t$.

Proof. That the elements of $\mathcal{B}$ are nonzero and span C(K) follows at once from the fact that $\mathcal{S}$ spans C(K). If $[\sigma^*] \in \mathcal{S}$ is equal to a linear combination of some preceding elements then so is any $[\tau^*] \in \mathcal{S}$, $\tau \supset \sigma$. This follows by taking the exterior product of the given linear combination and $[(\tau \setminus \sigma)^*]$. So B is a simplicial complex.

Next, note that there is a direct sum decomposition, $C(K) = A_K = \sum A^{j_1,\ldots,j_t}$, where $A^{j_1,\ldots,j_t}$ denotes forms which vanish unless j_i of the entries are from V_i, $1 \le i \le t$. Each element of $\mathcal{B}$ lies in (precisely) one of these direct summands. Furthermore $\mathcal{B} \cap A^{0,\ldots,j_i=q,\ldots,0} = \{[\sigma^*] : \sigma \in B,\ \sigma \subseteq \beta_i,\ |\sigma| = q\}$ must be linearly independent because, otherwise, $\le_L$ being a total order on this subset of $\mathcal{B}$, one of its elements will be a linear combination of the preceding. Hence any basis $\bar{\mathcal{B}}$ of C(K) contained in $\mathcal{B}$ must contain all such subsets of $\mathcal{B}$.

replace

A basis $\bar{\mathcal{B}}$ of above kind is thus indexed by a simplicial set $\bar{B}$ nested between the simplicial complexes B and B_0. If $\bar{\mathcal{B}}$ is chosen by means of lexicographic deletions arising from any total order of β extending $\leqslant$, then $\bar{B}$ too is a simplicial complex. However it is convenient to use other kinds of $\bar{\mathcal{B}}$ also.

Remark. A partial order analogous to $\leqslant_L$ has been used by Björner, Frankl, and Stanley [3]. Note that when $t = N$, $\leqslant_L$ is $=$ and B is isomorphic to K. At the other extreme, when $t = 1$, $\leqslant_L$ is a total order, and once again $\mathcal{B}$ is a basis of C(K). Such total orders have found some interesting uses -- see e.g. Stanley [26] and Kalai [8] -- since the pioneering work of Macaulay [16].

We will refer to B as a "lexicographic basis" of K even though $\mathcal{B}$ itself is usually not a basis of C(K). However it is possible many times to use some condition on K to infer the linear independence (of some part) of $\mathcal{B}$:

(3.2) If K is a join of its t full monochromatic subcomplexes K_i, $\text{vert}(K_i) = \kappa_i$, then the cardinality of B can be no bigger than that of K, and so $\mathcal{B}$ is a basis of C(K). Note that now B too is a join of its monochromatic subcomplexes B_i, $\text{vert}(B_i) = \beta_i$, $1 \leqslant i \leqslant t$.

(3.3) Note that if K is a join then all color types $(n_1, \ldots, n_t)$, $n_i \leqslant (\dim K_i)+1$, have to occur in K. In fact $\mathcal{B}$ is linearly independent even under the weaker condition that, if a color type $(n_1, \ldots, n_t)$ occurs in K then all $\alpha_1 \cup \ldots \cup \alpha_t$, $\alpha_i \subseteq \kappa_i$, $\alpha_i \in K$, $|\alpha_i| = n_i$, should be in K.

(3.4) In §3 we will consider another condition which will ensure that B is initially joined. By this we mean that all simplices of type $\sigma = \sigma_1 \cup \ldots \cup \sigma_t$, $\sigma_i \subseteq \beta_i$, $\sigma_i \in B$, are in B provided at least one σ_i contains the minimum vertex b_{i1} of β_i, and that the corresponding subset $\mathcal{B}_{init}$ of $\mathcal{B}$ determined by these simplices is linearly independent.

In order to interpret the coboundary operators of C(K) in C(B) we need suitable monomorphisms $C(K) \longrightarrow C(B)$. Let $\bar{\mathcal{B}} = \{[\sigma^*]_K : \sigma \in \bar{B}\} \subseteq \mathcal{B}$ be a basis (3.1) of C(K), and let $\{[\sigma^*]_B : \sigma \in B\}$ be the canonical basis (2.4) of C(B). We will now use these bases to define a linear map $L: C(K) \longrightarrow C(B)$.

If $\sigma \in \bar{B}$ has the partition $\sigma = \sigma_1 \cup \ldots \cup \sigma_t$ then $L[\sigma^*]_K$ will be the exterior product of the $L[\sigma_i^*]_K$, $1 \leq i \leq t$. So it is enough to consider the case $\sigma \subseteq \beta_i$. Let $b_{i1} = \min \beta_i$. Then set $L[\sigma^*]_K = [\sigma^*]_B + \sum_{\theta <_L \sigma} c_\theta [\theta^*]_B$, where θ's and c_θ's are so chosen that $b_{i1} \cup \theta \in B$ and $[(b_{i1} \cup \sigma)^*]_K = \sum_{\theta \leq_L \sigma} c_\theta [(b_{i1} \cup \theta)^*]_K$. Note that θ's and c_θ's are uniquely determined (upto sign) by σ. Thus such a map L is canonically attached to any lexicographic basis $\bar{B}$ of K.

