POINCARÉ'S PAPERS ON TOPOLOGY

BY

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Notes* of Lectures given in 1993-94

in the

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*not yet finalized

These are notes of most of the lectures (upto and including the *Quatrième Complément*) given in the seminar on Poincaré's *Analysis Situs* during 1993-94. The notes of the lectures on the *Cinquième Complément* are yet to be typed up; these, as well as Poincaré's Last Geometric Theorem, will also be included in the finished version. Most of the following material, i.e. pp. 53-237, is in the original or "Zeroth" edition (= lectures as delivered), however pp. 1-52 are from the in-progress revised or "First" edition. I have also included the talk which I gave in Nancy in May, 1994.

K. S. Sarkaria

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PREFACE †

These are notes (not yet finalized) based on lectures given in the **Chandigarh Topology Seminar of 1993-94**. The aim of these lectures was to give a review of the beautiful and far-sighted contributions of Henri Poincaré to topology, together with a discussion of their subsequent evolution and developement.

As **Leray** says, in his preface to the second part of *Tome* VI of Poincaré's *Oeuvres* (1953), Poincaré **"crèa toute la Topologie algèbrique moderne"** in his papers. In fact, Poincaré's great paper *"Analysis Situs"*, and the first two of its five *Complèments*, alone cover essentially almost *all* of what is usually contained in modern Algebraic Topology texts, *and much more*.

The *Troisième* and *Quatrième Complèments* of this paper went on to deal with the fundamental and homology groups of complex surfaces, and these results of (Picard and) Poincaré were later generalized by Lefschetz to all smooth complex projective varieties. These theorems have played an important (witness Deligne's proof of Weil's "Riemann Hypothesis") rôle in twentieth century mathematics.

The *Cinquième Complèment* gave birth to a celebrated example of a 3-manifold, which has played a ubiquitous rôle in the most enchanting parts (work of Kirby-Siebenmann, Freedman, Donaldson, etc.) of modern topology, and besides introduced a powerful method, now called "Morse Theory", for the study of the topology of smooth manifolds.

We conclude with *Poincaré's Last Geometric Theorem*, which stemmed from his work on the 3-body problem, and which continues to be a stimulus for on-going research on symplectic structures, etc.

The reader will **not** *find here a* **translation** *of Poincaré's papers.* (Some portions are however translated in full, e.g. the *Introduction* of *"Analysis Situs"*.) Our object being to give a clear exposition of Poincaré's *ideas*, we have attempted to give instead a **TEXT** which is mathematically clearer and conciser than the original, without being unfaithful to the *spirit* of the latter. (However we emphasize that, in

† written for the as yet (pp. 1-52) unfinished "first edition".

order to get a real *feel* of this great master, the reader should, *and is strongly urged to*, read the originals also.)

In this process, we have frequently inter-leaved modern notation, terminology, and even inserted some modern proofs, into the text, when we felt that this did not interfere materially with the *essence* of the original contribution.

The rest of our commentary is given separately from this text, in the shape of **NOTES**, numbered (a), (b), ... , *which appear at the end of each section.*

The main title of each chapter is that of the corresponding original paper. Likewise, the main titles of the sections (§§) of each chapter are that of the sections of the corresponding original paper.

However **we have made changes in the presentation of the material within each section** to ensure quicker readability. Thus, *within* each section, the order, the numbering, the arrangement, and the labelling of the material, as well as almost all the diagrams, *are our own.* In particular, we have chosen to **highlight** some statements as Definitions, Propositions, Theorems, Corollaries and Remarks, as against Poincaré, whose writing style was consistently informal. (There were however two results in Poincaré's first *Complément*, pertaining to matrices and determinants, which were highlighted as *"Theorèms"* !)

Besides giving comments on the relationship of Poincaré's results to later developements, our notes contain **some new results** — e.g. a generalization of Poincaré's classification theorem re some 3-manifolds of the type $\mathbb{R}^3/G$, and an enumeration of such manifolds via class numbers of some algebraic number fields, a combinatorial Hodge decomposition theorem, a definition of a new homology using characters, etc. — some new proofs of old results, and many conjectures and questions.

The success of this seminar, as well as the contents of some of these notes, owe much to the enthusiasm of its other participants, especially Prof. I. B. S. Passi, Dr. Dharam Singh, Dr. D. B. Rishi, Ms. Gurmeet Kaur and Mr. Keerti Vardhan. (For example a new proof of Poincaré's result re "orientable determinants" is joint work with D. B.

Rishi.) I extend to all of them, as well as to our chairman, Prof. N. Sankaran, my heartfelt thanks for their assistance. Thanks are due also to Profs. S. G. Dani, R. N. Gupta, and R. V. Gurjar for providing references and/or helpful comments.

CHAPTER I

SUR L'ANALYSIS SITUS

C. R. de l'Acad. Sci. 115 (1892), 663-666.

One knows what is meant by the connectivity of a surface, and the important role which this notion plays in complex function theory, even though it is borrowed from a totally different branch of mathematics, i.e. the geometry of situation or **Analysis Situs**.[(a)]

It is because researches in this subject might have some applications outside of Geometry that there is interest in generalizing them to spaces of more than three dimensions. **Riemann** was well aware of this, and desirous of generalizing his beautiful discovery, he had given some thought to higher-dimensional spaces from the point of view of Analysis Situs, but has unfortunately left us only some very incomplete fragments on this subject. His results were later rediscovered and extended by **Betti** who associated to an n-dimensional hypersurface or variety in (n+1)-space n-1 numbers which measure its connectivities in dimensions 1 through n-1.[(b)]

Those who repudiate Geometry of more than three dimensions would surely have labelled this result useless and frivolous, except for the fact that our colleague Monsieur **Picard** has made use of these Betti numbers in his work in Analysis and ordinary Geometry of surfaces.

However the question is not settled. One may ask whether conversely these Betti numbers determine the variety from the viewpoint of Analysis Situs, i.e. whether two varieties having the same Betti numbers are always related by a continuous deformation ? This is so for (surfaces in) 3-space, and one would be tempted to believe that the same holds in all dimensions. *It is the opposite which is true.*

For this we define the **fundamental group** G of the hypersurface as follows. Consider any generic system of multiple-valued locally defined continuous functions $F_1, \ldots, F_p$ on the hypersurface, having the property that if we follow any branch over any *infinitely small* (= small enough) loop we return to the same values. Then G is the (abstract, discrete) group of all permutations of the branches which ensue if we follow branches over all *finite* loops.[(c)]

Clearly this group is preserved as we deform the variety. The converse, though less evident, is also true for closed varieties, i.e. *what determines a closed hypersurface from the point of view of Analysis Situs is its fundamental group.*[(d)]

Therefore, we are led to the question : *if two closed varieties have the same Betti numbers, do they always have isomorphic groups ?*

Our examples will be closed hypersurfaces of 4-space parametrized by functions X_1, X_2, X_3, X_4 of three variables invariant under the discontinuous group G of motions of 3-space generated by

$$(x, y, z) \longrightarrow (x+1, y, z),$$
$$(x, y, z) \longrightarrow (x, y+1, z), \text{ and}$$
$$(x, y, z) \longrightarrow (\alpha x+\beta y, \gamma x+\delta y, z+1),$$

where α, β, γ, and δ are four chosen integers with $\alpha\delta - \beta\gamma = 1$.[(e)]

We have used the same letter G because it is easily verified that this is indeed the fundamental group of such a hypersurface. Of course both G and the hypersurface depend on the choice of the 4 integers or the corresponding linear transformation $T \in SL(2,\mathbb{Z})$.

We will show that *two such discontinuous groups* G_1 *and* G_2 *are isomorphic if and only if* T_1 *and* T_2 *are in the same conjugacy class of the group* $GL(2,\mathbb{Z})$.[(f)]

There is an *infinity* of such conjugacy classes.

On the other hand we will check that *the first Betti number of such a hypersurface can be only* 3, 2, *or* 1, and these cases happen respectively if and only if (i) $\alpha = \delta = 1$ and $\beta = \gamma = 0$, (ii) not this but $\alpha + \delta = 2$, and (iii) generically. In addition we will show that the second Betti number is 3 in all cases.[(g)]

The above might throw some light on the theory of complex surfaces and render less strange a result of **Picard** which says that the first Betti number of a closed generic algebraic surface is zero.[(h)]

NOTES

(a) All this is due to RIEMANN, who defined the **connectivity** of a surface S to be the least number of closed curves by cutting which we can disconnect S. It equals $1 + \dim H_1(S;\mathbb{Z})$, i.e. it is *one more* than the modern **Betti number** of S.

Since **Poincaré's Betti numbers** were designed as generalizations of connectivity, they were all one more than the modern ones. *We will however reduce them by one and thus only use the modern Betti numbers.*

We will see later (in the first *Complément*) that **Betti's own numbers** were still different : the ith one of these was one more than the least number of elements required to generate $H_i(M;\mathbb{Z})$.

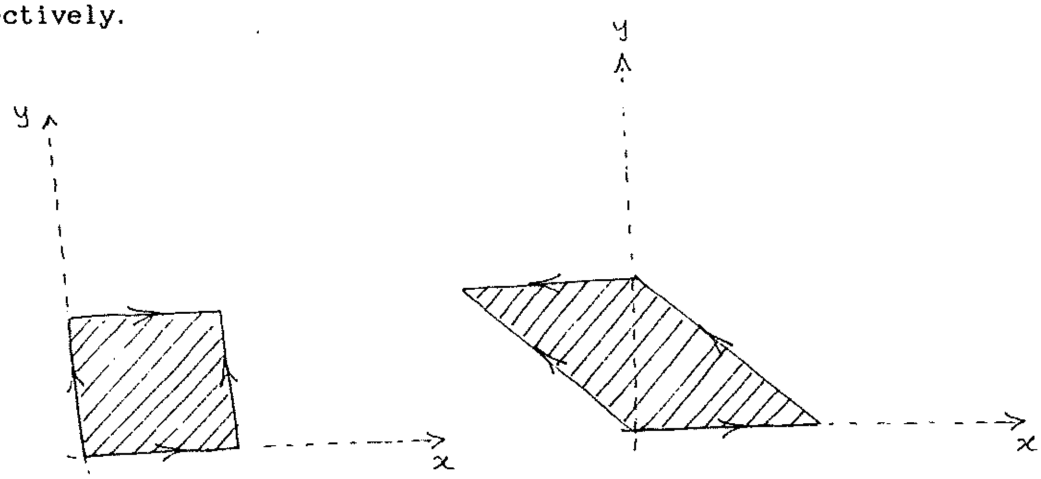
(b) As will become clear later (see II.3.3 and II.3.d) by a **variety** Poincaré meant one which is *differentiable* and *smooth* (i.e. a manifold) and the phrase "**in** (n+1)-**space**" was essentially redundant : e.g. it did *not* mean embedded, and throughout his focus will only be on the variety, and almost never on the existence or nature of its self-intersections in (n+1)-space. So e.g. the phrase "closed hypersurface of 4-space" of a later paragraph is best understood simply as "closed 3-manifold".

(c) Our summary of Poincaré's definition shows that he is thinking of $\pi_1(V)$ as the group of covering transformations of a universal covering space U of V situated in $V\times\mathbb{R}^p$, the multiple-valued function F being the inverse of the projection $U \longrightarrow V$.

(d) *This "converse" is false* : the closed 4-manifolds S^4 and $S^2\times S^2$ (which can also be both embedded in 5-space) are non-homeomorphic, even though they both have a trivial fundamental group.

(e) If one allows $\alpha\delta - \beta\gamma = \pm 1$ then one also gets some non-orientable manifolds.

Since GL(1,$\mathbb{Z}$) = {± 1}, there are *only two* analogous groups G in dimension two, viz. $\langle (x,y) \longmapsto (x+1,y),\ (x,y) \longmapsto (x,y+1)\rangle$ and $\langle (x,y) \longmapsto (x+1,y),\ (x,y) \longmapsto (-x,y+1)\rangle$. Their fundamental domains are as shown below, and thus these 2-manifolds are the **torus** and the **Klein bottle** respectively.



However if one allows G ⊂ Diff($\mathbb{R}^2$) to have *fixed points*, then one can realize *all* 2-manifolds as $\mathbb{R}^2$/G. As creator of the theory of automorphic functions, Poincaré was well aware of this, and so might have been hoping for a similar structure theorem for 3-manifolds ?

Poincaré's interest in these 3-manifolds was also due to his work on dynamical systems — these 3-manifolds are the mapping tori of automorphisms of the torus — and PICARD's work on complex surfaces.

(f) Though this is the main result of this initial announcement, it would not be correct (see e.g. II.0.b) to call it "*Poincaré's first topological theorem*" (as against "*Poincaré's last geometric theorem*" of his last full-length paper) : we'll see later that it holds for all T ∈ GL(2,$\mathbb{Z}$) and has a natural connection with Algebraic Number Theory.

(g) *This is false :* Poincaré's own Duality Theorem (see II.3.4) will show that the second Betti number equals the first Betti number.

We remark that Poincaré will also give in the following paper some more mundane examples of homotopically inequivalent 3-manifolds having the same Betti numbers, e.g. $\mathbb{R}P^3$ and S^3.

(h) From its very inception, Poincaré's *Analysis Situs* was heavily influenced by PICARD's work on complex surfaces. In fact, in the *Troisième Complément*, Poincaré will re-define his 3-manifolds by a polynomial equation $z^2 = F(x,y)$, where the complex variable y is constrained to be on some closed curve. The *Quatrième Complément* will also be devoted to complex surfaces.

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CHAPTER II

ANALYSIS SITUS

Journal de l'École Polytechnique, 1 (1895), 1-121.

§ 0. Introduction.

Today, nobody doubts that n-dimensional Geometry has objective reality. The beings of hyperspace can be defined precisely, just like those of ordinary space, and even if we can't visualize them, we can conceive of them, and study them. Thus, for example, one may criticize a Mechanics of more than three dimensions as lacking objective reality, but the same cannot be said of Hypergeometry.

Geometry in fact does not have as sole *raison d'être* the description of objects which we can sense : it is above all the analytical (= logical) study of a group; nothing prevents us thus from embarking on a study of analogous and more general groups.

But why, one might ask, can't one stick to an analytical language instead of replacing it by a geometrical one, which surely loses all its advantages as soon as the senses cannot intervene ? It is because this new language is more concise; it is because the analogy with ordinary geometry creates associations of fertile ideas and may suggest useful generalizations.

But maybe these reasons are not sufficient ? It is not enough in fact that a science be legitimate : it is necessary that its utility be incontestable. **So many are the objects which solicit our attention, that only the most important have the right to obtain it.**

Indeed, there are parts of Hypergeometry which are not very interesting : there are, for example, the researches on the curvature of (hyper)surfaces in n dimensional space. One is sure from the very beginning of obtaining the same results as in ordinary Geometry, and thus one undertakes a long voyage only to see the same scenery which one encountered at home.

There are some problems where analytical language would be totally inconvenient.

One knows the utility of geometrical figures in complex function theory, and in evaluating complex line integrals, and one badly misses their assistance when one wants to study, for example, the functions of two complex variables.

Let us try to fathom the nature of this assistance; firstly, the figures bolster the infirmity of our spirit by calling to its aid our senses; but it is not this alone. It has been often repeated that **Geometry is the art of reasoning well with figures not well-made**; yet these figures, if they are not to mislead us, must satisfy certain conditions; their proportions can be grossly different, but the relative positions of their various parts must not be in disorder.

The aim of these figures is thus to make us conversant of certain relations between the objects of our study, and these relations are those which pertain to a branch of Geometry called **Analysis Situs** which describes the relative situation between some points, lines, and surfaces, without bothering about their sizes.

There are similar relations between the beings of Hyperspace; there is thus an Analysis Situs in more than three dimensions, as has been demonstrated by **Riemann** and **Betti**.

This science makes us knowledgeable about these kinds of relations, even though now this knowledge is not intuitive, since our senses are no longer involved. **Thus this science is going to, to some extent, render the same service which we demand ordinarily of the figures of Geometry**.

I will restrict myself to three examples.

The classification of algebraic curves by means of their genus is based, following **Riemann**, on the classification of closed real surfaces, made from the viewpoint of Analysis Situs. An immediate induction now tells us that the classification of algebraic surfaces and the theory of their birational transformations is intimately tied to the classification of closed real (hyper)surfaces in 5-space from the viewpoint of Analysis Situs. M. **Picard**, in a work which has been hailed by the *Académie des Sciences*, has already stressed this point.[(a)]

Besides, in a series of memoirs published in the *Journal de Liouville* and entitled *"Sur les courbes définis par les équations différentielles"*, I have used ordinary 3-dimensional Analysis Situs to study (second order) differential equations. The same researches have also been pursued by M. **Walther Dyck**. One sees easily that a generalized Analysis Situs would permit us to similarly treat higher order equations, and in particular those of Celestial Mechanics.[(b)]

M. **Jordan** has analytically determined the groups of finite order which are contained in the linear group of n variables. M. **Klein** had previously, by a geometrical method of rare elegance, solved the same problem for the linear group of two variables. Could'nt one extend the method of M. Klein to a group of n variables, or even *an arbitrary continuous group* ? I have'nt been able to do this so far, but I have thought long on this question, and it appears to me that the solution should depend on a problem of Analysis Situs and that the generalization of the celebrated theorem of **Euler** should play a role in this.[(c)]

I do not think therefore that I have, in writing this memoir, laboured on some work having no utility; I regret only that it is so long; but, when I have attempted to constrain myself, I have tended to become obscure; so I have preferred to be a little garrulous.

NOTES

(0.a) Upto birational equivalence a non-singular complex curve is determined by the connectivity of its Riemann surface.

Here, by a **complex curve** is meant a polynomial equation $f(x,y) = 0$ over $\mathbb{C}$, **non-singular** means that $\partial f/\partial x$ and $\partial f/\partial y$ are never both zero, and two complex curves are deemed **birationally equivalent** iff their equations are related to each other by a rational change of variables.

The solutions of each such equation $f(x,y) = 0$ had been visualized by RIEMANN (1857) as points of an *abstract* orientable closed 2-manifold V — its **Riemann surface** — as follows.

For almost all x, the polynomial equation $f(x,y) = 0$ has $\deg_y f$ distinct solutions $y = \phi(x)$. Let $B_i \in \hat{\mathbb{C}}$ be the finitely many exceptions, O any other point, and consider the 2-cell obtained by **cutting** the 2-sphere $\hat{\mathbb{C}}$ along the lines OB_i :

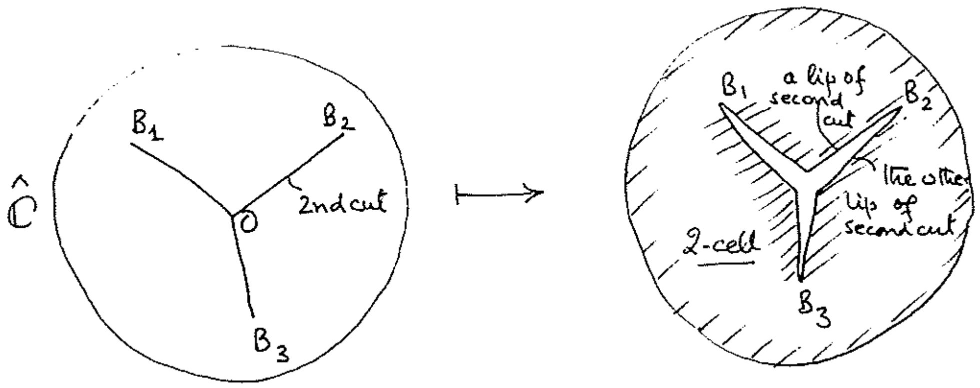

Fig. (0).

Then V is obtained from $\deg_y f$ disjoint copies of this 2-cell by identifying a **lip** of a cut of any copy, to the other lip of the same cut of another copy, as per the analytic continuation of the function $\phi(x)$.

Hoping for a similar result, PICARD had likewise visualized the solutions of a non-singular complex polynomial equation $f(x,y,z) = 0$ as an *abstract* — the "in 5-space" of the text is redundant — closed 4-manifold V, and had calculated the first Betti number of V.

Further results regarding the fundamental and homology groups of these **complex surfaces** V will be given in the *Troisième* and *Quatrième Compléments* of this paper.

(0.b) The index of any tangent vector field on a surface equals the Euler characteristic of the surface. This is one of the many remarkable results which Poincaré had obtained before 1885 in the course of his extensive dynamical investigations.

Here, we recall that the vector field is assumed to have only isolated singularities, and its **index** is the sum of the **winding numbers** (or **degrees** : see 8.c) of the maps $S^1 \longrightarrow S^1$ obtained by normalizing the vector field on small circles enclosing each singularity.

We note that the solutions of a second order **ordinary differential equation** $F(y, \frac{dy}{dt}, \frac{d^2y}{dt^2}) = 0$ can be visualized as the trajectories of a vector field on the 2-manifold defined by $F(u,v,w) = 0$ (and some inequalities) : thus the aforementioned **index theorem** gives information about the solutions of this O.D.E. Likewise its generalization to higher dimensional manifolds, duly established later by HOPF, gives some information about O.D.E.'s of order ≥ 3. We remark that similar results are now known also for partial differential equations.

(0.c) Poincaré is of course *wrong* in asserting that the (next-to-impossible) problem of **classifying the finite subgroups of** $GL(n,\mathbb{C})$ had been solved : even the easier task of classifying the finite subgroups of $GL(n,\mathbb{Z})$ remains to be accomplished.

However JORDAN (1878) had proved many interesting results about finite groups — e.g. that **there is a constant λ depending only on** n **such that any finite subgroup of** $GL(n,\mathbb{C})$ **contains an Abelian normal subgroup of index less than λ** — and (excepting two groups of orders 168 and 169 which he missed) had classified all the finite subgroups of $GL(3,\mathbb{C})$. This case n = 3 was much harder than the case n = 2, which had been done previously by KLEIN. The classification is now known for n ≤ 7 : for references, and other information on this subject see DIXON.

Covered this: results of Quillen et al

I do not know if there are (as Poincaré hopes) topological proofs of Jordan's results, but the generalization of **Euler's formula** V-E+F = 2 (between the numbers of vertices, edges and faces of a polyhedron) alluded to by Poincaré will be found in § 16 of this very paper.

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§ 1. Prèmiere définition des variétés.

(Smooth differentiable affine varieties.)

(1.1) Definition. A nonempty subset V of n-space which is defined by p equations $F_{\alpha}(x_1, \ldots, x_n) = 0$ and q inequalities $\phi_{\beta}(x_1, \ldots, x_n) > 0$, where the functions F and ϕ are continuously differentiable, will be called an (n-p)-dimensional **variety (of the first kind)** if the rank of the matrix $[\partial F_{\alpha} / \partial x_i]$ is equal to p at all points of V.[a]

When a variety is defined only by inequalities (i.e. when p = 0) then it is called a **domain** (such a V is an open subset of $\mathbb{R}^n$). Furthermore, varieties which are one-dimensional, resp. not one-dimensional but having codimension one, are called **curves**, resp. (hyper) **surfaces**.

A variety will be called **bounded** if the distance of all points from the origin is less than some constant.

(*We emphasize that, from now on, the unqualified word* "variety" *will always stands for a* "variety of the first kind".)

(1.2) We will usually confine ourselves to (path) **connected** varieties. This because **any variety can be decomposed into some** (possibly infinitely many) **connected varieties.**[b]

For example, the plane curve shown below is the disjoint union of the two connected curves obtained by adjoining to its defining equation either the inequality $x < 0$ or else $x > 0$.

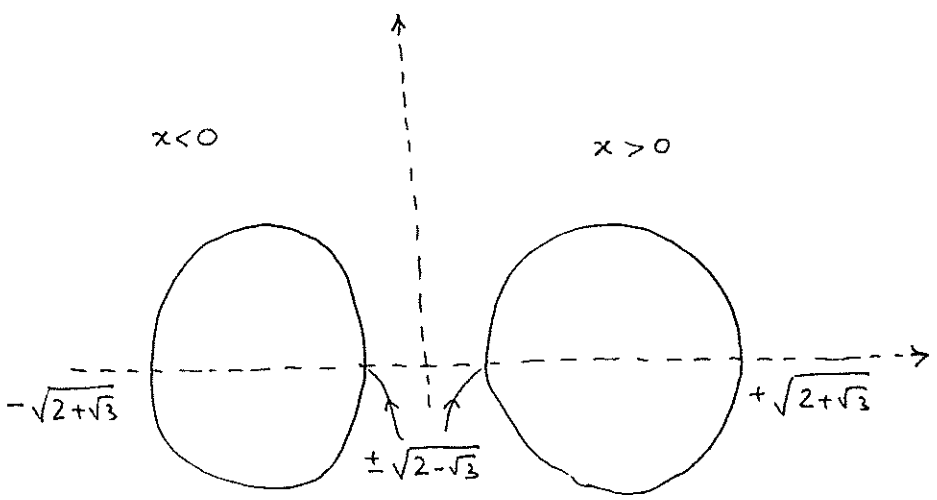

Fig. (1). $y^2 + x^4 - 4x^2 + 1 = 0.$

(1.3) Definition. By the **complete boundary** of the above variety V we will mean the set of all points of n-space satisfying $\{F_\alpha = 0,\ 1 \le \alpha \le p,\ \phi_\beta = 0;\ \phi_\gamma > 0,\ 1 \le \gamma \ne \beta \le q\}$ for some $1 \le \beta \le q$.

However sometimes we'll think of the largest (non-singular) (n-p-1)-dimensional variety contained in this set as the true **boundary** of V. A **closed variety** will be one which is connected, bounded, and which has an empty (true) boundary.[c]

NOTES

(1.a) Though here Poincaré considers C^1 functions, *it will be assumed from now on that all functions* (e.g. F and ϕ) *are infinitely differentiable* : in fact Poincaré himself will (from § 3 on) quite often demand that some functions be even *real analytic*.

Poincaré will check in (3.5) that the **non-singularity** of V, i.e. the **Jacobian criterion**, $\mathrm{rank}[\partial F_\alpha / \partial x_i] \equiv p$ on V, ensures that V is **smooth**, i.e. that each of its points has a neighbourhood diffeomorphic to (n-p)-space : in fact $\mathrm{rank}[\partial F_\alpha / \partial x_i] \equiv p$ *on* V *implies* V *is an* (n-p)-*manifold, even if the number of equations* $F_\alpha = 0$ *is more than* p.

However, the fact that the number of equations is exactly p, i.e. that V **is the intersection of** p **hypersurfaces**, implies in addition that this (n-p)-manifold V is of a special kind (see § 8).

Nowadays, a V defined as in (1.1), but by possibly more than p equations, would be called a (differentiable and non-singular) **quasi-affine variety**, or more precisely a variety *of* the open affine subset U defined by the q inequalities $\phi_\beta > 0$; and simply an **affine variety** in case there are *no* inequalities.

We note that the Jacobian criterion can be subsumed within the defining inequalities. On the other hand, note also that, *for many purposes, we don't need to consider inequalities at all, because* **any quasi-affine variety of n-space is diffeomorphic** (see § 2) **to an affine variety of** (n+1)-**space**, viz. that defined by the old equations $F_\alpha(x_1, \ldots, x_{n+1}) = 0$, and the new equation $x_{n+1}.f_U(x_1, \ldots, x_n) - 1 = 0$, where $f_U : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a differentiable function which is nonzero precisely on U, the open set defined by the q inequalities $\phi_\beta > 0$.

Here we have used the well-known fact that any given closed set of $\mathbb{R}^n$ is equal to $f^{-1}(0)$ for some differentiable function $f: \mathbb{R}^n \to \mathbb{R}$ (e.g. for an affine variety $\{F_\alpha = 0\}$ we can take $f = \sum_\alpha (F_\alpha)^2$). Thus *the class of possibly singular affine differentiable varieties of n-space is simply enormous* : it consists of all the closed sets of the (usual) **topology** of n-space !

However it contains the very interesting subclass of possibly singular **affine algebraic varieties,** i.e. those V's which are defined by some *polynomial* equations $F_\alpha(x_1, \ldots, x_n) = 0$, and (assuming that these F_α's generate all such equations) the points of V at which the Jacobian has maximal rank are called its *non-singular points.*

The notion of dimension extends in a natural way to any such singular variety V : one defines dim(V) to be the length of any maximal chain of **irreducible** subvarieties (i.e. those which are not unions of two proper subsets themselves definable by polynomial equations) of V. If dim(V) = n-p, and V is defined by just p polynomial equations (cf. 1.1), then V is called a **complete intersection**.

The **complexification** $V_\mathbb{C} \supseteq V$, i.e. the subspace of $\mathbb{C}^n$ consisting of all complex solutions of the defining (real) polynomial equations $F_\alpha = 0$ of the algebraical variety V, is often easier to study than V itself. However, if the polynomials F_α are over $\mathbb{Z}$, then interest centers most not on this bigger space $V_\mathbb{C}$, but on the smaller subsets of **integral or rational points** $V_\mathbb{Z} \subseteq V_\mathbb{Q} \subseteq V$. To extract the maximum information about $V_\mathbb{Q}$, *it is useful to have an Analysis Situs over each completion of the rationals*, e.g. the p-adic numbers $\mathbb{Q}_p$: however in these notes we'll deal only with the real completion $\mathbb{R}$ (and its algebraic closure $\mathbb{C}$).

(1.b) This follows because V, and thus each component of V, **has a**

trivial normal bundle (cf. § 8 below) : so each component is the zero set of some differentiable function $\mathbb{R}^n \to \mathbb{R}^p$ of rank ≡ p.

(1.c) Note that Poincaré's "complete boundary" is somewhat incomplete in the sense that it does not contain points which satisfy all the equations F = 0, *more than one* of the equations $\phi = 0$, and the remaining inequalities.

We note that though "closed varieties" have compact closures, they need not be compact themselves : e.g. consider the closed 2-dimensional variety of 3-space defined by $x_1^2 + x_2^2 + x_3^2 - 1 = 0$ and $1 - x_3 > 0$.

As against this, we'll always use **closed manifold** to mean a manifold (without boundary) which is compact.

§ 2. Homéomorphisme.

(= Diffeomorphism.)

(2.1) Consider the "group" formed by all differentiable (in both directions) bijections between pairs of open subsets of a euclidean space : the science whose object is the study of this and some other analogous "groups" is called **Analysis Situs.**[a]

(2.2) Definition. Two varieties of n-space will be called **diffeomorphic** iff there is a bijection between them which extends to a differentiable bijection between open euclidean sets obtained by replacing their defining equalities $F_\alpha = 0$ by some inequalities $-\varepsilon < F_\alpha < +\varepsilon$. A similar definition can be given for more complicated figures, made up of many varieties, of n-space.[b]

NOTES

(2.a) Poincaré considers instead the larger category of all differentiable maps, between open subsets of n-space, which have nonsingular Jacobians everywhere. We note that such maps need not have *inverses*, e.g. consider $z \longmapsto e^z$ on all of $\mathbb{R}^2 = \mathbb{C}$.

So we have taken the liberty of modifying his definition slightly : but note that even now, *multiplication is not always defined*, and that there are *many identity elements* : so our "group" is still only a **groupoid** (a special type of category).

Poincaré's definition of Analysis Situs is evidently inspired by KLEIN whose *Erlanger Program* (1872) had pointed out that each known geometry could be considered as the study of a concomitant group.

(2.b) Such a modern definition — cf. MILNOR — would be simply to declare any two closed euclidean sets diffeomorphic iff they are isomorphic in the **category of affine varieties** (differentiable and possibly singular), i.e. one whose morphisms are restrictions of differentiable maps between affine spaces. (Replacing "differentiable" by "polynomial" one gets the subcategory of affine *algebraic* varieties.)

However note e.g. that *there is no homeomorphism of the positive* x-*axis to the entire* x-*axis which extends to a continuous map from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$: this shows why, as morphisms for the bigger **category of quasi-affine varieties**, one uses restrictions of differentiable maps between all affine open sets.

We note also that for *compact* (non-singular) varieties of n-space, the apparently stronger definition of Poincaré is implied by the above categorical one : to see this use compactness and the fact that varieties (of the first kind) have trivial normal bundles (see 8.b).

The fact that Poincaré did not demand that his differentiable bijection be that of the *entire* ambient n-space indicates clearly that he was aware of **knotting**.

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[Mi] J. W. MILNOR, *Topology from the Differentiable Viewpoint,* Univ. of Virginia (1966).

§ 3. Deuxième définition des variétés.

(Manifolds.)

(3.1) In the following we'll consider m-dimensional varieties ν which consist of all points of n-space satisfying a system of n equations $x_i = \theta_i(y_1, \ldots, y_m)$ with rank $[\partial\theta_i / \partial y_j] \equiv m$, and some inequalities $\psi(y_1, \ldots, y_m) > 0$.[(a)]

For example, the system of three equations $x_1 = (R + r.\cos y_1).\cos y_2$, $x_2 = (R + r.\cos y_1).\sin y_2$ and $x_3 = r.\sin y_1$ defines a **torus** in 3-space :

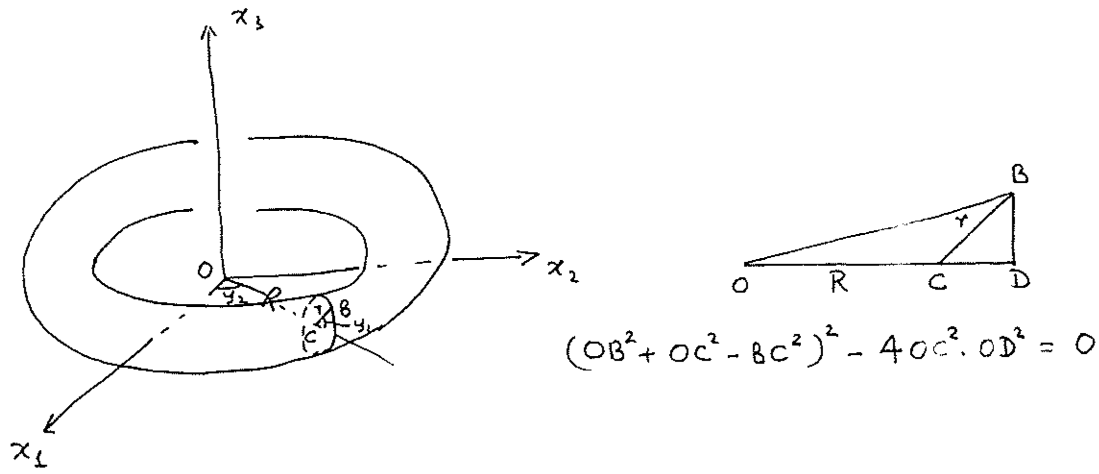

Fig (3.i). $(x_1^2 + x_2^2 + x_3^2 + R^2 - r^2)^2 - 4R^2(x_1^2 + x_2^2) = 0.$[(b)]

In the following we'll mostly use *connected* ν's (we can in fact even assume them to be m-cells) and, unlike in the above example, our θ's will be one-one.

(3.2) Without loss of generality we can, and will, assume in the above, and likewise for the definition of (1.1), that all equations are **real analytic** : this follows because we can always replace θ by an arbitrarily close real analytic function θ'.[(c)]

With this understood, two m-dimensional varieties ν of type

(3.1) will be called **analytic continuations** of each other iff their intersection $\nu \cap \nu'$ is also an m-dimensional variety of type (3.1).

Note that two varieties ν coincide iff their parameters y_1, ... , y_m and z_1, ... , z_m are related by an analytic diffeomorphism; so, more generally, they are analytic continuations of each other, iff there is a partially defined analytic diffeomorphism between their parameters.

We now use analytic continuation to vastly extend the applicability of definition (3.1) as follows.

(3.3) Definition. By a connected m-**manifold** we'll understand any *connected network* of varieties ν, i.e. a graph whose vertices are connected varieties of the type (3.1), with two vertices contiguous in the graph iff they are analytic continuations of each other.[(d)]

We will now check that all varieties are manifolds, and for this we will use the following well-known result.

(3.4) Inverse Function Theorem. *If the* n *real analytic equations* $y_i = F_i(x_1, \ldots ,x_n)$ *are such that their functional determinant is nonzero at* x, *then they have real analytic solutions* $x_i = \theta_i(y_1, \ldots ,y_n)$ *valid in some neighbourhood of* F(x).[(e)]

(3.5) Theorem. *Varieties are manifolds.*[(f)]

Proof. Let P be any point of an (n-p)-dimensional analytic variety V, defined as in (1.1) by p equations $F_\alpha(x_1, \ldots , x_n) = 0$ and some inequalities $\phi(x_1, \ldots , x_n) > 0$.

To check that V is a manifold it obviously suffices to find an (n-p)-dimensional variety ν_P of the type (3.1) such that $P \in \nu_P \subseteq V$.

For this, we choose any p additional analytic functions F_{p+1}, ... , F_n of n variables, which vanish at P, and are such that the functional determinant of all the n functions F_i is nonzero at P. Using (3.4) we now solve the n equations $u_i = F_i(x_1, \ldots ,x_n)$ to get real analytic solutions $x_i = \theta_i(u_1, \ldots ,u_n)$ in some neighbourhood of $F(P) = 0$ specified by some inequalities $\lambda(u_1, \ldots ,u_n) > 0$. By making this neighbourhood smaller, if need be, we will assume also that these inequalities imply the defining inequalities $\phi(x_1, \dots, x_n) > 0$ of V.

So the n equations $x_i = \theta_i(0, \dots, 0, y_1, \dots, y_{n-p})$ and the inequalities $\lambda(0, \dots, 0, y_1, \dots, y_{n-p}) > 0$ are satisfied by P and imply the defining p equations $F_\alpha(x_1, \dots, x_n) = 0$ and the defining inequalities $\phi(x_1, \dots, x_n) > 0$ of V : they thus give us a ν_P such that $P \in \nu_p \subseteq V$. *q.e.d.*

However conversely, as we'll show later in (8.2), all manifolds are *not* varieties (of the first kind).

NOTES

(3.a) We note that *all such systems of equations and inequalities do not define varieties* : e.g. if θ is not assumed 1-1, ν can be a figure eight, or, as shown below, a non-orientable manifold (see § 8).

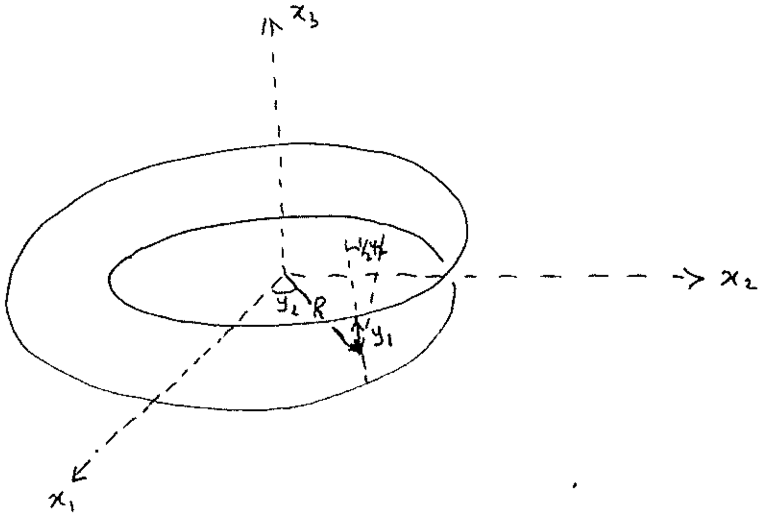

Fig. (3.ii). *The Möbius strip* $x_1 = (R - y_1 \sin \frac{1}{2} y_2)\cos y_2$, $x_2 = (R - y_1 \sin \frac{1}{2} y_2)\sin y_2$, $x_3 = y_1 \cos \frac{1}{2} y_2$, $|y_1| < R - \varepsilon$.

However, if the domain of the **parameters** $y_1, \dots, y_m$ is restricted to a sufficiently small open m-ball, then the new system will have a one-one θ, and one can check — cf. proof of (3.5) — that ν is an open m-**cell** of n-space, i.e. a variety of n-space diffeomorphic to m-space.

Thus *all such systems of equations and inequalities do define an* m-*manifold in* n-*space in the sense of* (3.3).

We remark that in the paper Poincaré talks about θ being 1-1 only at the *end* of this section.