(3.5) Lower triangular isomorphisms L. <u>If B is initially joined (3.4) then one has vector space monomorphisms $L: C(K) \longrightarrow C(B)$ which obey $L\delta_\omega = \delta_\omega L$ for any linear combination ω of the elements $b_{i1}^* \in V^*$, $b_{i1} = \min \beta_i$, $1 \leq i \leq t$.</u>

<u>Proof</u>. Choose a basis $\bar{\mathcal{B}} \subseteq \mathcal{B}$ such that $\mathcal{B}_{init} \subseteq \bar{\mathcal{B}}$, and using this define $L: C(K) \rightarrow C(B)$ as above. Let $\omega = b_{i1}^*$. Both sides of $L\delta_\omega[\sigma^*]_K = \delta_\omega L[\sigma^*]_K$, $\sigma \in \bar{B}$, are zero if $b_{i1} \in \sigma$. If $b_{i1} \notin \sigma$ then the right side equals $\bigwedge_{j \neq i} L[\sigma_j^*]_K \wedge [(b_{i1} \cup \sigma_i)^*]_B$ if $b_{i1} \cup \sigma_i \in B$, and $\bigwedge_{j \neq i} L[\sigma_j^*]_K \wedge \sum_{\theta <_L \sigma_i} c_\theta [(b_{i1} \cup \theta)^*]_B$ if $b_{i1} \cup \sigma_i \notin B$. To check that the left side has the same values we use the definition of L and the fact that $\delta_\omega[\sigma^*]_K = [(b_{i1} \cup \sigma)^*]_K \in \mathcal{B}_{init}$ if $b_{i1} \cup \sigma_i \in B$, and, if $b_{i1} \cup \sigma_i \notin B$, then $[(b_{i1} \cup \sigma)^*]_K = \sum_{\theta <_L \sigma_i} c_\theta [(b_{i1} \cup \theta \cup \sigma \setminus \sigma_i)^*]_K$ where $(b_{i1} \cup \theta \cup \sigma \setminus \sigma_i) \in \mathcal{B}_{init}$. The linear map L is one-one because its matrix with respect to the 2 bases is lower triangular with 1's on the diagonal.

<u>Remark</u>. Even for graded ideals I of A(V) other than those of the type I_K, analogous vector space isomorphisms $L: A(V)/I \rightarrow C(B)$ can be constructed to simplicially interpret some coboundary operators $\delta_\omega: A(V)/I \rightarrow A(V)/I$, $\delta_\omega[\theta] = [\omega \wedge \theta]$, $\omega \in V^*$.

<u>4. Forms with rational functions as coefficients</u>

<u>Note</u>. We will now continue the above discussion for the case when the field F of coefficients is the field of all rational functions in N algebraically independent variables over a prime field F_p, p = char (F).

We assign a distinct variable to each of the N vertices of K. If v_{ij}, $1 \leq i \leq t$, $1 \leq j \leq N_i$, denotes the jth vertex of κ_i (in some chosen total order of κ_i), then the corresponding variable will be denoted X_{ij}. So $F = F_p(\{X_{ij}\})$.

Let $\beta = \{b_{ij}\}$, $1 \leq i \leq t$, $1 \leq j \leq N_i$, be any partially ordered compatible basis of

V. The product partial order $\leq$ on the subsets of β is the partial order, intermediate between inclusion $\subseteq$ and $\leq_L$, defined as follows: $\sigma \leq \tau$ iff one can choose a strictly increasing function from σ into τ. We will now show that there are lexicographic bases B of C(K) which are well behaved with respect to this partial order.

(4.1) Shifted model B_K of C(K). The lexicographic basis B_K of C(K) determined by the compatible basis $\left\{ b_{ij} = \sum_{1 \leq k \leq N_i} (X_{ij})^k v_{ik} \right\}$ of V is closed with respect to the product partial order, i.e. $\tau \in B_K$ and $\sigma \leq \tau$ implies $\sigma \in B_K$.

Proof. For each $1 \leq i \leq t$, $\det (X_{ij})^k \neq 0$. So $\beta = \{b_{ij}\}$ is indeed a basis of V. Next, note that any permutation π of the variables extends uniquely to a field automorphism π of F over F_S, the subfield of symmetric rational functions. This induces F_S-linear automorphisms π of the vector space V and the algebra A_K by $\sum_{v \in K} c_v v \mapsto \sum_{v \in K} \pi(c_v) v$ and $\sum_{\alpha \in K} c_\alpha [\alpha^*] \longmapsto \sum_{\alpha \in K} \pi(c_\alpha)[\alpha^*]$ respectively. Any permutation $\beta \to \beta$ which preserves each β_i, say $b_{ij} \mapsto b_{ij'}$, occurs as the restriction of such an F_S-linear automorphism $V \to V$, viz. that which arises from the permutation of variables $X_{ij} \mapsto X_{ij'}$. Let $\sigma \leq \tau \in B_K$. Choose a permutation π of β which is strictly increasing on each σ_i and $\beta_i \setminus \sigma_i$, and which maps each σ_i into τ_i. If $[\sigma^*]$ were equal to some linear combination $\sum_{\theta <_L \sigma} c_\theta [\theta^*]$ of preceding elements, then, by applying the F_S-linear algebra automorphism π of A_K, we see that $[\pi(\sigma)^*] = \sum_{\theta <_L \sigma} \pi(c_\theta)[\pi(\theta)^*]$. But this is not possible because $\pi(\sigma) \subseteq \tau \in B_K$ and $\theta <_L \sigma \Rightarrow \pi(\theta) <_L \pi(\sigma)$. So $\sigma \in B_K$.

Remark. The above argument was inspired by the "permutation lemma" used by Björner and Kalai [4]. We are calling a set of subsets of a poset a shifted complex if it is closed with respect to the product partial order. Note that for a toset this coincides with the usage of Björner and Kalai; at the other extreme for the partial order = it coincides with the notion of a simplicial complex. Note also that a change of the defining partial order of β from $\leq$ to $\pi(\leq)$ only replaces $\mathcal{B}_K$ by its image $\pi(\mathcal{B}_K)$ under the induced F_S-linear automorphism π of the algebra A_K. Thus, upto simplicial isomorphism, B_K depends only on the vertex-colored simplicial complex K and the field characteristic p. A similar construction works for any F provided the symmetric group on N letters occurs as a group of automorphisms of F.

(4.2) To consider a very simple example let K be the 2-colored hexagon of Figure 1. We assert that its shifted model B_K is as in Figure 2.