(3.b) A general method — cf. HIRSCH — for writing a **polynomial equation representing a surface of any given genus** p is illustrated below. It is based on the observation that, for ε small, the boundary of any ε-neighbourhood, of a closed space curve having p-1 double points, is a surface of genus p.

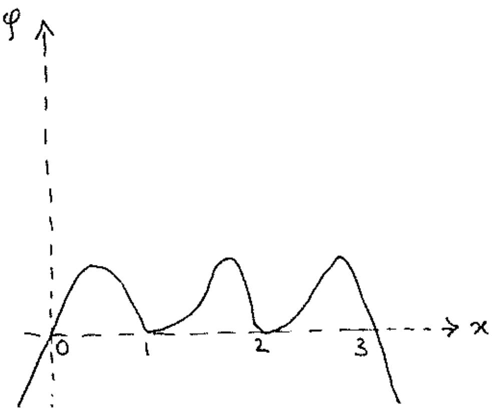

Fig. (3.iii). *Graph of function* $\phi(x) = x.(x-1)^2.(x-2)^2.(3-x)$.

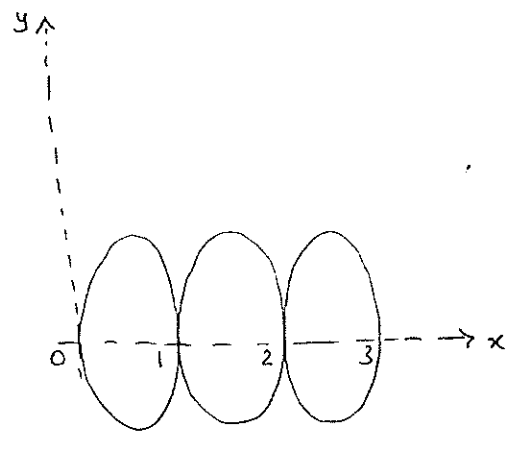

Fig. (3.iv). *Plane curve* $y^2 - \phi(x) = 0$.

Poincaré's definition, for it can be obtained at once by identifying, in the disjoint union of the varieties ν, all overlaps $\nu\cap\nu'$ *of those pairs* of varieties which are analytic continuations of each other (the required ϕ's being now provided by the inverses of the θ's occuring in the definition of the ν's).

On the other hand note that some other intersections $\nu\cap\nu'$ need not be m-varieties of type (3.1). More precisely, **the union in n-space of all the varieties ν of the network is the image of an immersion of the aforementioned M in n-space.**

We have called Poincaré's network of varieties a *manifold* (and we'll use the same letter M for it) simply because for Poincaré this **immersion** (= locally 1-1 smooth map) $M \longrightarrow \mathbb{R}^n$ is only extra baggage, and his focus will always be only on the abstract manifold M.

We note that, like its modern counterpart, Poincaré's definition is exactly similar for the continuously or infinitely differentiable or real or complex analytic cases ... ; however a special feature of the analytic case is that the concomitant immersion $M \longrightarrow \mathbb{R}^n$ is determined uniquely by any of its **germs** ν.

We remark that *the idea of an abstract manifold* (probably due to RIEMANN : see 0.a) was "well-known" in Poincaré's time, however it entered into mathematical *books* only starting with the book of WEYL.

This notion is very useful, and in fact forced on us, because many natural constructions (e.g. that of Riemann himself) lead us out from the class of manifolds in euclidean spaces, to manifolds which are not in any euclidean space.

However we remark that sometimes, even for the intrinsic study of a manifold, it is useful to give oneself *the convenience of an ambient euclidean space*, since then the manifold gets *readily* equipped with some geometrical and analytical tools frequently useful for its study.

Moreover, there is no loss of generality in doing this, because WHITNEY has shown that **any abstract n-manifold is diffeomorphic to a differentiable and non-singular affine variety** (see 1.a) **of 2n-space.**

(3.e) *Proof.* Let $F_i(x_1, \ldots, x_n)$ be n **formal power series** in n variables, having no constant terms, and such that the nxn matrix formed by the coefficients of their linear terms is nonsingular. Then it is easy to see that there are unique formal power series solutions $x_i = \theta_i(y_1, \ldots, y_n)$ having the same properties, the coefficients of the series θ_i being some **universal polynomial** functions of the coefficients of the original series F_i.

It remains to check that, if the series F are convergent in a neighbourhood of the origin, then the formal inverse series θ are also convergent in some neighbourhood of the origin. This we'll do using **Poincaré's method of dominant functions.**

The case when each F_i is a **geometric series** is easy, because now we have explicit formulas for their sums. Using these we can — cf. GOURSAT — explicitly solve $x_i = F_i$, thus obtaining formulas giving the sums of the series θ_i in a neighbourhood of the origin.

original reference is Ch 2 of Poincaré's own: "Les méthodes nouvelles...", vol. I

For the general case we can find, perhaps in a smaller neighbourhood, convergent geometric series G_i with positive coefficients which dominate the corresponding coefficients of the series F_i. But then, by virtue of the *universal* nature of the polynomials mentioned above, the convergent power series solutions of $y_i = G_i$ dominate the formal power series solutions $x_i = \theta_i$ of $y_i = F_i$: so the power series θ_i are convergent in a neighbourhood of the origin. *q.e.d.*

An analogous argument works for complex analytic maps, and one can establish an inverse function theorem also for continuously or infinitely differentiable maps. However the case of polynomial maps is much more difficult.

The point is that, if the polynomial equations $y_i = F_i$ have *local* polynomial solutions $x_i = \theta_i$, then these are also *global* solutions, and so $\det(\partial F_i/\partial x_j)$, being an identically nonzero polynomial, must be a nonzero constant. The converse is a well-known open problem.

Jacobian Conjecture. Let $y_i = F_i(x_1, \ldots, x_n)$ be n polynomial equations with $\det(\partial F_i/\partial x_j)$ a nonzero constant, then they have polynomial solutions $x_i = \theta_i(y_1, \ldots, y_n)$.

For more re this problem, and the combinatorics of the universal polynomials mentioned above, see BASS-CONNELL-WRIGHT.

(3.f) The method of dominant functions can be used also to prove (for the analytic case) the following generalization of (3.4).

An implicit function theorem. *If we are given* p *equations in* p+m *variables such that the functional determinant with respect to some* p *of the variables is nonzero, then we can locally solve for these* p *variables in terms of the remaining* m.

We note that (3.5) follows at once by applying the above to the p equations $F_\alpha(x_1, \ldots, x_n) = 0$, since we can solve for p of the x's in terms of the remaining n-p which can serve as our y's.

In the terminology of (1.a), (3.5) amounts to checking that **any nonsingular point of $F^{-1}(0)$ is smooth.** The converse of this is false : smooth points need not be non-singular.

However, it is true that **any smooth point of an irreducible complex algebraic variety $F^{-1}(0)$ is nonsingular** : see MILNOR.

We remark that Poincaré's (3.5) can be easily strengthened in many diverse ways — cf. (1.a) and § 8 — and these *"implicit function theorems"* are of fundamental importance in differential topology.

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§ 4. Variétes opposées.

(Orientation of varieties.)

(4.1) Definitions. We will equip each variety V with a **transverse orientation** determined by the order in which its defining equations $F_\alpha = 0$ are written. So interchanging any two equations gives, *not* V, but the **opposite variety** $-V$, and more generally, given any nonsingular matrix $A_{\alpha\beta}$ of functions, we will assume that the ordered set of equations $\sum_\alpha A_{\beta\alpha}.F_\alpha = 0$ gives V, resp. $-$ V, iff $\det(A_{\alpha\beta})$ is positive, resp. negative.

Furthermore, each variety ν of type (3.1) with θ one-one will be equipped with an **orientation** determined by the order in which its parameters $y_1, \dots, y_m$ are written. So interchanging two of them gives, *not* ν, but the opposite variety $-\nu$, and more generally, if the parameters undergo a transformation $y_1, \dots, y_m \longmapsto z_1, \dots, z_m$, we'll assume that the resulting variety is ν or $-\nu$, depending on whether the transformation's functional determinant is positive or negative.[a]

(4.2) Convention. The above two concepts will be tied to each other by stipulating that if $\nu_P \subseteq V$ as in (3.5), then ν_P has the correct orientation iff the nxn functional determinant mentioned in (3.5) is positive.[b]

(4.3) Oriented boundary ∂V. We will assume that each of the (n-p-1)-dimensional non-singular varieties occuring in the boundary (see 1.3) of a transversely oriented variety V, is equipped with the transverse orientation determined by writing the equations of V in order and putting the new equation $\phi = 0$ in the very end.

NOTES

(4.a) In modern terms we would say : the choice of an orientation of $\mathbb{R}^p$ fixes, via $F : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, an orientation of the trivial normal bundle of $V = F^{-1}(0)$ in $\mathbb{R}^n$, and an orientation of m-space fixes, via the one-one θ, an orientation of $\nu = \mathrm{Im}(\theta)$.

(4.b) In other words, the transverse orientation of $\nu_P \subseteq V$, *followed* by the orientation of ν_P, should yield the orientation of $\mathbb{R}^n$.

§ 5. Homologies.

(5.1) Definition. Suppose there is, in a given manifold M, a variety V whose oriented boundary consists of k_i copies of the variety ν_i for $1 \le i \le a$, and s_j copies of the variety $-\mu_j$ for $1 \le j \le b$. Then we'll write

$$k_1.\nu_1 + \ldots + k_a.\nu_a \simeq s_1.\mu_1 + \ldots + s_b.\mu_b,$$

and refer to this relation as a **homology** of M ; moreover we'll handle homologies *just like equations* : so the sum of any two homologies will also be deemed to be a homology, and we can take any term to the other side provided we change its sign.[(a)]

(5.2) We'll sometimes write $k_1.\nu_1 + \ldots + k_a.\nu_a \simeq \varepsilon$ to indicate that the sum of the varieties of M written on the left is homologous to a sum of varieties contained in the boundary of M.[(b)]

NOTES

(5.a) Example. To see that varieties can occur in ∂V with coefficients other than ± 1, consider e.g. V = {an open Möbius strip minus the arc ν_1} ⊂ M = $\mathbb{R}^3$.

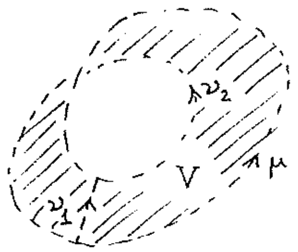

Fig (5).

Then $\partial V = 2\nu_1 + \nu_2 - \mu$, and so $2\nu_1 + \nu_2 \simeq \mu$ is a homology of $\mathbb{R}^3$.

In modern terms Poincaré's homologies can be understood as follows:

We linearly extend the definition (4.3) to **chains**, i.e. finite integral linear combinations of bounded varieties of V, to obtain the **boundary operator** $\partial : C_{q+1}(V) \longrightarrow C_q(V)$, where $C_q(V)$ denotes the Abelian group of all q-dimensional chains. Then $c \simeq 0$ iff $c \in Im(\partial)$.

Somewhat confusingly, Poincaré will also use the *different notion* which is obtained if we interpret "just like equations" of definition (5.1) as allowing division by nonzero integers. He clarified this distinction later in the *Complément* and called these **homologies with division**. We'll call these **rational homologies**, since now it makes sense to allow the coefficients to be rationals, and will denote them by $\simeq_{\mathbb{Q}}$. This notion too can be interpreted as above, by using now the graded vector space of **rational chains** $C(V;\mathbb{Q})$, i.e. all finite rational linear combinations of bounded varieties of V.

(5.b) We remark that Poincaré will make only a fleeting use in § 9 of this notion of **homology rel** bd(M). To make the given definition complete and intrinsic, we must define **boundary** bd(M) **of a manifold**, and this should be done in such a way that bd(M) is a subset of M.

So, we will assume here that M is a **manifold-with-boundary**, i.e. a space locally diffeomorphic to a euclidean closed half space, with bd(M) being all points of M which do not have a neighbourhood homeomorphic to an open euclidean set.

With this understood, we can now interpret these relative homologies as above, in terms of the quotient boundary operator $\partial : C_{q+1}(M, bdM) \longrightarrow C_q(M, bdM)$, where $C_q(M, \partial M) = C_q(M)/C_q(\partial M)$.

§ 6. Nombres de Betti.

(6.1) **Definition.** The cardinality of a maximal **linearly independent** set — i.e. one for which there is no non-trivial homology between its members — of closed r-dimensional varieties contained in M will be called the rth **Betti number** $b_r(M)$ of M.[(a)]

(6.2) **Example.** *Let* M *be a 3-dimensional domain whose boundary is the disjoint union of* n *closed surfaces* S_i, *then* $b_1(V) = \frac{1}{2}\sum_i b_1(S_i)$ *and* $b_2(V) = n - 1$.[(b)]

NOTES

(6.a) As noted in (I.a) we have modernized the definition slightly: Poincaré's numbers are *one more* than those of (6.1).

From the definition it is *immediate* that **the numbers** $b_r(M)$ **are diffeomorphism invariants**, however it is not at all obvious that they are **finite** for M compact. We'll see later that they are indeed finite, and in fact that **they coincide with the modern Betti numbers**.

Yet more is true ... (6.1) can be shown equivalent to the definition obtained by replacing "closed r-varieties" by "closed r-chains", i.e. $c \in C_r(M)$ such that $\partial(c) = 0$, where ∂ is as in (5.a). So $b_r(M)$ **is the rank of the free part of the rth homology group** $\ker(\partial)/\mathrm{im}(\partial)$ **of** ∂, this being well-defined because one can check the all-important $\partial\circ\partial = 0$.

Once again, it is *immediate* that **this homology group** $H_r(M; \mathbb{Z})$ **is a diffeomorphism invariant**, but not quite obvious that **it can be identified with the singular homology of** M. For a manifold-with-boundary we can likewise define the **relative homology group** $H_r(M, \mathrm{bd}M; \mathbb{Z})$ as $\ker(\partial)/\mathrm{im}(\partial)$, where now ∂ is as in (5.b).

As noted in (I.a) the numbers defined by BETTI himself were different from Poincaré's : in modern terms, he had considered **the least number of elements required to generate** $H_r(M;\mathbb{Z})$, rather than the rank of the free part of this group.

(6.b) *Proof.* Like in (1.b) we'll think of each S_i as the boundary of a thickening of a bouquet of $\frac{1}{2}\, b_1(S_i)$ circles. The given domain has thus the homotopy type of $S^3 \setminus \{\text{union of these bouquets}\}$, and the homology of this spherical complement can be calculated easily using the **Alexander duality** of § 9. *q.e.d.*

We remark that it was well-known since RIEMANN that $b_1(S)$ is *even* for any closed surface S, and the number $\frac{1}{2}\, b_1(S)$ was called the **genus** of the surface.

§ 7. Emploi des intégrales.

(Differential forms.)

(7.1) Definition. The integral

$$\int_V \sum \omega_{\alpha_1 .. \alpha_r}(x_1, \dots , x_n).dx_{\alpha_1} \dots dx_{\alpha_r} , \quad 1 \le \alpha_i \le n ,$$

over any r-dimensional variety V of n-space, will be defined to be

$$\sum_\nu \int \sum \omega_{\alpha_1 .. \alpha_r}(x_1, \dots , x_n).\det(\partial x_{\alpha_i}/\partial y_j).dy_1 \dots dy_r ,$$

where $V = \sum \nu$ is a partition (see 3.5 and 8.2) of V into some compatibly oriented varieties ν of the type (3.1), and for each ν, the multiple integral is evaluated, using the equations $x_i = \theta_i(y_1, \dots , y_r)$ of ν, between the limits of y_i prescribed by the inequalities of ν.[a]

We'll always assume that the components $\omega_{\alpha_1 .. \alpha_r}$ are **skewsymmetric** in their indices, i.e. they merely change sign when two of the indices α_i are interchanged.[b]

(7.2) Poincaré's Lemma. *The integrals* $\int_V \omega$ *are zero for all closed varieties* V *of* n-*space if and only if the* $\binom{n}{r+1}$ *cyclic sums*

$$\sum (-1)^{r.i}.\partial/\partial x_{\alpha_i}[\omega(\alpha_{i+1}, \dots ,\alpha_{r+1},\alpha_1, \dots ,\alpha_{i-1})],$$

are identically zero, i.e. iff $d\omega = 0$ throughout $\mathbb{R}^n$:. for a proof see my paper in Acta Math, vol. 9.[c]

(7.3) Remark. By using the methods of this Acta paper (i.e. the generalized **Stokes' formula**) it follows that the $\binom{n}{r+1}$ conditions $d\omega = 0$, in a neighbourhood of an m-manifold M of n-space, are *sufficient* to ensure that for each homology $\partial c \simeq 0$ of M we have $\int_{\partial c}\omega = 0$. However they are not *necessary* : they can be replaced by only $\binom{m}{r+1}$ analogous conditions at all points of M.[d]

(7.4) Proposition. *For any* ω *as in* (7.3) *one can find at* *most* $b_r(M)$ *numbers such that the integral* $\int_V \omega$ *of* ω *over any closed r-variety* V *of* M *is a linear integral combination of these numbers.*

Proof. If we take $t > b_r(M)$ such integrals $\{\int_{V_1} \omega , \dots , \int_{V_t} \omega\}$, there there is a non-trivial homology $n_1V_1 + n_2V_2 + \dots + n_tV_t \simeq 0$ between the corresponding closed varieties V_1, and so by (7.3) we have a non-trivial linear integral dependence $n_1 . \int_{V_1} \omega + \dots + n_t . \int_{V_t} \omega$. Thus the additive subgroup of $\mathbb{R}$ generated by such integrals is free of rank $\le b_r(M)$, and so has a $\mathbb{Z}$-basis containing $\le b_r(M)$ elements. *q.e.d.*

In other words, the indefinite integral $\int\omega$ of any r-form ω with $d\omega = 0$ has at most $b_r(M)$ **periods** (= elements of above $\mathbb{Z}$-basis).

Further it can be shown that **this bound is the best possible**, i.e. there exists such an ω having $b_r(M)$ periods. For r = 1, m-1, this interpretation of the numbers $b_r(M)$ was given by BETTI himself.**(e)**

NOTES

(7.a) *The definition does not depend on the partition* $V = \sum \nu$: this follows from the **change of variables formula** for multiple integrals, because any two ν's are related by a positive functional determinant.

(7.b) There is no loss of generality in assuming this because it is clear that the skew-symmetrization of the integrand

$$\omega = \sum \omega_{\alpha_1 .. \alpha_r}(x_1, \dots , x_n) . dx_{\alpha_1} \dots dx_{\alpha_r} ,$$

has the same integrals. **We'll in fact identify ω with its indefinite integral** $V \longmapsto \int_V \omega$, i.e. we will think of ω as a **differential form** of $\mathbb{R}^n$. For example, the integrals of dydx being the negative of that of dxdy, we'll assume dxdy = -dydx, and this **wedge product** will usually be denoted $dx \wedge dy$.

(7.c) Here, for a function $\omega(x_1, \dots , x_n)$, the **total differential** df is defined by

$$d\omega = \frac{\partial \omega}{\partial x_1} . dx_1 + \dots + \frac{\partial \omega}{\partial x_n} . dx_n ,$$

and, more generally, the **exterior derivative** of a degree r form ω is the degree r+1 form defined by

$$d\omega = \sum d\omega_{\alpha_1 . . \alpha_r} \wedge dx_{\alpha_1} \wedge \dots \wedge dx_{\alpha_r} .$$

Sketch proof of (7.2). Since $d \circ d = 0$, $d\omega = 0$ is necessary for the integrability of the *partial differential equations* $d\theta = \omega$: in the cited paper, Poincaré had checked conversely that (for $\mathbb{R}^n$) this **integrability condition** ensures that one has $d\theta = \omega$ for some θ (and it is this statement which is now usually called "Poincaré's Lemma").

Then (7.2) follows easily by using the generalized Stokes' formula, which too was established by Poincaré in the same paper :

$$\int_{\partial V} \omega = \int_V d\omega .$$

This formula is deduced by an inductive argument starting from the case m = 1, i.e. the **fundamental theorem of calculus**. *q.e.d.*

(7.d) In modern terms these "$\binom{m}{r+1}$ analogous conditions" can be formulated as follows :

First recall that, by the **tangent space** TM_x at the point x of the manifold M, we understand the m-dimensional vector subspace of $\mathbb{R}^m$ spanned by $\frac{\partial x}{\partial y_1}$, ... , $\frac{\partial x}{\partial y_m}$, where $x = \theta(y_1, \dots , y_m)$ is any parametrization of a neighbourhood of x.

Then that, a degree r real (or complex) **differential form** $\omega \in \Omega^r(M)$ of M assigns smoothly, to each point x of M, a skewsymmetric r-linear map $TM_x \times \dots TM_x$ (r times) $\longrightarrow \mathbb{R}$ (or $\mathbb{C}$). With respect to a local parametrization, the local 1-forms dual to the local vector fields $\{\frac{\partial x}{\partial y_1}$, ... , $\frac{\partial x}{\partial y_m}\}$ are denoted $\{dy_1, \dots , dy_n\}$, and so *locally* such an ω can again be written uniquely as $\sum \omega_{\alpha_1 . . \alpha_r}(y_1, \dots , y_m) . dx_{\alpha_1} \dots dx_{\alpha_r}$.

The **exterior derivative** $d : \Omega^r(M) \longrightarrow \Omega^{r+1}(M)$ can now be defined exactly as in (7.c), since an easy verification shows that the choice of the local parametrization is immaterial.

With this understood, the required "$\binom{m}{r+1}$ conditions" amount to demanding that ω **should be a differential** r-**form of** M **such that** $d\omega = 0$.

(7.e) We note that (the indefinite integral $\int\omega$ of) any differential r-form ω determines a **cochain** $\omega \in C^r(M)$, i.e. a $\mathbb{Z}$-linear map $\omega : C_r(M) \to \mathbb{R}$ (or $\mathbb{C}$), and if $d\omega = 0$ this cochain vanishes on $\mathrm{Im}(\partial)$, and so induces a $\mathbb{Z}$-linear map $\omega : H_r(M; \mathbb{Z}) \to \mathbb{R}$ (or $\mathbb{C}$) : Poincaré is asserting that, for some ω, **the group of residues** $\omega(H_r(M)) \subset \mathbb{R}$ attains its maximum $\mathbb{Z}$-dimension $b_r(M)$.

Example. The **Cauchy residue formula** shows that the period group of the complex valued differential 1-form $\omega = (1/z).dz$ of $\mathbb{C} \setminus \{0\}$ is the additive subgroup of $\mathbb{C}$ generated by $2\pi i$ and so has rank $b_1(\mathbb{C} \setminus \{0\}) = 1$.

We remark that Poincaré's assertion (of which he offers no proof) follows easily from (and is in fact equivalent to) **de Rham's theorem.** This theorem says that the groups kerd/imd, defined by the exterior derivative d, coincide with the **cohomology** $H^*(M; \mathbb{R}$ or $\mathbb{C})$ of M, i.e. the groups $\ker\delta/\mathrm{im}\delta$ defined by the **coboundary operator** $\delta : C^r(M) \to C^{r+1}(M)$, $(\delta a)(c) = a(\partial c)$, acting on *all* real or complex cochains of M. So e.g. Poincaré's Lemma tells us that $H^r(\mathbb{R}^n; \mathbb{R}) = 0$ for all $r \geq 1$.

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§ 8. Variétés unilatères et bilatères.

(Orientability of manifolds.)

(8.1) Definition. An m-manifold M (see 3.5) will be called **orientable** iff we can assign an orientation (see 4.1) to each of the varieties ν of its connected network (= graph), in such a way that the mxm determinant $\det(\partial y_i/\partial y'_j)$ is positive whenever ν is contiguous to ν'.

Since the above determinant is obviously always nonzero, M can fail to be orientable iff *either*, its graph has a contiguous pair $\{\nu,\nu'\}$ with the determinant not of the same sign in all the components of $\nu\cap\nu'$, *or else*, has a **one-sided circuit** $(\nu_1, \ldots , \nu_q)$, i.e. one for which making the determinant between ν_i and ν_{i+1} positive for $1 \le i \le q-1$, makes the determinant between ν_q and ν_1 negative. (See also §6 of "Second Complément.)

Now assume M oriented, and increase the size of its network by adding a new ν^* which also parametrizes a portion of the space M. We choose a ν which overlaps ν^*, and orient ν^* in such a way that its determinant with ν is positive. Then it is easy to check that the determinant of ν^* with *all* overlapping ν's is positive. Thus **orientability is a property of the space** M, rather than of the network of varieties ν covering it.[a]

(8.2) Theorem. *Varieties are orientable manifolds.*[b],[c]

Proof. In (3.5) we saw that any (non-singular) $m = n-p$ dimensional variety V of n-space is an m-manifold. If we orient the parametrizations ν_P constructed in (3.5), as per convention (4.2), then a determinantal calculation (or the remark 4.b) shows that they are compatible to each other in the sense of (8.1). *q.e.d.*

So an (open) **Möbius strip**, being non-orientable, cannot occur as a variety V in any n-space, even though it is of course a 2-dimensional manifold.

We remark also that if a *closed* m-manifold embeds in (m+1)-space, then it is necessarily orientable.[(d)]

NOTES

(8.a) A *maximal network*, i.e. one obtained by adding *all* such ν^*, is nowadays called the **differentiable structure** (or analytic structure etc.) of M.

(8.b) However **orientability is not sufficient to ensure that a manifold** M **can occur as a variety** (of the first kind) in some n-space.

To see this note that if $M = F^{-1}(0)$ for some smooth function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ having rank p at all points of M, then we can smoothly choose, for each $x \in V$, p linearly independent **normal vectors** $v_i(x)$ of $(TM_x)^{\perp}$, the orthogonal complement in $\mathbb{R}^n$ of the tangent space (see 7.d) to M at x.

So, since since adding this **trivial normal bundle** $TM^{\perp}$ to the tangent bundle TM gives a trivial bundle $M \times \mathbb{R}^n$ it follows that **the characteristic classes of such an** M **must be zero** : cf. MILNOR.

For example, *the* **complex projective plane** $\mathbb{C}P^2$, *which is orientable, but has non-trivial characteristic classes, cannot occur as a variety in any* n*-space.*

The aforementioned **characteristic classes** are important invariants of M which can be defined as follows. One embeds M in any $\mathbb{R}^N$ with N big, and considers the map $x \longmapsto TM_x$ of M into the **Grassmann manifold** G(N,m) of all m-dimensional vector subspaces of $\mathbb{R}^N$. This map induces a homomorphism $H^*(G(N,m)) \longrightarrow H^*(M^m)$, and the cohomology classes lying in the image of this map are called the characteristic classes of M.

It can also be shown conversely that *any* M *whose tangent bundle is* **stably parallelizable**, *i.e. becomes trivial after adding a suitably high dimensional trivial bundle, does occur as a variety in some* n*-space.*

Also note that if $F : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ has rank p at all points of $F^{-1}(0)$, then $F^{-1}(0)$ is a *closed set* of $\mathbb{R}^n$ and an *embedded submanifold* of $\mathbb{R}^n$, so e.g. $F^{-1}(0)$ cannot be a cylinder with a point missing or a figure eight.

And, conversely, one can check that *any embedded closed submanifold of $\mathbb{R}^n$ having a trivial normal bundle is of this type* $F^{-1}(0)$.

However — see notes (1.a) and (3.d) — definition (1.1) suffices to give all manifolds, *provided* we modify it by allowing $\geq$ p equations.

(8.c) For any smooth map $f: N^n \to P^p$ between manifolds, the **derivative** at x, $Tf : TN_x \to TP_{f(x)}$, is the linear map which images the basis vectors $\frac{\partial\theta}{\partial y_i}$ (see 7.d) of TN_x to $\frac{\partial(f\circ\theta)}{\partial y_i}$: note that the chain rule for partial derivatives shows that T **is a functor** from the category of pointed manifolds to the category of vector spaces.

An $x \in N$ is a **regular point** of f if the derivative of f at x is surjective, and a $y \in P$ is called a **regular value** of f if any x with $f(x) = y$ is a regular point of f.

The argument of (3.5) shows that *the level surfaces of* f *constitute a* **foliation** *of the open set of its regular points*, i.e. a partition into **leafs** (= submanifolds) such that near each point one has local coordinates $u_1, \dots, u_{n-p}, y_1, \dots y_m$ in which each slice y = const. is contained in some leaf. (Moreover for this foliation, the leaves are closed sets of N and have a trivial normal bundles in N.)

If N is compact, and rank(Tf) is identically p, then it is easy to see that this foliation is a **fibration**, i.e. is a union of open sets of the type {leaf} × {p-ball}. (Generalizing earlier work of HADAMARD, who had dealt with the case n = p, EHRESMANN showed that this conclusion is true even for non-compact N, provided f is **proper**, i.e. such that the inverse image of each compact set is compact.)

On the other hand, if rank(Tf) is *not* identically p, then the closed (n-p)-manifolds occuring as **the inverse images** $f^{-1}(y)$ **of regular values** of f need not be diffeomorphic to each other. However PONTRJAGIN made the fundamental observation that **they are cobordant to each other, and their cobordism class depends only on the smooth homotopy class of** F. Here, by a **cobordism** between two manifolds, we mean a manifold of dimension one more whose boundary is their disjoint union.

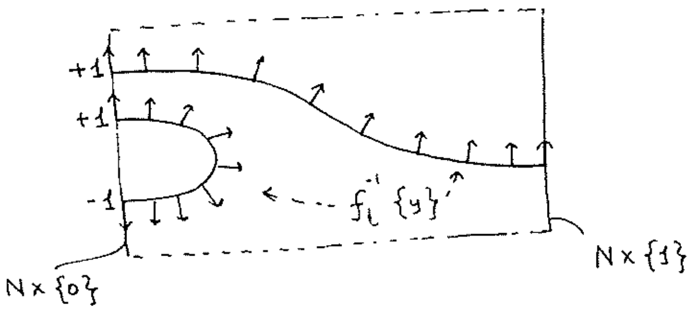

Fig.8.

For example, the above picture shows how the inverse image of the regular value of a homotopy f_t can provide such a cobordism. It shows also that these cobordisms can be **framed**, i.e. their normal bundle in N × I can be equipped with a trivialization, which restricts to given trivializations of the normal bundles of the two manifolds in N.

Pontrjagin proved that **there is a bijection between the framed cobordism classes of codimension** p **submanifolds of** N **and smooth homotopy classes of maps of** N **into the** p**-sphere**.

For the *case* n = p, and N orientable, resp. non-orientable, the framed cobordism class of the finite set $f^{-1}(y)$ is determined by $\#(f^{-1}y) = \sum\{\mathrm{sgn}(x) : x \in f^{-1}(y)\}$, resp. $\#(f^{-1}y)$ mod 2 (see fig. 8), where $\mathrm{sgn}(x) = \pm 1$ depending on whether Tf_x preserves or reverses orientation. So now the above result reduces to the following older one.

Hopf's Theorem. *The homotopy classes of maps* f *from an* n*-manifold* N *into the* n*-sphere are classified by their* **degree**

$$\deg(f) = \#(f^{-1}y),$$

or *their* degree mod 2, *depending upon whether the manifold* N *is* *orientable or not.*

Moreover, these degrees determine, and are determined by the map induced by f in nth homology with integral or mod 2 coefficients. (This point enables one to generalize the definition of degree, and so of intersection numbers — see 9.1 and 9.a — to continuous maps.)

These remarks illustrate the importance of the *"implicit function theorem"* (3.5), and its generalizations (8.2) etc., in topology.

(8.d) *Proof.* If f is any **embedding** (= a one-one smooth function) from a closed m-manifold M into (m+1)-space, then **Alexander duality** with mod 2 coefficients (see § 8) shows that the complement of f(M) in (m+1)-space has *two* components. So we can smoothly assign a unit normal vector to each point of f(M). *q.e.d.*

We note that now Poincaré refers to a Möbius strip in 3-space, and apparently to any embedded m-manifold of (m+1)-space, as a (hyper) *"surface"* : however we'll use this word only in the sense of (1.1).

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§ 9. Intersection des deux variétés.

(Intersection numbers.)

(9.1) Definition. Let ν be a p-dimensional variety, of a given oriented n-manifold M, which intersects an (n-p)-dimensional variety ν' of M finitely many times, and in a *transverse* way. We assign to each of these intersections x the number +1, resp. −1, iff the orientation of ν at x followed by that of ν' agrees, resp. disagrees, with that of M. The sum of these numbers will be denoted $N(\nu,\nu')$, and called the **intersection number** of ν with ν' in M.[a]

We note that $N(\nu,\nu')$ changes sign if the orientation of any one of the three manifolds $\{M,\nu,\nu'\}$ is reversed, and that

$$N(\nu',\nu) = (-1)^{\dim\nu.\dim\nu'} N(\nu,\nu').$$

In (9.2) *and* (9.3) *below we'll work within a fixed oriented manifold* M, which will be either compact, or else the interior of a compact manifold-with-boundary $\bar{M}$ (see 5.b) in which case we'll denote $bd(\bar{M})$ by bd(M).

(9.2) Theorem. *If there exists a* p*-variety* C *with* $\partial C = 0$ *rel boundary, such that* $\sum_i k_i.N(C,V_i)$ *is nonzero for given closed* (n-p)*-varieties* V_i *, then we cannot have* $\sum_i k_iV_i \simeq_{\mathbb{Q}} 0$ *; and conversely, if this homology does not hold, then such a* C *can be found.*

Proof **(for case** p = 1 **only).** To establish the direct part it obviously suffices to check that, if W is any connected open subset with $\partial W = V_1 + \dots + V_t$, and C is any transversal oriented curve with $\partial C = 0$ rel bd(M), then $N(C,V_1) + \dots + N(C,V_t) = 0$.

To see this note that, *if* C *is closed*, then it must go as many times from the complement of $\bar{W}$ into W, as it goes from W into this complement. Furthermore, the same is true also *if* C *is an arc having both extremeties on* bd(M), because any such C begins and ends ouside $\bar{W}$. So, in all cases, $N(C,\partial W)$ is the sum of an equal number of +1's and −1's, and thus is zero.

Conversely, suppose that the homology $\sum_i k_i V_i \simeq_{\mathbb{Q}} 0$, $k_i \in \mathbb{Q}^*$, does not hold in M. If one of the V_i's is homologous to a rational combination of the others we can always replace it by this combination without affecting the value of the rational number $\sum_i k_i . N(C, V_i)$. So we can assume without loss of generality that there is *no* non-trivial homology amongst these V_i's.[(b)]

This implies, *if M is without boundary*, that the complement W of $V_1 \cup \ldots \cup V_t$ in M must be connected, for otherwise the ∂ of any component of this complement will furnish a non-trivial homology between some of these V_i's. And, *if* M *has a nonempty boundary*, it implies likewise that each component W_y of this complement must be incident to the boundary of M.

So we can join the extremeties y and z, of a small arc yxz cutting V_1 transversely at x, to either each other in W, or else to two points of bd(M) via two arcs in W_y and W_z respectively. This gives us a C with $\partial C = 0$ mod bd(M) for which $\sum_i k_i . N(C, V_i) = k_1$ is nonzero. *q.e.d.*

(9.3) Lets now extend the above argument to $p \geq 2$:

We will assume that M is a variety defined by some equations $F_\alpha = 0$ (and some inequalities) and that the p-dimensional $C \subset M$ is determined by n-p additional equations $F'_\gamma = 0$.

As for the (n-p)-dimensional varieties $V_i \subset M$, we'll assume that their points satisfy p-1 common additional equations $\Phi_\nu = 0$ — i.e. that they all lie on the (n-p+1)-dimensional variety $W \subseteq M$ determined by the equations $F_\alpha = 0$ and $\Phi_\nu = 0$ — and that each of the *codimension one* sub-varieties V_i of W is determined by one more equation $F''_i = 0$.

We note now that $N(C, V_i) = N(C \cap W, V_i)$, and that $C \cap W$ is a *curve* of W. Secondly we note that if we have $\sum k_i . V_i \simeq_{\mathbb{Q}} 0$ in W, then the same homology is true also in M. The converse is of course not true; but, if this homology holds in M, then we can always, by suitable choosing the functions Φ, find *some* W of the above type, in which it holds.

So, by applying the case p = 1 to W, it follows that Theorem (9.2) holds even when $p \geq 2$.[(c)]

(9.4) Corollary (POINCARE DUALITY). *For any closed n-dimensional orientable manifold* M *we have* $b_p(M) = b_{n-p}(M)$ *for* $0 \le p \le n$.

I believe that this result has never been claimed before; nevertheless it is known to many persons, who have even made some applications of it.[(d)]

Proof. We orient M and choose in it maximal sets of independent (see 4.1) p- and (n-p)-dimensional closed oriented varieties $\{C_1, \ldots, C_\lambda\}$ and $\{V_1, \ldots, V_\mu\}$, where $\lambda = b_p(M)$ and $\mu = b_{n-p}(M)$.

In case the number λ of linear equations $\sum_i x_i.N(C_j, V_i) = 0$ is less than the number μ of unknowns x_i, they would have a non trivial rational solution $x_i = k_i$. Then (by the direct part of 9.2) we will have $\sum_i k_i.N(C, V_i) = 0$ for *all* closed r-dimensional C's. So (by the converse part of 9.2) we would have $\sum_i k_i.V_i \simeq_{\mathbb{Q}} 0$ in M. Since this is not so we must have $\lambda \ge \mu$.

Likewise $\mu \ge \lambda$. *q.e.d.*

(9.5) Corollary. *For any orientable closed* n-*manifold* M *with* n *even and* n/2 *odd the middle Betti number* $b_{n/2}(M)$ *is even.*

In the proof we'll make use of the

Definition. Given any closed oriented n-manifold M *with* n *even*, and any $b = b_{n/2}(M)$ independent closed (n/2)-subvarieties V_1, V_2, ... of M, we have the bxb **intersection matrix** $N = [N(V_i, V_j)]$. Here, the V_i's are assumed transverse to each other, and $N(V_i, V_i)$ denotes the intersection number of V_i with a transverse homologue.[(e)]

Proof. Since n/2 is odd the intersection matrix is a skewsymmetric $b \times b$ matrix (see 9.1). So if b were odd its determinant would be zero. So we would be able to find rationals k_i not all zero such that $\sum_j k_j.N(V_i, V_j) = 0$. So (cf. proof of 9.4) we would have $\sum_j k_j.N(C, V_j) = 0$ for all (n/2)-varieties C, which implies $\sum_j k_j.V_j \simeq_{\mathbb{Q}} 0$ in M, a contradiction. *q.e.d.*

We will show later by means of examples that this result is not true if M is not orientable, or if n/2 is even.[(f)]

NOTES

(9.a) *In the paper the same definition is given more formally in terms of determinants.*

For example, for the case $M = \mathbb{R}^n$, and ν and ν' as in (3.1), Poincaré defines *for each* $(x,x') \in \nu \times \nu'$, the number $S(x,x') \in \{-1,0,+1\}$ as the sign of the nxn determinant

$$\begin{vmatrix} \partial x_i/\partial y_j \\ \partial x_i'/\partial y_k' \end{vmatrix}, \quad 1 \le i \le n, \ 1 \le j \le p, \ 1 \le k \le n-p,$$

and then sets $N(\nu,\nu') = \sum \{S(x,x') : x = x'\}$.

Likewise, when M, ν, and ν' are varieties in t-space, defined by G = 0, G = 0 = F, and G = 0 = F' respectively, he sets $N(\nu,\nu') = \sum \{S(x,x') : x = x'\}$, where now, for each $(x,x') \in \nu \times \nu'$, $S(x,x')$ denotes the sign of the txt determinant $|\partial G_\gamma/\partial x_i \ \partial F'_\beta/\partial x_i \ \partial F_\alpha/\partial x_i|$.

However, the requirement that each intersection x be **transverse**, i.e. that the above determinants be nonzero there, or equivalently, that at each such x the tangent space to M be spanned by the vectors tangent to ν or ν', is not explicitly made in the paper.