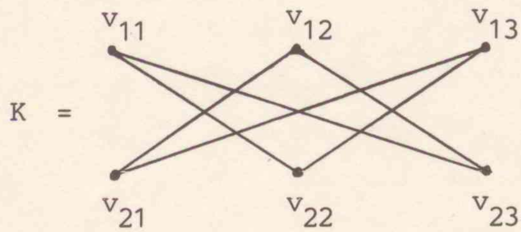

Fig.1

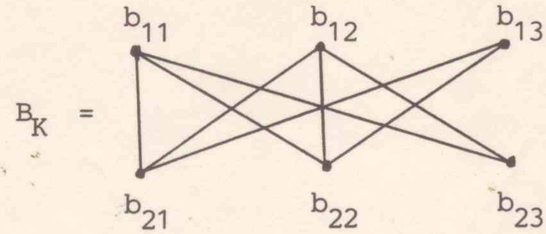

Fig.2

To see this note that, with β as in (4.1), one has $[(b_{1j}b_{2k})^*]_K = \sum_{1\leq q\neq m\leq 3} (X_{1j})^q(X_{2k})^m [(v_{1q}v_{2m})^*]_K$ for all $1\leq j,k\leq 3$. It can be verified that ~~Since~~ all 6×6 minors of the 9×6 coefficient matrix of these equations are nonzero, so it follows that any 6 of the forms $[(b_{1j}b_{2k})^*]_K$ are linearly independent in $A_K^2 = C^1(K)$. (In general ~~More generally a similar argument shows that~~ if K has $f(n_1,\ldots,n_t)$ simplices of color type $(n_1,\ldots,n_t)$, then a ~~any~~ subset of $\{[\sigma^*]_K : \sigma\subseteq\beta\ ,\ |\sigma_i| = n_i\ \forall i\}$ of cardinality $\leq f(n_1,\ldots,n_t)$ need not ~~has to be~~ linearly independent in C(K).) Using this it follows easily that the B_K of Figure 2 determines a maximal set of 2-forms for which no element is a linear combination of the preceding 2-forms (in the lexicographic partial order). Note that $\mathcal{B}_K$ is not a basis of C(K) but that it is initially joined and $\mathcal{B}_{init}$ has 5 elements of degree 2.

Let B be any simplicial complex with vertices $\beta = \beta_1 \cup \ldots \cup \beta_t = \{b_{ij} : 1\leq i\leq t,\ 1\leq j\leq N_i\}$. We define a linear map $U\colon C(B)\to C(B)$ by specifying its values on the elements $[\sigma^*]_B$, $\sigma\in B$, of the canonical basis (2.4) of C(B). If $\sigma\in B$ has the partition $\sigma = \sigma_1\cup\ldots\cup\sigma_t$ then $U[\sigma^*]_B$ will be the exterior product of the $U[\sigma_i^*]_B$, $1\leq i\leq t$. So it is enough to consider the case $\sigma\subseteq\beta_i$. Then put $U[\sigma^*]_B = [\sigma^*]_B$ if $b_{i1}\notin\sigma$ and $U[\sigma^*]_B = \sum_{1\leq j\leq N_i} [((\sigma\setminus b_{i1})\cup b_{ij})^*]_B$ if $b_{i1}\in\sigma$.

(4.3) Upper triangular isomorphisms U. If B is a shifted complex on the poset β, then the vector space isomorphism U just defined obeys $U\delta_{\omega_B} = \delta U$, $\omega_B = b_{11}^* + \ldots + b_{t1}^*$.

Proof. U is indeed an isomorphism because its matrix is upper triangular with 1's on the diagonal. Let $\omega_i = b_{i1}^* + \ldots + b_{iN_i}^*$, $1\leq i\leq t$. We will now verify that $U\delta_{b_{i1}^*}[\sigma^*]_B = \delta_{\omega_i}U[\sigma^*]_B$ for all $\sigma\in B$. (Adding these formulae gives $U\delta_{\omega_B} =$

δU.) If $b_{i1} \varepsilon \sigma$ then both sides work out to be zero, while if $b_{i1} \notin \sigma$ and $b_{i1} \cup \sigma \varepsilon B$ then both sides work out to be $\bigwedge_{j \neq i} U[\sigma_j^*]_B \wedge \sum_{1 \leq j \leq N_i} [(b_{ij} \cup \sigma_i)^*]_B$. It remains to consider the case when $b_{i1} \notin \sigma$ and $b_{i1} \cup \sigma \notin B$. Now the left side is obviously zero. On the other hand, $\delta_{\omega_i} U[\sigma^*]_B$ is a linear combination of terms $[\theta^*]_B$, with θ's all bigger (in the product partial order) than $b_{i1} \cup \sigma$. So, B being shifted, all these terms are also zero.

Remark. The argument shows infact that $U\delta_{\omega_B} = \delta U$ iff B obeys $b_{ij} \varepsilon \theta \varepsilon B \Rightarrow (\theta \setminus b_{ij}) \cup b_{i1} \varepsilon B$. When $t = 1$ such complexes are called "near cones" by Björner and Kalai [4]. At the other extreme if $t = N$ then any B satisfies this condition. One has $\dim H_{\omega_B}(B) \geqslant \dim H(B)$ always, with equality holding iff the above condition is satisfied.

We can now complete the description of colored shifting:

(4.4) Shifting isomorphisms T_K. Let D be the diagonal isomorphism (2.5) of C(K) determined by the elliptic 1-form $\omega_{B_K} = \sum_{\substack{1 \leq i \leq t \\ 1 \leq k \leq N_i}} (X_{i1})^k v_{ik}^*$, and let L and U be as in (3.5) ~~(so B_K is being assumed initially joined)~~ and (4.3). Then $T_K = ULD: C(K) \to C(B_K)$ is a vector space monomorphism obeying $T_K \delta = \delta T_K$.

Remark. Using this one can find a colored shifted $\bar{B}_K \subseteq B_K$ having the same "colored face vector" and "colored Betti vector" (obvious definitions) as K. Extremal problems involving these vectors, as K runs over some class of simplicial complexes, can thus be reduced to the corresponding class of shifted complexes. For $t = 1$ this fact has been used by Björner and Kalai [4] to give (amongst other things) elegant proofs of the well known theorem of Kruskal [12] and Katona [10], and its homological analogue [21]. We will show elsewhere that similar corollaries can be deduced also for $t > 1$. In this paper however we will give a somewhat different application.

5. Equivariant shifting

Note. The assumptions of nos. 3 and 4 continue to be in force. Furthermore, now we will work with a fixed total order in each κ_i, $1 \leq i \leq t$, and $\leq$ will refer only to this partial order on $\kappa = \kappa_1 \cup \ldots \cup \kappa_t$.

The transformation group G of such a K consists of all permutations g of

$\{1,2,\ldots,t\}$ for which there exists a (necessarily unique) simplicial isomorphism $g\colon K \to K$ preserving $\leq$ and such that $g(\kappa_i) = \kappa_{g(i)}$ for all $1 \leq i \leq t$. Recall that our forms have coefficients in $F = F_p(\{X_{ij}\})$, $1 \leq i \leq t$, $1 \leq j \leq N_i$, and that F_S denotes the subfield of all symmetric rational functions over F_p. The twisted action of any $g \in G$ on $C(K)$ is provided by the F_S-linear algebra automorphism of $C(K) = A_K$ defined by $g(X_{ij}) = X_{g(i)j}$ and $g[\alpha^*]_K = [(g\alpha)^*]_K$ for all $\alpha \subseteq \kappa$.

(5.1) Equivariance. <u>The shifted model B_K of $C(K)$ can be equipped with a simplicial G action in such a way that the shifting monomorphism T_K (4.4) commutes with the induced twisted actions of G on the cochains of K and B_K.</u>

<u>Proof</u>. $\mathcal{B}_K$ is the "basis" of $C(K)$ obtained from the spanning set $\{[\sigma^*]_K : \sigma \subseteq \beta\}$, $\beta = \{b_{ij} = \sum_{1 \leq k \leq N_i} (X_{ij})^k v_{ik}\}$, by deleting all elements which are linear combinations of lexicographically lesser elements. Here $v_{ij} \in \kappa_i$ is the jth vertex in the given total order of κ_i. So the F_S-linear algebra isomorphism $g\colon C(K) \to C(K)$, $g \in G$, images b^*_{ij} to $b^*_{g(i)j}$. And $[\sigma^*]_K$ is deleted iff $g[\sigma^*]_K$ is deleted. We define $g\colon B_K \to B_K$ by $[g(\sigma)^*]_K = g[\sigma^*]_K$. This induces an F_S-linear algebra isomorphism $g\colon C(B_K) \to C(B_K)$ by $g(X_{ij}) = X_{g(i)j}$ and $g[\sigma^*]_B = [(g\sigma)^*]_B$ for all $\sigma \subseteq \beta$.

The algebra isomorphism $D\colon C(K) \to C(K)$ is induced (see (2.5) and (4.4)) by the linear isomorphism $v_{ik} \mapsto (X_{i1})^k v_{ik}$ of V. So $D\,g = g\,D$. If $\mathcal{B}_{init}$ extends to a G-invariant basis $\overline{\mathcal{B}} \subseteq \mathcal{B}_K$ (for the sake of simplicity we will simply incorporate this into the definition (3.4) of "initially joined"), then the equations $L\,g = g\,L$ and $U\,g = g\,U$ follow immediately from the definitions of L and U. So $T_K\,g = g\,T_K$ for all $g \in G$.

<u>Remark</u>. Upto a simplicial G-isomorphism, the G-complex B_K is independent of the G-invariant $\leq$ used in its construction. Note also that the untwisted algebra isomorphisms $g\colon C(K) \to C(K)$, $g[\alpha^*]_K = [(g\alpha)^*]_K$ $\forall\, \alpha \subseteq \kappa$, yield $g(b^*_{ij}) = b^*_{g(i)j}$ only if one imposes conditions $X_{ij} = X_{g(i)j}$ between the indeterminates (this can be thought of as a shifting of the quotient K/G only). However such lexicographic bases B are usually not shifted and one does not have $U\,\delta_{\omega_B} = \delta\,U$.

§3. Embeddability

6. Deleted joins

A simplicial complex K is said to embed in the space $\mathbb{R}^m$ if there exists a continuous one-one map from K to $\mathbb{R}^m$. If $m > 2n$ where $n = \dim K$, then any general position linear map from K to $\mathbb{R}^m$ is one-one. On the other hand for $m \leq 2n$ there are some non-trivial obstructions to embeddability which we now proceed to discuss.

We recall that the join A.B of 2 disjoint simplicial complexes is made up of all simplices $\alpha \cup \gamma$, $\alpha \in A$, $\gamma \in B$. The deleted join K_* of a simplicial complex K is the subcomplex of ${}^1K.{}^2K$, the join of 2 disjoint copies of K, consisting of all simplices ${}^1\alpha \cup {}^2\gamma$, $\alpha \in K$, $\gamma \in K$, $\alpha \cap \gamma = \phi$. We will usually write K.K instead of ${}^1K.{}^2K$ and (α, γ) instead of ${}^1\alpha \cup {}^2\gamma$. There is a free $\mathbb{Z}_2$-action on K_* given by $(\alpha,\gamma) \mapsto (\gamma,\alpha)$.

(6.1) A simplicial complex K embeds in $\mathbb{R}^m$ only if there exists a continuous $\mathbb{Z}_2$-map from its deleted join K_* to the antipodal m-sphere S^m.