Note also that the *degree* $\deg(f) = \#(f^{-1}(y))$ of a map $f : N^n \to P^p$ (defined in 8.c) can be interpreted as the intersection number of the fundamental cycle of N^n with its 0-chain $\sum\{\mathrm{sgn}(x).x : x \in f^{-1}(y)\}$, and conversely, in the notation of (8.c), we have $N(\nu,\nu') = \#(\Theta^{-1}(\Delta))$, where $\Theta = \theta \times \theta'$, and $\Delta = \{(x,x) : x \in M\}$ is the **diagonal** of M^2.

(9.b) We note that the argument of this paragraph would'nt have worked with integral homologies. In fact *for* $p \ge 2$ *the analogue of Theorem* (9.2) *for integral homologies is false* : e.g. we can have a non-bounding V with $2V \approx 0$ and so $N(C,V) = 0$ for all transversals C.

In the next paragraph Poincaré will assume that if the boundary of any component of the complement of $V_1 \cup \ldots \cup V_t$ contains a *part* of some V_i then it must contain *all* of V_i : this is not true (see below) but can be arranged at this point by replacing the V_i's by suitable homologues.

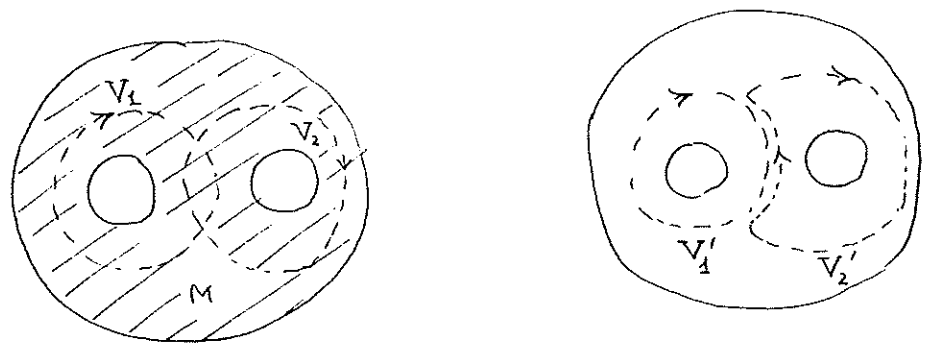

Fig. 9.

Poincaré is also going to use the fact that *each* V_i *occurs at most twice in each component's boundary*, and that when it occurs twice these occurences cancel out in the component's oriented boundary. Note that this implies that any codimension one homology is a linear combination of some having coefficients ± 1 only, and that $H_{n-1}(M^n;\mathbb{Z})$ **is free.**

(9.c) By slightly "enlarging" a variety of M having oriented boundary $\sum_i k_i V_i$, one does get an immersed (n-p+1)-dimensional manifold-with-boundary W which contains all the V_i's in its interior, and since $\sum_i k_i V_i \simeq 0$ *in* W, the direct part of the case p = 1 of (9.2) now yields $\sum_i k_i . N(C, V_i) = \sum_i k_i . N(C \cap W, V_i) = 0$.

Thus Poincaré has sketched a correct proof of the direct part of (9.2) *for all* p. *However, for the converse, there are serious problens with his sketched argument.*

To see this, lets assume we do not have $\sum_i k_i V_i \simeq_{\mathbb{Q}} 0$ in M. To start the argument rolling, we need a compact (n-p+1)-dimensional immersed W containing all the V_i's in its interior : but, *in the absence of a homology between the* V_i*'s, it is not clear why such a W should exist?*

However, since varieties are null-cobordant (see 8.c), we can find an (n-p+1)-manifold with boundary $\overline{W}$ having disjoint copies $\overline{V}_i$ of the V_i's in its interior, and so the image of a suitable map from $\overline{W}$ into M, which extends the maps $\overline{V}_i \longrightarrow V_i$, might provide us with such a W ...

So lets grant such a W. Since we certainly do not have $\sum_i k_i V_i \simeq_{\mathbb{Q}} 0$ on this W, the converse part of the case p = 1 of (9.2) gives us a curve c of W with $\partial c = 0$ rel bd(W) such that $\sum_i k_i . N(c, V_i)$ is nonzero. Still the proof is far from finished, because *it is not at all clear why we must have a transversal closed* p-*variety* C ⊂ M *with* C∩W = c ?

The above two gaps in Poincaré's argument were pointed out by **Heegaard** (1899), who in fact considered Poincaré duality (9.4) to be false, and gave a "counter-example" to this effect !

For Poincaré's response to this criticism see the *Complément* : briefly, he conceded that the second gap was serious, but showed that Heegaard's example was fallacious, and gave a new (and correct !) proof of a stronger duality theorem (equivalent to the modern group-theoretic formulation).

(9.d) Notable amongst these was PICARD who had stated this duality clearly, and used it for his study of non-singular complex surfaces.

Though (9.4) is stated in the paper only for closed manifolds, Poincaré must have been aware that the same argument shows that (9.2) also has as corollary the LEFSCHETZ DUALITY, $b_{n-p}(\text{int}M) = b_p(M, \text{bd}M)$, *for any oriented manifold-with-boundary* M. Still more generally (if one continues to ignore the lacunae noted in 9.c) the arguments of (9.2) – (9.4) also give $b_{n-p}(M \setminus A) = b_p(M, A)$, for any pair (M, A) of compact spaces whose difference M \ A an orientable n-manifold.

This yields ALEXANDER DUALITY, i.e. the relationship between the Betti numbers of a closed subset A of a sphere S^n, and those of its complement $S^n \setminus A$: to see this note that $b_p(S^n, A) = b_{p-1}(A)$, except if p = n, when it is one more.

Poincaré was certainly aware e.g. of the JORDAN CURVE THEOREM, i.e. the case n = 2 and $A \cong S^1$ of Alexander duality, and (6.2) and (6.b) suggest that he might have been aware of the general statement too.

(9.e) *Given* b_{n-p} *independent* V_i*'s, we can choose a* **dual basis** *of the* C_j*'s such that* $N(C_j, V_i)$ *coincides with the identity matrix* δ_{ij} (the proof of 9.2 shows this for p = 1). This shows that (for n even) *the intersection matrix* $N \in \text{Aut}(\mathbb{Z}^b) = GL(b,\mathbb{Z})$, i.e. $\det(I) = \pm 1$.

Though the matrix $N \in GL(b,\mathbb{Z})$ depends on the choice of the b independent closed varieties V_i of M, its **congruence class** $\mathcal{N} = \{PNP'\} \subseteq GL(b,\mathbb{Z})$ *is an invariant of the oriented manifold* M. Thus any property of N which is shared by all members of $\mathcal{N}$ — or, à fortiori, by all members of the bigger congruence class $\mathcal{N}_{\mathbb{F}} = \{PNP' : P \in GL(b,\mathbb{F})\}$, for any field $\mathbb{F} \supseteq \mathbb{Z}$, e.g. $\mathbb{F} = \mathbb{Q}$, $\mathbb{Q}_p$ or $\mathbb{R}$ — is an invariant of M.

For n/2 *odd* (the case considered by Poincaré), and $\mathbb{F}$ a field $\supseteq \mathbb{Z}$, *no new invariant is given by* $\mathcal{N}_{\mathbb{F}}$, since we can always find a $P \in GL(b,\mathbb{F})$ such that the skewsymmetric N becomes $\text{diag}(\,..\,, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \,..\,)$, so the class $\mathcal{N}_{\mathbb{F}}$ is determined just by the size or **rank** b of N.

However, for n/2 *even* (the case *not* considered by Poincaré) *the matrix* N *is symmetric, and has many other well-known congruence invariants besides rank :*

For example its **parity** (N is called *even* if all its diagonal elements are even, and *odd* otherwise), its **signature** (i.e. the number of positives *minus* the number of negatives in any diagonal matrix of $GL(b,\mathbb{R})$ congruent to it), and the fact whether it is **definite** or not (i.e. whether its signature is ± b), etc. : see e.g. SERRE or MILNOR-HUSEMOLLER for more regarding congruence invariants.

We remark that **Poincaré's duality theorem is only the first of many striking results about the intersection matrix**, e.g.,

WHITEHEAD : *a closed simply connected* (see § 12) *4-manifold* M *is uniquely determined, upto homotopy type, by the integral congruence class of its intersection matrix* N.

ROCHLIN : *if a closed simply connected 4-manifold* M *has an even intersection matrix* N*, then* 16 *divides its signature.*

DONALDSON : *if a closed simply connected 4-manifold* M *has a definite intersection matrix* N, *then it must be integrally congruent* to ± I.

FREEDMAN : *Any symmetric integral matrix* N *with determinant* ± 1 *is the intersection matrix of some simply connected closed 4-dimensional* topological *manifold* M.

Using these results, and information about the congruence classes of GL(n,ℤ), one obtains e.g. more than 100 million distinct simply connected closed *topological* 4-manifolds M with $b_2(M) = 32$ and N definite, out of which only 2 can admit a *differentiable* structure !

We remark, in this context, that in § 13, Poincaré will analogously classify some closed 3-manifolds by the *conjugacy classes* of GL(2,ℤ).

(9.f) Two simple examples which show this are the *Klein bottle* and the *complex projective plane.*

Note that (9.6) generalizes the fact that the b_1 of a closed orientable surfaces is even : another (deeper) generalization of this is that *all the odd Betti numbers* $b_{2i+1}(M)$ *of a closed Kähler manifold* (e.g. a non-singular projective variety) *are even.*

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§ 10. Platonic 3-manifolds.

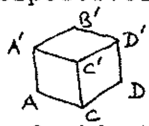
Generalizing from the accepted practice for surfaces Poincaré assumes the following without proof.

Proposition 8. *For any closed manifold* M, *one can find some polytope(s)* P *having an even number of facets, and a differentiable surjection* ι : P $\longrightarrow$ M *which is one-one, but for the fact that* **facets of** P **are identified in pairs** F ≡ F' (in an obvious sense).

Orientability criterion : to ensure $\partial M = 0$, each identification F ≡ F' should be such that it *reverses* the induced orientations of F, F' ⊆ ∂P. (So, for dim(M) = 3, if we "walk" on P along ∂F keeping F to our left, then the corresponding walk on P along $\partial F'$ should keep F' to our right.)

For example, the opposite sides of the **square** 

can be identified as AC ≡ BD, AB ≡ CD to obtain the **torus**, and this is in fact the *only* identification of opposite sides of the square which obeys the orientability criterion. (The other identifications AC ≡ DB, AB ≡ CD and AC ≡ DB, AB ≡ DC give non-orientable 2-manifolds, viz. the **Klein Bottle** and the **real projective plane** respectively.)

On the other hand, for the **cube** P = 

, since we can also *rotate* facets, there are *many* ways of identifying opposite facets without violating the orientability criterion, e.g. the following five given by Poincaré.

	Example 1	Example 2	Example 3	Example 4	Example 5
ABDC ≡	A'B'D'C'	B'D'C'A'	B'D'C'A'	B'D'C'A'	D'C'A'B'
ACC'A' ≡	BDD'B'	DD'B'B	DD'B'B	BDD'B'	D'B'BD
ABB'A' ≡	CDD'C'	DD'C'C	C'CDD'	CDD'C'	D'C'CD

Clearly Ex. 1 is $\mathbb{R}^3$ mod $\mathbb{Z}^3$, i.e. the **3-torus**, while Ex.5 is the **real projective 3-space** (: we note that Poincaré uses instead of Ex.5 the equivalent antipodal identification of the boundary of an **octahedron**, the Platonic solid dual to the cube).

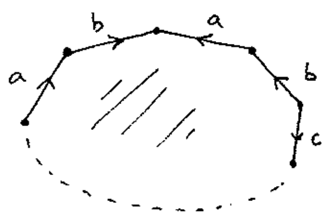
[**Triangulability.** *Prop. 8 implies and is implied by the fact that* M *has a differentiable* **(finite) cell subdivision** (in an obvious sense) : in later sections Poincaré uses *this* version of Prop. 8. (We'll denote by P/ι the cell subdivision of M given by identification classes of cells of P under ι.)

A cell subdivision is called **simplicial** (resp. **simple**) iff each cell of dimension i (resp. codimension i) is incident to i+1 cells of dimension (resp. codimension) one less, and these determine the cell uniquely. A simplicial subdivision is also called a **triangulation**.

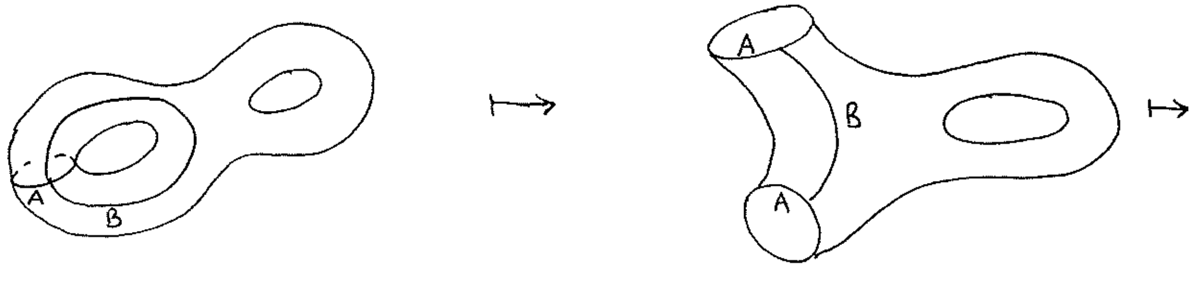
It is easily seen that *any cell subdivision of* M *can be modified to obtain a simplicial or simple subdivision.* So Prop.8 is equivalent to a result proved later by **Whitehead**, viz. that *differentiable manifolds have a differentiable triangulation.*]

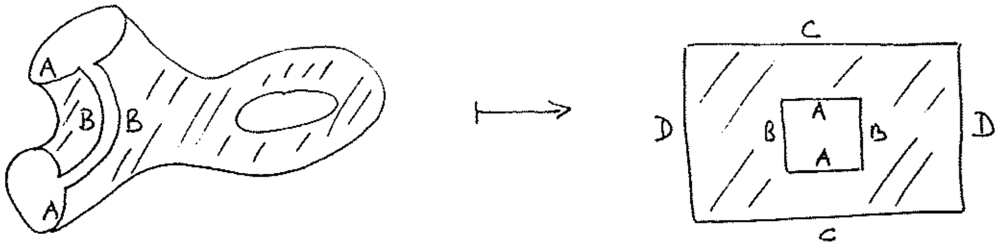
[**Triangulability of surfaces had led to their classification** : one begins by checking that *any direction-reversing identification of pairs of sides of* any *even polygon gives a surface*, and then, by means of some operations (see e.g. **Lefschetz**'s book) one modifies the polygon to one of the following 4g-gons with pairs of sides identified as per the **commutator relation**

$$aba^{-1}b^{-1}cdc^{-1}d^{-1} \,..\, = 1 :$$

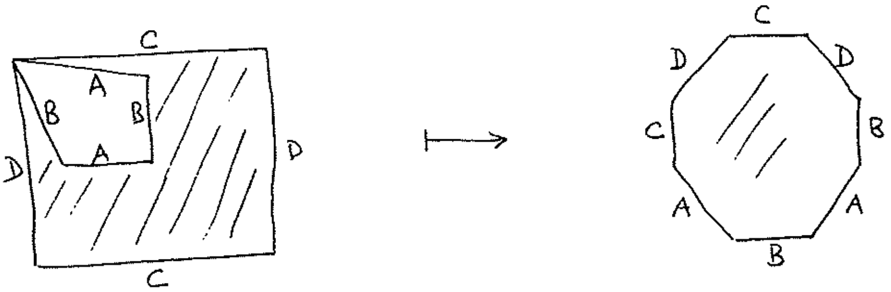


This is the surface with g handles. To see this for g = 2 cut a surface with 2 handles as follows to get a square annulus from which the surface can be recovered by making the indicated identifications :





By homogeneity, the above hole can be anywhere, so this annulus can be replaced by the required octagon as follows :



Poincaré must have hoped that a similar procedure would classify 3-manifolds : though this problem is *very* hard, recent work shows that this hope remains alive.]

[**There are only five regular solids**, viz. the tetrahedon, cube, octahedron, icosahedron, and dodecahedron. We will deduce this gem of antiquity (it was known to Plato !) from the following.

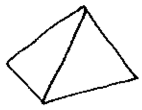
Euler's formula. *For any cell subdivision of the 2-sphere one has*

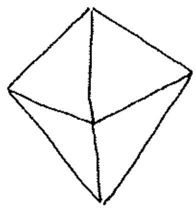
$$v - e + f = 2,$$

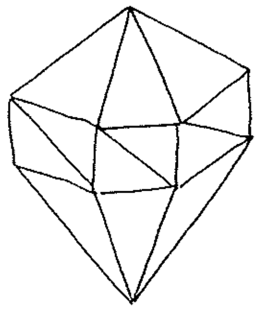
where v, e, *and* f *are the number of vertices, edges, and faces of the subdivision.* (More generally, it was known in 1895 that, for any cell subdivision of a surface with g handles, the **Euler characteristic** $v - e + f$ equals $2 - 2g$.)

If cell subdivision is regular, r (resp. s) edges are incident to each face (resp. vertex) of the subdivision, so we also have $rf = 2e = sv$. Multiplying Euler's equation by rs, and using these, we get $e(2r - rs + 2s) = 2rs$. So $2r - rs + 2s$ is positive : this, and $r, s \geq 3$, clearly imply $\{r,s\} = \{3\}$, $\{3,4\}$ or $\{3,5\}$.

The tetrahedron is both simple and simplicial, the cube and the dodecahedron are simple, while their duals, the octahedron and the icosahedron are simplicial : thus **the only three regular triangulations of the 2-sphere are as follows** :







In the *Cinquième Complement* of this paper, there is another (and *much* more important) example of a 3-manifold which too is built from a Platonic solid, i.e. the **Poincare manifold** $\mathcal{P}$, obtained by identifying antipodal facets of a dodecahedron after a rotation of $2\pi/5$.

Poincaré showed in the *Cinquième Complement* that $H_i(\mathcal{P}) \cong H_i(S^3)$ $\forall$ i even though $\mathcal{P}$ is not diffeomorphic to S^3.]

Given a cell subdivision, the (open) **star** of an (open) cell σ consists of all cells of which σ is a face. The intersections of these cells, with the boundary of a small transversal disk of the ambient space with centre in σ, constitutes the **link** of σ.

It can be shown that **links of any smooth triangulation of a manifold are spheres**. Also, it is easily seen that links of cells of *any* P/ι, of codimensions $\leq$ 2, are always spheres. But in general there are singularities in codimension $\geq$ 3.

Proposition 9. *An identification ι of pairs of facets of a 3-polytope* P *gives a 3-manifold iff the Euler characteristic of the link of each vertex of* P/ι *is* 2.

Further, of the five examples tabulated above, Example 2 *is not a manifold* (it has precisely two singularities and their links are 2-tori) but *the remaining four* Examples 1, 3-5, *are all orientable* 3-*manifolds.*

(It will be later shown that all these manifolds are distinct.)

Proof. The absence of codimension ≤ 2 singularities means that the links of the possible singularities of M, i.e. the vertices of P/ι, must be surfaces. Their Euler characteristic being 2, these surfaces must be 2-spheres. So there are no singularities.

For any vertex $\alpha \in P/\iota$, let v_α, e_α, and f_α denote the number of vertices, edges, and faces in the link of α. It is easily seen that f_α = cardinality of the class α, e_α = half the sum of the number of facets incident to each vertex of the class α, and v_α = number of classes of edges incident to vertices of class α, taking care to count such a class twice if both extremities are in the class of vertices α.

A calculation shows now that $v_\alpha - e_\alpha + f_\alpha$ is always 2 for our examples, except for the two vertices of Example 2 when it is 0. *q.e.d.* (We checked that opposite facet identifications of cube give in all seven orientable 3-manifolds.

[This calculation also reveals that $f_0 - f_1 + f_2 - f_3$ (where f_i denotes the number of i-dimensional cells in P/ι) is 0 for Examples 1, 3-5, but for Example 2 this number equals 2.

In fact we have always $f_3 = 1$ and $f_2 = 3$, and f_1 and f_0 can be computed by enumerating the identification classes under ι of the edges and vertices of P. For instance for Ex. 2 we have $f_1 = 2 = f_0$, the identification classes being {AB, B'D, C'C, B'A', AC, DD'}, {AA', DC, C'A', B'B, C'D', DB} and {A, B', C', D}, {B, D', C, A'}.

An alternative criterion due to **Thurston** says in fact that P/ι *is a 3-manifold iff* $f_0 - f_1 + f_2 - f_3$ *is zero.* We note further that Poincaré's and Thurston's criteria are valid even if the identifications do not satisfy the orientability criterion.]

[If we take two **cones** aX and bX having only the base manifold X in common, then the space $S(X) = aX \cup bX$ is called the **suspension** of X. It is intuitively obvious (and true !) that SX is a topological (n+1)-manifold if and only if X is an n-sphere (and then of course SX is an (n+1)-sphere).

However **Edwards** has proved the remarkable fact that **the double**

suspension $S(S(\mathcal{P}))$ **of** $\mathcal{P}$ **is homeomorphic to a 5-sphere**: note that the **topological triangulation** of the 5-sphere thus obtained has $S(\mathcal{P})$, which is not even a manifold, occuring as a link in it !

However it is always true, even for topological triangulations, that links are *homotopy equivalent* to spheres.

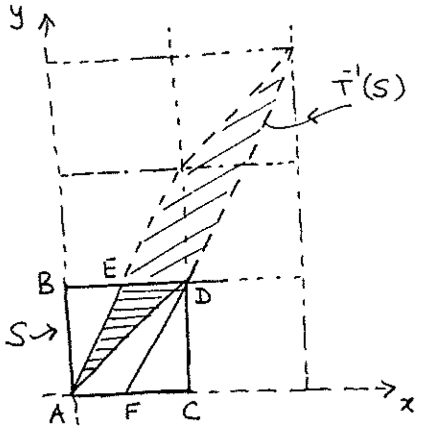
Also closed or compact **topological manifolds need not have a topological triangulation** : it is known that $\mathcal{P}$ embeds topologically in 4-space (even though it does not embed differentiably) and the closure of the bounded component of $\mathbb{R}^4 \setminus \mathcal{P}$ can not apparently be subdivided into finitely many cells !]

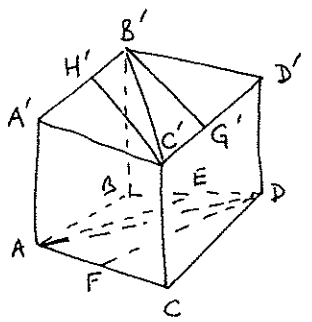
§ **11.** Each integral matrix $T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ with determinant 1 determines the following **subdivision** P_T **of the cube** P :

Thinking of T^{-1} as a linear map of the x-y plane, we subdivide the bottom facet S into cells σ obtained by intersecting it with the integral translates of the parallelogram $T^{-1}(S)$. For each σ there is a unique integral translate σ' of $T(\sigma)$ which lies in the top facet, and we'll subdivide the top facets using these cells σ'. The vertical facets are left un-subdivided.

Poincaré considers the example $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ when S subdivides into triangles ABC and BCD, and the top into corresponding triangles A'D'C' and B'A'D', and points out that the number of cells of P_T increases with the size of the entries of the matrix T.

The case $T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is illustrated below :





σ	σ'
ABE	C'B'G'
AED	A'H'C'
ADF	B'D'G'
FDC	H'B'C'

Example 6. The identification ι of the facets of any subdivided cube P_T according to $\sigma \equiv \sigma'$, ACC'A' ≡ BDD'B' and ABB'A' ≡ CDD'C', satisfies the orientability condition.

Further we see that P_T/ι coincides with the manifolds of Examples 1 and 4 respectively for T = I and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. More generally Poincaré now checks the following.

Proposition 10. *For any* $T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2,\mathbb{Z})$, P_T/ι *represents a closed orientable 3-manifold.*

Proof. Let G_T be the group of affine linear transformations of 3-space generated by $(x, y, z) \mapsto (x+1, y, z)$, $(x, y, z) \mapsto (x, y+1, z)$, and $(x,y,z) \mapsto (\alpha x+\beta y, \alpha x+\beta y, z+1)$.

We note that G_T is a **free and discontinuous group** of transformations of 3-space, i.e. none of its non-identity elements has a fixed point, and none of its **orbits** $\{g(x) : x \in G\}$ has a limit point. Moreover each x has a neighbourhood which is disjoint from all its g-images for $g \neq 1$, so the quotient space $\mathbb{R}^3/G_T$ of orbits is locally homeomorphic to 3-space, i.e. it is a 3-manifold.

In fact P_T/ι *is homeomorphic to* $\mathbb{R}^3/G_T$ *:*
For this we note that the unit cube P is a **fundamental domain** of G_T, i.e. each orbit has at least one member in P, and not more than one in its int(P). Moreover, for each bottom facet σ of P_T, there is a unique $s \in G_T$ such that $s^{-1}(P) \cap P = \sigma$, and then $s(\sigma) = \sigma'$. Thus ι identifies all boundary points of the cube which are in the same orbit. *q.e.d.*

[The **mapping torus** of a diffeomorphism $\tau : M \longrightarrow M$ is the manifold obtained by the identifications $(p,0) \equiv (\tau(p),1)$ in the cylinder $M \times [0,1]$. We note that a $T \in GL(2,\mathbb{Z})$, i.e. a linear automorphism of $\mathbb{R}^2$ mapping $\mathbb{Z}^2$ onto $\mathbb{Z}^2$, induces an automorphism $\tau : \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$, and *the above manifold is the mapping torus of this toral diffeomorphism* (with the condition $\det(T) = 1$ being equivalent to its orientability).

Poincaré will check that Exs. 3, 5 are *not* of the above type $\mathbb{R}^3/G$: their fundamental groups are finite, and they are covered not by $\mathbb{R}^3$ but by the 3-sphere. One can also think of Examples 3 and 5, as well as $\mathcal{P}$, as SO(3) mod a suitable finite subgroup.

In the *Troisième Complémwnt*, Poincaré reconsiders Ex. 6 (and more generally a mapping torus of *any* closed surface) starting from another, algebraic geometrical, definition.]

[The case $|tr(T)| > 2$ (i.e. when T has two distinct real eigenvalues λ_1

and λ_2, $0 < |\lambda_1| < 1 < |\lambda_2|$) of above toral diffeomorphisms τ is *very important* because it has led to lots of interesting things:

For example, its **zeta function** $\exp(\sum_{m\geq 1} \frac{1}{m}.N_m.z^m)$, which "counts" the numbers N_m of **periodic points** x, $\tau^m(x) = x$, of various periods m, turns out to be a *rational function* (= $\frac{1 - \lambda_1\lambda_2 z}{(1 - \lambda_1 z)(1 - \lambda_2 z)}$), and it is was found more generally that the same is true for any **hyperbolic diffeomorphism**, i.e. one for which there is a continuous splitting of the tangent spaces into two subspaces, of which one "expands" and the other "contracts" under the diffeomorphism.

By patching together the directed line segments (p,t), $0 \leq t \leq 1$, of M × [0,1], under the identifications $(p.0) \equiv (\tau(p),1)$, we obtain a **flow** on the mapping torus of τ. An analogous zeta function, counting the number of **closed trajectories** of different lengths of this flow, turns out to be a *meromorphic function*, and it is known more generally that the same is true for any **hyperbolic flow**, i.e. those whose complementary tangent spaces have an analogous splitting. (For the case of a "geodesic flow" of the bundle of unit tangent vectors of a surface of "negative curvature", this gives **Selberg zeta function.**)

Again, **Anosov** has shown that hyperbolic diffeomorphisms are **structurally stable**, i.e. all diffeomorphisms "near" to them are topologically conjugate to them, and this result has played a key role in **Smale's** study of generic diffeomorphisms and flows: *see B.A.M.S. of* 1967.]

§ 12. Fundamental group.

The system $\mathcal{F}$ of λn partial **differential equations**

$$\frac{\partial y_\alpha}{\partial x_i} = \mathcal{F}_{\alpha,i}(x_1, \ldots, x_n; y_1, \ldots, y_\lambda),$$

where the λn differentiable functions $\mathcal{F}_{\alpha,i}$, $1 \le \alpha \le \lambda$, $1 \le i \le n$, are defined over an open set $D \times \mathbb{R}^\lambda$, is said to be **integrable** if

$$\frac{\partial \mathcal{F}_{\alpha,i}}{\partial x_j} + \sum_\beta \frac{\partial \mathcal{F}_{\alpha,i}}{\partial y_\beta}.\mathcal{F}_{\beta,j} = \frac{\partial \mathcal{F}_{\alpha,j}}{\partial x_i} + \sum_\beta \frac{\partial \mathcal{F}_{\alpha,j}}{\partial y_\beta}.\mathcal{F}_{\beta,i}$$

throughout $D \times \mathbb{R}^\lambda$. The following was known regarding these in 1895.

Proposition 11 (EXISTENCE THEOREM OF DEAHNA-FROBENIUS). *For each* $(p,q) \in D \times \mathbb{R}^\lambda$, *the above integrable system of differential equations* $\mathcal{F}$ *has a unique solution* $y_\alpha = F_\alpha^{(p,q)}(x_1, \ldots, x_n)$, *valid over a sufficiently small connected neighbourhood* $N_{(p,q)} \subseteq D$ *of* p, *and such that* $F_\alpha^{(p,q)}(p_1, \ldots, p_n) = q_\alpha$.

We now *assume* that $\mathcal{F}$ is such that the above neighbourhoods $N_{(p,q)}$ of p can be chosen independently of q, and denote them by N_p. Then, for points p_1, p_2 of D so near that N_{p_1} and N_{p_2} intersect, we have a well-defined **substitution** (= bijection) $S_{12} : \mathbb{R}^\lambda \longrightarrow \mathbb{R}^\lambda$ such that

$$F^{(p_1,q)} = F^{(p_2,S_{12}(q))}$$

on $N_{p_1} \cap N_{p_2}$.

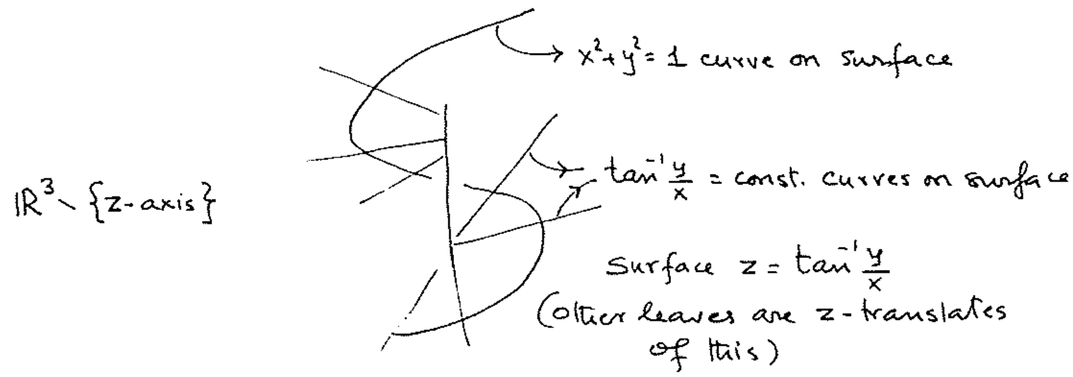
Now let W be a manifold in D with a chosen **base point** b. On any **loop** (= oriented closed curve) C of W through b, choose a finite number of such nearby points $b = p_1, p_2, \ldots, p_{t-1}, p_t = b$, and consider the substitution $S_{12}S_{23} \ldots S_{(t-1)t}$ obtained by performing the above substitutions one after the other.

Since this substitution is independent of the choice of the p_i's, we'll denote it by S_C. We note further that, if C_1C_2 is the loop obtained by following C_1 by C_2, then $S_{C_1C_2} = S_{C_1}S_{C_2}$: thus all such substitutions form a group $g_{\mathcal{F}}$, which is called the **monodromy group**, of the integrable differential equations $\mathcal{F}$, on the manifold W, at the point b.

[*Example.* On $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}$ the differential equations $\mathcal{F}$:

$$\frac{\partial z}{\partial x} = \frac{-y}{x^2 + y^2}, \qquad \frac{\partial z}{\partial y} = \frac{x}{x^2 + y^2},$$

are integrable, with local solutions $z = \tan^{-1}(\frac{y}{x}) + k$ defined on domains $N_{(x,y)}$ not depending on z. Further, these solutions patch together into helical surfaces, which are z-translates of each other, and which partition $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}$ into 2-dimensional **leaves** as follows.



Since going around the unit circle $S^1 \subset \mathbb{R}^2 \setminus \{0\}$ once (as shown) gives the leaf-preserving substitution $z \mapsto z + 2\pi$, the monodromy $g_{\mathcal{F}}$ of $\mathcal{F}$ over this circle is isomorphic to the group $\mathbb{Z}$ of integers.]

[More generally, the n-dimensional **plane field** $\mathcal{F}$ of $D \times \mathbb{R}^\lambda$, complementary to the **fibers** $\{p\} \times \mathbb{R}^\lambda$ of the projection $D \times \mathbb{R}^\lambda \mapsto D$, which is prescribed by the functions $\mathcal{F}_{\alpha,i}$, is tangent to the leaves (e.g. helical surfaces for above case) of a **foliation** $\mathcal{F}$ of $D \times \mathbb{R}^\lambda$, iff the differential equations $\mathcal{F}$ satisfy the integrability conditions.

An n-plane field $\mathcal{F}$ which is transverse to the fibers, and such that $\mathcal{F}_i(p, L(q)) = L(\mathcal{F}_i(p,q))\ \forall\ L \in GL(\alpha, \mathbb{R})$, is said to be a (linear) **connection**. If a connection is integrable then it is easy to see that the local solutions $F^{(p.q)}$ of $\mathcal{F}$ are indeed defined on domains N_p

independent of q.

We note that $\mathcal{F} \longrightarrow D$ (with the foliation $\mathcal{F}$ equipped with the leaf topology), or for that matter even its restriction to a *single leaf* $L_{\mathcal{F}}$ of $\mathcal{F}$, is an example of a **covering** of D : i.e. each component, of the pre-image of any sufficiently small connected $N_p \subseteq D$ is mapped homeomorphically onto N_p.

Definitions similar to the one above can be made for any *manifold* $W \subseteq D$ by looking at portions of the leaves, etc., lying above W : thus for the example above, the portion of the helical surface $L_{\mathcal{F}}$ above S^1 is a *covering* of the unit circle, and it can be seen that it is equivalent to $\mathbb{R} \longrightarrow S^1 \subseteq \mathbb{C}$ defined by $x \mapsto \exp(2\pi i x)$.

We note also that the above definition of $g_{\mathcal{F}}$ generalizes at once to *any* covering $\mathcal{F} \longrightarrow W$, and members of $g_{\mathcal{F}}$ are called deck or **covering transformations**. Poincaré's formulates this definition as follows :

He thinks of a covering $\mathcal{F}$ as a **multiple valued function** F (e.g. $\tan^{-1}(y/x)$ in above example) having, for each small $N_p \subseteq W$, differentiable **branches** $F^{p,q} : N_p \longrightarrow \mathbb{R}^\lambda$ (= *sections* over N_p of $\mathcal{F} \longrightarrow W$) which take *distinct* values $F^{p,q}(x)$ for each $x \in N_p$, and defines $g_{\mathcal{F}}$ to consist of all permutations S_C of these branches resulting from "following" them around all loops C at b.]

The point to note is that if C is any **lacet**, i.e. a path starting from b, followed by a small loop, followed by the opposite path ending at b, then the substitution S_C reduces to the identity substitution.

Poincaré now sets $C_1 + C_2 \equiv C_1C_2$ (: so this + may not be commutative) and, motivated by the above observation, he puts $C \equiv 0$ for all lacets C. More generally an **equivalence** $A \equiv B$, where A and B are formal integral combinations of loops at b, is obtained, starting from these elementary ones, by using the *rules* : $A \equiv B \Leftrightarrow B \equiv A$, $A \equiv B$ and $C \equiv D \Rightarrow A + C \equiv B + D$ (but maybe not $A + C \equiv D + B$), $2A \equiv A + A$, $2A - 3B \equiv 0 \Leftrightarrow 2A \equiv 3B$, etc.

Proposition 12. *The 1-dimensional homologies of a connected W are its abelianized equivalences* (thus e.g. if $4A - 3B + 7A + B \equiv 0$, then $11.A -$

$2B \simeq 0$).

Proof. This follows from the definitions of 1-homologies and equivalences, and the fact that any oriented closed curve C is homologous to the loop ACA^{-1}, where A is any path from the base point b to a point of C. *q.e.d.*

[Poincaré surprisingly cites $\partial\Sigma \equiv 0$, where Σ is *any* oriented 2-manifold of W, as an instance of an equivalence ! *This is false*, and in general one only has the homology $\partial\Sigma \simeq 0$ (: consider e.g. a torus-with-hole W).

He notes also that, unlike for homologies, a base point is involved in the definition of equivalences : however note that for connected W's the isomorphism class of the group defined below is independent of the chosen base point.]

For *any* monodromy group $g_{\mathcal{F}}$, we obviously have (1) $C \equiv C_1 + C_2 \Rightarrow S_C = S_{C_1}S_{C_2}$ and (2) $C \equiv 0 \Rightarrow S_C = \mathrm{Id}$. But we may also *"imagine"* a **fundamental group** G (of W at b) of substitutions S_C satisfying (1) and the stronger (2') $C \equiv 0 \Leftrightarrow S_C = \mathrm{Id}$. Poincaré notes that the natural epimorphism from G onto a $g_{\mathcal{F}}$ can be 1-1, but is in general not so, because some loop C, which is *not* decomposable into lacets, may still give the identity substitution in $g_{\mathcal{F}}$.

Poincaré's "imagine" can be interpreted merely as the existence of an *abstract group* with required relations amongst some generators, or else as a much stronger statement asserting *the existence of integrable differential equations* $\mathcal{F}$ *whose* $g_{\mathcal{F}}$ *obeys* (2'), or an intermediate statement asserting *the existence of a* **universal** *covering* $\mathcal{F} \longrightarrow$ W, *i.e. one for which* (2') *holds*, or equivalently $g_{\mathcal{F}} \cong G$.
The following was probably known to Poincaré (because e.g. he states in the next section that the fundamental group of the manifold of § 11 coincides with the monodromy group G_T of the covering $\mathbb{R}^3 \longrightarrow \mathbb{R}^3/G_T$).

Proposition 13. *A covering* $\mathcal{F} \longrightarrow W$, *with* $\mathcal{F}$ *connected, is universal iff* $\mathcal{F}$ *is* **simply connected**, i.e. $\vec{C} \equiv 0$ *for all loops* of $\mathcal{F}$.

Proof. Note first that each loop C at b **lifts** to *curves* $\bar{C}$ starting at each of the points $\bar{b}$ above b, and ending at $S_C(\bar{b})$. Thus S_C is the identity iff *all* the lifts of a loop C are themselves loops. But, since $\mathcal{F}$ is *connected*, it can be seen, that this is equivalent to asking that just *one* lift of C is a loop.

If $\mathcal{F}$ is not simply connected, take a $\bar{C}$ in it which is not decomposable into lacets. Since lacets lift to lacets, its image C in W can not be decomposable into lacets either. So $S_C : \mathcal{F} \longrightarrow \mathcal{F}$ is the identity map, even though C is not equivalent to 0, which contradicts (2').

The converse is clear because only a trivial C can lift to a loop $\bar{C}$ in the simply connected $\mathcal{F}$. *q.e.d.*

[Thus *the fundamental group* G *is a free and discontinuous group of transformations of the universal covering* and we have $\mathcal{F}/G \cong W$. Also it is easy to see that any diffeomorphism $W \longrightarrow W'$ lifts to a diffeomorphism $\mathcal{F} \longrightarrow \mathcal{F}'$ commuting with the covering transformations. So $\mathcal{F}/G \cong \mathcal{F}/G'$ *iff* G *and* G' *are conjugate in the group of all diffeomorphisms of* $\mathcal{F}$.

Poincaré had already proved the *existence* of a universal covering for *surfaces* some years before, so it is likely that he *did* intend his "imagine" in a strong sense. (Existence, and even characterization, of differential equations $\mathcal{F}$ obeying (2') might well be known now ?)