Proof. We recall that each point of a join of spaces, X.Y, either lies in one of the 2 'ends' X, Y, or is an interior point of a unique line segment having one end in X and the other in Y. Any continuous map f from K to $\mathbb{R}^m$ determines a continuous $\mathbb{Z}_2$-map $f^{(2)}: {}^1K.{}^2K \to {}^1\mathbb{R}^m.{}^2\mathbb{R}^m$, $f^{(2)}(u.{}^1x + v.{}^2y) = u.{}^1(f(x)) + v.{}^2(f(y))$. If f is one-one then $f^{(2)}$ restricts to a $\mathbb{Z}_2$-map $f^{(2)}: K_* \to \mathbb{R}^m_*$, where $\mathbb{R}^m_*$ equals ${}^1\mathbb{R}^m.{}^2\mathbb{R}^m$ minus the fixed points $\frac{1}{2}.{}^1z + \frac{1}{2}.{}^2z$, $z \in \mathbb{R}^m$. So it suffices to check that $\mathbb{R}^m_*$ has the $\mathbb{Z}_2$-homotopy type of S^m.

To see this symmetrically contract the 2 ends ${}^1\mathbb{R}^m$, ${}^2\mathbb{R}^m$, to 2 points to see that $\mathbb{R}^m_*$ has the $\mathbb{Z}_2$-homotopy type of the suspension of the subspace of $\mathbb{R}^m_*$ consisting of points of the type $\frac{1}{2}.{}^1x + \frac{1}{2}.{}^2y$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, $x \neq y$. But this subspace is $\mathbb{Z}_2$-homeomorphic to the deleted product $\mathbb{R}^m_\#$ of $\mathbb{R}^m$, i.e. the $\mathbb{Z}_2$-subspace of ${}^1\mathbb{R}^m \times {}^2\mathbb{R}^m$ ($= \mathbb{R}^m \times \mathbb{R}^m$) consisting of all points of the type (x,y), $x \neq y$. And, that $\mathbb{R}^m_\#$ has the $\mathbb{Z}_2$- homotopy type of S^{m-1}, follows by projecting orthogonally on the m-dimensional orthogonal complement of the diagonal vector subspace and normalising.

The involution ν of K_* induces an involution $\nu: C(K_*) \to C(K_*)$. We denote by

$C_s(K_*) = \sum_{i \geq 0} C^i_s(K_*)$ the subcomplex of symmetric cochains c, $\nu c = c$. Note that 1_{K_*} (see no. 1 for notation) is a zero dimensional symmetric cocycle of K_*.

A similar notation will be used for any $\mathbb{Z}_2$-simplicial complex E, ν. Let us suppose that char (F) = 2. Frequently (e.g. for $E = K_*$, and more generally for any free $\mathbb{Z}_2$-complex) it so happens that the following sequence is exact in dimensions ≥ 0 :

$$(6.2) \qquad 0 \rightarrow C_s(E) \hookrightarrow C(E) \xrightarrow{Id + \nu} C_s(E) \rightarrow 0.$$

In all such cases we have the corresponding long exact cohomology sequence,

$$(6.3) \quad H^0_s(E) \xrightarrow{Sm} H^1_s(E) \rightarrow \ldots \rightarrow H^q_s(E) \rightarrow H^q(E) \rightarrow H^q_s(E) \xrightarrow{Sm} H^{q+1}_s(E) \rightarrow \ldots .$$

The jth iterate of its connecting homomorphism, $Sm^j: H^0_s(E) \rightarrow H^j_s(E)$, images the symmetric cohomology class $[1_E] \in H^0_s(E)$ to a cohomology class $Sm^j[1_E] \in H^j_s(E)$, which will be called the jth Smith class of the $\mathbb{Z}_2$-complex E.

(6.4) A simplicial complex K embeds in $\mathbb{R}^m$ only if the (m+1)th Smith class of its deleted join K_* is zero.

Proof. The definition of Smith classes makes sense even for the complex of singular cochains of any free $\mathbb{Z}_2$-space. Further, the cochain complex of a $\mathbb{Z}_2$-simplicial complex is $\mathbb{Z}_2$-cochain homotopy equivalent to the bigger complex of all singular cochains of the space of E. Thus the continuous $\mathbb{Z}_2$-map of (6.1) induces a homogenous degree zero map $H_s(S^m) \rightarrow H_s(K_*)$ commuting with the connecting homomorphisms Sm. Also note that the quotient space $K_*/\mathbb{Z}_2$ is always connected, and that this map is an isomorphism in dimension zero. So the vanishing of the the (m+1)th Smith class of the m-dimensional space S^m implies that of K_* .

See Wu [30] for more details regarding such arguments.

Note. Keeping our application (no. 7 below) in mind, we assume from here on that the Smith classes of E are being considered over the field of rational functions (cf. no. 4) over the prime field of 2 elements, $F = F_2(\{X_w\})$, $w \in$ vert E. Also, that the $\nu : C(E) \rightarrow C(E)$ used in their definition is the twisted F_S-linear action (cf. no. 5) given by $\nu(X_w) = X_{\nu(w)}$, $\nu[\alpha^*]_E = [(\nu\alpha)^*]_E \; \forall \; \alpha \subseteq$ vert E.

We now give examples when these obstruction classes are nonzero. These examples--

here σ^i_j denotes the j-skeleton of an i-simplex σ^i-- are due to van Kampen [28].(The non-embeddability of the following complexes in $\mathbb{R}^m$ was also proved independently by Flores [5].) The Smith theoretic interpretation of van Kampen's obstructions being used here is due to Wu [30], and the proof given below is inspired by [30], pp. 115-118.

(6.5) Van Kampen-Flores Theorem. The (m+1)th Smith class of $(\sigma^{m+2}_n)_*$, $n \leq m \leq 2n$, is nonzero.