Another point which supports this view is that, *in* 1895 *the notion of an* abstract *group was used very warily* : but for this fact, Poincaré would surely have, for the parallel case of *homologies*, also mentioned the concomitant *homology group* !]

[The above curve-lifting definition of $g_{\mathcal{F}}$ applies even to *non-integrable* linear connections $\mathcal{F}$: this follows because PICARD'S EXISTENCE THEOREM for ordinary differential equations supplies us with curves tangent to any vector field contained in $\mathcal{F}$.

We note that one no longer has $C \equiv 0 \Rightarrow S_C = Id$ for these **holonomy groups** $g_{\mathcal{F}}$, and they are generally much bigger than the $g_{\mathcal{F}}$'s of an integrable $\mathcal{F}$, and can even be as big as *all* of $GL(\lambda, \mathbb{R})$, i.e. there may not be *any* proper subspace V_b of $\{b\} \times \mathbb{R}^\lambda$ which is preserved by $g_{\mathcal{J}}$.

In case there is such a $V_b \subset \{b\} \times \mathbb{R}^\lambda$, curve-lifting will gives us an equidimensional V_p over each $p \in D$, and the connection will be tangent to the subspace $V = \bigcup V_p$ of $D \times \mathbb{R}^\lambda$, which is in general a twisted (i.e. not diffeomorphic to $D \times V_b$) *vector bundle* over D.]

[These days the fundamental group of W at b is written $\pi_1(V,b)$, with the higher **homotopy groups** $\pi_j(V,b)$, $j \geq 2$, being defined iteratively as follows : if ΩV is the **space of all loops** of V at b, with the constant loop β as its base point, then $\pi_1(\Omega V,\beta) = \pi_2(V,b)$, etc.]

§ 13. In case a manifold can be recognized as the orbit space of a known discontinuous group G of diffeomorphisms of a *simply connected* Y, then G, being the monodromy group of the universal covering $Y \longrightarrow Y/G$, is also the fundamental group of Y/G.

The above method applies to the manifolds of § 11 (and so Exs. 1 and 4 of § 10) and also to Ex. 5 of § 10 since it is visibly diffeomorphic to $S^3/2$, where by 2 we denote the group of order two generated by the antipodal involution of the 3-sphere.

Besides, Poincaré also gives the following method.

Proposition 14 (FUNDAMENTAL GROUP OF A TRIANGULATED MANIFOLD). *For* $W \cong P/\iota$ *as in* § 10, *choose a base point* $p \in \mathrm{int}(P)$, *and conjugate points* $f \in \mathrm{int}(F)$, $f' \in \mathrm{int}(F')$, *in each pair* $\{F,F'\}$ *of identified facets of the polytope* P.

(a) *For each facet* F, *let* $\mathcal{F}$ *be the loop consisting of the line segment from* p *to* f, *followed by the line segment from* f' *to* p. *Then* $\{\mathcal{F}\}$ *is a set of* **fundamental loops**, i.e. any loop at p is in the equivalence class of some integral linear combination (repetitions allowed) of these loops. (In fact only half these loops are needed because $\mathcal{F}' \equiv -\mathcal{F}$.)

(b) *Furthermore, codimension* 2 *cells of* P/ι, *i.e. identification classes*

$$\{(F_t)'\cap(F_1),\ (F_1)'\cap(F_2),\ \dots\ ,\ (F_{t-1})'\cap(F_t)\}$$

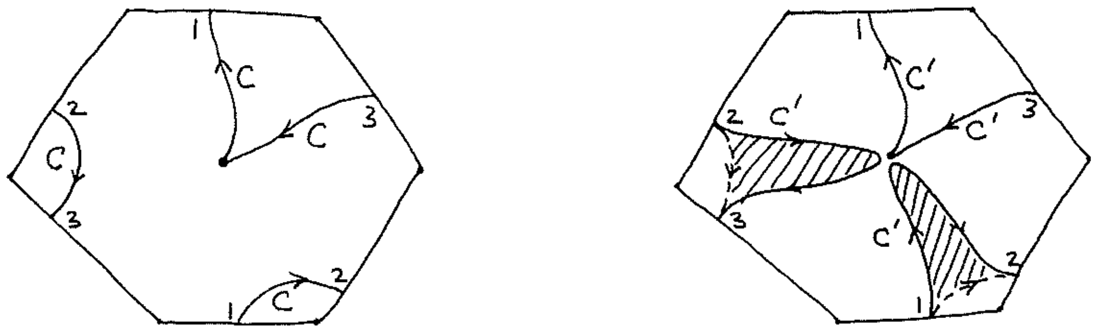
of codimension 2 *faces of* P, *give the* **fundamental equivalences**

$$\mathcal{F}_1 + \mathcal{F}_2 + \dots + \mathcal{F}_t \equiv 0,$$

between these loops, which (together with the aforementioned $\mathcal{F}' \equiv -\mathcal{F}$) imply all equivalences.

Thus G is known via generators and relations, and abelianizing these gives [the first homology group and so] the first Betti number. If W is an orientable manifold, then Poincaré duality yields [the codimension one homology group and so] the codimension one Betti number. So for an oriented manifold of dimension $\le$ 3, we can compute *all* its Betti numbers.

Proof. (a) Any equivalence class of loops contains a general position loop. These have only isolated transverse intersections with the (m-1)-cells only, and so can be decomposed as shown below into a sum of loops of the type $\pm \mathcal{F}$.

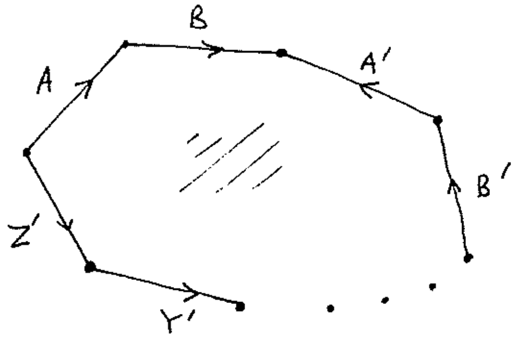


$C \simeq C'$

(b) The required equivalence follows because $\mathcal{F}_i \equiv pv_{i-1}v_ip$, the v_i's being the consecutive vertices of the polygonal link of the codimension 2 cell of P/ι in question.

Furthermore, any equivalence between our loops is determined by a map of the 2-disk into W. Since we can assume this map to be in general position, it will have only isolated transverse intersections with the codimension 2 cells of P/ι : and so the equivalence can be written as a sum of as many fundamental equivalences. *q.e.d.*

[As an application of the above method, let us consider again the case of a **surface with** g **handles**, represented by a 4g-gon P with the identifications ι as shown.



Now P/ι has only *one* codimension 2 cell, namely the identification class of all vertices of P :

$$\{(Z)'\cap(A),\ (A)'\cap(B)',\ (B)\cap(A)',(A)\cap(B),\ \dots\}.$$

So the fundamental group G is generated by the loops $\{\mathcal{A},\ \mathcal{B},\ \dots\}$ subject to the sole relation

$$\mathcal{A} - \mathcal{B} - \mathcal{A} + \mathcal{B} + \mathcal{C} - \dots \equiv 0.$$

Thus *for* $g \geq 2$ *the fundamental group of a surface is non-abelian.* Furthermore, the abelianization of the above equivalence is just $0 \simeq 0$, so the first homology group of the surface is the free abelian group on the 2g generators $\{\mathcal{A},\ \mathcal{B},\ \dots\}$, and so $b_1 = 2g$ as previously stated.]

Proposition 15 (CUBICAL MANIFOLD $\mathcal{C}$). *The fundamental group of* Ex. 3 of §10 *is of order eight, being isomorphic to the group generated by* **i**, **j**, *and* **k**, *subject to* **k** = **ij**, **i** = **jk**, **j** = **ki**, *and* **ikj** = 1.

Furthermore, there is an isomorphic subgroup **8** *of diffeomorphisms of the 3-sphere, and our manifold is diffeomorphic to the orbit space* $S^3/\mathbf{8}$.

Proof. Let X, Y, and Z denote the facets of our cube P which lie on the coordinate planes x = 0, y = 0, and z = 0 respectively, and let X', Y', and Z' be the opposite facets. The identifications of Example 3 lead to the following identification classes of edges :

{XZ, Z'Y', YX'} {YX, X'Z', ZY'}

{ZY, Y'X', XZ'} {Y'X, X'Z, Z'Y}

So the fundamental group of this orientable 3-manifold is generated by X, Y, and Z, subject to the corresponding four relations

$$Z - Y - X \equiv 0 \qquad X - Z - Y \equiv 0$$

$$Y - X - Z \equiv 0 \qquad X + Z + Y \equiv 0$$

Thus $X \mapsto \mathbf{i}$, $Y \mapsto \mathbf{j}$, $Z \mapsto \mathbf{k}$, gives an isomorphism of this additive group with the multiplicative one determined by the requisite relations $\mathbf{k} = \mathbf{ij}$, $\mathbf{i} = \mathbf{jk}$, $\mathbf{j} = \mathbf{ki}$, $\mathbf{ikj} = 1$. These imply $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2$, $\mathbf{i}^4 = 1$. To see this note that $\mathbf{ikj} = 1$ gives $\mathbf{k} = \mathbf{i}^{-1}\mathbf{j}^{-1}$, which in $\mathbf{k} = \mathbf{ij}$ gives $\mathbf{i}^{-2} = \mathbf{j}^2$. Likewise $\mathbf{i}^{-2} = \mathbf{k}^2$. Next, solve the 4 equations for $\mathbf{i}$ to get $\mathbf{kj}^{-1}$, $\mathbf{jk}$, $\mathbf{k}^{-1}\mathbf{j}$ and $\mathbf{j}^{-1}\mathbf{k}^{-1}$, and multiply to check $\mathbf{i}^4 = 1$, and thus also $\mathbf{i}^{-2} = \mathbf{i}^2$. *Any group element can be written uniquely as* $\mathbf{i}^a\mathbf{j}^b$, $0 \le a \le 3$, $0 \le b \le 1$. For, we can replace $\mathbf{k}$ by $\mathbf{ij}$, and then each $\mathbf{ji}$ by $\mathbf{i}^3\mathbf{j}^3$ to get $\mathbf{i}^a\mathbf{j}^b$. Since both $\mathbf{i}$ and $\mathbf{j}$ are of order 4, we can clearly keep clearly $0 \le a, b \le 3$; but further we can also replacing $\mathbf{j}^2$ by $\mathbf{i}^2$ and $\mathbf{j}^3$ by $\mathbf{i}^2\mathbf{j}$, so in fact it suffices to keep $0 \le b \le 1$. That the listed eight elements are distinct is easily checked.

Now recall that this group of **Hamilton** had figured in the definition of **quaternions**, which are points (x, y, z, t) of 4-space considered as combinations $x + \mathbf{i}y + \mathbf{j}z + \mathbf{k}t$, and multiplied using the group operations and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$. (This is analogous to considering points (x,y) of 2-space as **complex numbers** $x + iy$ and multiplying them using the group $i/\langle i^4 = 1)$ and $i^2 = -1$.)

So left quaternionic multiplication by $\mathbf{i}$ corresponds to $(x, y, z, t) \mapsto (-y, x, -t, z)$ because $\mathbf{i}.(x + \mathbf{i}y + \mathbf{j}z + \mathbf{k}t) = \mathbf{i}x - y + \mathbf{k}z - \mathbf{j}t$. Likewise, left multiplications by $\mathbf{j}$ and $\mathbf{k}$ correspond respectively to $(x, y, z, t) \mapsto (-z, t, x, -y)$ and $(-t, -z, y, x)$. These generate the order eight group **8** of diffeomorphisms of the unit sphere $S^3 \subset \mathbb{R}^4$.

The elements of **8** preserve the boundary of the 4-cube $\{(x,y,z,t) : -1 \le x, y, z, t \le +1\}$, and replacing our 3-cube P by the facet $t = 1$, which is a *fundamental domain* of **8**, we can check that the identifications ι are equivalent to identifying points of ∂P belonging to the same orbit. This shows that our manifold is diffeomorphic to $S^3/\mathbf{8}$. *q.e.d.*

[The homologies $Z - Y - X \simeq 0$, $X - Z - Y \simeq 0$, $Y - X - Z \simeq 0$, $X + Z + Y \simeq 0$, show that the *first homology group of* $\mathscr{C}$ is the abelian group on X and Y subject to $2X \simeq 0$ and $2Y \simeq 0$, i.e. it is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Thus *the Betti numbers of* $\mathscr{C}$ *are* $b_1 = b_2 = 0$, *i.e. the same numbers as* S^3 *or* $\mathbb{R}P^3$; however the homology groups of these three manifolds are distinct.

We remark that by recognizing right away that $\mathscr{C}$ is $S^3/\mathbf{8}$, we could have avoided generators and relations : however Poincaré works out even Exs. 1 and 4-5 (but not the non-manifold of Ex.2) by above method, even though they are all obviously of type Y/G with Y simply connected.]

[With quaternionic multiplication S^3 becomes a Lie group, and if we think of **8** as its *subgroup* generated by **i**, **j**, and **k**, then $\mathscr{C}$ consists of all *left cosets* of this subgroup. (Likewise, the *dodecahedral manifold* $\mathscr{P}$ of the fifth complement is the left coset space $S^3/\mathbf{120}$ of another finite subgroup of order 120 which has trivial abelianization, and so $S^3/\mathbf{120}$ will have even the same homology *groups* as S^3.)

Recall also that under $x + iy + jz + kt \leftrightarrow \begin{bmatrix} x+iy & z+it \\ -z+it & x-iy \end{bmatrix}$ quaternionic multiplication corresponds to multiplication of matrices of this type, and since such a matrix has determinant 1 iff $x^2 + y^2 + z^2 + t^2 = 1$, we see that *the Lie group* S^3 *is isomorphic to* SU(2) (and so **8** and **120** can also be regarded as subgroups of SU(2).]

Proposition 16. *The orientable closed 3-manifold* $\mathbb{R}^3/G_T$ *of* § 11 *has Betti numbers* $b_1 = b_2 = 3$ *iff* $T = I$; *otherwise* $b_1 = b_2 = 2$, *or* 1, *depending on whether* tr(T) *equals* 2 *or not.*

So, for Ex. 1 we have $b_1 = b_2 = 3$, while for Ex. 4 , $b_1 = b_2 = 1$.

Proof. Let $T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2,\mathbb{Z})$, and let the three generators $(x, y, z) \mapsto (x+1, y, z)$, $(x, y+1, z)$, and $(\alpha x + \beta y, \gamma x + \delta y, z+1)$ of our discontinuous group G (= G_T) be denoted by C_1, C_2, and C_3 respectively.

We note the following **commutation relations** between these generators :

$$C_1 + C_2 \equiv C_2 + C_1.$$
$$C_1 + C_3 \equiv C_3 + \alpha.C_1 + \gamma.C_2.$$
$$C_2 + C_3 \equiv C_3 + \beta.C_1 + \delta.C_2.$$

groups.]

We assert that *the fundamental group* G *of* $\mathbb{R}^3/G$ *is isomorphic to the group generated by* C_1, C_2, *and* C_3 *subject to the above relations.*

To see this note that any relation can be brought, by using the above commutation rules into the form $m_1.C_1 + m_2.C_2 + m_3.C_3 \equiv 0$. But it is easy to check that the transformation $m_1.C_1 + m_2.C_2 + m_3.C_3$ of 3-space is the identity iff the integers m_1, m_2, and m_3 are all zero. So this relation has just become $0 \equiv 0$.

Abelianizing the above equivalences we get the trivial homology $0 \simeq 0$ and two more :

$$(\alpha - 1).C_1 + \gamma.C_2 \simeq 0.$$
$$\beta.C_1 + (\delta - 1).C_2 \simeq 0.$$

So *the first homology group of* $\mathbb{R}^3/G$ *is the abelian group generated by* C_1, C_2, *and* C_3 *subject to the above relations.*

We note now that both these homologies are trivial, i.e. become $0 \simeq 0$, iff $T = I$. So in this, and only this, case the first homology group is the free abelian group on 3 generators and $b_1 = b_2 = 3$.

For $T \neq I$, the above homologies are proportional iff the determinant $\begin{vmatrix}\alpha-1 & \gamma \\ \beta & \delta-1\end{vmatrix}$ vanishes, i.e. iff $tr(T) = \alpha + \delta = 2$. So in this case, and only in this case, the Betti numbers are $b_1 = b_2 = 2$.

In all other cases the homologies are non-trivial and non-proportional, and so we have $b_1 = b_2 = 1$. *q.e.d.*

[One can also calculate the first (and thus any) *homology group* of $\mathbb{R}^3/G_T$, this being $\mathbb{Z} \oplus \mathbb{Z}/s\mathbb{Z} \oplus \mathbb{Z}/t\mathbb{Z}$, where the **elementary divisors** $s \geq 0$, $t \geq 0$, $s|t$, of $\begin{bmatrix}\alpha-1 & \gamma \\ \beta & \delta-1\end{bmatrix}$, are obtained by diagonalizing this matrix via elementary row and column operations over $\mathbb{Z}$: explicitly, one has $s =$ h.c.f.$(\alpha-1, \gamma, \beta, \delta-1)$ and $st = \pm \det\begin{bmatrix}\alpha-1 & \gamma \\ \beta & \delta-1\end{bmatrix}$.

This shows that Ex. 4, i.e. *the manifold* $\mathbb{R}^3/G_T$ *where* $T = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}$, *has the same homology groups as* $\mathbb{R}^3/G_{T'}$, *where* $T' = \begin{bmatrix}2 & 1 \\ 3 & 2\end{bmatrix}$, even though these manifolds are (see below) distinct and have non-isomorphic fundamental

§ 14. A classification theorem.

Poincaré begins this section by recalling an older result (Prop. 17 below) of his regarding surfaces, perhaps to serve as motivation for the examples of 3-manifolds being discussed.

We remark that A LOT was known about surfaces (see following notes) in 1895, and some of it is needed to understand Prop. 17.

[*The story had started no doubt with diophantine equations* ... But, the integral (or rational, or even the real) zeros of a polynomial $P(Z,W)$ being harder to understand, attention had turned to the associated **complex curve** $R = \{(w,z) : w \in \hat{\mathbb{C}},\ z \in \hat{\mathbb{C}},\ P(w,z) = 0\}$. This R is generally a closed **Riemann surface** (= *2-manifold with a complex structure*) and can be visualized as the graph of the multiple-valued **algebraic function** $w(z)$ which solves $P(w,z) = 0$, with the projection ζ : $R \to \hat{\mathbb{C}}$, $\zeta(w,z) = z$, being a (finitely) **branched covering** of R over the extended complex plane $\hat{\mathbb{C}}$ (= complex projective line $\mathbb{C}P^1$).

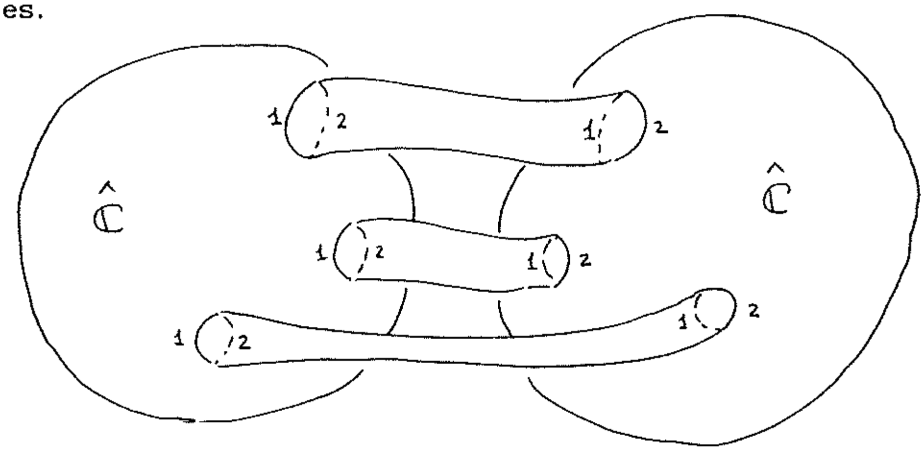
By this we mean that for each $x \in R$, there is an integer $n(x) \geq 1$, such that ζ becomes $z \longmapsto z^{n(x)}$ in suitable complex coordinates near x and $\zeta(x)$: thus this notion generalizes the previous one of (unbranched) *covering*, which corresponds to the case $n(x) \equiv 1$. (Since $\hat{\mathbb{C}}$ is simply connected, this case occurs iff $R \cong \hat{\mathbb{C}}$; also note that since R is compact, we have n(x) bigger than 1 for only *finitely* many x.)

Example. *The complex curve* R *defined by* $w^2 = (z-a_1).\ \ldots\ .(z-a_t)$ *is a 2-sphere with* $g = [(t-1)/2]$ *handles.*

(So, if we assume the classification theorem of surfaces, it follows that any orientable 2-manifold admits a complex structure.)

To see this draw in $\hat{\mathbb{C}}$ the g+1 line segments (a_1,a_2), (a_3,a_4), ... ,(a_{2g-1},a_{2g}), and (a_t,∞) or (a_{t-1},a_t) depending on whether t is odd or even. Our square root function w(z) has two branches, each single valued if we do not cross these lines, and going one into the other, the moment we cross any of these lines.

Thus the graph R of w(z) is obtained by taking two copies of the above, and identifying each dotted line of one copy with the corresponding solid line of the other. Since the algebraic function w(z) is à fortiori analytic, this 2-manifold R has a natural complex structure. Moreover, as shown below, R is diffeomorphic to a surface with g handles.



The projection $\zeta : R \longrightarrow \hat{\mathbb{C}}$ is the identity map on each copy : so each point of $\hat{\mathbb{C}}$ has two pre-images under ζ, except for the 2g+2 points a_1, a_2. ... , a_t, and also ∞ if t is odd, which have one pre-image each. One has n(x) = 2 at these 2g+2 points of R, and n(x) ≡ 1 elsewhere.

We note that this $\zeta : R \longrightarrow \hat{\mathbb{C}}$ is of **degree** d = 2, i.e. $|\zeta^{-1}(z)| \equiv 2$ almost always, except at the β = t+1 or t (for t odd or even) **branch points**, so the **genus formula** 2 − 2g = 2d − β holds : *in fact this formula holds for all algebraic curves.*

Conversely it was known that **any closed Riemann surface is the graph of some polynomial equation** f(z,w) = 0 **over** $\mathbb{C}$. This followed from Poincaré's theory of *automorphic functions* : these generalized the notion of *elliptic functions*, in which particular case the defining equation can be written in the form $w^2 = 4.z^3 - g_2.z - g_3$ given by

Weierstrass.

One says that a Riemann surface is **definable over** $\mathbb{F} \subseteq \mathbb{C}$ if there is a defining equation $f(z,w) = 0$ with coefficients in $\mathbb{F}$.]

[It was known furthermore that, for each algebraic function $w(z)$, there is an **Abelian integral** $\bar{w}(z) = \int^z f(u,w(t)).dt$ (i.e. with $f(u,v)$ *rational*) such that the graph of the multiple-valued analytic function $\bar{w}(z)$, together with its finitely many poles and branch points, gives likewise a *simply connected* (and usually non-closed) Riemann surface $\bar{R}$, whose projection $\bar{\zeta}$: $\bar{R} \longrightarrow \hat{\mathbb{C}}$, $\bar{\zeta}(z,\bar{w}) = z$, is (usually) an *infinitely branched covering*, i.e. at some points it can also look locally like $z \mapsto e^z$. Furthemore, we can arrange that $\bar{\zeta}$ is the composition of an (unbranched) **complex analytic universal covering** γ: $\bar{R} \longrightarrow R$, $\gamma(z,\bar{w}) = (z,w)$, and the (finitely) branched covering ζ: $R \longrightarrow \hat{\mathbb{C}}$. (Exemplify.)

To classify Riemann surfaces one thus starts with the simply connected case, for which there is the following celebrated result.

(At this point, we should recall pertinent electromagnetism, and give, instead of the following, a *complete* physical proof.)

RIEMANN MAPPING THEOREM. *If a Riemann surface is simply connected then it must be complex analytically homeomorphic to either the extended complex plane* $\hat{\mathbb{C}}$, *or the finite complex plane* $\mathbb{C}$, *or the unit disk* $\Delta = \{z : |z| < 1\}$.

The following heuristic argument assumes that the result is known (see e.g. **Ahlfors** book) for the case of simply connected domains of $\hat{\mathbb{C}}$.
Proof. Given a complex analytic function on $\mathbb{C}$, the level curves of its real and imaginary parts are orthogonal. We imagine one of these to be the trajectories of a **steady state electrical current**, and the other to be the equipotential lines.

A **unit charge** at some point of our surface must lead to currents without closed trajectories : this follows because, the surface being simply connected, we would otherwise have more charges enclosed within such closed orbits. These flow lines and equipotential curves give the required conformal 1-1 map on a simply connected domain of $\hat{\mathbb{C}}$. *q.e.d.*]

[*It is curious that some of the most celebrated results of the nineteenth and twentieth centuries have dealt respectively with the following amazingly similar problems* : **does a given 2-manifold, resp. n-manifold, admits a complex, resp. differentiable, structure, and, if so, how many ?**

For example, the Riemann mapping theorem tells us that *the sphere admits a unique complex structure*, while *the plane admits precisely two complex structures* : this last follows because $\mathbb{C}$ and Δ are diffeomorphic, but not conformally equivalent, because, by Liouville's theorem, a holomorphic function $\mathbb{C} \longrightarrow \Delta$ is constant.

Also (anticipating its much harder XXth century analogue !) *there are 2-manifolds which do not admit a complex structure* : this follows easily because, by virtue of the Cauchy-Riemann equations, the 2x2 Jacobian of a holomorphic map is positive, and so *a Riemann surface always comes with an orientation*, so the underlying 2-manifold must be orientable.

Conversely **Gauss** had (essentially) shown, *without appealing to any classification*, that **all orientable 2-manifolds admit complex structures**: this and Riemann's theorem thus imply that any closed simply connected 2-manifold is diffeomorphic to the 2-sphere. (More generally we'll see below that *the classification theorem of surfaces follows from the theorems of Gauss and Riemann.*)

In the fifth complement of this paper, Poincaré posed the analogous question : *is any closed simply connected 3-manifold diffeomorphic to the 3-sphere* ? This celebrated problem, now called the **Poincaré conjecture**, is still open.
But, surprisingly, the analogous problems in higher dimensions are now largely solved !! E.g. it is known — due to the work of **Smale**, ... , **Freedman** — that, *any closed n-manifold,* $n \geq 4$, *which has the same homotopy type as the n-sphere, is homeomorphic to the n-sphere.* But *not* necessarily diffeomorphic, as has been shown, by **Milnor**, for an infinity of $n \geq 7$. On the other hand, for n belonging to another infinite set

{5, 6, 8, ...}, even this stronger conclusion is true : it is unknown if this is so also for $n = 4$.

Regarding Euclidean spaces it is known that all but one of them admit a *unique* differentiable structure : the exceptional **4-space admits uncountably many differentiable structures** (which have as yet not been organized into a nice "moduli space" : cf. below).

The above followed by combining Freedman's work with that of **Donaldson** mentioned before : *curiously, just as in the Riemann mapping theorem, ideas from physics played a big role in Donaldson 's theorem also* !]

It follows easily from the Riemann Mapping Theorem that Γ **is a fundamental group of a closed Riemann surface** R **iff it is a fixed point free and discontinuous group of complex analytic homeomorphisms, of** $\hat{\mathbb{C}}$ **or** $\mathbb{C}$ **or** Δ**, having a compact quotient,** *and the classification of closed Riemann surfaces, upto complex analytic homeomorphism, is equivalent to the determination of all such subgroups, upto conjugation, in the group of all complex analytic homeomorphisms of* $\hat{\mathbb{C}}$ *or* $\mathbb{C}$ *or* Δ.

Before taking up Poincaré's remarks re the main case $\bar{R} = \Delta$ we'll look at the other two cases.

Case $\bar{R} = \hat{\mathbb{C}}$: *now* $\Gamma = 1$, *and* R *is complex analytically homeomorphic to* $\hat{\mathbb{C}}$.

If a holomorphic transformation of $\hat{\mathbb{C}}$ images $\mathbb{C}$ to $\mathbb{C}$, i.e. if it is an *entire function*, then its degree can not be greater than 1, otherwise it will take each value more than once in every neighbourhood of infinity. So, in this case, it is of the type, $z \mapsto az+b$, $a \neq 0$. Otherwise, it maps some $k \in \mathbb{C}$ to infinity, and so must be $z \mapsto 1/z-k$, followed by a transformation of this type.
Thus each holomorphic transformation of $\hat{\mathbb{C}}$ $(= \mathbb{C}P^1)$ *is a* (projective or) **fractional linear transformation**

$$z \longmapsto \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma \neq 0.$$

Since this has a fixed point, viz. a solution of the quadratic $z.(\gamma z + \delta) = \alpha z + \beta$, it follows that $\Gamma = 1$.

(Note that we can assume $\alpha\delta - \beta\gamma = 1$, and that the group $\mathrm{Hol}(\hat{\mathbb{C}})$ of holomorphic transformations of $\hat{\mathbb{C}}$ is isomorphic to $SL(2,\mathbb{C})/\{\pm I\}$.)

Case $\bar{R} = \mathbb{C}$: *now* Γ *is the free abelian group generated by two linearly independent translations, and* R *is a* **complex torus** $R = \mathbb{C}/\Gamma$. (We'll later organize all these possible complex structures of the 2-torus into a nice "moduli space".)

Our $\Gamma \subset \mathrm{Hol}(\mathbb{C})$ can not contain transformations $z \mapsto az+b$, with $a \neq 1$, because these have a fixed point $z = b/(a-1)$ in $\mathbb{C}$: so Γ must consist exclusively of some translations $z \longmapsto z + b$. If one of these translations, "b_3" $\in \Gamma$, is not rationally dependent on two others "b_1", "b_2" $\in \Gamma$, then, by Kronecker's theorem, the conjugates of the iterates of "b_3" are dense in the parallelogram with sides b_1 and b_2, and so Γ won't be discontinuous. So there are at most two rationally independent translations in Γ; and indeed there must be two, since otherwise the quotient is clearly not compact.

We note that we would have got the same answer had we set out to find all *groups of rigid motions of the* **euclidean plane** *which are discontinuous, fixed point free, and with a compact quotient* : possibly this observation led Poincaré to solve the case $\bar{R} = \Delta$ as follows ?

Case $\bar{R} = \Delta$. By above *any* closed Riemann surface with **genus**, i.e. number g of handles bigger than one is of the type Δ/Γ, where Γ is a discontinuous and fixed point free subgroup of $\mathrm{Hol}(\Delta)$.

[Being simply connected $\hat{\mathbb{C}}$ is *not* of this type, besides *there is no complex structure on the 2-torus other than the ones considered above.* In fact a famous theorem of **Gauss-Bonnet**, applied to a normalized hyperbolic metric (see below) on Δ says that *the Euler characteristic of such a* Δ/Γ *must equal the negative of its hyperbolic area* : so, since a torus has zero Euler characteristic it can not be of this type.]

Any member of $\mathrm{Hol}(\Delta)$ is a fractional linear transformation of the type

($\mathbb{C}/\langle z \mapsto e^{\pi/3\, i} z,\ z \mapsto z+1 \rangle \cong$ Zeeman Dunce Hat

$\mathbb{C}/\langle z \mapsto z \pm 1 \pm i,\ z \mapsto \bar{z} \rangle \cong$ Möbius strip)

$$z \longmapsto \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \quad |\alpha|^2 - |\beta|^2 \neq 0.$$

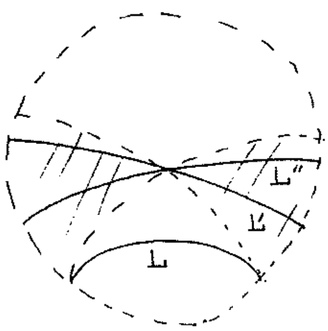
To see this we check first that the above transformations do belong to Hol(Δ) and act transitively on Δ. So it suffices to verify that any f ∈ Hol(Δ) *with* f(0) = 0 is of the above form, i.e. such that f(z) = a.z, where $|a|$ (= $|\alpha/\bar{\alpha}|$) = 1. To see this note that Schwarz Lemma gives $|f(z)| \le |z|$ and $|f^{-1}(z)| \le |z|$, i.e. $|f(z)| = |z|$, which implies that the holomorphic function f must be a rotation about 0.

Using the above formula one can now check that (besides being orientation and angle-preserving) *the transformations of* Hol(Δ) *map any circular arc of* Δ, *which is perpendicular to* ∂Δ, *to another such circular arc* (note that diameters of Δ are also such arcs).

Poincaré's great insight of 1880 (the one he had while stepping into a carriage !), was that Hol(Δ) could be regarded as the group of all orientation preserving rigid motions of the following non-Euclidean plane called a **hyperbolic plane** :

The **lines** *are circular arcs of* Δ *perpendicular to its boundary, and the* **angle** *between any two lines is the ordinary angle.*

Here, by **non-Euclidean plane** we mean that all but one of the usual axioms (e.g. *"two distinct points determine a unique line"*, *"if two distinct lines meet, they determine a unique point"*, etc.) of school geometry are still valid, the exception being **the parallel postulate** : *"given a point not on a line, there is a unique parallel line through it"*. Now, instead, we have an *infinity* of parallel lines :



L ∥ L′

L ∥ L″

The importance of the above insight lies in the fact that now all the

tools of school **geometry**, which do not use the parallel postulate, are available to us for Δ !

In particular, using any chosen line segment as a "yard-stick", and our rigid motions' group Hol(Δ), we can define a **hyperbolic distance function** on Δ having the usual properties.

Note here that, under our group Hol(Δ), the "yard-stick" shrinks indefinitely as it moves towards ∂Δ : thus *the boundary of* Δ *is at an infinite hyperbolic distance from its interior points.*

[Poincaré also interpreted Hol($\hat{\mathbb{C}}$) as the group of orientation preserving rigid motions of a **hyberbolic 3-space** and used it for the more difficult study of the discontinuous or **Kleinian subgroups** $\Gamma \subset$ Hol($\hat{\mathbb{C}}$). (See also his *"Science and Hypothesis"*) But by and by 2-dimensional, but more analytic, alternate methods were discovered. However **Thurston** has now revived the more natural geometric method of Poincaré to obtain beautiful results about 3-manifolds, in the same spirit as the study of 2-manifolds done below.]

With above geometric interpretation of Hol(Δ) in hand, Poincaré analyzed **Fuchsian groups** (= discontinuous subgroups Γ of Hol(Δ)) as follows :

We can define (just as in the euclidean case) a **fundamental domain** P for Γ as follows : we select any point $p_0 \in \Delta$ which is not a fixed point of any member of Γ, and take the closure of the set of all points which are strictly nearer to p_0 than any of their conjugates under Γ.

Next we check that this P is **convex** : it is the intersection of the **closed half spaces** H_{ij} containing p_0 and bounded by the right bisectors h_{ij} of segments joining conjugates p_i, p_j of p_0.

We now turn to the *topological boundary of above* P *in* Δ. By the **sides** of P we will mean, maximal open intervals of the lines h_{ij}, which lie in this boundary, and do not contain any fixed point of a $g \in \Gamma$ of order 2 which preserves h_{ij}. The remaining points of this boundary are called the **vertices** of P. (Note that the boundary of P *in* the plane is in general bigger, since its intersection with ∂Δ may be nonempty : these are called the **limit points** of P.)

Since the **tesselation** $T = \{g(P) : g \in \Gamma\}$ is a subdivision of Δ into non overlapping cells which fill it up completely, it follows that for each side s of P, there is a *unique* $g \in \Gamma$ such that g(P) is the neighbour of P in this tesselation which shares the side s with it. Using this we equip P with the **pairwise side identification** $\iota : s \equiv g^{-1}(s)$.

Under this identification each vertex v of P belongs to an identification class. If this class is finite, the sum of the interior angles of P at these vertices will be called the **angle-sum** of P at v.

Proposition 17. *A discontinuous group* Γ *of* Δ *has a compact quotient* Δ/Γ *iff the above fundamental domain* P *is in the interior of* Δ.

Furthermore P *is a (hyperbolic) polygon, and the angle-sum at each of its vertices* v *is* $2\pi/n_v$, *where* n_v *is an integer* ≥ 1, *and* $\Delta \longrightarrow \Delta/\Gamma \cong P/\iota$ *is a finitely branched covering, with* $n(g(v)) = n_v \ \forall g \in \Gamma$, *and* $n(x) = 1$ *elsewhere. Thus such a* Γ *is fixed point free (i.e. the fundamental group of* Δ/Γ*) iff all these angle sums are equal to* 2π.

Moreover, two such Γ*'s are conjugate in the group* Diff(Δ) *of diffeomorphisms of* Δ *iff the quotients* R/Γ *are diffeomorphic and this happens iff they have the same genus* g.

[Our statement, as well as proof, are variants of what is in the paper : in some respects Poincaré's version is stronger.]

Proof. The (continuous) hyperbolic distance function of Δ induces a similar distance function on the quotient, which must be bounded because the quotient is compact. So P is bounded with respect to the hyperbolic distance of Δ, and thus contained in a compact subset of Δ.

Next, Γ being discontinuous, we note that the portions of the lines h_{ij} which meet a given compact subset of Δ must be separated from each other by a positive distance. So only finitely many of them can meet the boundary of P, and thus P is a polygon. The quotient P/ι, of this even polygon under the aforementioned side identification, easily identifies with Δ/Γ.

Furthermore, as we go around the star of v in our tesselation T, we find that the angles subtended at v by these conjugates of P, run one by one through the angles of P at the vertices in the identification class of v. However, just one such run through this identification class may not complete the star, we might have to repeat n_v times : so in general the angle-sums are of the type $2\pi/n_v$. The assertions regarding branching and fixed point free Γ's is now clear.

To see the last part we note the fundamental domain P (and so the tesselation T) are by no means unique. For example we may **cut** off any part of P, and then **paste** a congruent part to get another *(but usually non-convex)* fundamental domain. So, using a procedure mentioned before, we can replace P by a **normal fundamental (4g)-gon** Q, say one with *opposite* sides identified.

Given two groups Γ and Γ' with the same g, we choose for them two normal fundamental 4g-gons Q and Q' as above, and then a diffeomorphism $\bar{f}$: Q $\longrightarrow$ Q' preserving side identifications. This extends uniquely to a diffeomorphism $\bar{f} : \Delta \longrightarrow \Delta$ commuting with Γ and Γ, and clearly such an $\bar{f}$ exists iff there is a diffeomorphism $f : \Delta/\Gamma \longrightarrow \Delta/\Gamma'$. *q.e.d.*

We remark that Poincaré had also shown conversely that if P is an even hyperbolic polygon, pairs of whose sides are congruent under some motions $g_i \in Hol(\Gamma)$ in such a way that the angle-sums are all of the above kind, then the group $\Gamma \subset Hol(\Delta)$ which is generated by these g_i's is also of the above kind.

Using this one can work out the conjugacy classes of such groups Γ *in* $Hol(\Delta)$, and thus classify the complex structures of any surface.

[For example the complex structures of a 2-torus can be organized into a **moduli space** $\cong \mathbb{C}$ as follows :

To each Γ spanned by two independent translations of $\mathbb{C}$, associate the **lattice** (= additive subgroup of $\mathbb{C}$) given by the orbit L of 0. If (z_1, z_2) is any basis of L, with order so chosen that $z = z_2/z_1$ satisfies

$\mathrm{Im}(z) > 0$, then any other such **oriented basis** is given by $(\alpha z_1 + \beta z_2, \gamma z_1 + \delta z_2)$ where $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2,\mathbb{Z})$.

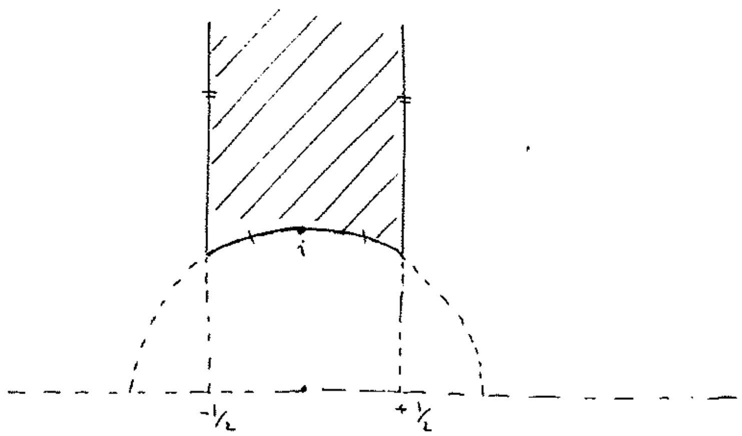
We note next that the groups Γ and Γ' commute with an $f \in \mathrm{Hol}(\mathbb{C})$, iff some oriented basis (z_1', z_2') of Γ' is of the type (az_1, az_2), where (z_1, z_2) is an oriented basis of Γ : this follows because we know that f is of the type $f(z) = az + b$.

Thus if we associate to each Γ the subset $S = \{z = z_2/z_1\}$ of the **upper half plane** $H = \{z : \mathrm{Im}(z) > 0\}$, we see that Γ is conjugate to Γ' iff S and S' have a point in common. But then we must in fact have S = S' : this follows because S is an orbit of the **modular group** $PSL(2,\mathbb{Z})$, i.e. the subgroup of Hol(H) consisting of all transformations of the type

$$z \longmapsto \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2,\mathbb{Z}).$$

Thus we have one-one correspondences between the sets of all complex structures on the 2-torus, all conjugacy classes of subgroups Γ of $\mathrm{Hol}(\mathbb{C})$ generated by two independent translations, and the set $H/PSL(2,\mathbb{Z})$, which we'll see below **is a Riemann surface conformally equivalent to** $\mathbb{C}$.

In fact the picture (see below) of the fundamental domain of the modular group, shows at once that $H/PSL(2,\mathbb{Z})$ is *diffeomorphic to 2-space*, and that the quotient map $j : H \longrightarrow H/PSL(2,\mathbb{Z})$ is finitely branched with $n(x) = 2$ at pre-images of i, $n(x) = 3$ at pre-images of ρ, and $n(x) = 1$ elsewhere :



That $H/PSL(2,\mathbb{Z})$ is *conformally equivalent to* $\mathbb{C}$, and not Δ, however requires some extra work : one needs to construct a **modular function** j :

$H \longrightarrow \mathbb{C}$, i.e. a surjective holomorphic function commuting with the action of the modular group. Such a well known function was constructed first by **Dedekind**.

We recall here that H itself is conformally equivalent to Δ, e.g. under $z \mapsto \frac{1 + iz}{1 - iz}$. Thus Dedekind's j-function shows that **even $\mathbb{C}$ is covered by Δ if we allow branched coverings**, likewise one also has branched coverings of $\hat{\mathbb{C}}$ by Δ. Two tesselations of Δ exhibiting these facts are illustrated below :

Poincaré also used this H as a model for the hyperbolic plane in place of Δ (and likewise **upper 3-space** H^3, with a hyperbolic metric, to study $\mathrm{Hol}(\hat{\mathbb{C}})$). We note that Hol(H) ($\cong$ Hol(Δ)) consists of

$$z \longmapsto \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2,\mathbb{R}),$$

and so is isomorphic to $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}$. Likewise $\mathrm{Hol}(\hat{\mathbb{C}}) \cong PSL(2,\mathbb{C})$.

The moduli spaces of complex structures on a surface with $g \geq 2$ handles were worked out analytically by **Teichmuller**, **Ahlfors**, and **Bers**, and then geometrically by **Thurston** : they are essentially (6g-6)-dimensional disks.]

[**Any closed Riemann surfaces is isomorphic to Δ/Γ for some discontinuous** *(but possibly with fixed points)* **group $\Gamma \subset \mathrm{Hol}(\Delta)$**. This follows because we can exhibit Δ as a branched covering of $\hat{\mathbb{C}}$ by composing an unbranched covering $\Delta \longrightarrow R$ of a Riemann surface with $g \geq 2$ handles with a branched covering (say as a graph of an algebraic function) $R \longrightarrow \hat{\mathbb{C}}$, and to exhibit Δ as a branched covering of a complex torus $\mathbb{T}$ we can compose the branched covering $j : \Delta \longrightarrow \mathbb{C}$ with the quotient map $\mathbb{C} \longrightarrow \mathbb{T}$.

Refinements of the above result are important in number theory :

Bely's Theorem : *a complex torus* $\mathbb{T}$ *is definable over* $\overline{\mathbb{Q}}$ *iff it is of the type* Δ/Γ *for some finite index subgroup* Γ *of the modular group* $PSL(2,\mathbb{Z}) \subset Hol(\Delta)$.

Taniyama's Conjecture : *if a complex torus* $\mathbb{T}$ *is definable over* $\mathbb{Q}$ *then it is of type* Δ/Γ *for some congruence subgroup* Γ.

Here by a **congruence subgroup** we mean a subgroup of $PSL(2,\mathbb{Z})$ bigger than some subgroup of the type $\Gamma(N) = \{g \in PSL(2,\mathbb{Z}) : g \equiv I \bmod N\}$. Very recently **Wiles** has given a proof of a special case of Taniyama's conjecture. Wiles' result has aroused a lot of interest because it was already known, by work of **Frey**, **Serre** and **Ribet**, that it implies **Fermat's Last Theorem** !!

As the following list of results shows, Wiles' Theorem is but the latest instance of *the extensive and fascinating interplay between Riemann surface topology and number theory.*

(1) **Faltings** proved that *if a Riemann surface* R *of a rational polynomial equation* $F(z,w) = 0$ *has* ≥ 2 *handles, then the subset* $R_{\mathbb{Q}} \subset R$ *of rational solutions is finite.*

Amongst the tools needed to prove this is the classical result of **Hurwitz**, viz. that, *if* R *has* $g \geq 2$ *handles, then the group* $Hol(R)$ *of its holomorphic transformations, is finite, and in fact of order* $84(g-1)$. (As against this $Hol(\hat{\mathbb{C}}) = PSL(2,\mathbb{C})$, and $Hol(\mathbb{T})$ of a complex torus $\mathbb{T} = \mathbb{C}/\Gamma$ contains at least all the translations of $\mathbb{C}$.)

(2) Faltings theorem was conjectured by **Mordell** who had himself proved a famous result for the genus one case : *if* R *has one handle, then there is a finite subset* $S_0 \subseteq R_{\mathbb{Q}}$ *such that* $\bigcup_{i\geq 0}(S_i)$ *where* S_{i+1} *is obtained from* S_i *by adding the third intersections with* R *of straight lines*

determined by pairs of points of S_i. Mordell's theorem was conjectured in a 1901 paper of **Poincaré**.

On the other hand **Siegel** has shown that, *even in the genus one case, the subset of integral points* $R_{\mathbb{Z}} \subset R$ *is finite* (and Faltings proof involves also a generalization of Siegel's result).

For the genus zero case case $R_{\mathbb{Z}}$ can be infinite e.g. for $x^2 + y^2 = 1$ (and at other extreme $R_{\mathbb{Q}}$ can be empty e.g. for $x^2 + y^2 = 3$) but now the knowledge of just *one* point of $R_{\mathbb{Q}}$ determines them all : as the second intersections with R of straight lines passing through this point.

(3) **Hasse** and **Weil** proved that *if* R *has* g *handles, and* R_p *is the set of solutions of its defining equation mod a prime* p, *then* $|R_p|$ *differs from* p-1 *by at most* $2g(p)^{1/2}$. Note here that R_p is not even a subset of R !

Weil also conjectured a higher dimensional generalization of above result, the pursuit of which led **Grothendieck** to launch a massive programme, which finally culminated in **Deligne**'s proof of Weil's conjecture.

(4) For the case of genus 1, the choice of a rational base point makes $R = \mathbb{T}$ into a **group** (= $\mathbb{C}/\Lambda$ where Λ is a lattice) and $R_{\mathbb{Q}}$ into a subgroup thereof. Mordell's theorem is equivalent to saying that $R_{\mathbb{Q}}$ *is a finitely generated abelian group.*

It is unknown whether the rank of the free part of this group can be arbitrarily big, however **Mazur** has shown that its torsion is bounded : in fact *the torsion part of* $R_{\mathbb{Q}}$ *is one of the following groups :* $\mathbb{Z}/N\mathbb{Z}$ with $1 \le N \le 10$ or $N = 12$, or else $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ with $1 \le N \le 4$.

Note that any **isogeny**, i.e. holomorphic endomorphism, of the group $\mathbb{T}$, is induced by a map $z \longmapsto cz$ of $\mathbb{C}$, which maps Λ to Λ, and that these isogenies form a ring under addition and composition. For most $\mathbb{T}$'s the c's have to be integral and this ring is $\cong \mathbb{Z}$.

In case there is a non-integral c (e.g. $z \longmapsto iz$ for the lattice Λ generated by 1 and i) it can be easily seen to be non-real : $\mathbb{T}$'s having such a **complex multiplication** have the property that by attaching their points of finite order one can construct interesting abelian field extensions of some algebraic number fields.]

Poincaré now turns to the classification theorem announced in his 1892 *Comptes Rendus* note.

Proposition 17 (CLASSIFICATION OF MAPPING TORI). *The manifolds* $\mathbb{R}^3/G_T$ *and* $\mathbb{R}^3/G_{T'}$ *are diffeomorphic iff* T *is conjugate in* $GL(2,\mathbb{Z})$ *to either* T' *or its inverse.*

[This is a *corrected* version of the result stated in the paper : Poincaré's conjugation is (apparently) in $SL(2,\mathbb{Z})$, and he makes no mention of the inverse.]

Proof of "if". In fact the mapping torus of *any* diffeomorphism τ: M $\longrightarrow$ M is diffeomorphic to that of any conjugate $\tau' = \upsilon\circ\tau\circ\upsilon^{-1}$: M $\longrightarrow$ M : because the diffeomorphism $(x,t) \mapsto (\upsilon(x),t)$ of $M \times [0,1]$ maps the pair of points $((x,0),(\tau(x),1))$ onto the pair of points $((\upsilon(x),0), (\tau'(\upsilon(x),1))$, and the two mapping tori are obtained respectively by identifying these pairs of points.

Again the involution $(x,t) \mapsto (x,1-t)$ of $M \times [0,1]$ maps the pair of points $((x,0),(\tau(x),1))$ onto the pair of points $((\tau(x),0),(x,1))$, and the identifications of the latter pairs gives the mapping torus of τ^{-1}.

$\mathbb{R}^3/G_T$ and $\mathbb{R}^3/G_T$ being diffeomorphic, their fundamental groups G_T and G_T are isomorphic, i.e. we have also the *"if"* of the following assertion.

Proposition 17a. *The groups* G $(= G_T)$ *and* G' $(= G_{T'})$ *are isomorphic iff* T *is conjugate to* T' *or its inverse in* $GL(2,\mathbb{Z})$.

Proof of "only if" (of Prop. 17a, and so of Prop. 17). We note (see proof of Prop. 16) that G', being isomorphic to G, we can choose generators C_1, C_2, C_3 *of* G', such that C_1 and C_2 span a normal subgroup G_{12} which is free abelian of rank 2, and

$$C + C_3 \equiv C_3 + T(C) \quad \forall\, C \in G_{12},$$

where $T : G_{12} \longrightarrow G_{12}$ denotes the isomorphism $C_1 \mapsto \alpha.C_1 + \gamma.C_2$, $C_2 \mapsto \beta.C_1 + \delta.C_2$, defined by $T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in GL(2,\mathbb{Z})$.

Replacing C_3 *by its negative, replaces* T (in above relations) *by its inverse :* because, on applying above equivalence to $T^{-1}(-C)$ we get $-T^{-1}(C) + C_3 \equiv C_3 - C$, i.e. $C - C_3 \equiv -C_3 + T^{-1}(C)$ $\forall C \in G_{12}$. (This shows again that G_T is isomorphic to $G_{T^{-1}}$).

Replacing C_1, C_2, *by* $U(C_1)$, $U(C_2)$, *where* $U \in GL(2,\mathbb{Z})$, *replaces* T *by* UTU^{-1} *:* because the isomorphism of G_{12} defined by $U(C_1) \mapsto UTU^{-1}(U(C_1)$ and $U(C_2) \mapsto UTU^{-1}(U(C_2)$ coincides with T.

If trace of T' is not ± 2, *then the following sequence of such replacements, of the generators* C_1, C_2, C_3 *of* G', *replaces* T *by* T' *:*

Since C_1, C_2, C_3 are generators of G', we can write them uniquely as

$$C_1 = a_3.C_3' + a_1.C_1' + a_2.C_2',$$
$$C_2 = b_3.C_3' + b_1.C_1' + b_2.C_2',$$
$$C_3 = c_3.C_3' + c_1.C_1' + c_2.C_2',$$

where the determinant of the coefficient matrix is necessarily ± 1.

In case b_3 is not zero, we find a $U = \begin{bmatrix} p & r \\ q & s \end{bmatrix} \in SL(2,\mathbb{Z})$, such that $p.a_3 + q.b_3 = 0$. (For this, first choose integers r and s such that $r.b_3 + s.a_3$ equals h.c.f. of a_3 and b_3 : then there exist integers p, q satisfying the required conditions $p.r - q.s = 1$ and $p.a_3 + q.b_3 = 0$.) Replacing C_1 and C_2 by $U(C_1)$ and $U(C_2)$, we see that we need only consider the *case when* $b_3 = 0$.

Since tr(T') *is not* ± 2, $b_3 = 0 \Rightarrow a_3 = 0$ (so $c_3 = \pm 1$, $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \pm 1$):

Otherwise, if $b_3 = 0$ and $a_3 \neq 0$, the integral combinations of C_1 and C_2 which are also integral combination of C_1' and C_2', must be multiples of C_2, i.e. the normal free abelian subgroup $G_{12} \cap G'_{12}$ has rank 1 and is generated by C_2. So $-C_3' + C_2 + C_3' \in G_{12} \cap G'_{12}$ is a multiple of C_2.

Choose a vector space basis ξ, η of $G'_{12}\otimes\mathbb{C}$ such that $T'(\xi) = s.\xi$ and $T'(\eta) = s^{-1}.\eta$, where s and s^{-1} are the eigenvalues of T'. If $C_2 = \mu.\xi + \rho.\eta$, we have $- C_3' + C_2 + C_3' = T'(C_2) = s.\mu.\xi + s^{-1}.\rho.\eta$, which is a multiple of C_2 iff $s = s^{-1}$.

If $c_3 = -1$ we replace C_3 by its negative, so we need only to consider *the case* $b_3 = 0 = a_3$, $c_3 = 1$, *and* $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \pm 1$.

Now, if $U^{-1} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \neq I$, we replace C_1, C_2 by $U(C_1)$, $U(C_2)$, so it remains only to consider *the case when* $C_1 = C_1'$, $C_2 = C_2'$, *and* $C_3 = C_3' + c_1.C_1 + c_2.C_2$.

In this case, the commutation rules of C_1', C_2', C_3' show $C_1 + C_3 = C_3 + \alpha'.C_1 + \gamma'.C_2$ and $C_2 + C_3 = C_3 + \beta'.C_1 + \delta'.C_2$, whereas we are given $C_1 + C_3 = C_3 + \alpha.C_1 + \gamma.C_2$ and $C_2 + C_3 = C_3 + \beta.C_1 + \delta.C_2$: so now $T = T'$.

When tr(T) $\neq \pm 2$, the same argument works with G, G' interchanged.

When neither T **nor** T' **has trace different from** $\pm$ 2, *we proceed as follows :*

Any $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2,\mathbb{Z})$ *with trace* 2 *or* −2 *is conjugate to an element of the type* $\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$ *or* $\begin{bmatrix} -1 & h \\ 0 & -1 \end{bmatrix}$: for, if $\gamma \neq 0$, a $U = \begin{bmatrix} p & r \\ q & s \end{bmatrix} \in SL(2,\mathbb{Z})$ with $2\gamma.p - (\alpha-\delta).q = 0$, makes the bottom left corner of

$$\begin{bmatrix} s & -r \\ -q & p \end{bmatrix} . \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} . \begin{bmatrix} p & r \\ q & s \end{bmatrix}$$

i.e. $-q.(\alpha.p + \beta.q) + p.(\gamma.p + \delta.q) = \gamma.p^2 + (\delta - \alpha).pq - \beta.q^2$, equal to zero, because this quadratic has a double root $(\alpha - \delta)/2\gamma$. Using this as lemma we will now get some group-theoretical information about G_T.

If tr(T) = 2, *then either* T = I, *and so each element of* G_T *commutes with any other, or else there is, upto sign, a unique such primitive element.* Further, in case tr(T) = −1, there is *no* such nonzero element, but now either T = −I, when each element of G_T commutes with the *double* of any other, or else there is, upto sign, a unique such primitive element.

(By *primitive* we mean a nonzero element which is not a multiple of another.)

The above follows because G can be generated by C_1, C_2, C_3 obeying $C_1 + C_2 = C_2 + C_1$, $C_1 + C_3 = C_3 \pm C_1$, and $C_2 + C_3 = C_3 + h.C_1 \pm C_2$: if $h \neq 0$, $\pm C_1$ are the required distinguished primitive elements.

So, since $G \cong G'$, $tr(T) = 2 \Leftrightarrow tr(T') = 2$, with $T = I \Leftrightarrow T' = I$, and $tr(T) = -2 \Leftrightarrow tr(T') = -2$, with $T = -I \Leftrightarrow T' = -I$.

When $T = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$ and $T' = \begin{bmatrix} 1 & h' \\ 0 & 1 \end{bmatrix}$, with h and h' nonzero, we choose *in* G' ($\cong$ G) generators C_1, C_2, C_3 obeying above relations. The uniqueness of the distinguished primitive elements implies that $C_1 = \pm C_1'$: and by replacing C_1, C_2 by $-C_1$, $-C_2$ we can assume in fact $C_1 = C_1'$. Thus

$$\begin{aligned} C_1 &= C_1', \\ C_2 &= b_1.C_1' + b_2.C_2' + b_3.C_3', \\ C_3 &= c_1.C_1' + c_2.C_2' + c_3.C_3', \end{aligned}$$

where $b_2.c_3 - c_2.b_3 = \pm 1$ Using this, and the commutation rules of C_1', C_2', C_3', it turns out that $C_2 + C_3 = C_3 \pm h'.C_1 + C_2$, whereas we are given $C_2 + C_3 = C_3 + h.C_1 + C_2$: so $h' = \pm h$.

When $T = \begin{bmatrix} -1 & h \\ 0 & -1 \end{bmatrix}$ and $T' = \begin{bmatrix} -1 & h' \\ 0 & -1 \end{bmatrix}$, with h and h' nonzero, a similar argument again enables us to reduce to the case $h' = \pm h$.

The result follows because $\begin{bmatrix} 1 & -h \\ 0 & 1 \end{bmatrix}$ (resp. $\begin{bmatrix} -1 & -h \\ 0 & -1 \end{bmatrix}$) is the inverse of $\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$ (resp. $\begin{bmatrix} 1 & -h \\ 0 & 1 \end{bmatrix}$). *q.e.d.*

[The *above* pairs of inverses are in fact conjugate in GL(2,$\mathbb{Z}$), for

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -h \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & h \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -h \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

However note that, in general, this is not so, e.g. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and its inverse $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are *not* conjugate in GL(2,$\mathbb{Z}$).]

Since tr(T) is the same for $T \in SL(2,\mathbb{Z})$, or its inverse T^{-1}, or any conjugate UTU^{-1}, and can obviously take infinitely many values, Prop. 17 gives *an infinite list of closed orientable 3-manifolds, which are* *diffeomorphic iff their fundamental groups are isomorphic.*

[Since $<C_1, C_2> \cong \mathbb{Z}^2$ and $G_T/<C_1, C_2> \cong \mathbb{Z}$, G_T is an **extension** of $\mathbb{Z}^2$ by $\mathbb{Z}$; also conversely, whenever $0 \rightarrow \mathbb{Z}^2 \rightarrow G \xrightarrow{a} \mathbb{Z} \rightarrow 0$ is exact, then $G \cong G_T$, where $T: \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ is the restriction of the inner automorphism of G induced by any element whose a-image is 1.

Thus Prop. 17a *classifies all extensions of* $\mathbb{Z}^2$ *by* $\mathbb{Z}$ *upto group isomorphism.* (If we consider instead isomorphisms of the extensions which *preserve* the original group $\mathbb{Z}^2$, then the same classification holds again, and this weaker result is much easier to prove, and generalizes to extensions by $\mathbb{Z}$ of *any* group H.)

It seems that Prop. 17a *generalizes to extensions of* $\mathbb{Z}^n$ *by* $\mathbb{Z}$, and might be true for many groups H other than $\mathbb{Z}^n$ also. Likewise one has topological generalizations of Prop. 17.]

Poincaré was unable to prove (as claimed in his 1892 note) that *any* two orientable 3-manifolds having isomorphic fundamental groups are diffeomorphic. Regarding this, and the question of deciding when a finitely generated group is the fundamental group of some orientable 3-manifold, he now states prophetically that such problems will *"exigeraient de difficiles études et de longs développements"*.

[Besides 17a, Poincaré also gives some other algebraical side-results. Also, he again mentions the "Picard paradox" of generic surfaces having lower Betti numbers.]

§ 15. Sometimes it is convenient to describe a manifold by a combination of the implicit and parametric methods.

For example, a $(n-p+q-\lambda)$-dimensional manifold U in n-space might consist of points x which satisfy some p equations $F_\alpha(x;\ y) = 0$, where this system of equations depends on a parameter y which itself runs on a manifold W in q-space satisfying λ constraints $\phi_\beta(y) = 0$.

Or else, another $(q - \lambda)$-dimensional manifold V of n-space might be parametrized, $x = \theta_i(y)$, by these $y \in W$. In case this parametrization is globally valid, and the functions θ are invariant with respect to a discontinuous group G of diffeomorphisms of W, then we have $V \cong W/G$.

Example 7 (REAL PROJECTIVE PLANE). Let $W = S^2$, the unit sphere of 3-space defined by $x^2 + y^2 + z^2 = 1$. The restriction of the map θ: $\mathbb{R}^3 \longrightarrow \mathbb{R}^6$, $\theta(x, y, z) = (x^2, y^2, z^2, xy, yz, zx)$ to W is of rank 2, and is such that the inverse image of each point of $V = \theta(W)$ consists of two antipodal points. So $V \cong S^2/2$, 2 being the fixed point free order two group generated by the antipodal involution $(x, y, z) \longmapsto (-x, -y, -z)$. We note that *this* **real projective plane** $V \cong S^2/2$ *is contained in the 5-dimensional affine subspace of* $\mathbb{R}^6$ given by the condition that the sum of the first three coordinates is 1. (Also in 4-sphere obtained by intersecting with unit S^5.)

Besides, as Poincaré points out, note also that *the non-orientability* of V stems from the fact that the antipode of a point having spherical coordinates (ϕ, θ) has coordinates $(\phi + \pi, \pi - \theta)$, and this has negative Jacobian -1.

[Note on the other hand, that the antipode map of S^3 is *orientation preserving*, which corresponds to the fact that the real projective space $S^3/2$ (see Ex.5) was an *orientable* 3-manifold : more generally *real projective n-space* $\mathbb{R}P^n \cong S^n/2$ is orientable iff n is odd.

Note also that there is a similar **Veronese embedding** θ which realizes the real projective (n-1)-space in $\frac{n(n+1)}{2}$- space. However this dimension is nowhere near the least possible : **Whitney** has shown that *any closed n-manifold can be realized in (2n)-space.*

This number 2n is in general the best possible : for example it is known that *closed non-orientable 2-manifolds do not embed in 3-space*, and, for all $k \geq 1$, *the real projective* 2^k*-space does not embed in an m-space with m less than* 2^{k+1}. However, the problem of finding the least m such that a given n-manifold embeds in m-space is usually a hard problem.]

Example 8 (SYMMETRIC SQUARES OF SPHERES). Let $W = S^{q-1} \times S^{q-1}$, the product of (q-1)-spheres in (2q)-space $\mathbb{R}^{2q} = \mathbb{R}^q \oplus \mathbb{R}^q$, consist of all (y,z) such that $y_1^2 + \dots + y_q^2 = 1$ and $z_1^2 + \dots + z_q^2 = 1$. Consider now the map $\theta: \mathbb{R}^{2q} \longrightarrow \mathbb{R}^{q(q+3)/2}$ defined by

$$\theta(y_1, \dots, y_q; z_1, \dots, z_q) = (y_i+z_i; y_iz_i ; y_iz_k + z_ky_i).$$

Since the inverse image of any point of $V = \theta(W)$ is either a **diagonal** point (y,y), or a pair {(y,z), (z,y)} of points of W, we see that $V \cong \mathrm{Sym}(S^{q-1})$ or $\mathrm{Sym}^2(S^{q-1})$, the **symmetric square** (= the space of all subsets of cardinality ≤ 2) of S^{q-1}.

Proposition 19. *For all* $q \geq 3$, $\mathrm{Sym}(S^{q-1})$ *is a closed* (2q-2)*-dimensional pseudomanifold, being in fact a closed 4-manifold for* q = 3. *For* q = 2 *it is a 2-manifold with boundary.*

Further, these pseudomanifolds are non-orientable for q *even, and orientable for* q *odd, and all their Betti numbers are trivial, except the* (q-1)*th number which is equal to* 1.

[Poincaré seems (see below) to assert that one obtains closed *manifolds* for all $q \geq 3$: however this is false, for we'll see that *there are singularities for all* $q \geq 4$.

Further $\mathrm{Sym}(S^1)$ *is a* **Moebius strip** *with boundary*, while $\mathrm{Sym}(S^2)$ *is diffeomorphic to the* **complex projective plane** $\mathbb{C}P^2$, i.e. the quotient of $\mathbb{C}^2 \setminus \{0\}$ under the equivalence relation $(z_1,z_2) \simeq (az_1,az_2)$.]

Proof. This time the order two group, generated by $(x,y) \mapsto (y,x)$, *is fixed point free only on the complement of the diagonal* (q-1)*-sphere* $\Sigma^{q-1} \subset S^{q-1} \times S^{q-1}$. So à priori we are sure only that $W \setminus \theta(\Sigma)$ is an

open (2q-2)-manifold. However, for $q \geq 3$, the *possible singular set* $\theta(\Sigma)$ has codimension ≥ 2, so then V is a closed pseudo manifold.
Poincaré works out the local nature of $\mathrm{Sym}(S^1)$ *near any point of the circle* $\theta(\Sigma)$ *as follows :*

Consider the image, under the above map $\theta : S^1 \times S^1 \longrightarrow V$, $(y_1, y_2, z_1, z_2) \longmapsto (x_1 = y_1+z_1, x_2 = y_1z_1, x_3 = y_2+z_2, x_4 = y_2z_2, x_5 = y_1z_2+y_2z_1)$, of a small neighbourhood of the diagonal point (0, 1, 0, 1). This is diffeomorphic to the portion of *a small neighbourhood of the origin of the* x_1-x_2 *plane which lies on or above the parabola* $x_1^2 = 4x_2$. To see this note that $(y_1-z_1)^2 = x_1^2 - 4x_2$, to work out y_1, z_1, in terms of x_1, x_2; then y_2, z_2; and finally x_3, x_4, and x_5.

Poincaré works out the local nature of a complementary planar section of $\mathrm{Sym}(S^2)$ *near any point of the 2-sphere* $\theta(\Sigma)$ *as follows :*

Consider the intersection of the codimension 2 plane $x_1 = 0 = x_3$ of 9-space, with the image under the map $\theta : S^2 \times S^2 \longrightarrow V$, $(y_1, y_2, y_3, z_1, z_2, z_3) \longmapsto (x_1 = y_1 + z_1, x_2 = y_1z_1, x_3 = y_2+z_2, x_4 = y_2z_2, x_5 = y_1z_2+y_2z_1, x_6 = y_3+z_3, x_7 = y_3z_3, x_8 = y_1z_3 +y_3z_1, x_9 = y_2z_3+y_3z_2)$, of a small neighbourhood of the diagonal point (0, 0, 1, 0, 0, 1). This is homeomorphic to a *neighbourhood of the origin of the half cone* $4x_2x_4 - x_5^2 = 0$, $x_2 \leq 0$, $x_4 \leq 0$, *in* x_2-x_4-x_5 *space.* To see this calculate y_1 and y_2 (from which y_3 also follows) by $x_2 = -y_1^2$ and $x_4 = -y_2^2$; then z_1 and z_2 to be their negatives (and from this z_3); and check that we have a 1-1 parametrization of the stated section (e.g. $x_5 = y_1z_2+y_2z_1$ holds because right side equals $-2y_1y_2$ whose square is $4x_2x_4$ which equals the square of the left side.)

[*An easier argument which shows the topological nature of the singularities, for all* $q \geq 2$, *is the following :*

Note first that on the q-dimensional subspace of (2q)-space complementing its diagonal, i.e. on $y + z = 0$, the map $(y,z) \longmapsto (z,y)$, coincides with the antipodal map $(y,z) \longmapsto (-y, -z)$. From this it is seen that the **link** in W, of each point of $\theta(\Sigma)$, is the **join** of a (q-2)-sphere (its link in $\theta(\Sigma)$), and a real projective (q-2)-space (its link in the complementary q-1 directions).

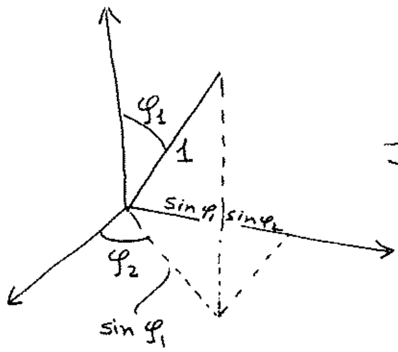
This link $S^{q-2} * \mathbb{R}P^{q-2}$ is, for all $q \geq 3$, a closed pseudomanifold, but is *not* (because of say homological reasons) a *sphere unless* $q = 3$, in which case $S^1 * \mathbb{R}P^1 \cong S^1 * S^1 \cong S^3$. For $q = 2$, $S^0 * \mathbb{R}P^0 = S^0 * \{pt\}$ is a closed 1-cell.

We note that a similar argument shows in fact that *the symmetric square* $\mathrm{Sym}(M^2)$ *of* **any** *2-manifold* M *is a 4-manifold.*

We note also that the above $\theta : S^2 \times S^2 \longrightarrow \mathbb{C}P^2$ is *yet another type of branched covering* : on a tubular neighbourhood ($\cong \mathbb{C} \times \Sigma$) of the 2-sphere Σ this map identifies (z,s) with $(-z,s)$, i.e. is equivalent to $(z,s) \longmapsto (z^2,s)$.]

Orientability. In local coordinates the involution $(y,z) \mapsto (z,y)$ reads $(\phi_1, \ldots, \phi_{q-1}; \phi'_1, \ldots, \phi'_{q-1}) \mapsto (\phi'_1, \ldots, \phi'_{q-1}; \phi_1, \ldots, \phi_{q-1})$, and so has Jacobian $(-1)^{q-1}$. Thus this involution of $S^{q-1} \times S^{q-1}$ is orientation preserving, and so $\mathrm{Sym}(S^{q-1})$ is orientable iff q is odd.

From now on these $(\phi_1, \ldots, \phi_{q-1})$ and $(\phi'_1, \ldots, \phi'_{q-1})$ will be assumed to be *spherical coordinates*, and we will denote by J and J' the corresponding spherical *volume elements* :



$$J = (d\varphi_1)(\sin\varphi_1 \, d\varphi_2)(\sin\varphi_1 \sin\varphi_2 \, d\varphi_3)\cdots$$

We choose a diagonal point (u,u) and an oriented **latitude** $U_1 = \{(v,u) : v \in S^{q-1}\}$ and **longitude** $U_2 = \{(u,v) : v \in S^{q-1}\}$ through it.

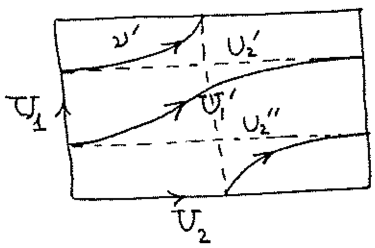
We'll now check that the Betti numbers of $S^{q-1} \times S^{q-1}$ *are all zero except for* $b_0 = 1$, $b_{q-1} = 2$ (with all (q-1)-cycles generated by U_1 and U_2), *and* $b_{2q-2} = 1$:

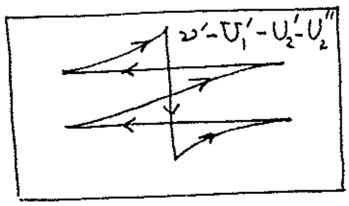
By duality it is enough to work out the numbers in dimensions $\leq$ q-1. Further, all manifolds of dimension *less* than q-1 can be deformed into *the* (2q-2)-*ball* $(S^{q-1} \times S^{q-1}) \setminus (U_1 \cup U_2) \cong (S^{q-1}\setminus\{u\}) \times (S^{q-1}\setminus\{u\})$: so we are left only to deal with the dimension q-1.

To see that U_1 *and* U_2 *are homologically independent*, Poincaré uses the fact that, for any irrational λ, the periods of the closed (q-1)-form $J_1 + \lambda . J_2$ on U_1 and U_2 are integrally independent.

Poincaré next checks that any general position *oriented* (q-1)-*manifold* ν *is homologous to* $m.U_1 + n.U_2$, where m, resp. n, is the algebraical (i.e. accounting for intersection number being +1 or −1) number of times it cuts U_2, resp. U_1.

To see this he draws (see the 2-torus case illustrated below) through each intersection point with U_1, a parallel U_2' to U_2, and through each intersection point with U_2, a parallel U_1' to U_1. The sum of all these parallels is homologous to $m.U_1 + n.U_2$. Also we can obviously replace ν by a homologous ν' which *coincides* with these parallels in a small vicinity of each of these intersections. So the difference of ν' and the sum of these parallels, is a cycle in the ball $(S^{q-1} \times S^{q-1}) \setminus (U_1 \cup U_2)$, and is thus homologous to zero.





Poincaré's argument for the Betti numbers of V = $Sym(S^{q-1})$ *runs as follows* (the calculation of b_0, b_{2q-2} being clear, he works in other dimensions only):

Note that U_1 and U_2 coincide in this quotient, so they give rise to a single (q-1)-cycle U of V. This U is homologically non-trivial because the closed form J+J' has a nonzero integral over it.

Next he says that we can **lift** any closed oriented manifold ν of V to a manifold ω of W, by choosing continuously, for each point of ν, *one* of the two corresponding points of W. This ω can be closed or non closed.

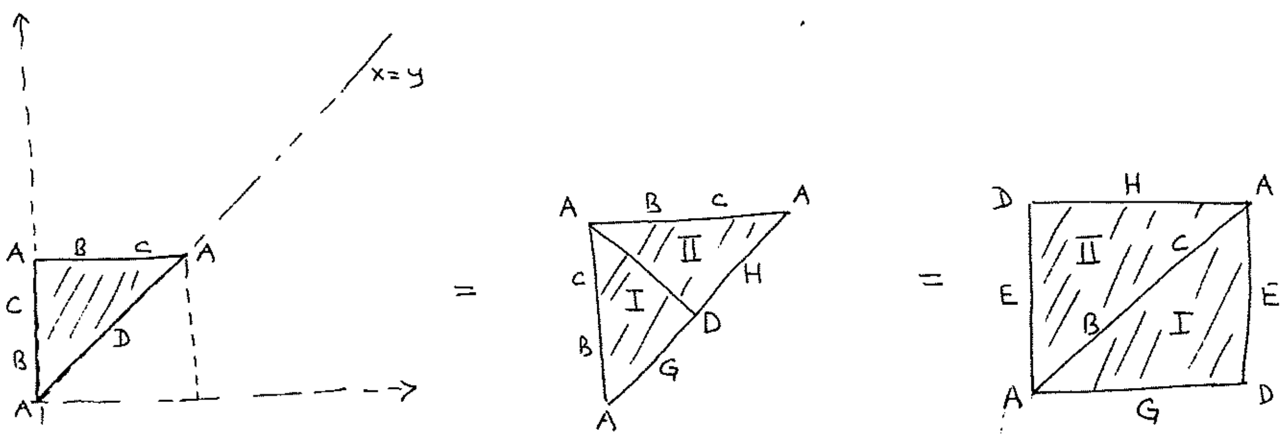
Case ω closed : now ω is homologous to 0, or to an integral combination $m.U_1+n.U_2$, and so ν is homologous to 0 or to $(m+n).U$.

Case ω nonclosed : by duality (??) we need to work in dimensions $\le$ q-1 only. So $\partial(\omega)$, which consists of pairs of symmetrical points of W, is of dimension $\le$ q-2. We can symmetrically deform it into the diagonal (q-1)-sphere Σ, and then cap it off in Σ. Thus we can replace ν by a homologous ν' which can be lifted to a closed ω', and so we can apply the above case.

[The argument of the last para is doubtful : e.g. V is usually not even a manifold, and half the time non-orientable, so duality theorem does not apply to V. It does to W of course, but how does that imply above reduction to dimensions $\le$ q-1 ? Also he does'nt spell out the above deformation, though it can probably be done.

However we can use instead the obvious fact that *the double* $2.\nu$ *of any closed manifold* ν *of* V *lifts to a closed manifold* ω. Thus the Betti numbers of V are as given by Poincaré.]

[The fact that Sym(S^1) is a Moebius strip is clear from the following cutting and pasting :



To see $\mathrm{Sym}(S^2) \simeq \mathbb{C}P^2$ we proceed as follows ....] *q.e.d.*
see below

[More generally **the symmetric group** Σ_n **of** n **letters acts by permutations on the n-fold product of** $S^2 \cong \mathbb{C}P^1$ **and the quotient** $((S^2)^n) / \Sigma_n$ **is diffeomorphic to** $\mathbb{C}P^n$.

An inductive proof of this using **Newton's Theorem** *on elementary symmetric functions* runs as follows ...

Note that this argument used the fact that $\mathbb{C}$ is algebraically closed. In fact $((\mathbb{R}P^1)^n) / \Sigma_n$ is *not* $\mathbb{R}P^n$, e.g. for n = 2 we saw above that it is a Möbius strip, and for n = 3 it can be seen to be a solid torus. It would be interesting to work this out for all $n \geq 4$ also.

Note also that $\mathrm{Sym}^n(X)$ (see below) is somewhat different from $((X)^n)/\Sigma_n$ for $n \geq 3$.]

By Prop. 7 the middle Betti number of an orientable closed manifold of dimension 4k+2 is even. Above V^4 ($\cong \mathbb{C}P^2$) is orientable and with middle Betti number 1, so (as we pointed out before) Prop. 7 does not extend to orientable closed manifolds of dimension 4k.

[Poincaré says that *all* the above V^{4k}'s provide such examples, but this is not so : for $k \geq 2$, these are *not* manifolds ! Likewise he says that the V^{4k+2}'s, which are non-orientable and have middle Betti number 1, show that Prop. 7 does not extend to non-orientabile closed (4k+2)-manifolds : again the examples are wrong, because these are *not* manifolds. But, as we pointed out before, the Klein Bottle can be used instead to show this.]

[Other interesting examples like the ones considered above :

(1) **Massey** and **Kuipers** showed that S^4 *is the quotient of* $\mathbb{C}P^2$ *under complex conjugation* $[z_1, z_2] \longleftrightarrow [\bar{z}_1, \bar{z}_2]$, the quotient map $\mathbb{C}P^2 \longrightarrow S^4$ being a 2-fold covering branched like $z \longmapsto z^2$ on a real projective plane contained in S^4.