Proof. Let r = 2n-m-1, q = m-n. Totally order the vertices of σ^{m+2} (= ${}^1\sigma^{m+2}$) by giving them the labels 1,2,...,m+3. Let τ (resp. θ) denote the simplex formed by the first r+1 (resp. the remaining 2q+3) vertices. Then σ^{m+2}_n contains $\tau^r_r \cdot \theta^{2q+2}_q$ and so $(\sigma^{m+2}_n)_*$ contains $(\tau^r_r \cdot \theta^{2q+2}_q)_* \cong (\tau^r_r)_* \cdot (\theta^{2q+2}_q)_*$. This last simplicial complex has two (m+1)-simplices incident to each m-simplex. Thus it is a symmetric mod 2 cycle of $(\sigma^{m+2}_n)_*$

Now, if one assigns the negative integral labels -1,-2,...,-(m+3), to the corresponding vertices of the second copy ${}^2\sigma^{m+2}$, then one gets an odd number (in fact exactly 2q+3) (m+1)-simplices of this cycle of the alternating type $\{+n_0, -n_1, +n_2, \ldots, \pm n_{m+1}\}$, $0 < n_0 < n_1 < n_2 \ldots < n_{m+1}$. But one can verify that the symmetric mod 2 cochain supported on such alternating simplices and their antipodes is an (m+1)-cocycle representing the (m+1)th Smith class $Sm^{m+1}[1_E]$. (See [23]: one uses the fact that , in each dimension, the coboundary of the sum of the alternating simplices, equals the sum of the next higher dimensional alternating simplices and their antipodes.) So this class is nonzero because it takes a nonzero value on the aforementioned mod 2 cycle.

Remark. Note that $(\tau^r_r)_*$ is the octahedral r-sphere. On the other hand a lemma of Flores [5] tells us that $(\theta^{2q+2}_q)_*$ is $\mathbb{Z}_2$-homeomorphic to the antipodal (2q+1)-sphere. So the mod 2 cycle used in the above proof is infact a $\mathbb{Z}_2$-triangulation of the antipodal (m+1)-sphere. Ky Fan [14] showed that if the vertices of any such triangulation are assigned the labels $\{\pm 1, \pm 2, \ldots\}$, in such a way that antipodal (resp. contiguous) vertices are assigned antipodal (resp. non-antipodal) labels, then there is always an odd number of alternating (m+1)-simplices. We used a particular instance of such a coloring in the above proof.

Thus a simplicial complex containing σ_n^{m+2}, $n \leq m \leq 2n$, is non-embeddable in $\mathbb{R}^m$. Infact the same conclusion is true with a much weaker notion than containment.

(6.6) Corollary. If there exists a $\mathbb{Z}_2$-cochain homomorphism $T: C(K_*) \to C((\sigma_n^{m+2})_*)$, $n \leq m \leq 2n$, obeying $T(1) = 1$, then the $(m+1)$th Smith class of K_* is nonzero.

Proof. Let $\Delta = \sigma_n^{m+2}$. The coboundary operators $C^{-1}(K_*) \xrightarrow{\delta} C^0(K_*)$ and $C^{-1}(\Delta_*) \xrightarrow{\delta} C^0(\Delta_*)$ image the unit element 1 to 1_{K_*} and 1_{Δ_*} respectively. Since $T\delta = \delta T$ and $T(1) = 1$ we thus have $T(1_{K_*}) = 1_{\Delta_*}$. Further, since T commutes with the $\mathbb{Z}_2$-actions, there are induced maps $T: H_s^i(K_*) \to H_s^i(\Delta_*)$, $i \geq 0$, commuting with the connecting homomorphisms Sm. So $T\, Sm^{m+1}[1_{K_*}] = Sm^{m+1}[1_{\Delta_*}]$, which is nonzero by (6.5). Thus $Sm^{m+1}[1_{K_*}] \neq 0$.

We will now use this result and shifting to establish some necessary conditions on the face (or f-) vectors of embedded simplicial complexes.

7. Heawood Inequalities

For any t-colored simplicial set A, $f(A; q_1, \dots, q_t)$ will denote the number of simplices of color type $(q_1, \dots, q_t)$, i.e. those having q_i vertices of color i, $1 \leq i \leq t$. Note that for t=1, $f(A; q) = f_{q-1}(A)$, the number of (q-1)-simplices of A.

The complexes K.K and K_* will be 2-colored by assigning the color 1 (resp. 2) to the vertices of ${}^1K$ (resp. ${}^2K$).

(7.1) Lemma. If $f_{i-1}(K) > C.f_{i-2}(K)$ then $f(K_*; i,j) > (C-j).f(K.K; i-1,j)$.

Proof. Let $K_{(r)}$, $r \geq 0$, denote the simplicial set consisting of all $(\sigma, \theta) \in$ K.K with $|\sigma \cap \theta| = r$. Each simplex (σ, θ) of $K_{(r)}$ of color type (i,j) is incident to r simplices of $K_{(r-1)}$ of color type (i-1,j) viz. those obtained by deleting a vertex of $\sigma \cap \theta$ from the first component. Conversely, each simplex (τ, θ) of $K_{(r-1)}$ of color type (i-1,j) is incident to at most j-r+1 simplices of $K_{(r)}$ of color type (i,j) viz. those obtained by adding a vertex of $\theta \setminus \tau$ to the first component. So $r.f(K_{(r)}; i,j) \leq (j-r+1).f(K_{(r-1)}; i-1,j)$ for all $r \geq 1$.

Hence
$$f(K_*; i,j) = f_{i-1}(K)\, f_{j-1}(K) - \sum_{r \geq 1} f(K_{(r)}; i,j)$$
$$> C\, f_{i-2}(K)\, f_{j-1}(K) - \sum_{r \geq 1} \frac{j-r+1}{r} f(K_{(r-1)}; i-1,j)$$

$$\geqslant C\, f_{i-2}(K)\, f_{j-1}(K) - j \sum_{s\geqslant 0} f(K_{(s)};\, i-1,j)$$

$$= (C-j).f(K.K;\, i-1,j).$$

We now consider the result mentioned in §1.