(2) **Borsuk** and **Bott** showed that the **symmetric cube** $\mathrm{Sym}^3(S^1)$, i.e. *the*

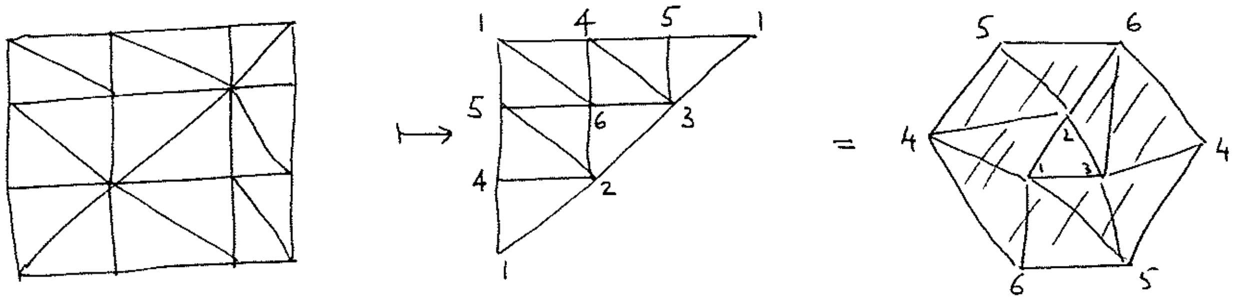
Map given by $\mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^2$, $([z_1, z_2], [w_1, w_2]) \mapsto [z_1 w_1, z_1 w_2 + z_2 w_1, z_2 w_2]$ is surjective and induces diffeom. of $\mathrm{Sym}\, S^2$ with $\mathbb{C}P^2$. Likewise $\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$ (n times) divided out by action of symmetric n group is $\mathbb{C}P^n$. (Only pt. used is that complex polynomial is determined by its roots.)

space of all subsets of S^1 *of cardinality* ≤ 3, *is diffeomorphic to* S^3, the quotient map $S^1 \times S^1 \times S^1 \longrightarrow S^3$ being a 6-fold covering branched like $z \longmapsto z^6$ on a circle contained in S^3.

Moreover the above 3-sphere can be visualized as the union of two solid tori, obtained by dividing $S^1 \times S^1 \times S^1$ out by the obvious action of the group of all permutations of 3 letters.

(For other references on symmetric powers, e.g. the general theorem of **Dold-Thom**, see **Steenrod**'s *"Reviews"*.)

(3) The fact that $\mathrm{Sym}(S^1)$ is a Möbius strip has a *simplicial refinement*: the **6-vertex Mobius strip** results if we divide out a suitable 9-vertex torus by its simplicial involution $(a,b) \leftrightarrow (b,a)$.



Note that by adding the missing triangle we get the **6-vertex real projective plane**, which can be obtained also as the quotient of an **icosahedron** by its antipodal involution. Further it can be seen that neither the Möbius strip nor $\mathbb{R}P^2$ can be triangulated by less than 6 vertices, and that each has a unique triangulation with 6 vertices.

An analogous simplicial refinement of $\mathrm{Sym}^2(S^2) \simeq \mathbb{C}P^2$ (starting from the tetrahedral triangulation of S^2) has been used by **Bier** and **Brehm** to give a *ten-vertex triangulation of the complex projective plane*.

However this time 10 is not least possible : **Kuehnel** found that $\mathbb{C}P^2$ **has a unique minimal triangulation with 9 vertices** and this can be obtained from a simplicial refinement of the Massey-Kuipers' theorem. (Probably the Borsuk-Bott theorem also has an interesting simplicial refinement.)

In general it is a very hard problem to find the least number of vertices required to triangulate a given manifold. For example, *for surfaces other than the 2-sphere, the main point in the proof of the* **map color theorem** *is to construct a triangulation with the least number of vertices : these constructions of* **Ringel, Youngs** *and others are now well understood via branched coverings.*

However, for the case of the 2-sphere, the main point is to show that four colors *suffice* (to distinguish neighboring vertices of any triangulation) and there is still no conceptual proof of this **four color theorem.**]

[Note on the role of symmetric squares in **Van Kampen**'s embedding theory.]

§ 16. **Euler characteristic.**

A **subdivision** P of a closed p-manifold V is a partition of V into finitely many **regions** (= nonclosed manifolds) $\{\nu_p\}$, $\{\nu_{p-1}\}$, ..., of dimensions p, p–1, ... , and we'll denote by N the alternating sum,

$$N = \alpha_p - \alpha_{p-1} + \dots \pm \alpha_0,$$

where α_q denotes the number of the ν_q's.

A subdivision P will be called a **cell subdivision** if each ν_q, resp. $\partial\nu_q$, is diffeomorphic to a q-cell, resp. (q-1)-sphere.

Proposition 20 (INVARIANCE OF EULER CHARACTERISTIC). *All cell subdivisions* P *of a manifold* V *have the same* N.

[This is a *corrected version* of the result stated in the paper. Poincaré claims the above even when each $\bar{\nu}_q$ is a *simply connected* manifold-with-boundary, however this generalization is false :

E.g. if $V = \mathbb{C}P^2$ or $S^2 \times S^2$, $\nu_4 = \text{int}(D)$ and $\nu_4' = \text{ext}(D)$, where D is a closed 4-disk $D \subset V$, and the lower dimensional regions $\{\nu_3\}$, $\{\nu_2\}$, $\{\nu_1\}$, $\{\nu_0\}$ constitute a cell subdivision of $\partial D \cong S^3$, then P is of above kind and has N = 2, but for any cell subdivision of V one has N = 3 or 4.

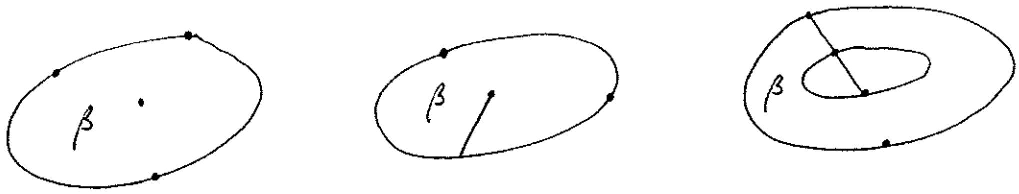
(Poincaré probably *intended* this cellular version, because, *at the time* he wrote §§ 16-18, he apparently believed that the only simply connected q-manifolds-with-boundary were closed q-cells ? This is like his 1892 assertion that a closed q-manifold (of (q+1)-space) is determined by its fundamental group, which of course he found out that he could'nt prove, by the time he had finished writing up § 14. So maybe §§ 16-18 were written before § 14, and this mistake was left in simply because of inattention ?)

However the above result does remain true when the $\bar{\nu}_q$'s are just homotopically, or even homologically, trivial, and in fact even under some still weaker conditions.]

Poincaré's first invariance proof. We note that any subdivision satisfies the condition

(*) "all ν_q, $q \le p-1$, are incident to at least two ν_{q+1}'s",

unless the boundary of some region β is not a manifold :



By putting a new vertex in some cells if need be (a process which clearly does'nt change N) we can in fact work only with cell subdivisions having no **singular region**, i.e. a q-region, $q \le p-2$, incident to just two (q+1)-dimensional regions. We will assume this below for P, but not for the P_i, $i \ge 0$.

We have to check that N is the same for any two cell subdivisions P and P_0 of V. For this it suffices to consider *the case when* P_0 *is* **finer** *than* P, i.e. when each region of P_0 is contained in some region of P. This follows because, given any two cell subdivisions of V, we can easily find another which is finer than both of them.

Now Poincaré describes **an algorithm**, which, starting from P_0, successively gives (possibly non cell-)subdivisions P_{i+1}, $i \ge 0$, each having two regions less than the preceding subdivision P_i :

START by searching P_i for a ν_0 which is singular, and if such a ν_0 is found, **erase** it (thus making its union with the two incident ν_1's into one region) to get P_{i+1}.

Otherwise, search P_i for a ν_1 which is incident to just two ν_2's, and if such a ν_1 is found, erase it (thus making its union with the two incident ν_2's into one region) to get P_{i+1}.

.... (and so on) ....

Till finally we come to the case when we search P_i for a ν_{p-1} which separates two ν_p's *which are in the same region of* P, and if *such* a ν_{p-1} is found we erase it (thus making its union with the two incident ν_p's into one region) to get P_{i+1}.

If no such ν_{p-1} is found the algorithm STOPS.

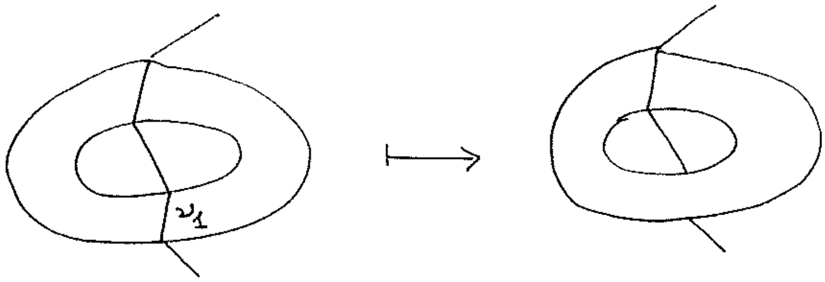
Clearly such erasures preserve N. Further, Poincaré asserts that the algorithm stops at P_i iff P_i = P, so *"q.e.d."*

[**An analysis of the above proof.** (1) Since the union of the erased ν_q and its only two incident ν_{q+1}'s is also a non closed manifold, each step $P_i \mapsto P_{i+1}$, $i \geq 0$, does lead to a new subdivision of V.

(2) Also each of these subdivisions is finer than P. If an erasure of a region of dimension $q \leq p-2$ is made, this follows from the fact that its two incident (q+1)-regions are *automatically* in same region of P, otherwise P would have a singular region. And, if an erasure of a ν_{p-1} is made, this follows because of the extra condition that we took care to insert, for this case, in our algorithm.

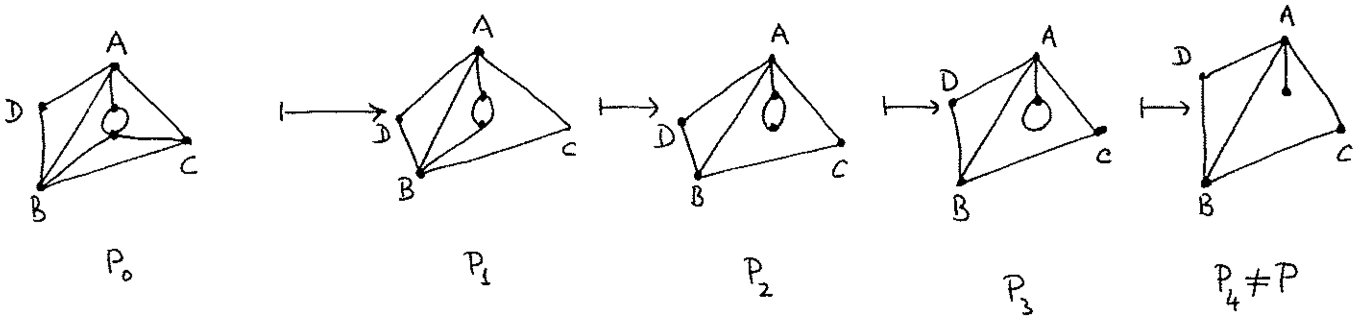
(3) If algorithm stops at P_i, *and* P_i *satisfies the condition* **(*)**, then P_i = P. This is so because there are no singular regions. And, all (p-1)-regions must be contained in (p-1)-regions of P, otherwise they are erasable. This then implies that all (p-2)-regions must be contained in (p-2)-regions of P, otherwise they would be singular, and so on.

(4) Note that **(*)** is true for P_0, but $P_i \mapsto P_{i+1}$ need not preserve **(*)**. In fact if $P_i \mapsto P_{i+1}$ involves an erasure of a ν_t, then it will preserve **(*)** for all $q \neq t$, but might give some t-region incident to just one (t+1)-region :



(5) Above example also shows that $P_i \mapsto P_{i+1}$ might change a cell subdivision into a non cell subdivision. (Poincaré mentions this "objection" at the end of this "proof" but does not actually *say* that it falsifies — see (6) below — his argument.) In case our algorithm *does* go though cell subdivisions, then of course (*) will hold at each stage and the algorithm will stop only when P has been reached.

(6) Let P denote the tetrahedral boundary ABCD, and let P_0 be obtained by subdividing the triangle ABC further as shown below. Then Poincaré's algorithm may run as follows stopping before P is reached :



(7) Poincaré's first invariance proof is based on trying to show something more than "P_0 **is finer than** P" iff "**P can be obtained from** P_0 **by some sequence of erasures**", but most probably even *this is false* : in fact the problem of finding a correct combinatorial reformulation of "P_0 is finer than P" is still open even for triangulations.

(8) Poincaré's second invariance proof will be based roughly on showing instead that the relation "P **and** Q **have a common finer subdivision**" is same as the "**equivalence relation generated by erasures**" : *this seems true* and resembles a later theorem of **Newman**.

(9) Poincaré's second proof also contains a recipe — see Prop. 23

below — for cell-subdividing any closed manifold : a complete proof of **triangulability of (differentiable) manifolds** was however given much later by **Whitehead.**]

Proposition 21. *For any cell subdivision of a* p*-sphere one has* N = 2 *if* p *is even and* N = 0 *if* p *is odd.*

Proof. By invariance N is independent of the cell subdivision P of the p-sphere. Let P be the boundary of the (p+1)-**cube** of (p+1)-space enclosed within the hyperplanes $x_i = \pm 1$, $1 \le i \le p+1$.

The q-faces of P are obtained by taking any p+1-q of the x_i's and setting them equal to +1 or -1, so $\alpha_q = 2^{p+1-q} \cdot \binom{p+1}{q}$. Hence

$$(1-2)^{p+1} = 1 - \alpha_p + \ldots \pm \alpha_0 = 1 - N,$$

and thus $N = 1 - (-1)^{p+1}$. *q.e.d.*

Poincaré's second invariance proof will be by an *upward induction on the dimension* p *of* V, with the inductive hypothesis used as follows.

Proposition 22. *Let* Prop. 20, *and so* Prop. 21, *be true for dimensions less than* p. *Then*

$$\gamma_p - \gamma_{p-1} + \ldots \pm \gamma_{q+1} = 1 + (-1)^{p-q-1},$$

where γ_t, $t > q$, *is the number of* t*-regions of any cell subdivision* P *of a closed* p*-manifold* V, *containing a fixed* q*-region* ν_q *on their boundary.*

Proof. Choose a point in the interior of ν_q, a complementary (n–q)-dimensional plane of the ambient n-space passing through this point, and a small (n-q)-dimensional ball in it with the chosen point as centre.

The intersection of this ball with V is a (p-q)-dimensional ball, whose boundary L is a (p-q-1)-sphere. Further, each of the γ_t t-cells of P incident to ν_q intersects L in a distinct (t-q-1)-cell. (If all $\bar{\nu}$ had *only* been assumed simply connected, these intersections need *not* be simply connected.)

Thus L, which is called the **link** of ν_q in P, becomes a cell subdivided sphere of dimension p-q-1. Since $p-q-1 < p$, the desired equality follows by applying Prop. 21. *q.e.d.*

By a **quadrillage** of n-space we will understand a cubical subdivision of n-space determined by n pencils of non-accumulating hyperplanes parallel to the coordinate planes, $x_i = a_{i,k}$, $1 \le i \le n$.

Proposition 23 (TRIANGULABILITY OF MANIFOLDS). *Let* V *be a closed* p-*manifold in* n-*space. Then there is a quadrillage of* n-*space such that the intersection of each of its* (n-t)-*cubes* D_{n-t} *with* V *is a* (p-t)-*cell* ν_t, *and these cells constitute a cell subdivision* Q *of* V.

Poincaré gives no details, but it is likely that the above is correct.

Poincaré's second invariance proof. We want to check that N(P) = N(R) for any two cell subdivisions P and R of V.

For this we choose a quadrillage Q of V, which has to satisfy certain conditions w.r.t. P and R which we'll specify later. Let P' be the common subdivision of P and Q obtained by intersecting their cells (likewise analogous common subdivision R' of R and Q). We'll prove N(P') = N(P) and N(P') = N(Q) (and likewise N(R') = N(R) and N(R') = N(Q)).

Proof of N(P') = N(P) *:* We go from P' to P by *erasing the hyperplanes* $x_i = a$ *one by one*. Let δ_q denote the number of q-cells of P' on this plane, δ'_q the number adjacent to the $(x_i < a)$-side of the plane, and δ''_q the number adjacent to the $(x_i > a)$-side of this plane.

Then Poincaré asserts that $\delta'_q = \delta_{q-1} = \delta''_q$ (with also $\delta'_0 = 0 = \delta''_0$ and $\delta_p = 0$). Assuming this, the erasure of the hyperplane decreases each α_q by $\delta_q + \delta_{q+1}$, and since the alternating sum over q of these numbers is zero, N remains same.

Proof of N(P') = N(Q). By making the **mesh** of the quadrillage Q small enough we can ensure that the interior c of each cell of Q intersects only *one* least dimensional cell ν_q of P : thus the cells of P intersecting c are precisely those that are in the **star** of ν_q, i.e. have ν_q on their boundary.

We now go from P' to Q as follows :

We first erase all cells of P' which are in p-cells c of Q but which have lesser dimension than p. So in each c we are erasing the least dimensional ν_q and, for each $p > t > q$, γ_t incident cells of dimension t. Moreover the number of p-cells within c was γ_p before and 1 after. Thus the total decrease in N is $-1 + \gamma_p - \gamma_{p-1} + \ldots \pm \gamma_{q+1} \mp 1$, which by Prop. 23 is zero.

Next we erase all cells of P' which are in (p-1)-cells of Q but which have lesser dimension than p-1, and so on. The same verification shows N remains same at each step. *q.e.d.*

[$\delta'_q = \delta_{q-1} = \delta''_q$ can certainly be ensured by letting P' be the intersection of a rectilinear *copy* of P with a general position quadrillage of the ambient space. And likewise R' will come from a rectilinear copy of R. But then we are not talking of the *same* Q but rather of *two* different Q's : thus there seems to be some difficulty with this proof here.]

§ 17. Having convinced himself of the invariance of the **Euler characteristic** N, Poincaré now pushes ahead with its calculation.

Proposition 24. *Closed odd dimensional manifolds have* $N = 0$.

Proof. The **generalized face number** $\beta_{\lambda\mu}$ of a cell subdivision P is the sum, over all $\nu_\lambda \in P$, of the number of μ-cells incident to ν_λ : note that $\beta_{\lambda\lambda} = \alpha_\lambda$ and $\beta_{\lambda\mu} = \beta_{\mu\lambda}$.

We will now sum the following triangular **tableau** :

$$\begin{array}{llllll}
\beta_{p,p-1} & -\beta_{p,p-2} & +\beta_{p,p-3} & -\beta_{p,p-4} & + \dots \\
 & +\beta_{p-1,p-2} & -\beta_{p-1,p-3} & +\beta_{p-1,p-4} & - \dots \\
 & & +\beta_{p-2,p-3} & -\beta_{p-2,p-4} & + \dots \\
 & & & \dots\dots\dots & \dots \\
 & & & & \dots
\end{array}$$

The sum of the first row is the sum of the N's of the bounding (p-1)-spheres of the α_p p-cells of P, so (by Prop. 21) it equals $2\alpha_p$. Likewise that of second row is zero and that of third is $2\alpha_{p-2}$, etc. Thus the sum of the tableau is twice $\alpha_p + \alpha_{p-2} + \dots$.

On the other hand the sum of the qth column is the sum of the α_q expressions of Prop. 22 corresponding to the links of the q-cells ν_q of P. So their sums are $2\alpha_{p-1}$, 0, $2\alpha_{p-3}$, 0, ... Thus the sum of the tableau is also equal to twice $\alpha_{p-1} + \alpha_{p-3} + \dots$.

Equating the two values one gets $N = 0$. *q.e.d.*

[We note that for p even, row summation of above tableau gives same numbers as column summation, so no further information about N.

We note also that Poincaré's column summation contains as a special case some equations which are usually attributed to later mathematicians :

This is the case when P is a **triangulation** and so $\beta_{\mu\lambda} = \alpha_\lambda . \binom{\lambda+1}{\mu+1}$ for all

$\mu \le \lambda$: this follows because each λ-cell is now a simplex with $\lambda+1$ vertices, and any subset of $\mu+1$ vertices determines an incident μ-cell.

So now the column summations of the above proof read,

$$\alpha_p . \binom{p+1}{\mu+1} - \alpha_{p-1} . \binom{p}{\mu+1} + \ldots \pm \alpha_{\mu+1} . \binom{\mu+2}{\mu+1} = (1+(-1)^{p-\mu+1}) . \alpha_\mu ,$$

i.e. the so-called **Dehn-Sommerville-Klee equations** of a closed manifold.

By Prop. 21, it follows that *for spheres*, these equations hold also for $\mu = -1$ if we make the convention $\alpha_{-1} = 1$ (i.e. that P has a unique empty simplex). Then these equations are collectively equivalent to the **functional equation** $\zeta(z) = \zeta(1-z)$, where

$$\zeta(z) = \alpha_p . z^{p+1} - \alpha_{p-1} . z^p + \ldots \pm \alpha_0 . z \mp 1 = 0.$$

Thus the **zeros of $\zeta(z)$**, which of course occur in complex conjugate pairs, are symmetrical with respect to the line Re(z) = 1/2, and, *for many (but not all) spherical triangulations it is true also that these zeros are in fact either real or on this line* !

This suggests that *the* L *functions of modular curves* may be interpretable in terms of the combinatorics of the corresponding tesselation of Δ ?]

§ 18. **Euler-Poincare' Formula.**

In this last section Poincaré completes the calculation of N. Note that since his Betti numbers b_i (see § 6) were invariant by definition, the following result contains Prop. 20. Also, by virtue of the Poincaré duality (see § 9) it contains Prop. 24.

The following argument is in fact **another invariance proof**, and indeed the one which affected later developements most : it implicitly gives a *new definition of Betti numbers depending on cell subdivision*, and Poincaré identifies these numbers with the old invariant Betti numbers.

Proposition 25 (EULER-POINCARE FORMULA). *For any cell subdivision* P *of a closed* p*-manifold* V *one has*

$$\alpha_p - \alpha_{p-1} + \alpha_{p-2} - \dots = b_p - b_{p-1} + b_{p-2} - \dots .$$

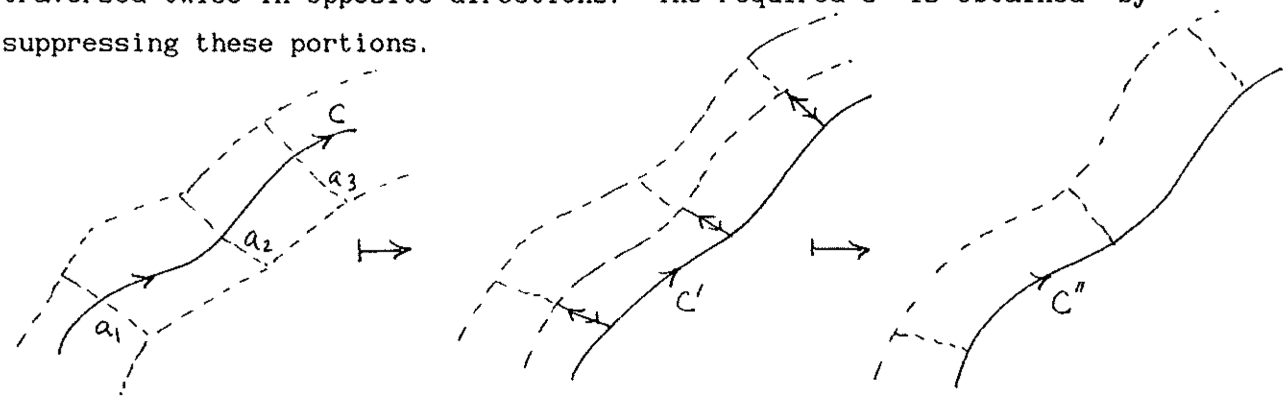
To make the argument clear we start with the known case of surfaces which Poincaré attributes (in § 16) to **de la Jonquière**.

Proof for case p = 2. Assign to each of the α_0 vertices of P any number, and to each of its oriented α_1 edges the difference δ of the numbers of its two vertices. These α_1 numbers δ depend on the α_0 numbers, and conversely determine them upto an additive constant, so *there are in all* $\alpha_1 - \alpha_0 + 1$ *linear relations between the* δ'*s*.

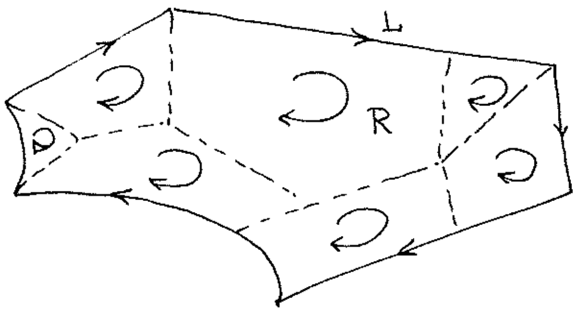
Moreover, *these linear relations are given by setting equal to zero, the algebraic sum of the* δ'*s, of some cycle* K *of edges of* P : this is easy and equivalent to the analogue of Prop. 25 for the one-skeleton of P !

We'll now count these relations in a different way. Firstly, each of the oriented α_2 faces of P furnishes such a cycle, viz. its *perimeter* Π. Secondly, from any chosen b_1 homologously independent cycles C of V, we construct *cycles* C" *of edges of* P as follows :

We divide C into arcs $a_i a_{i+1}$ each representing a crossing of a face of P. Replacing each such arc by a homologous arc on the bounding polygon (see below) of this face gives us a homologous cycle C' which consists of edges of P *and* some portions of edges of P, these latter being traversed twice in opposite directions. The required C" is obtained by suppressing these portions.



We assert that any relation between the δ's is a linear combination of the $\alpha_2 + b_1$ relations given by the Π's and the C"s. To see this, let K be any cycle of edges of P. Adding a suitable linear combination of the C"s to it we get L, which is homologous to zero. Being a cycle of edges of P, this L must be the *boundary of a sum* R *of faces of* P, and the relation corresponding to L, is the sum of the relations corresponding to the perimeters Π of these faces making up R:



[Note that we just checked, for the above case, that the following **new definition of Betti numbers** (to which we alluded to in the beginning) is valid :

b_i *is the maximal number, of linear combinations with zero boundary of i-faces of* P, *such that no nontrivial linear combination of these is a boundary of a sum of* (i+1)*-faces of* P.

This is indeed true in general and the basis of most computations of Betti numbers.]

The manifold being closed and orientable, the sum of *all* the oriented perimeters is zero, but no *partial sum* of the Π's is zero. And equally, no other linear combination of the Π's and C"s is zero, for then the latter would not be homologically independent. Thus our new count shows $\alpha_2 + b_1 - 1$ linearly independent relations between the δ's. Equating this with $\alpha_1 - \alpha_0 + 1$ gives required formula. *q.e.d.*

Proof for case p = 3. Define δ's as above, and once again note that there are $\alpha_1 - \alpha_0 + 1$ relations between them. Then again take the α_2 perimeters Π of 2-faces, and definition of the b_1 cycles C" of edges of P is also almost similar. Only this time $a_i a_{i+1}$ represent crossings of 3-faces of P.

Once again it is true that these $\alpha_2 + b_1$ cycles Π and C" generate all the relations amongst the δ's : we argue as before only it takes more effort now to see that once again L bounds a sum of 2-faces of P. We start with *any* R with L as boundary. Its portions r in each 3-cell are then deformed to homologous surfaces of their faces which is a sum of some complete 2-cells and some portions. In this R' these "portions" occur twice with opposite orientation. Suppressing them we get R" which is a sum of 2-cells of P and bounds L.

We now turn to the linear independence of these $\alpha_2 + b_1$ relations ε = 0 between the δ's. If there is a dependence relation, the C"s can't be involved in it, because then they won't be homologically independent. Amongst the Π's there are now many relations. For example the α_3 *boundaries* Φ of the 3-cells each give a relation, viz. that the sum of the Π's of its 2-faces is zero. Then as above we can make b_2 closed homologically independent (pseudo) 2-manifolds D", made up exclusively from 2-cells of P, and we have b_2 relations ε = 0 corresponding to the sums of the perimeters Π of the 2-cells of each D". Duplicating above argument it follows that any ε = 0 is a linear combination of these $\alpha_3 + b_2$ relations. Besides the obvious dependence amongst these, viz. that the sum of the Φ's is zero, there is no other. So in all we have $\alpha_3 + b_2 - 1$ dependencies.

Correcting for them we see that the number of linearly independent relations amongst the δ's is $\alpha_2 + b_1 - (\alpha_3 + b_2 - 1)$. Equating it with the old value $\alpha_1 - \alpha_0 + 1$, we get the required formula $\alpha_3 - \alpha_2 + \alpha_1 - \alpha_0 = 1 - b_2 + b_1 - 1$ (which in turn is zero because of Poincaré duality $b_1 = b_2$. *q.e.d.*

The cases $p \geq 3$ are proved by exactly similar argument.

[Poincaré was simultaneously trying to prove three things :

(1) That the new, and apparently dependent on P, definition of Betti numbers, is in fact *independent* of P.

(2) That this new definition gives the *same* numbers b_i as the old definition of § 6.

(3) That, *for any fixed* P, the alternating sum of the face numbers α_i equals the alternating sum of the (newly defined) Betti numbers b_i of P.

Of these, his sketch for the *algebraical result* (3) was essentially complete, though fancier and more and more general variations of this important fact have been given later by **Hopf**, **Lefschetz**, and many others.

A full proof of the *invariance theorem* (1) was given about 20 years later by **Alexander**.

A proof that the new P-dependent (Betti numbers, and even) homology groups (which are known to coincide with "singular" or "Cech" homology groups) are the same as the *geometric homology groups* (implicit in § 6 of this paper) has apparently not appeared in print yet, though the book by **Buoncristiano-Rourke-Sanderson** does contain a very similar result.]

CHAPTER III

SUR LES NOMBRES DE BETTI

C.R.de l'Acad. d. Sci., 128 (1899), 629-630

The duality theorem of *"Analysis Situs"* has been termed inexact by **Heegaard**. However an example given by Heegaard (as well as an example in *"Analysis Situs"* itself) *only* shows this provided one uses numbers as defined by using the first of the following two definitions.

The qth Betti number of V *is the maximum number of distinct closed oriented closed* q-*manifolds* ν_q *of* V, where
(1) for **Betti**, the ν_q's were "distinct" if no oriented (q+1)-manifold ω has boundary $\partial(\omega)$ equal to the union of some ν_q's, while
(2) in *"Analysis Situs"*, the ν_q's were deemed "distinct" if no oriented (q+1)-manifold ω has boundary $\partial(\omega)$ equal to the union of some ν_q's *with repetitions of* ν_q*'s allowed.*

With this second definition, i.e. *as stated in "Analysis Situs", the duality theorem is true* : a new polyhedral proof of this will be given in a longer paper.

CHAPTER IV

COMPLEMENT A L'ANALYSIS SITUS

Rend. d. Cir. Math. di Pal., 13 (1899), 285-343

§ I. **Introduction.** This paper was written in response to **Heegaard**'s 1898 criticism of Poincaré's (and **Picard**'s) duality theorem, viz. *for any closed orientable* m*-manifold* V *the* q*th and* (m–q)*th Betti numbers are equal.*

Example 3 of "Analysis Situs, i.e. $S^3/8$, has, as we saw, 1-manifolds C which don't bound (and whose multiple 4C bounds) while it can be shown (although Poincaré did'nt actually do it) that all the oriented 2-manifolds in $S^3/8$ do bound. (An example given by Heegaard also shows the same.) Thus *duality is false with Betti's original definition of his numbers.* (see §4 of "Sec Complém.")

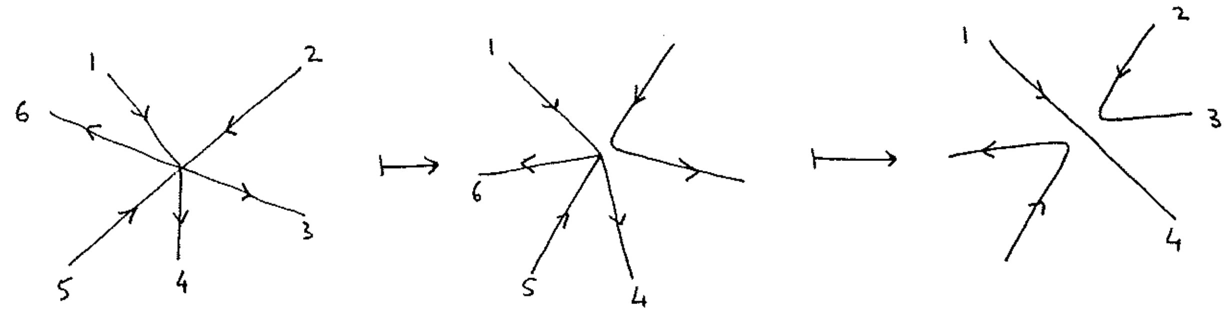
The corrected definition of **Betti numbers**, as given in "Analysis Situs", was the following :

$b_q(V)$ was defined as the maximal number of independent closed (oriented) q-manifolds of V, where by **independent** we mean that there is no non-trivial homology between these manifolds, where in turn by a **homology** we meant a relation, written as

$$\nu_1 + \dots + \nu_t \simeq 0,$$

and signifying that the left side (in which the q-manifolds ν_i might repeat) is the codimension one boundary $\partial(\omega)$ of an (oriented) (q+1)-manifold.

[Here an (oriented) q-manifold (or a **q-chain**) c of V is being called **closed** iff its codimension one oriented boundary $\partial(c)$ is zero (in modern parlance one might call c a **q-cycle** of V). This (smooth) c is usually not compact, while its closure $\bar{c}$, which is compact, is usually not smooth : it is usually not even a (closed) **pseudomanifold**, i.e. it can have codimension one singularities. However by **resolving** these, in the manner shown below, we can obtain a *nearby* pseudomanifold :



In § XI Poincaré will attempt to show that given any q-cycle c we can find a cell subdivision in which c becomes cellular : such *triangulability* questions are still of great interest.

But most later treatments of invariance of homology simply evade triangulability by experimenting with different definitions of "q-chain": e.g. an obvious way to avoid § XI would be to call c ⊆ V a "q-chain of V" iff it is a (cellular) chain of some cell subdivision.

Such "evasive" strategies, of which perhaps **Eilenberg**'s idea of *singular* "q-chains" was the most far-reaching, have now extended the applicability of homology theory far beyond smooth manifolds.]

As before, we'll add and subtract homologies, as well as multiply them by integers, in the obvious ways. Besides, Poincaré now makes the explicit *convention* that they will also be divided by nonzero integers, i.e. he uses **homologies over** $\mathbb{Q}$ (: to make this clearer one can use $2\nu_1 + 3\nu_2 \simeq_{\mathbb{Q}} 0$ to denote that some integral multiple $2r\nu_1 + 3r.\nu_2$ bounds, while the old $2\nu_1 + 3\nu_2 \simeq 0$ still means that $2\nu_1 + 3\nu_2$ itself bounds).

[Betti's and Poincaré's numbers were *one more* than the numbers above : this was so because they arose as generalizations of **Riemann's connectivity of a surface** S, i.e. the number of closed cuts required to disconnect S, and this is one more than the modern $b_1(S)$ used above.]

At this point Poincaré answers (see notes of "Analysis Situs") the two specific criticisms of Heegaard of his original *intersection theoretic* *proof of duality* : he accepts one of these, and also remarks that his old flawed "proof" apparently "works" even with Betti's definition !

In the following Poincaré will give a new *combinatorial proof of duality*, based on a *new definition of Betti numbers* which (à priori) depends on a cell subdivision, but will be shown *independent* of the subdivision and *same* as the definition recalled above : so he will (attempt to) carry out the entire programme which we outlined in remarks to § 18 before !

§ II. **Incidence numbers** ε^q_{ij} of a cell subdivision P (of a closed oriented m-manifold V) having, for each $0 \le q \le m$, the oriented q-cells $\nu_q = a^q_1, \dots, a^q_{\alpha_q}$ (m-cells having orientation of V) are defined thus :

$$\varepsilon^q_{ij} = \begin{cases} 0 \text{ if } a^{q-1}_j \text{ is not on } \partial(a^q_i), \text{ and} \\ +1 \text{ or } -1 \text{ if } a^{q-1}_j \text{ is on } \partial(a^q_i) \text{ with same or opposite orientation} \end{cases}$$

(besides we'll make the conventions $\varepsilon^{m+1}_{1j} = 1$ and $\varepsilon^0_{i1} = 1$). These **schema**, i.e incidence matrices ε^q (of 0's, +1's and -1's), are not arbitrary but satisfy the following necessary conditions.

[Poincaré only requires that ν_q's be "simply connected" but, as noted before, he probably *intended* "cells" (at some other places, his more general "simply connected" needs to replaced by "n-sphere, $n \ge 2$").]

Proposition 1. *For a fixed* i *and* k *the products* $\varepsilon^q_{ij}\varepsilon^{q-1}_{jk}$ *are either all zero, or else all but two are zero, these being* +1 *and* -1. *Hence we always have*

$$\varepsilon^q\varepsilon^{q-1} = 0.$$

Proof. For $q = m+1$ the result follows because there are precisely two m-cells a^m_j incident to a given a^{m-1}_k, and for $q = 1$ the result follows because each edge a^1_i has precisely two vertices.

For other values of q the result follows because the product $\varepsilon^q_{ij}\varepsilon^{q-1}_{jk}$ is clearly zero unless a^q_i has a^{q-2}_k on its boundary, in which case precisely two (q-1)-cells of the (q-1)-sphere $\partial(a^q_i)$ have a^{q-2}_k on their boundary. Hence $\sum_j\varepsilon^q_{ij}\varepsilon^{q-1}_{jk} = 0 \ \forall$ i, k, i.e. the product matrix $\varepsilon^q\varepsilon^{q-1} = 0$. *q.e.d.*

[*A new homology ?* Let $G = \{g_1, g_2, \dots, g_{|G|}$ be any finite Abelian group, with its elements totally ordered in some way, and suppose also that a non-trivial character κ of G is given. Now define the boundary $\partial_\kappa(v_1, v_2, v_3 \dots)$ of any sequence of vertices by $\sum_i \kappa(g_j,\ j = i \bmod |G|)\ (v_1, v_2, \dots, \hat{v}_i, \dots)$. In other words we have just imitated the definition of ordinary boundary which corresponds to the case $G = \mathbb{Z}/2\mathbb{Z}$

when of course there is just one non-trivial character. In the general case it can be checked that *the* $|G|$*th power of* ∂_κ *is zero*, so the groups $\ker(\partial_\kappa{}^p)/\mathrm{im}(\partial_\kappa{}^q)$ are defined whenever $p+q = |G|$. We will give some results regarding these (apparently new ?) homology groups later.]

Exercise. On the boundary V of a **simplex** in n-space, defined by $x_1 = 0$, ... , $x_n = 0$, $x_1 + \ldots + x_n = 0$, let a^q_j denote the i-cell obtained by omitting some q+1 of these n+1 equations, say the α_1th, ... , α_{q+1} th equations, and equip it with the orientation prescribed by the order of the remaining equations. Then

$$\varepsilon^{q+1}_{ij} = \mathrm{sgn}\ (\beta - \alpha_1)\ \ldots\ (\beta - \alpha_{q+1}),$$

if a^{q+1}_i results by omitting in addition the βth equation. Using this Poincaré verifies Prop. 1 for this cell subdivision P of V.

[To put the formula of Prop. 1 in its modern garb let C_q denote the (integral or rational) span of the q-cells, and define the **boundary operator** $\partial : C_q \longrightarrow C_{q-1}$ by $\partial(a^q_i) = \sum_j \varepsilon^q_{ij} a^{q-1}_j$: then above formula says $\partial \circ \partial = 0$: *"boundaries have no boundaries"*. Simple as it is, this is arguably the *"most important formula of this century"* : one defines homology groups $(\ker\partial)/(\mathrm{im}\partial)$ starting from it, and it is true that most of the famous results of this century have involved homology.]

We note that $\partial : C_q \longrightarrow C_{q-1}$ is explicit in this paper, however Poincaré uses the notation $c \equiv f$, $c \in C_q$, $f \in C_{q-1}$, i.e. a **congruence involving q-cells and** (q-1)**-cells**, instead of the modern $\partial(c) = f$. So $c \equiv 0$, $c \in C_q$, a **congruence involving only q-cells**, means that that c is a **cycle**, i.e. that $\partial(c) = 0$; and more generally $c \equiv f$, with $c \in C_q$ *and* $f \in C_q$ means $c - f \equiv 0$, i.e. that c-f is a cycle.