(7.2) Theorem. <u>If $C = C(n,m)$ is big enough, then a simplicial complex K satisfying $f_n(K) \geqslant C.f_{n-1}(K)$ is non-embeddable in $\mathbb{R}^m$, $n \leqslant m \leqslant 2n$.</u>

<u>Proof</u>. Without loss of generality we can assume that K is a $\subseteq$- minimal simplicial complex satisfying the given inequality. Then each $(n-1)$-simplex must be incident to more than C n-simplices, and so $(i+1)f_i(K) > (n-i+C)f_{i-1}(K)$ for $1 \leqslant i \leqslant n$. It will suffice to assume that <u>$f_i(K) > (n+2m+5)f_{i-1}(K)$ for $1 \leqslant i \leqslant n$</u>.

Let B_{K_*} -- with set of vertices $\beta = \beta_1 \cup \beta_2$ -- be the shifted model (4.1) of the 2-colored simplicial complex K_*. Note that $f(\,St_v B_{K_*};\, i,j)$ is no bigger than $f(\,B_{K_*};\, i-1,j)$ or $f(\,B_{K_*};\, i,j-1)$, depending on whether $v \in \beta_1$ or $v \in \beta_2$. (We are using the standard notation for <u>stars</u>, $St_v A = \{\alpha:\, v \in \alpha \in A\}$.) Hence the number of $(2n+1)$-simplices of B_{K_*} containing at least one vertex from the first $m+2$ vertices of β_1 or β_2 is no bigger than $2(m+2).f(B_{K_*};\, n,n+1)$. On the other hand, the total number of $(2n+1)$-simplices of B_{K_*},

$$
\begin{aligned}
f_{2n+1}(B_{K_*}) &= f(B_{K_*};\, n+1,n+1) \\
&\geqslant f(K_*;\, n+1,n+1) && \text{by (3.1),} \\
&> 2(m+2).f(K.K;\, n,n+1) && \text{by (7.1),} \\
&\geqslant 2(m+2).f(B_{K_*};\, n,n+1) && \text{by (3.2).}
\end{aligned}
$$

So B_{K_*} contains a $(2n+1)$-simplex which does not contain any of the first $m+2$ vertices of β_1 or β_2. Since B_{K_*} is shifted, it follows that it must contain all the simplices of the simplicial complex $\sigma_n^{m+2}.\sigma_n^{m+2}$ determined by the first $m+3$ vertices of β_1 and β_2. We will denote by $R\colon C(B_{K_*}) \rightarrow C((\sigma_n^{m+2})_*)$ the $\mathbb{Z}_2$-cochain epimorphism obtained by restricting the forms of B_{K_*} to the subcomplex $(\sigma_n^{m+2})_*$.

To check that B_{K_*} is initially joined (3.4) note that $B_{init} = St_{b_{11}}(B.B) \cup St_{b_{21}}(B.B)$. So the number of simplices of color type (i,j) lying in B_{init} is no bigger than $f(B.B; i-1,j) + f(B.B; i,j-1) = f(K.K; i-1,j) + f(K.K; i,j-1)$, which by (7.1) is less than $f(K_*; i,j)$. Hence by (4.2) $\mathcal{B}_{init}$ is linearly independent in C(K). The simplices of B_{K_*} (and B_{init}) are either of the type (θ,θ) or occur in pairs (θ,τ), (τ,θ). Choose any simplicial set $\bar{B} \supseteq B_{init}$, which contains either none or both members of each pair, and which is such that $f(\bar{B}; i,j) = f(K_*; i,j)\ \forall\, i,j$. By (4.2) this is a $\mathbb{Z}_2$-basis of C(K), and can be used to define the lower triangular map L (3.5).

We now use (6.6) with $T = R\, T_{K_*}$ where T_{K_*} is the $\mathbb{Z}_2$-cochain monomorphism of (4.2) and (5.1), to conclude that the (m+1)th Smith class of K_* is nonzero. So K is non-embeddable in $\mathbb{R}^m$ by (6.4).

We can now establish some conjectures of [20] and [21] as easy corollaries:

(7.3) Least valences of embedded complexes. Let $\delta_{n-1}(K)$ denote the least number of n-simplices incident to an (n-1)-simplex of K. Then $\delta_{n-1}(K)$ is bounded as K runs over all simplicial complexes embedded in $\mathbb{R}^m$ or S^m, $n \leq m \leq 2n$.

Proof. Follows from (7.2) because $\delta_{n-1}.f_{n-1}(K) \leq (n+1).f_n(K)$.

It would be interesting to determine the numbers $\delta_{n-1}(S^m) = \sup\{\delta_{n-1}(K): K \subseteq S^m\}$, $n \leq m \leq 2n$, exactly. Note that $\delta_0(S^2) = 5$ and is attained at the icosahedron.

As in [21], $c_{n-1}(K)$ will denote the (n-1)th weak chromatic number of K, i.e the least number of colors which can be assigned to the (n-1)-simplices of K in such a way that no n-simplex has all its faces of the same color. The well known theorem of Ramsey [18] says that $\lim_{N\to\infty} c_{n-1}(\sigma^N_n)$ is infinite. On the other hand for embedded complexes we have the following finiteness theorem.

(7.4) Ramsey colorings of embedded complexes. The weak chromatic number $c_{n-1}(K)$ is bounded as K runs over all simplicial complexes embedded in $\mathbb{R}^m$ or S^m, $n \leq m \leq 2n$.

Proof. Infact $c_{n-1}(K) \leq \delta_{n-1}(S^m) + 1$. To see this use induction on $f_{n-1}(K)$ and the fact that K has an (n-1)-simplex which is incident to at most $\delta_{n-1}(S^m)$ n-simplices.