The conditions of Prop. 1 *are by no means sufficient to characterize incidence matrices which can arise from cell subdivisions of closed manifolds* :

e.g. as noted before the (open) **star** (= **aster**) of any cell, i.e. the union of all cells containing it, must be itself an open cell.

Poincaré poses the problem of combinatorially characterizing the set of all incidence matrices belonging to a given manifold V : we note that a later theorem of **Newman** does give a description of this set provided we know *one* cell subdivision of V.

Poincaré also asks if two manifolds having the same incidence matrices are diffeomorphic : this seems to have a positive answer since the diffeomorphism can be defined following the manner in which the cells fit each other.

§ III. **Betti number** $b_q(P)$ **of a cell subdivision** P of a manifold V is the maximum number of independent closed oriented q-dimensional manifolds (= q-cycles) of V, *which are made up of cells of* P.

Note that *Poincaré is not requiring the homologies to be cellular* : $b_q(P)$ has been defined just like $b_q = b_q(V)$ except for the last condition. So we have $b_q \geq b_q(P)$).

Proposition 2 (EULER-POINCARE FORMULA). *Let* P *is a cell subdivision, having* α_q *q-cells,* $0 \leq q \leq m$, *of an oriented* m-*manifold. Then*

$$b_m(P) - b_{m-1}(P) + b_{m-2}(P) - \dots = \alpha_m(P) - \alpha_{m-1}(P) + \alpha_{m-2}(P) - \dots .$$

Proof. The set of cellular q-cycles consists of all $c \in C_q(P)$ such that $\partial(c) = 0$ (or $c \equiv 0$).

If $c \in C_q(P)$ is such that $\partial(f) = c$ for some $f \in C_{q+1}(P)$ then c (which is closed by Prop. 1) is obviously homologous to 0. In § VI we'll prove the following converse :

If a cellular q-*cycle bounds some* (q+1)-*manifold, then it also bounds a cellular* (q+1)-*manifold.*

It thus follows that the image of $\partial : C_{q+1}(P) \longrightarrow C_q(P)$ constitutes *all* the homologies $c \simeq 0$ possible between cellular closed q-manifolds of V.

Poincaré now defines α_q' = "number of q-cells of P distinct upto congruence between q-cells", and α_q'' = "number of q-cells of P distinct upto homology", i.e. $\alpha_q' = \dim(C_q/\ker\partial)$ and $\alpha_q'' = \dim(C_q/\mathrm{im}\partial)$.

So $\alpha_q - \alpha_q'' = \dim(\mathrm{im}\partial) = \alpha_{q+1}'$ and $b_q(P) = \alpha_q'' - \alpha_q'$ (also check $\alpha_m = \alpha_m' + 1$ and $\alpha_0'' = 1$) and the required formula follows at once by calculating the alternating sum of the equations $\alpha_q = \alpha_{q+1}' + \alpha_q''$. *q.e.d.*

§ IV. **Subdivision map** C(P) $\longrightarrow$ C(P') is defined by

$$a^q_j \longmapsto \sum_k B(q,q,j,k)$$

(Poincaré uses = in place of $\longmapsto$), where P' is a (cell) **subdivision of a subdivision** P of some manifold, and B(q,h,j.k) = *the* k*th of the* q-*cells of* P' *which* **belongs** to *the* j*th* h-*cell of* P (i.e. *not* contained in any lower dimensional cell of P : note that this cell of P is uniquely determined by the cell of P'), and the cells B(q,q,j,k) are assumed oriented compatibly with the q-cell of P to which they belong.

Proposition 3. *Above linear map* C(P) $\longrightarrow$ C(P') *is a* **chain map,** *i.e. it commutes with the boundary operators of* P *and* P'.

In the paper above statement, whose proof is obvious, is formulated of course in terms of congruences.

The necessary condition of Prop. 3 is by no means sufficient, i.e. we can have a surjective chain map C(P) $\longrightarrow$ C(Q), imaging each q-cell to a union of q-cells, without Q being isomorphic to a subdivision of P (e.g. take P = a triangulated 2-sphere, and Q = P \ {a triangle} ∪ {a 2-torus minus a triangle}).

§ V. **Proposition 4** (INVARIANCE THEOREM). *For any cell subdivision* P *of a closed oriented* m*-manifold* V *we have* $b_q(P) = b_q$ *for all* $0 \le q \le m$.

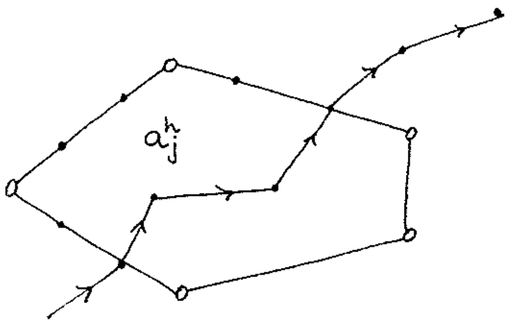
Proof. Consider any cell subdivision P' finer than P and let $\sum \alpha. B(q,h,j,k) \equiv 0$ be a q-cycle of P'. Take a maximum h occuring in it, and let $S\alpha. B(q,h,j,k)$ denote the *portion* of this cycle contained in a maximum dimensional a_j^h.

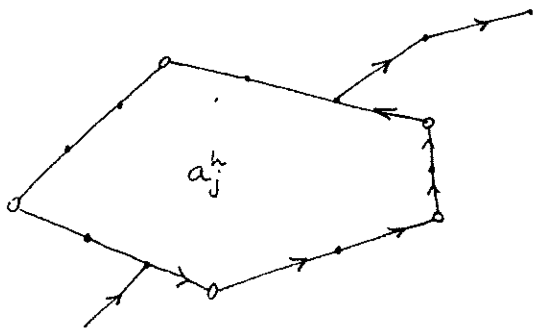
The boundary of the cycle's portion contained in a_j^h *must be contained in the boundary of* a_j^h. This follows from the fact that $\partial(S\alpha. B(q,h,j,k))$ equals the boundary of $\sum\alpha. B(q,h,j,k) \setminus S\alpha. B(q,h,j,k)$, which is contained in some cells of P of dimensions $\le h$ and other than a_j^h.

If $h > q$ the (q-1)-cycle $\partial(S\ \alpha. B(q,h,j,k))$ of the (h-1)-sphere $\partial(a_j^h)$ bounds a q-manifold of this sphere. So, *by using the result to be proved in* § VI *below*, it also bounds a q-chain of P' lying on this sphere.

[Note that the triviality of the requisite Betti number of the sphere in fact follows if we inductively assume the theorem in dimensions less than m and use the result of § VI.]

If we replace the portion $S\alpha. B(q,h,j,k)$ of $\sum\alpha. B(q,h,j,k)$ by this q-chain we get a new q-cycle of P' homologous to $\sum\alpha. B(q,h,j,k)$. By repeating this construction a finite number of times we see that *our* q*-cycle is homologous to one with all* h*'s equal to* q.





For a q*-cycle of the type* $\sum \alpha. B(q,q,j,k) \equiv 0$, *the coefficients* α, *of the portion* $S\alpha. B(q,q,j,k) \equiv 0$ *contained in an* a_j^q, *are all equal to each* *other.* This follows from the fact that the boundary of this portion has to be on the boundary of a_j^q, and any two q-cells of P' belonging to this q-cell of P can be joined to each other by a sequence of such q-cells, each sharing a (q-1)-face with the preceding.

Thus the original q-cycle of P' has been shown homologous to the chain subdivision of a q-cycle $\sum \alpha_j a_j^q$ of P. Thus we have shown $b_q(P') \le b_q(P)$. Since $b_q(P') \ge b_q(P)$ is obvious we obtain $b_q(P') = b_q(P)$.

So *the numbers* $b_q(P)$ *do not depend on the cell subdivision* P. This follows because any two cell subdivisions P and Q have a common finer cell subdivision P', and thus $b_q(P) = b_q(P') = b_q(Q)$.

Given any t *homologously independent closed* q-*manifolds of* V *we will show in* § XI *below that there exists a cell subdivision* Q *of* V *with respect to which these* t *manifolds become cellular.* So it follows that $t \le b_q(Q) = b_q(P)$. So $b_q \le b_q(P)$. Since $b_q \ge b_q(P)$ is obvious this gives the required $b_q = b_q(P)$. *q.e.d.*

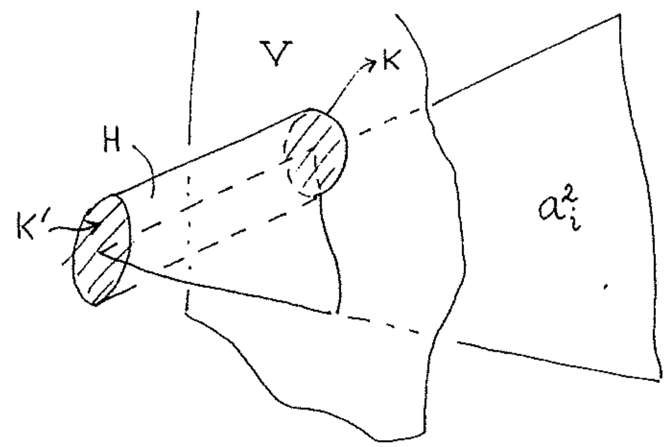
§ **VI.** The following was used in §§ III, V.

Proposition 5 (CELLULAR HOMOLOGIES SUFFICE). *If a cellular q-cycle bounds some (q+1)-manifold, then it also bounds a cellular (q+1)-manifold.*

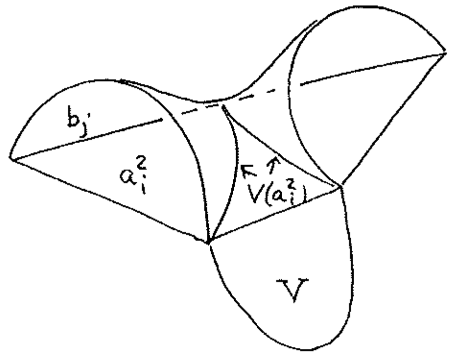
Proof (for a cell subdivision P of a 3-manifold W only). If a cellular 2-cycle $\sum b_j^2$ is the boundary of some 3-manifold $V \subseteq W$, then V must be *already* cellular. This follows because, W being a 3-manifold, the set-theoretic boundary bd(V) equals $\sum b_j^2$, and if a 3-cell had points of V and also of its complement then it would also have a point of bd(V).

Let us now consider the case of a cellular 1-cycle $\sum b_j^1$ which is the boundary of a 2-manifold $V \subseteq W$. Without loss of generality we can assume that this V is in *general position* with respect to P. We will now show how to modify V so that it becomes cellular.

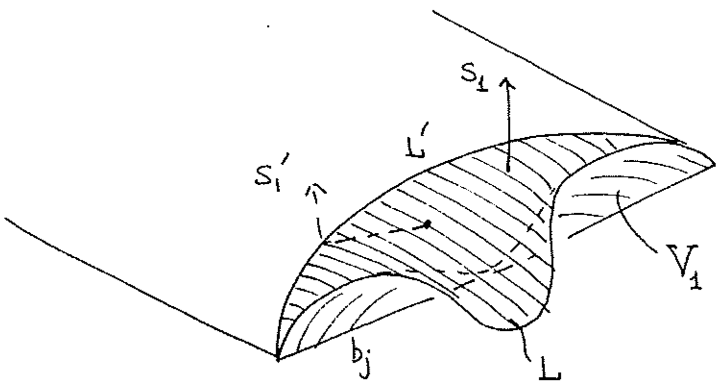
If the portion $V(a_i^2)$ of V contained in it consists of a finite number of arcs meeting the boundary of the 2-cell in finitely many points not situated on any b_j, then by replacing V by V' = V – K + H + K' (see fig. below) we can ensure that any such point coincide with a vertex of the 2-cell :



For an a_i^2 having some b_j's as edges, it may still happen that $V(a_i^2)$ consists of these b_j's, and a finite number of arcs meeting the boundary of the 2-cell in finitely many **nodal points** situated within these b_j's :



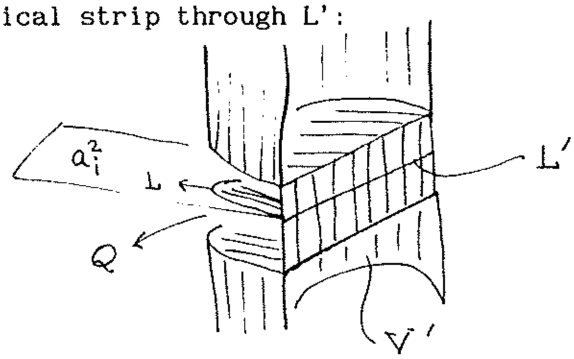
To move a nodal point, situated on an edge b_j of a_i^2 to a vertex of a_i^2 we proceed as follows. On V draw an arc L very near to b_j but meeting it only at its two endpoints. Let S_1 be the surface obtained by "rotating" this arc to a similar arc L' on another 2-cell of P through b_j. The construction $V \longmapsto V' = V - V_1 + S_1 + S_1'$ (see below) now either removes the nodal point of a_i^2 or moves it to a vertex of the 2-cell :



This follows because if, for V', there were a nodal point of a_i^2 on b_j then there must be an arc on $V' \cap a_i^2$ ending in this nodal point but this is impossible because this arc is eventually in S_1' which shares only b_j with a_i^2.

[Note that this new V' is *not* in general position with respect to P since it shares the region S_1' with a 2-cell. However by perturbing it slightly we can ensure this, and also that no new intersections with edges of P are created.]

Our *new* V cuts each 2-cell in arcs L having end points on its vertices. To move each such L to the boundary of the 2-cell choose a 2-cell $Q \subseteq a_i^2$ whose boundary is L plus an arc L' on the boundary of a_i^2 joining the end points of L. Then delete a small "vertical" strip of V passing through L, and replace it by two cells like Q situated "above" and "below" it, plus a new vertical strip through L':



For the new V thus obtained the portion $V(a_i^3)$ of V in each 3-cell bounds a 1-cycle made of the edges of the 3-cell. We now replace $V(a_i^3)$ by a 2-chain of $\partial(a_i^3)$ which bounds this 1-cycle. Doing this for each 3-cell we finally obtain the required cellular V which has the same boundary as the original V. *q.e.d.*

Note : *for the sake of simplicity Poincaré works out much of the following also for the case of a closed oriented 3-manifold only.*

§ VII. The reciprocal (or dual) polyhedron P^*.

The derived P' of a cell subdivision P is the cell subdivision obtained by subdividing the cells of P, in any order in which their dimensions are non decreasing, by coning the already subdivided boundary of each cell a_i^r over an interior point $P(a_i^r)$.

[A cell subdivision P is a **poset** (= partially ordered set) under the relation "is a face of" and its derived P' consists of all simplices $\{P(a_i^r), P(a_j^s), P(a_k^t), \ldots\}$ where $a_i^r < a_j^s < a_k^t < \ldots$.

More generally the derived P' of *any* poset P is the (abstract) **simplicial complex** (= *a finite set of finite sets closed under* $\subseteq$) of all its totally ordered subsets $\{a_i, a_j, a_k \ldots\}$, $a_i < a_j < a_k < \ldots$ (and this P' can be visualized geometrically by representing its vertices by suitable points $P(a_i)$ of some euclidean space).

We note that *the Betti numbers etc. of any poset are defined to be those of its derived simplicial complex* : this agrees with the old definition for cell complexes P because $b_q(P) = b_q(P')$ (a very simple case of the invariance theorem of which Poincaré will give new proofs below).]

To each cell a_i^{3-r} (remember that Poincaré is working in a 3-manifold) of P we associate the **dual cell** $b_i^r \subseteq P'$ as follows : $b_i^0 = P(a_i^3)$ and b_i^r is the cone over $P(a_i^{3-r})$ of the union of all b_j^s corresponding to cells a_j^{3-s} of P which contain a_i^{3-r} on their boundary.

The fact that b_i^r *is an* r-*cell is equivalent to saying that the* **link** *of* a_i^{3-r} *is an* (r-1)-*sphere* : this follows because the aforementioned union of the b_j^s's is homeomorphic to this link.

[Also note that given any poset P and an element p thereof, *there are two important kinds of subcomplexes of the derived* P', viz. the ones which consists of all simplices $\{a_i, a_j, a_k \ldots\}$ with $p \geq a_i > a_j > a_k > \ldots$, and the others which consists of all simplices $\{a_i, a_j, a_k \ldots\}$ with $p \leq a_i < a_j < a_k < \ldots$.

When P is a cell subdivision of a manifold then these subcomplexes of P' cover the (closed) cells of P and their duals. As Poincaré points out later, *everything pertaining to duality holds for all posets* P *for which all subcomplexes of* P' *of the above kind are cells.*]

There is a unique way of *orienting the dual cells* such that the orientation of each a_i^{3-r} followed by that of its dual b_i^r agrees with the given orientation of the 3-manifold V. We will denote by P^* the cell subdivision comprising of these oriented dual cells b_i^r.

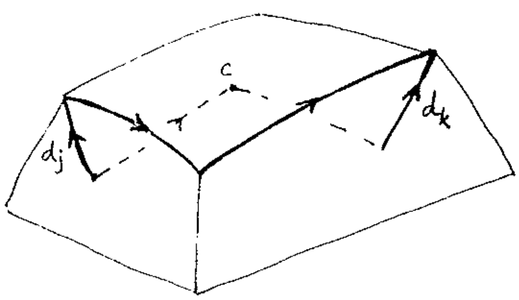
We observe that the 1-1 onto dual cell correspondence $P \longleftrightarrow P^*$ is *order-reversing*, i.e. that a_i^{3-r} is a face of a_j^{3-s} if and only if b_j^s is a face of b_i^r, and further that the *incidence matrices of* P^* *are the transposed incidence matrices of* P :

$$\varepsilon_{ij}^r(P^*) = \varepsilon_{ji}^{4-r}(P).$$

[So the poset P^* is isomorphic to the **opposite poset** of P, i.e. the same set with the partial order reversed. Note also that any poset and its opposite have the same derived, and so the same Betti numbers.

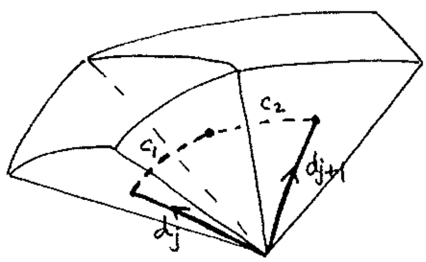
Also note 4-r in place of 3-r in above formula : this amounts to saying that the dual of the boundary operator $\partial : C_r(P^*) \longrightarrow C_{r-1}(P^*)$ is the **coboundary operator** $\delta : C^{3-r}(P) \longrightarrow C^{4-r}(P)$ defined by $(\delta f)(a_j^{4-r}) = f(\partial a_j^{4-r})$. (Here $C^q(P)$ denotes the group of all **q-cochains** of P, i.e functions from q-cells of P to integers or rationals.)]

A direct proof of $b_1(P^*) = b_1(P)$: Start with a 1-cycle $\sum b_i^1$ of P^*. By "rotating" it a bit we can write it as $\sum c_i$ where each c_i denotes a 1-cell in P' obtained by joining the barycentre of a 3-cell to those of two of its faces. We now use the fact that for each such c one has a homology $c \simeq d_j - d_k + A$ where A is an edge path on the boundary of the 3-cell of P to which c belongs :



Inserting such homologies we see that our $\sum b_i^1 = \sum c_i$ is homologous to a a 1-cycle $\sum A_k$ of P. So $b_1(P^*) \le b_1(P)$.

To see the opposite inequality start with a 1-cycle of P and write it as $\sum A_k$ where each A_k denotes an edge path on the boundary of a 3-cell. Replace each such A by a homologous $c - d_j + d_k$. Next note that each $d_j - d_{j+1}$ (coming from two 2-cells having a common vertex) is homologous to a sum of some c's (belonging to a sequence of 3-cells in the star of this vertex which connect the two 2-cells) :



So the given 1-cycle of P has been shown homologous to a sum of some c's. Rotating it a little we see that it is a 1-cycle $\sum b_i^1$ of P^*. Thus $b_1(P) \le b_1(P^*)$ and so $b_1(P) = b_1(P^*)$. *q.e.d.*

The above proof generalizes easily to give $b_q(P^*) = b_q(P)$ for all q (and in § X, Poincaré will give yet another direct proof of this).

[A proof simpler than above is to check that the Betti numbers are invariant under a single **elementary stellar subdivision** (i.e. putting a new vertex inside just one cell and coning) :

The result $b_q(P^*) = b_q(P)$ follows from this verification because P^* and P have the same derived, and a derived is a sequence of elementary stellar subdivisions.

Poincaré remarks also that $b_q(P^*) = b_q(P)$, together with the use of special kinds of triangulations, leads to a *simpler* proof of invariance, however this seems uncertain.]

§ VIII. **Reduction of incidence matrices.** By **arithmetical equivalence** we will understand the equivalence relation, in the set of all integer matrices, which is generated by the operations : *"add to some row (resp. column) some other row (resp. column)"*.

Proposition 6. *Any integral* $p \times q$ *matrix* A *is arithmetically equivalent to another integral* $p \times q$ *matrix* $H = [h_{ij}]$ *such that* $h_{ij} = 0$ *for* $j > i$, *or* $i > p$, *and also for* $i < j$ *provided* h_{jj} *and* h_{ii} *are relatively prime.*

Since the set of divisors of all $i \times i$ minors is invariant under above operations, the size of the largest nonzero minor of A, i.e. its **rank** r, must then be the same as the number of nonzero h_{ii}'s, and further, Δ_r = **greatest common divisor of all** $r \times r$ **minors** of $A = \prod \{|h_{ii}| : h_{ii} \neq 0\}$.

It appears that Poincaré had the following algorithm $A \longmapsto H$ in mind for the proof of the above *"bien connu"* result.

Proof. Amongst our **arithmetical operations** (generated by the ones mentioned above) we have all *signed transpositions* (i.e. interchange of two rows or columns with a change of sign of one of them) :

e.g. by adding the first row to the second $\begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ gives $\begin{bmatrix} R_1 \\ R_2+R_1 \end{bmatrix}$, which in turn is congruent to $\begin{bmatrix} -R_2 \\ R_2 + R_1 \end{bmatrix}$ since it is obtainable from it by adding the second row to the first, and finally, by adding the first row of this last matrix to its second row, we get $\begin{bmatrix} -R_2 \\ R_1 \end{bmatrix}$.

Using these transpositions we make the first row nonzero with its *smallest nonzero element* (in absolute value) at (1,1). In case it does not divide some other element of the row we add to the column of this element a suitable multiple of the first column to reduce the size of the smallest nonzero element of the row. Then bringing it to (1,1) we check again for divisibility. Eventually it will divide all other elements of this row and so they can be made all zero.

Next use transpositions to ensure second row nonzero and that the element at (2,2) is smaller than all elements to the right of it in the row. As above we test whether it divides them, ... , and so on.

Clearly this accomplishes the required triangularization. In case h_{ij}, $i < j$, is nonzero, with h_{jj} and h_{ii} relatively prime, we can make it zero by adding to the ith row a times the jth row and to the jth column b times the ith column, where the integers a and b are chosen so that $ah_{jj} + bh_{ii} = -h_{ij}$. (In particular if some $h_{ii} = 1$ then all other elements of the ith row become zero.) *q.e.d.*

[Poincaré was thus unaware that *complete diagonalization* is always possible (see **H.J.S.Smith,** *Phil. Trans. Roy. Soc.*, 151 (1861), 293-326 = *Collected Works, vol.*1, 367-409). However he rediscovered this fact later and gave a proof of it in § 2 of *Second Complément* (without any mention of Smith). We give below an algorithm for diagonalization which only involves a slight finesse in the one used above.

Proposition 6' (SMITH NORMAL FORM). *Any integral* p×q *matrix* A *is aritmetically equivalent to a unique* p×q *matrix of the type*

$$\begin{bmatrix} \mathrm{diag}(d_1,\ldots,d_r) & 0 \\ 0 & 0 \end{bmatrix},$$

where the **elementary divisors** d_i*'s are, with the possible exception of* d_r *for the case* $r = p = q$, *all positive, and such that each* d_i *divides* d_{i+1}.

Proof. Let m denote *the minimum absolute value* of (all) the nonzero elements of $A \neq 0$. We assert that A is *equivalent to a matrix whose* m *divides all elements.* Since an A with m = 1 is such, we see that it would suffice to check that, *if* m *does not have this divisibility property, then* A *is equivalent to a matrix with a smaller* m.

To do this bring ±m to the (1,1) spot by using above signed transpositions. *In case* m does not divide some other element in the first row (resp. column) we can reduce the absolute value of this element below m by adding to its column (resp. row) a suitable nonzero integral multiple of the first column (resp. row), this being clearly a permitted operation. *Otherwise*, we make all other elements of the first row and column zero by these operations, and then choose some other element which m does'nt divide : there must still be one, or else there was no such element to start with. Adding its row to the first row we bring it to the first row, and then reduce its absolute value below m as before.

The assertion being established we can now work with an A whose m divides all elements. We bring ±m to the (1,1) spot and make all other elements of the first row and column zero. Then we repeat the whole process for the matrix obtained by omitting the first row and first column ...

Thus we get an equivalent matrix of the required diagonal shape with the required successive divisibility of the diagonal elements. Only the requirements regarding their positivity need to be ensured.

If there is a row or column of zeros (this happens unless $r = p = q$) *we can change the sign of any row or column* by using two signed transpositions involving this line of zeros : so in this case we can make all the d_i's positive.

Even if there is no line of zeros by using two signed transpositions we can change the signs *of any pair* of diagonal elements. So we can ensure that *at most one* of these remains negative, and, in case this negative diagonal element is not already d_r, two more signed transpositions will ensure this also.

The uniqueness of the reduced matrix follows from the following characterization of its components :

(1) r is the *rank* of A, i.e. the largest i such that there is a nonzero ixi minor of A,
(2) each product $\Delta_i = d_1..d_i$ (or for $i = r$, $d_1..d_{r-1}|d_r|$) is the *highest common divisor of the* ixi *minors of* A, and

(3) d_r is positive unless $p = q = r$ and the *determinant* of A is

negative.

The above three statements follow because they are **invariant** under our operations, and true for the final matrix. *q.e.d.*

Thus a matrix is arithmetically equivalent to the identity matrix I_n *iff it belongs to* $SL(n,\mathbb{Z})$, *and this group is generated by the* $n^2 - n$ *matrices which have 1's on the diagonal and a single nonzero, and* $= 1$, *off-diagonal element* : this follows because pre and post multiplication by such *elementary matrices* is equivalent to the row and column operations which generate arithmetical equivalence. So, more generally, **a** $p\times q$ **matrix** A **of integers is arithmetically equivalent to** B **iff we can find** $P \in SL(p,\mathbb{Z})$ **and** $Q \in SL(q,\mathbb{Z})$ **such that** $PAQ = B$.

Identifying each $r\times t$ matrix M over $\mathbb{Z}$ with the $\mathbb{Z}$-linear map $\mathbb{Z}^r \longrightarrow \mathbb{Z}^t$ given by $x \longmapsto x.M$ we can reformulate yet again : *a group homomorphism* $A : \mathbb{Z}^p \longrightarrow \mathbb{Z}^q$ *is equivalent to* $B : \mathbb{Z}^p \longrightarrow \mathbb{Z}^q$ iff we can find sense preserving isomorphisms P and Q of $\mathbb{Z}^p$ and $\mathbb{Z}^q$ such that the diagram

$$\begin{array}{ccc} \mathbb{Z}^p & \xrightarrow{\ A\ } & \mathbb{Z}^q \\ P \big\uparrow \cong & & \cong \big\downarrow Q \\ \mathbb{Z}^p & \xrightarrow{\ B\ } & \mathbb{Z}^q \end{array}$$

commutes. This shows that the quotient groups $\mathbb{Z}^q/\text{im}(A)$ and $\mathbb{Z}^q/\text{im}(B)$ are isomorphic; so

$$\mathbb{Z}^q/\text{im}(A) \cong \oplus\{\mathbb{Z}/d_i\mathbb{Z} : |d_i| > 1\} \oplus \mathbb{Z}^{q-r},$$

from which **the fundamental theorem of finitely generated abelian groups**, i.e. that *these are direct sums of cyclic groups*, follows at once, because any such group generated by q elements is, by definition, isomorphic to a group of the type $\mathbb{Z}^q/\text{im}(A)$.

We note that if we also allow unsigned transpositions of rows or columns, i.e. if the above P and Q can be chosen from the bigger groups $GL(p,\mathbb{Z})$ and $GL(q,\mathbb{Z})$, then the Smith normal form has *all* $d_i > 0$.

Over $\mathbb{Q}$ the analogous notion of *algebraical* equivalence, i.e. $PAQ = B$ for some $P \in GL(p,\mathbb{Q})$ and $Q \in GL(q,\mathbb{Q})$, is still simpler, because now the normal form depends only on the rank.]

Just the algebraical invariant r suffices for the next result.

Proposition 7 (POINCARE DUALITY). *For any cell subdivision* P *of an* m*-manifold* V*, the qth Betti number satisfies*

$$b_q(P) = \alpha_q(P) - r(\varepsilon^q(P)) - r(\varepsilon^{q+1}(P)),$$

and thus coincides with the (m–q)*th Betti number of its dual* P^*.

So, using invariance theorem, we get $b_q(V) = b_{m-q}(V)$.

Proof. The formula follows at once because rank of the matrix ε^q (resp. ε^{q+1}) coincides with dimension of the image of the corresponding $\mathbb{Q}$-linear map $\partial_q\colon C_q \to C_{q-1}$ (resp. $\partial_{q+1}\colon C_{q+1} \to C_q$) and so its kernel has dimension $\alpha_q - r(\varepsilon^q)$.

So we have also $b_{m-q}(P^*) = \alpha_{m-q}(P^*) - r(\varepsilon^{m-q}(P^*)) - r(\varepsilon^{m-q+1}(P^*))$. But we know $\alpha_q(P) = \alpha_{m-q}(P^*)$, $(\varepsilon^q(P))^* = \varepsilon^{m-q+1}(P^*)$ and $((\varepsilon^{q+1}(P))^* = \varepsilon^{m-q}(P^*)$. So we have $b_q(P) = b_{m-q}(P^*)$. *q.e.d.*

[We note that, though it amounts to the above argument, the above formula is obtained in the paper as a corollary of an **algebraical version** of Prop.8 below, i.e. while reducing the following tableaux to their reduced form, one allows multiplication of a row or column by a nonzero rational : this enables one to make all $d_i = 1$.].

AN ALGORITHM FOR COMPUTING HOMOLOGY

Poincaré uses arithmetical operations on the following *initial* **qth tableaux of a polyhedron** (a matrix of size $(\alpha_q+\alpha_{q-1})\times(\alpha_q+\alpha_{q+1})$ built from two successive incidence matrices),

$$\begin{bmatrix} I_{\alpha_q} & (\varepsilon^{q+1})^* \\ (\varepsilon^q)^* & 0 \end{bmatrix},$$

taking care to add to each row or column of the tableaux a row of column of the **same kind** (first or second) only. Using these he obtains a sequence of *intermediate* tableaux

$$\begin{bmatrix} t_{ij} & t_{ij'} \\ t_{i'j} & 0 \end{bmatrix},$$

ending in a tableaux, having $t_{ij'}$ and $t_{i'j}$ in normal form.

(1) *We will continue Poincaré's tableaux reduction a little further.* After having normalized $t_{ij'}$ and $t_{i'j}$ we will also normalize the *submatrix* t_{ab} of t_{ij} formed by its last $\alpha_q - r(\varepsilon^{q+1})$ rows and last $\alpha_q - r(\varepsilon^q)$ columns: since only these rows and columns are involved in this step, $t_{i'j}$ and $t_{ij'}$ remain normalized during this stage. We'll refer to the tableaux so obtained as the **reduced qth tableaux** of P.

(2) *Also for us each* "normalization" *means diagonalization* (see Prop. 6') à la Poincaré's *Second Complément*, instead of the weaker triangularization (see Prop. 6) with which Poincaré makes do (except for the trivial algebraical version) in this paper.

Poincaré **equips** each of the first α_q rows of these tableaux with a q-chain c^q, and the remaining α_{q-1} rows with a (q-1)-chain c^{q-1}, as follows :

The chains associated to the rows of the initial tableaux are

$$a_1^q, \ \ldots \ , a_{\alpha_q}^q, \ a_1^{q-1}, \ \ldots \ , a_{\alpha_{q-1}}^{q-1},$$

and, when we add to the row having the chain c, the row having the chain c', the chain c' is changed to c'− c.

With above refinements (1) and (2) Poincaré's tableaux reduction gives the following sharper result.

Proposition 8 (CANONICAL BASIS OF P). *The reduced qth tableaux of* P *furnishes the basis* c_i^q , $1 \le i \le \alpha_q - r(\varepsilon^q)$ *for the integral* q*-cycles of* P, *and the multiples* $d_i(\varepsilon^{q+1}).c_i^q$, $1 \le i \le r(\varepsilon^{q+1})$, *of these cycles, constitute a basis for the integral* q*-boundaries of* P.

Proof. Clearly c_i^q is always a basis of C_q. And, since $t_{ij} \in SL(\alpha_q, \mathbb{Z})$, we see that $\sum_i t_{ij}\, c_i^q$, $1 \le j \le \alpha_q$, is also a basis of C_q for all tableaux.

For any intermediate tableaux the following equations hold :

$$\partial(\textstyle\sum_i t_{ij}\, c_i^q) = \textstyle\sum_{i'} t_{i'j}\, c_{i'}^{q-1}.$$

To see this note that, for the initial tableaux, these are the defining equations of the boundary operator,

$$\partial(a_j^q) = \textstyle\sum_{i'} \varepsilon_{ji'}^q\, a_{i'}^{q-1}.$$

Furthermore, each elementary column operation simply adds one equation to another while each elementary row operation is simply a re-arrangement

$$t.c + t'.c' = (t+t').c + t'.(c'-c)$$

within one side of each equation.

Using above equations of ∂ for the reduced tableaux, we see that no nontrivial combination of the first $r(\varepsilon^q)$ members of this basis of C_q is a cycle. So the remaining ones, which are cycles, give a basis

$$\left\{\textstyle\sum_i t_{ij} c_i^q : r(\varepsilon^q) < j \le \alpha_q\right\} \qquad (1)$$

(where $1 \le i \le \alpha_q$) for the q-cycles of P.

We next check that, for each intermediate tableaux, all q-boundaries are linear combinations of the expressions $\sum_i t_{ij'} c_i^q$: this follows as before because, for the initial tableaux, these expressions are $\sum_i \varepsilon_{j'i}^{q+1}\ a_i^q =$

$\partial(a_{j}^{q+1})$, and so generate all q-boundaries.

For the reduced tableaux only the first $r(\varepsilon^{q+1})$ of these expressions are nonzero. Further being multiples, $d_i(\varepsilon^{q+1}).c_i^q$, of basis elements, they are linearly independent, and so constitute a basis for the q-boundaries of P.

We now (finally !) use $\varepsilon^{q+1}\varepsilon^q = 0$ (i.e. $\partial\circ\partial = 0$) to see that c_i^q's are q-cycles for at least $1 \le i \le r(\varepsilon^{q+1})$. Subtracting suitable combinations of these from the q-cycles of (1) we get cycles (1)' which are as in (1) except that the summation is now only over $r(\varepsilon^q) < i \le \alpha_q$, i.e. all coefficients are from the submatrix t_{ab} which we normalized. So the nonzero cycles (1)' are nonzero multiples of some c_i^q where $r(\varepsilon^{q+1}) < i \le g$. Since the cycles c_i^q, $1 \le i \le g$, thus found are linearly independent and generate the cycles of the basis (1) it follows that they form another basis of q-cycles (and so $g = \alpha_q - r(\varepsilon^q)$. *q.e.d.*

[Note that Prop. 8 shows that **the qth homology group** of P is given by

$$H_q(P) \cong \oplus\{\mathbb{Z}/d_i(\varepsilon^{q+1})\mathbb{Z} : d_i(\varepsilon^{q+1}) > 1\} \oplus \mathbb{Z}^{b_q},$$

where $b_q = \alpha_q - r(\varepsilon^q) - r(\varepsilon^{q+1})$ (other applications will be given elsewhere).

Thus the homology of P is an algorithmically computable invariant of the sequence of its incidence matrices.

It should be interesting to similarly study invariants of *other* sequences (or arrays) of matrices e^q (maybe even with negative "Betti numbers" $b_q = \alpha_q - r(e^q) - r(e^{q+1})$!) which somehow reflect (like the incidence sequence) the way in which the cells of P fit together ?]

§ IX. **The intersection number** of a q-chain $V_1 = \sum_i \alpha_i a_i^q$ of P (*a cell subdivision of an oriented closed* m-*manifold*) and an (m–q)-chain $V_2 = \sum_i \alpha_i' b_i^{m-q}$ of the dual subdivision P^* satisfies

$$I(V_1, V_2) = \pm \sum_i \alpha_i . \alpha_i'.$$

This follows because each q-cell of P intersects only the dual (m–q)-cell of P^* and that too just once and always with the same orientation.

Proposition 9. *It is possible to find a* q-*cycle* V_1 *in* P *such that* $I(V_1, V_2)$ *is nonzero if and only if* $V_2 \simeq_{\mathbb{Q}} 0$ *does not hold in* P^*.

Proposition 9' : *The above assertion is true even if* " in P " *and* " in P^* " *are omitted.*

(In *Analysis Situs* we saw that this implies the duality theorem.)

Proof. Note that $\partial(V_1) = 0$ in P is equivalent to the equations $\sum_i \alpha_i \varepsilon_{ij}^q(P) = 0$ which are same as the equations $\sum_i \alpha_i \varepsilon_{ji}^{m-q+1}(P^*) = 0$.

On the other hand $V_2 \simeq_{\mathbb{Q}} 0$ does not hold in P^* iff there are no $\zeta_i \in \mathbb{Q}$ such that $\sum_j \zeta_j \varepsilon_{ji}^{m-q+1}(P^*) = \alpha_i'$ for all i, i.e. iff the matrix $\varepsilon^{m-q+1}(P^*)$ has less rank than the augmented matrix $[\varepsilon^{m-q+1}(P^*) \;\; \alpha']$, i.e. iff the null space of the latter matrix has smaller dimension, i.e. iff we can find $\alpha_i \in \mathbb{Q}$ (and so also $\alpha_i \in \mathbb{Z}$) such that $\sum_i \alpha_i \varepsilon_{ji}^{m-q+1}(P^*) = 0$ holds but $\sum_i \alpha_i \alpha_i' = 0$ does not hold. This proves Prop. 9.

The "only if" of Prop. 9' was correctly proved in *Analysis Situs*. For "if" (for which an incorrect proof was given in *Analysis Situs)* we can use § XI below to find a cell subdivision P of the m-manifold in which the given nonbounding V_2 is cellular, and then use Prop. 9 to find a V_1 of P^* such that $I(V_1, V_2)$ is nonzero. *q.e.d.*

Proposition 10. *The qth Betti number of* P *coincides with the one defined as per Betti's original definition* (see § I) *iff the greatest common divisor of the largest sized nonzero minors of the* (q+1)*th* *incidence matrix of* P *is* 1.

[The following shows that **Betti's qth number** coincides with *the least number of generators of the qth homology group.*]

Proof. This follows at once from Prop. 8 : the cycles c_i^q , $1 \le i \le r(\varepsilon^{q+1})$ bound iff $d_i(\varepsilon^{q+1}) = 1$. So Betti's qth number is obtained by adding to $b_q(P)$ the number of i's such that $d_i(\varepsilon^{q+1})$ is bigger than 1, and coincides with b_q iff the product Δ_r of these d_i's is 1. *q.e.d.*

§ X. The following proof indicates an *"arithmetical"* (= combinatorial) argument for $b_q(P) = b_q(P^*)$ which applies also to some abstract schemas.