Again let $c_{n-1}(S^m) = \sup\{c_{n-1}(K): K \subseteq S^m\}$. The Four Color Theorem $c_0(S^2) = 4$ of Appel and Haken [2] is equivalent to saying that $c_0(S^2)$ is attained at the minimal triangulation of the 2-sphere. We conjecture that the weak chromatic numbers of higher dimensional spheres are also attained at their minimal triangulations.

8. Concluding remarks

(8.1) Though (7.2) resolves in the affirmative the problem of Grünbaum [6] regarding the existence of linear inequalities, $f_n(K) < C_1 . f_{n-1}(K) + C_2 . f_{n-2}(K) + \ldots$ $\forall K \subset \mathbb{R}^m$, $n \leq m \leq 2n$, it is obviously only a first step towards a characterisation of the f-vectors of embedded simplicial complexes. For planar graphs $f_1(K) < 3.f_0(K) - 2$, and this is best possible. More generally for $K \subset R^{2n}$ it seems that $f_n(K) < (n+2). f_{n-1}(K)$. If so, then this would be best possible: For any $C < n+2$, if K is the n-skeleton of a (2n+1)-dimensional cyclic polytope with a sufficiently large number of vertices, then $f_n(K) \geq C.f_{n-1}(K)$. See McMullen and Shephard [17], pp.82-90, 112.

(8.2) We hope to show elsewhere that an n-complex, $n \neq 2$, is non-embeddable in $\mathbb{R}^{2n}$, iff, after a suitable shifting, it contains one of a finite explicitly given list of n-complexes. In this context, see also [24] where we have given a complete classification of all n-complexes, $n \neq 2$, which are critically non-embeddable in $\mathbb{R}^{2n}$. Note that such results are higher dimensional analogues of the well known graph planarity criterion of Kuratowski [13],and its recent generalisations to other 2-manifolds by Robertson and Seymour [19] et al. It seems infact that, mod suitable shiftings, such Kuratowski characterisations are valid not only for the non triviality of the Smith (or Stiefel-Whitney) classes, but for any characteristic class whatsoever.

(8.3) We will show elsewhere that there is an analogue of (7.2), and that thus the numbers $\delta_{n-1}(X^m)$ and $c_{n-1}(X^m)$ are finite, for any compact polyhedron X^m, $n \leq m \leq 2n$. The proof requires more topological background, and gives bounds in terms of the minimum number of vertices required to triangulate X^m. It seems likely that in the absence of some local homology, one can also give bounds in terms of the (global) homology of X^m. This is indicated by [21] where we gave a higher-dimensional generalization of the well known (square root) chromatic inequality of Heawood [7] for all pseudomanifolds X^m.

(8.4) The sth generalized Kneser graph $G_s(K)$ of a simplicial complex K is the one whose vertices are disjoint pairs of nonempty simplices (τ^s, θ), θ maximal, with (τ_1^s, θ_1) joined to (τ_2^s, θ_2) iff $\tau_1^s \subseteq \theta_2$ and $\tau_2^s \subseteq \theta_1$. We have proved [22] a combinatorial analogue of (6.1): If $G_s(K)$ has chromatic number $\leq m+1-2s$ then there exists a continuous $\mathbb{Z}_2$-map

$K_* \to S^m$. In particular, since $(\sigma_n^n)_*$ is an n-sphere, it follows that $G_s(\sigma_n^n)$, $n > 2s$, has chromatic number $> n-2s$: a result conjectured by Kneser [11] and established by Lovász [15]. We also note that the converse of (6.1), and thus also of (6.4) provided one uses integer coefficients, is known to be true under suitable dimensional restrictions. Such results (see [24] for more) go back to van Kampen [28]. Finally, note that there is an analogue of (7.4) for the chromatic number of $G_{n-1}(K^n)$ as K runs over all simplicial complexes embedded in $\mathbb{R}^m$, $n \leqslant m \leqslant 2n$.

(8.5) Formula (1.1) defines some useful coboundary operators even for vector valued forms, provided one now thinks of the $\omega(v_k)$'s as linear maps satisfying the integrability conditions $\omega(v_k)\,\omega(v_r) = \omega(v_r)\,\omega(v_k)$ $\forall$ k, r. To do this it suffices to consider polynomial (instead of the more usual "smooth") forms, i.e those with values (cf. no. 4) in $F[X_1, \ldots ,X_N]$ or some module (= vector bundle) over this. This is so because one can now use differential operators, and define invariants by means of their solution spaces, indices, etc. For example one has the de Rham operator defined by $\omega(v_k) = \frac{\partial(.)}{\partial X_k}$, its generalization $\omega(v_k) = \frac{\partial(.)}{\partial X_k} + \frac{\partial h}{\partial X_k}$ used by Witten [29], or, still more generally, the curvature zero vector bundle extensions of the de Rham operator considered by Sullivan [27], etc.

Further if char (F) = 0,then integration provides us with a degree 0 cochain map (this is = Stokes' Lemma) from the de Rham complex to F valued cochains of K, and this map induces an isomorphism in cohomology under which the exterior product is responsible for the cup (and some other non trivial) products in H(K). On the other hand note that the obvious identification (2.3), which has been seldom used since e.g. Alexander [1], only induces trivial products in H(K). If char $(F) \neq 0$ then one uses the natural duality between polynomial forms of K and equivariant cochains of K.K. ... to relate differential invariants and characteristic classes.

It seems useful also to consider another method for finding generic bases of A_K , in which one starts with all exterior products of a vector space basis for indecomposable forms, and then performs lexicographic deletions as in no.3. Infact the minimal models of Sullivan [27], which capture the rational homotopy types of de Rham algebras, are defined by means of such considerations.

A fuller account of some of these ideas will be given in [25] and elsewhere.

Acknowledgements

I am thankful to A. Björner for sending me a preprint of [4], and to A. Adler for a very stimulating conversation.

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