Proposition 11. *Let P be a cell subdivision of a closed oriented 3-manifold and P^* the dual subdivision. Then all homologies involving edges of P and P^* are linear combinations of homologies of the types*

$$b_i^1 \simeq a_k^0 b_j^0 - a_k^0 b_h^0 \quad \textit{and} \quad a_i^1 \simeq b_k^0 a_j^0 - b_k^0 a_h^0 ,$$

where $b_i^1 = b_h^0 b_j^0$ (resp. $a_i^1 = a_h^0 a_j^0$) is an edge of P^* (resp. P) and a_k^0 is a vertex of a_i^2 (resp. b_k^0 is vertex of b_i^2).

Proof. The boundary of any face of the 3-cell b_k^3 (resp. a_k^3) is a linear combination of homologies of the first (resp. second) type, so it follows that all homologies of P^* (resp. P) are also such combinations.

Since a 1-cycle of edges of both P and P^* is necessarily a sum of a 1-cycle of P and a 1-cycle of P^*, we only need to show that **any 1-cycle of P can be modified by means of above homologies to a 1-cycle of P^*** :

For this we partition the given 1-cycle into **edge paths**

$$\ldots a_h^0)(a_h^0 \ldots a_j^0)(a_j^0 \ldots a_t^0)(a_t^0 \ldots ,$$

lying on 3-cells $\ldots, a_k^3, a_m^3, \ldots$ of P respectively. Homologies of the second type change this cycle to

$$\ldots a_h^0)(a_h^0 b_k^0 a_j^0)(a_j^0 b_m^0 a_t^0)(a_t^0 \ldots ,$$

which can be repartitioned into edge paths as

$$\ldots a_h^0 b_k^0)(b_k^0 a_j^0 b_m^0)(b_m^0 a_t^0 \ldots. ,$$

and using homologies of the first type this changes to

$$\ldots b_k^0)(b_k^0 \ldots b_m^0)(b_m^0 \ldots. ,$$

where the edge paths are respectively on the 3-cells $\ldots, b^3_j, \ldots$ of the dual complex P^*. *q.e.d.*

Poincaré points out that the above argument works for any abstract 3-dimensional P whose vertex links have Betti numbers = 2 : thus he is anticipating the later generalization of duality to **homology manifolds**, i.e. P's whose vertex links have homology of a sphere.

§ XI. It is easily seen that a manifold can be cell subdivided if and only if it can be **triangulated**, i.e. subdivided into **simplices**. Poincaré now gives some ideas re the deeper problem of whether a subdivision exists at all.

Proposition 12. *Any (differentiable) manifold* V *can be triangulated.*

Attempted Proof. We assume triangulability in dimensions < p.

Poincaré assumes that the connected V is defined to be a union of manifolds ν each parametrized by a p-submanifold of some q-space defined by q-p equations. Then, using implicit function theorem, he reduces this to the modern definition of V being a union of manifolds ν' each paramtrized by p-space.

Then, within each ν', he defines ν'' to consist of points which are not in any of the other ν'. Poincaré says that "clearly" *one can express* V *as a union of manifolds* ν'' *having disjoint interiors* (but these ν'' need not be cells or even simply connected).

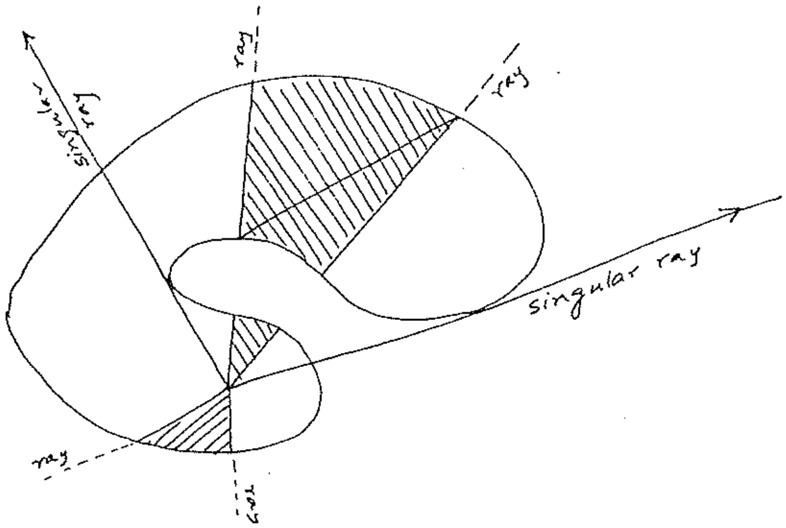
[However this is far from "clear" : the ν'' he just defined are certainly not such. So some modification — e.g. shrinking away common parts in some manner — is needed.]

In each of these ν'' parametrized by p variables $z_1, z_2, \ldots$, we choose an origin and consider (half) **rays** $ta_1, ta_2, \ldots, t > 0$. It will be assumed that they cut $\partial\nu''$ *finitely* many times (it seems very likely that this can be arranged by a slight perturbation of these ν'').

So, excluding the **special rays**, i.e. those which have points of tangency with $\partial\nu''$, all rays cut $\partial\nu''$ an *odd number* of times. The **special cone** of special rays cuts $\partial\nu''$ in a subvariety U of dimension ≤ p-2 which divides $\partial\nu''$ into some regions R.

Such an $R \subseteq \partial\nu''$ is called of the **first kind** if one (and so all) rays leave ν'' from it; otherwise of the **second kind**. Using the inductive hypothesis we triangulate all these (p-1)-dimensional regions of the first kind.

This triangulation is now extended to a cell subdivision of ν'' into pyramids, or **sections of pyramids** of (p-1)-simplices over the origin, furnished by unions of ray segments passing through them, and then each sectioned pyramid further subdivided into p simplices of dimension p :



These triangulations of the manifolds ν'' however suffer from the defect that the triangulations of the boundaries of two ν'' may not coincide. To overcome this difficulty we take a common simplicial subdivision of the various triangulations of each $\partial\nu''$. This results in a subdivision of the two simplicial **ends** of each simplex of ν'', and so the latter can be triangulated as the **join** of its subdivided ends. *? q.e.d. ?*

Poincaré says that we are now *"débarrassé des derniers doutes"* re triangulability, however this was really established beyond the shadow of any doubt only much later by **Whitehead**, **Cairns**, and others.

CHAPTER V

SECOND COMPLEMENT A L'ANALYSIS SITUS

Proc. Lond. Math. Soc., 32 (1900), 277-308

Introduction. Poincaré says that he'll doubtless be returning to the far-from-finished business of *"Analysis Situs"* many times, but for the moment it is only to simplify and clarify results already in hand.

§ 1. **Notational review.** We mentioned that by "$a_i^q \in P$ simply connected" Poincaré probably *intended* to say that a_i^q is a cell : this becomes all but certain now for he writes that this phrase means that (the boundary of) a_i^q is a *"hypersphère"* (see also the conjecture at the end).

The p-dimensional elements of any cell subdivision of our closed oriented p-manifold will be given the manifold's orientation. Also, as mentioned before, corresponding to each orientation of the lower dimensional cells of P, the lower dimensional cells of its dual P^* will be oriented in such a way that the incidence matrices satisfy

$$\varepsilon^{p-q+1}(P^*) = (\varepsilon^q(P))^* \quad \forall\, q\ .$$

Proposition 1. *With above orientations the intersection number* $N(a_i^q, b_i^{p-q})$ *is +1 or -1 with sign depending only on* q *as follows :*

q =	0	1	2	3	4	5	6	7	8	......
	+	−	−	+	+	−	−	+	+	

Proof. Let $\varepsilon_{ij}^q = \varepsilon_{ji}^{p-q+1} = 1$, and let the oriented a_j^{q-1} be given by the sequene of equations $F_1 = 0, \ \dots\ , F_{p-q} = 0,\ \psi = 0$, where omitting the last equation (and replacing it by $\psi > 0$) gives the oriented a_i^q.

Let the oriented b_j^{p-q+1} be given by the sequence of equations $\phi_1 = 0$, $\dots\ , \phi_{q-1} = 0$. We have $\psi = 0$ at the common point of b_j^{p-q+1} and a_j^{q-1}, and $\psi > 0$ at the point common to its face b_i^{p-q} and a_i^q. By adjusting ψ we can assume $\psi < 1$ in b_j^{p-q+1} and $\psi = 1$ on this face b_i^{p-q}. From $\varepsilon_{ji}^{p-q+1} = 1$ it now follows that the orientation of b_i^{p-q} is given by the sequence of defining equations $\phi_1 = 0, \ldots, \phi_{q-1} = 0, 1-\psi = 0$ (so replacing last equation by $1-\psi > 0$ gives $\psi < 1$ as desired).

By definition the intersection number $N(a_i^q, b_i^{p-q})$ is the sign of the pxp determinant formed by the partial derivatives of

$$F_1, \ldots, F_{p-q}, \phi_1 = 0, \ldots, \phi_{q-1} = 0, 1-\psi = 0 ;$$

while $N(a_j^{q-1}, b_j^{p-q+1})$ is the sign of the pxp determinant formed by the partial derivatives of

$$F_1, \ldots, F_{p-q}, \psi = 0, \ldots, \phi_{q-1} = 0.$$

The result follows because these two determinants (which are equal upto sign) have the same sign if and only if q is even. *q.e.d.*

§ 2. We now sketch Poincaré's diagonalization of integer matrices.

Proposition 2. *Any integer matrix can be diagonalized by pre- and post multiplying by two integer matrices of determinant* ± 1 .

Proof. Let us denote the rank and elementary divisors of the given integer sxt matrix A by r and d_i, $1 \le i \le r(A)$.

There exists $P \in GL(s,\mathbb{Z})$ *such that g.c.d. of the first row of* PA *is* d_1.

We'll argue out only the case $s = r$ and $d_1 = 1$ since the general case can be reduced easily to this.

For any $P \in GL(s,\mathbb{Z})$, the g.c.d. of the first row of PA divides the g.c.d. Δ_r of all rxr minors of A. So to ensure that it is 1 it will suffice to arrange that if any prime p divides Δ_r then it does not divide all elements of the first row of PA.

Since $d_1 = 1$ there is a row i(p) containing an element c not divisible by p. If the elements of the first row of P were all divisible by p excepting the i(p)th which equals the (p-2)th power of c mod p, then, since the (p-1)th power of c mod p is 1, an element of the first row of PA is not divisible by p.

Now choose any s relatively prime integers satisfying the above divisibility conditions with respect to the distinct prime factors p of Δ_r, and then any $P \in GL(s,\mathbb{Z})$ having these as its first row.

Next we find a $Q \in GL(t,\mathbb{Z})$ *such that* d_1 *is the leading element of* PAQ.

Choose any t relatively prime integers such that their scalar product with the first row of PA is d_1, and then choose a Q with these as first column.

Now we make all other elements of first row and column zero, and then repeat the above two steps for the remaining rows and columns.

The g.c.d. of the remaining elements being d_2, this will now come to (2,2) spot ... so we'll finally get the **normal form** diag(d_1, d_2, .. ,d_r, 0, ..) of A due to **Smith** (1861). *q.e.d.*

§ 3. **Arithmetical invariants of incidence matrices.** To calculate these, we diagonalize these matrices via arithmetical operations, but note that *one may also use (unsigned) transpositions of two rows or columns* (equivalently a row or column can be multiplied by -1).

[This follows because *the incidence matrices* ε^q_{ij} *are never of maximal rank* (since nonzero multiples of cells don't bound) and so their arithmetical invariants are also invariant under the (slightly) coarser equivalence of Prop. 2.

We'll allow this extra operation even for "polyhedra of the second type" (see § 4 below) having a nonsingular square incidence matrix. This makes sense because a subdivision gives a polyhedron (of the "first" or ordinary kind) and we are interested only in *those arithmetical invariants which are also invariant under subdivision* (e.g. the size of an incidence matrix is uninteresting).]

Poincaré **equips** the ith row (resp. jth column) of any e^q_{ij} obtained from ε^q_{ij} by using arithmetical operations with a q-chain c^q_i (resp. (q-1)-chain γ^{q-1}_j) as follows :

The chains associated to ε^q_{ij} are a^q_i (resp. a^{q-1}_j) and when we add to the row (resp. column) having the chain c (resp. γ), the row (resp. column) having the chain c' (resp. γ'), the chain c (resp. γ') is changed to c+c' (resp. $\gamma' - \gamma$).

Proposition 3. $\partial(c^q_i) = \sum_j e^q_{ij}\, \gamma^{q-1}_j$ *holds for all* e^q_{ij} *equivalent to* ε^q_{ij}.

Proof. The result follows because for ε^q_{ij} these are the defining equations $\partial(a^q_i) = \sum_j e^q_{ij}\, a^{q-1}_j$ of ∂, and each elementary row operation amounts to adding one such equation to another, while an elementary column operation amounts to a rearrangement $e\gamma + e'\gamma' = (e+e')\gamma + e'(\gamma'-\gamma)$ under the summation sign. *q.e.d.*

Applying this result to the diagonalized ε^q_{ij} , Poincaré now obtains the shorter proof (given before) of $b_q(P) = \alpha_q(P) - r(\varepsilon^{q+1}) - r(\varepsilon^q)$, which in turn yields $b_q(P) = b_{p-q}(P^*)$ because $(\varepsilon^q(P))^* = \varepsilon^{p-q+1}(P^*)\ \forall\ q$.

Also this shows that **Betti's qth number** *exceeds* (Poincaré's) qth Betti number $b_q(P)$ *by the number of elementary divisors of* ε^{q+1} *bigger than* 1, and the product of these divisors gives the number of "distinct" cycles whose multiples bound, i.e. $|Tor(H_q(P)|$. A manifold will be said to have no **torsion** (see § 6 for a justification of this terminology) iff this product is 1 .

[We see that Poincaré has thus given up on his **tableaux** and reverted to incidence matrices (which are in fact called "tableaux" in this paper, but we'll use *tableaux* only in the sense of the previous paper).

But, as we saw, *the complete reduction of the tableaux gives much more extra information than the diagonalization of the incidence matrices*, e.g. one gets a **Hodge basis** of P, and thus its **(co)homology groups** :

$$H_q(P) \cong \oplus\{\mathbb{Z}/d_i(\varepsilon^{q+1})\mathbb{Z} : d_i(\varepsilon^{q+1}) > 1\} \oplus \mathbb{Z}^{b_q} ,$$

$$H^q(P) \cong \oplus\{\mathbb{Z}/d_i(\varepsilon^{q})\mathbb{Z} : d_i(\varepsilon^{q}) > 1\} \oplus \mathbb{Z}^{b_q} .$$

And so also, by using above formulae and $(\varepsilon^q(P))^* = \varepsilon^{p-q+1}(P^*)$, one obtains the full **Poincaré duality of (co)homology groups** :

$$H_q(P) \cong H^{p-q}(P^*).$$

It is curious that Poincaré set up his tableaux only *to diagonalize their corners* : it seems reasonable to suppose that at some time he must have intended to reduce them further, but somewhere along the line he gave up on this good idea ! We'll give more details about the combinatorial Hodge basis mentioned above later.]

§ 4. **Computations.** Instead of using P's of the above or **first kind,** in which the cells a_i^q 's were "distinct", Poincaré now uses more general polyhedrons P of the **second kind,** in which the a_i^q 's are still cells but "not necessarily distinct".

This means that the closure of any cell is obtainable by making some identifications of the faces of a closed cell (so these P's are roughly todays *CW complexes*). The **incidence number** ε_{ij}^q of a P of the second kind is the sum of the the ordinary incidence numbers $\varepsilon_{ij'}^q$, as $a_{j'}^{q-1}$ runs over *all occurences* of a_j^{q-1} in a_i^q.

Proposition 4. *The formulae given above for Betti numbers etc., in terms of the invariants of incidence matrices, generalize to all* P's *of the second kind.*

Poincaré does'nt give a proof but an obvious approach is to see that proof given previously for subdivision invariance extends to such P's, and that by subdividing P can be made of the first kind.

However this proof does not extend to P's of the **third kind,** i.e. those for which the a_i^q 's are not necessarily cells, and it is easy to see that Prop. 4 is false for such P's.

Proposition 5. *One has* $H_0 \cong \mathbb{Z}$ *and* $H_3 \cong \mathbb{Z}$ *for* Exs. 1-5 *of* § 10 *of* "Analysis Situs", *and the remaining homology groups are as follows :*

	Example 1	Example 2	Example 3	Example 4	Example 5
H_2	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0	$\mathbb{Z}$	0
H_1	$\mathbb{Z}^2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Proof for Ex. 2. Our P has (for all examples) just one 3-celland three faces each occuring twice in its boundary with opposite orientation, so

$$\varepsilon^3 = [0 \quad 0 \quad 0].$$

Indexing the three rows and the two columns of ε^2 by the three faces ABDC=B'D'C'A', ABB'A'=DD'C'C, ACC'A'=DD'B'B, and the two edges AB=B'D'=C'C=B'A'=AC=DD', AA'=DC=C'A'=B'B=C'D'=DB we see that

$$\varepsilon^2 = \begin{bmatrix} 0 & 0 \\ 2 & -2 \\ 0 & 0 \end{bmatrix}.$$

To check e.g. that $\varepsilon^2_{12} = 0$ note that the second edge has two occurences, viz. DC and DB, in the boundary of the copy ABDC of the first cell, and these have opposite orientations. Again $\varepsilon^2_{21} = 2$ because the first edge has two occurences, viz. AB and B'A', in the copy ABB'A' of the second cell, and both are with positive orientation.

Likewise the two edges index the two rows and the two vertices A=B'=C'=D, B=D'=C=A' the two columns of the first incidence matrix,

$$\varepsilon^1 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

The three matrices are respectively of ranks 0, 1 and 1, and the elementary divisors of the last two are 2 and 1 respectively. So Ex. 2 has the stated homology.

The computations for the other examples are analogous. *q.e.d.*

[Poincaré ignores Ex. 2 while illustrating his method : this he did because Ex. 2 is *not* a 3-manifold and all his definitions are made *only* for manifolds. Actually **the (ordinary) homology of the underlying space of any cell complex** P is also defined to be H(P), for it is known that, even for all such spaces, this definition does not depend on P.

However as Ex. 2 shows, duality need not hold for non-manifolds : *so maybe Poincaré did not make this generalization for he felt that ordinary homology is not the "right" homology for non-manifolds ?*
We remark in this context that **Macpherson** has now changed the definition so as to take into account the the presence of singularities, and *for this new* **intersection homology**, *Poincaré duality does hold for all orientable pseudomanifolds* like Ex. 2.

We note that Exs. 4 and 1 ($\cong S^1 \times S^1 \times S^1$) are particular cases of Ex. 6 (so their homology is also given by Prop. 6 below) and Ex. 5 ($\cong \mathbb{R}P^3$) is defined in the paper by using antipodal identification of the boundary of an octahedron rather than that of a cube.

Another difference from the paper is of course that Poincaré only gives ranks and elementary divisors of the incidence matrices, but as we have seen, these immediately yield the (co)homology groups of P.

Note that Exs. 5, 4, and 3 ($\cong S^3/8$) *all* show that the 1- and 2-dimensional *Betti's* (as against Betti !) numbers of an orientable 3-manifold can be different.]

We recall that **Example** 6 (considered in §§ 11,13,14 of *"Analysis Situs"*) gives for each $T \in SL(2,\mathbb{Z})$ an orientable closed 3-manifold M (or M_T) and two such manifolds M and M' were shown to be diffeomorphic if and only if T is conjugate to T' or its inverse in $GL(2,\mathbb{Z})$. Furthermore, the fundamental group and Betti numbers of these manifolds were calculated. Poincaré now computes their torsion.

Proposition 6. *The homology of the orientable closed 3-manifolds* M_T, $T \in SL(2,\mathbb{Z})$, *of* Ex. 6, *is given by*

$$H_1 \cong \mathbb{Z} \oplus \mathbb{Z}/d_1(T-I)\mathbb{Z} \oplus \mathbb{Z}/d_2(T-I)\mathbb{Z} \quad \text{and} \quad H_2 \cong \mathbb{Z}^{b_1}.$$

Proof. The assertion that the second homology has no torsion is in fact true for all orientable closed 3-manifolds : see § 6 below.

To calculate H_1 we use § 13 of *"Analysis Situs"*, where it was shown that if $T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, then H_1 is isomorphic to the free abelian group generated by C_1, C_2, and C_3 subject to the relations

$$(\alpha-1).C_2 + \gamma.C_3 \simeq 0,$$
$$\beta.C_2 + (\delta-1).C_3 \simeq 0.$$

So, by diagonalizing the matrix T−I, of the coefficients of the above relations, we obtain the required result. *q.e.d.*

[It follows that amongst these infinitely many manifolds M_T there are *many examples of non-homeomorphic manifolds having the same homology groups* : e.g. the manifolds corresponding to $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ are such. However, as shown in "*Analysis Situs*", these manifolds are homeomorphic if and only if their fundamental groups are isomorphic.

In § 11 of "*Analysis Situs*" these M_T's were also obtained by a pairwise identification P of the faces of a suitably subdivided cube. So, if nonzero, $d_1(T-I)$ and $d_2(T-I)$ coincide with the elementary divisors of the incidence matrices of this P (to illustrate this Poincaré writes down these incidence matrices for the subcase $T = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$).]

Proposition 7 (HEEGAARD'S EXAMPLE). *All* $(x, y, z) \in \mathbb{C}^3$ *satisfying*

$$z^2 = xy \quad \text{and} \quad |x|^2 + |y|^2 = 1$$

form a closed connected orientable 3*-manifold with* $H_1 \cong \mathbb{Z}/2\mathbb{Z}$ *and* $H_2 \cong 0$.

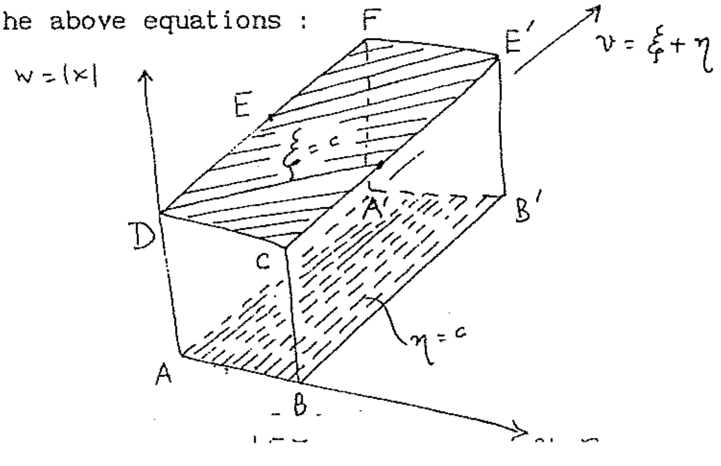
Proof. If (x,y,z) satisfies these three (real) equations, then $0 \le |x|$, $|y|$, $|z| \le 1$, so the set V of all such 3-tuples is compact. It is smooth because a calculation (which we omit) shows that the Jacobian of these three equations is always of maximum rank 3 on V.

To find a *cell subdivision* P *of the second kind for* V we begin by rewriting its defining equations as

$$|z|^2 = |x|.|y| \quad , \quad 2.\zeta = \xi + \eta \quad , \quad |x|^2 + |y|^2 = 1,$$

where $x = |x|e^{i\xi}$, $y = |y|e^{i\eta}$ and $z = |z|e^{i\zeta}$.

So V can be obained from the 3-cell $0 \le u = \eta \le 2\pi$, $0 \le v = \xi+\eta \le 4\pi$, $0 \le w = |x| \le 1$ by identifying boundary points for which x, y, and z are the same as per the above equations :



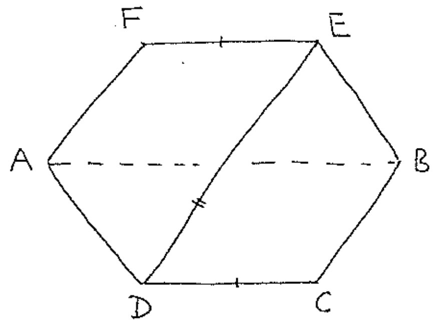
These identifications are as under :

(1) Each line segment $\eta = c$ of the bottom face $|x| = 0$ (resp. $\xi = c$ of the top face $|x| = 1$) gives the single point $(x,y,z) = (0, e^{ic}, 0)$ (resp. $(x,y,z) \equiv (e^{i\xi}, 0, 0)$)of V.

(2) Furthermore, each pair $(0, v, w)$, $(2\pi, v, w)$ of points of the box gives the same point $(x,y,z) = (we^{iv}, (1-w^2)^{1/2}, |w|^{1/2}(1-|w|^2)^{1/2}e^{iv/2})$ of V.

(3) Finally, each pair $(u, 0, w)$, $(u, 4\pi, w)$ of points gives the same point $(we^{-iu}, (1-w^2)^{1/2}e^{iu}, w^{1/2}(1-w^2)^{1/2})$ of V.

Doing only the identifications (1) on the above box we get a 3-ball with boundary cut up into four (curved) quadrilaterals :



Now (2) and (3) are equivalent to the identifications ADEF=BCDE and ABCD=ABEF of these quadrilaterals.

Doing these we get the required P : it has one 3-cell, the 2 faces ADEF=BCDE, ABCD=ABEF, the three edges AD=AF=BE=BC, AB, CD=EF=DE, and the 2 vertices A=B, C=D=E=F.

With respect to this order of the cells, the incidence matrices of P turn out to be as follows :

$$\varepsilon^3 = (0 \quad 0), \quad \varepsilon^2 = \begin{bmatrix} 0 & 0 & +2 \\ 0 & 1 & 1 \end{bmatrix}, \quad \varepsilon^1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

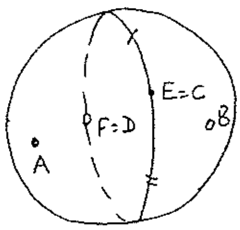
For example $\varepsilon^2_{11} = 0$ (resp. $\varepsilon^2_{13} = 2$) because the copy ADEF of the first face contains the two occurences AD and AF of the first edge (resp. DE and EF of the third edge) and these have opposite (resp. positive) orientations. Again ε^1_{11} is 1 (and not 0 as in paper) because there is only one occurence, viz. A, of the first vertex, in the copy AD of the first edge.

Normalizing the matrices and using the previous results we see that the homology of V is given by $H_3 \cong \mathbb{Z}$, $H_2 \cong 0$, $H_1 \cong \mathbb{Z}/2\mathbb{Z}$ and $H_0 \cong \mathbb{Z}$. *q.e.d.*

[We can use the method of § 12 of "*Analysis Situs*" on the above P to check that its fundamental group is also isomorphic to $\mathbb{Z}/2\mathbb{Z}$. However this (as well as some of the above) effort is wasted because

Heegaard's manifold is diffeomorphic to $\mathbb{RP}^3$ (= Poincaré's Ex. 5) !

To prove this note that ABCD=ABEF amounts to folding back and pasting the two quadrilaterals having edge AB in common in the last diagram. This results in a 3-ball with two poles A,B and an equator E(=C)F(=D)E(=C) marked on its boundary :



Under the final BCDE=ADEF these poles are switched and the equator gets rotated by π (e.g. great circle arc AF goes to BE, AE to BF, etc.) i.e. this is nothing but the antipodal identification on the bounding 2-sphere of this 3-ball : so $V \cong \mathbb{RP}^3$.

So the link at the origin, of the variety defined by $Q(x,y,z) = 0$, *where* Q *is any non-degenerate quadratic form over* $\mathbb{C}$, *is homeomorphic to* $\mathbb{RP}^3$: this follows because a unitary change of coordinates will make the equation $z^2 - xy = 0$.

In fact, still more generally, the link at the origin, of the complex hypersurface in $\mathbb{C}^n$ *given by* $Q(z_1, \dots, z_n) = 0$, *where* Q *is a non-degenerate quadratic form over* $\mathbb{C}$, *is homeomorphic to the unit tangent sphere bundle of the* $(n-1)$*-sphere.*

And this can be seen simply by noting that the equations $(z_1)^2 + \dots + (z_n)^2 = 0$ and $|z_1|^2 + \dots + |z_n|^2 = 1$ are equivalent to $(x_1)^2 + \dots + (x_n)^2 = 1/2 = (y_1)^2 + \dots + (y_n)^2$ and $x_1y_1 + \dots + x_ny_n = 0$ (here $z_k = x_k + iy_k$) ! (Lamotke's exposition of Lefschetz's thms. on complex varieties makes crucial use of this basic fact of "complex Morse theory".)

For more on the topology of links of isolated singularities of complex hypersurfaces see **Milnor**'s book on this topic.]

§ 5. **Invariance under subdivision.** The following argument generalizes that of §X of (the first) *Complément*.

Proposition 8. *Dual subdivisions* P *and* P^* *have the same Betti numbers and torsion coefficients* (i.e. $H_q(P) \cong H_q(P^*)\ \forall\ q$).

Proof. We note that the boundary of each **join** $a_k^{q+1}.b_i^{m+1}$, where a_k^{q+1} (resp. b_i^{m+1}) is a face of the cell a_i^{p-m-1} (resp. b_k^{p-q-1}) dual to b_i^{m+1} (resp. a_k^{q+1}) is given by the **product formula**

$$\partial(a_k^{q+1}.b_i^{m+1}) = \partial(a_k^{q+1}).b_i^{m+1} + a_k^{q+1}.\partial(b_i^{m+1}),$$

where ∂ is the **reduced boundary**, i.e. boundary of a vertex is taken to be 1 (= empty cell) rather than 0.

We will show that *upto addition of homologies of the type*

$$\partial(a_k^{q+1}).b_i^{m+1} + a_k^{q+1}.\partial(b_i^{m+1}) \simeq 0,$$

any cycle of P *(resp.* P^**) is homologous to a cycle of* P^* *(resp.* P*).*

For this note that the product formula implies that a **bidegree** (q,m+1) chain $\sum_{ki}\lambda_{ki}c_k^q.b_i^{m+1} = \sum_i C_i^q.b_i^{m+1}$ is a cycle iff each $C_i^q = \sum_k\lambda_{ki}c_k^q$ is a cycle. Further each of these cycles C_i^q, being in the closure of a_i^{p-m-1}, is the boundary of some (q+1)-chain D_i^{q+1} contained in this closed cell.

So the given cycle can be rewritten as $\sum_i\partial(D_i^{q+1}).b_i^{m+1}$, which can be changed, by adding homologies of the above type, to the bidegree (q+1,m) cycle $\sum_i D_i^{q+1}.\partial(b_i^{m+1})$.

Using a sequence of such homologies we see that any cycle of P^*, i.e. one of bidegree (0,t), can be changed to one of P, i.e. of bidegree (t,0). And likewise we can go in the opposite direction.

Thus we get a 1-1 onto correspondence between H(P) and $H(P^*)$. So P and P^* have the same Betti numbers. Further they have the same torsion coefficients because any order k > 0 element of H(P) which is given by a cycle whose kth but no lesser multiple bounds, corresponds to an order k element of $H(P^*)$. (Alternatively one can check that the correspondence is additive.) *q.e.d.*

Since $\varepsilon^q(P^*)$ is the transpose of $\varepsilon^{p+1-q}(P)$ it follows as an immediate corollary that *the* qth *torsion coefficients of* P *coincide with its* (p+1-q)*th* (not (p-q)th as misprinted in paper) *torsion coefficients* (this is essentially same as $H_q(P) \cong H^{p-q}(P)$).

Let us consider (cf. § 16 of *"Analysis Situs"*) a cell subdivision obtained from another by an **erasure** *of a* (q-1)*-cell incident to precisely two* q*-cells, which have no other* (q-1)*-cell in common.*

Poincaré notes that both P and P^* can be recovered from their common derived P' by a sequence of such erasures, thus Prop. 8 also follows from the following.

Proposition 9. *Homology is invariant under erasures of the above type.*

Proof. We note that the erasure of an a_i^{q-1} incident to (only) a_j^q and a_k^q (which have no other (q-1)-cell in common) is equivalent to the following **combinatorial operation** on the sequence $\varepsilon = \{\varepsilon^q\}$ of its incidence matrices :

If the addition of the kth row of ε^q to the jth row makes the ith column zero and creates no other new zeros, then omit the kth row and ith column from ε^q after doing this addition ; also omit the kth column from ε^{q+1} and the ith row of ε^{q-1}; keep all other matrices same.

We note that both α_q and α_{q-1} decrease by 1, and the numbers of cells of other dimensions remains same; also that operation affects only the incidence matrices ε^{q+1}, ε^q, and ε^{q-1}.

However neither the rank or the elementary divisors of ε^{q+1} change because the omitted kth column was identical to the jth. Likewise the image of the map $\partial: C_{q-1} \to C_{q-2}$ (corresponding to ε^{q-1}) is unaffected because the boundary of the omitted a_i^{q-1} is also the boundary of another q-chain on the boundary of a_j^q. However the rank of ε^q decreases by one but only an elementary divisor = 1 is lost in this process : this follows because one cannot obtain the cycle bounding a_j^q as the boundary of a chain which is the sum of the new bigger q-cell and the remaining q-cells of P.

Thus the homology groups are unchanged. *q.e.d.*

[A fundamental theorem of **Newman** says : *two triangulations of the same* V *are related by the equivalence relation generated by stellar subdivisions.*

Since one can recover a simplicial complex from a stellar subdivision by a sequence of erasures the above proposition, in conjunction with this Newman's theorem gives a proof of the invariance theorem.

Note that Newman's theorem implies that *the invariants of* V *are precisely all invariants of incidence matrices under the relation of* **combinatorial equivalence** *generated by the above operation.*]

with the preceding the (q-1)-cell associated to the in-between vertical. Starting with a_2^q, choose for each of these q-cells the same (resp. opposite) sign as for the preceding, according as the product of the two elements of the in-between vertical is +1 (resp. -1). Then the boundary of the "closed chain" $\sum_{1\le i\le t} \pm\ a_i^q$ of q-cells contains none of the (q-1)-cells associated to the verticals, *except* possibly the one shared by a_1^q with its predecessor a_t^q. This last named (q-1)-cell *is* in the boundary iff the product of the elements is +1 (resp. -1) according as the number of horizantals is odd (resp. even). *q.e.d.*

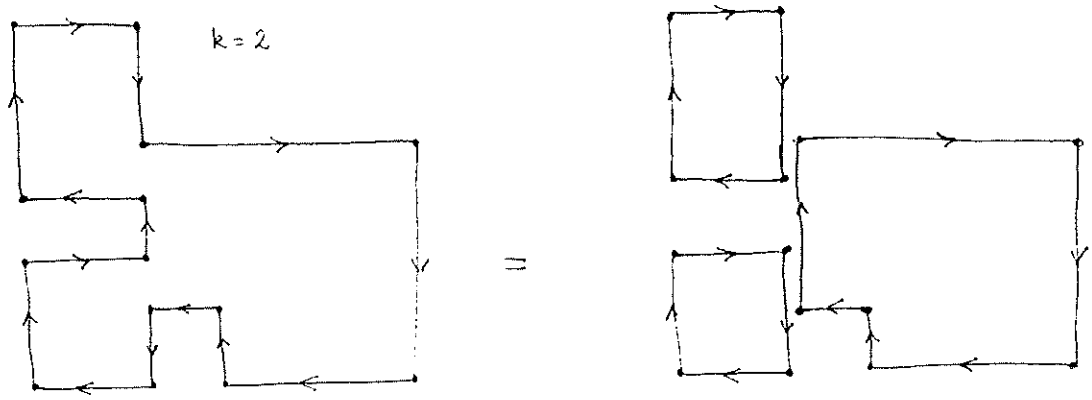
Proposition 11. *If a square matrix of* 0*'s,* +1*'s, and* -1*'s, has no one-sided "closed chains" then its determinant must be* 0, +1, *or* -1.

Proof. The stated property and |det| are invariant under an interchange of two rows/columns, as well as multiplication of a row/column by ±1. So we can restrict ourselves to the case when the given square matrix A has 1 in the (1,1) spot, and all the other nonzero elements of the first row are also 1.

In case the kth element, k ≥ 2, of the first row is 1, we make it zero by subtracting, from the kth column, the first column. This does not alter the determinant.

We assert that the resulting matrix A' *still consists only of* 0*'s,* +1*'s, and* -1'*, and also does not have a one-sided "closed chain".*
We will take k = 2. Let a be any element of the second column of A, a' the corresponding element of A', and a_1 the element of the first column in the same row. Consider the "closed chain" $11aa_1$, where 11 = horizantal joining the first two elements of the first row, of A. If a and a_1 are both nonzero, then $aa_1 = 1$, i.e. they are both +1 or both -1: this gives $a' = a - a_1 = 0$. So A' is also over {-1, 0, +1} as asserted.

Now consider any "closed chain" of nonzero elements of A'. It has an even number of elements on the new, i.e. second, column. If it has none, it is a "closed chain" of nonzero elements of A, and so two-sided. Otherwise it can be "factorized" as follows into "closed chains" having exactly two elements on the second column, with these being consecutive; and it is easy to see that it will suffice to show that each of these "components" is two-sided.



So we will assume that our "closed chain" has exactly two (nonzero) elements a', b' on the second column, and these are consecutive. We have to show that it is two-sided.

Since a', b' are both nonzero, exactly one of $\{a, a_1\}$, and likewise exactly one of $\{b, b_1\}$, is zero, and the other is ±1. If a and b are nonzero, then our "closed chain" is one of A, so two-sided. If a_1 and b_1 are nonzero, then it is a "closed chain" of matrix obtained from A by multiplying its first column by −1, so again two-sided.

Finally, if a = 0 and $b_1 = 0$, consider the "closed chain" of A formed by $a_1$11b, and the part of the "closed chain" of A' not containing a' and b'. This has one more vertical than the "closed chain" of A' being considered, and the product of its elements is the same except for a sign change. So the "closed chain" of A' is two-sided in all cases.

Use the above process to make all but the first element of the first row 1. Now continue with the smaller square matrix obtained by omitting the first row and the first column. The result follows because finally we are left with a singular matrix or [±1]. *q.e.d.*

[We note that *a matrix* A *over* {−1, 0, +1} *is arithmetically equivalent to* $\begin{bmatrix} I_{r(A)} & 0 \\ 0 & 0 \end{bmatrix}$ **via matrices over {−1, 0, +1}** *if and only if it has no one-sided "closed chains"* (interchange of two rows/columns is allowed).

To see "if" we proceed as in the above proof, noticing that all other elements of the first column can also be made zero in same manner, before omitting the first row and column.

To see "only if" we note that the above diagonal matrix has obviously no one-sided "closed chains", and from it deduce as above the same property for any matrix related to it by such operations.

We note that this is only *a sufficient condition that a matrix over* {-1, 0, +1} *have all elementary divisors* 1, such matrices need not be diagonalizable *via* matrices over {-1, 0, +1}.]

[There is an alternative argument which gives more.

Proposition 11'. *Let* Ω *be any set of complex numbers, containing* {-1, 0, +1}, *which is closed under multiplication, and let* A *be a square matrix, with entries from* Ω, *such that the alternating product of any nonzero "closed chain" having an even (resp. odd) number of horizantals is* +1 *(resp.* -1*). Then* $\det(A) \in \Omega$.

Here by **alternating product** we mean $a.b^{-1}.c.d^{-1}. \ldots$ (see fig. above). In the following we'll use the stated conditions only when there is *at most one horizantal/vertical on each row/column* : however note that the "factorization" pictured above then implies the condition for all "closed chains".

Proof. The given conditions are equivalent to saying the sums of some pairs of nonzero terms in the right side of

$$\det(A) = \sum_{\pi} (-1)^{\pi}.a_{1\pi(1)}a_{2\pi(2)} \cdots a_{n\pi(n)}$$

are zero.

To see this, interpret π and ϕ to be the **permutation matrices** given by the starting (resp. end) points of the horizantals of the "closed chain" and rewrite the condition $a.b^{-1}.c.d^{-1}. \ldots = \pm 1$ as $(a.c. \ldots) \mp (b.d. \ldots) = 0$. (The permutations keep rows/columns not occuring in the "closed chain" fixed.)

Dropping all such pairs we get either $\det(A) = 0$ or else the value of the determinant is equal upto sign to any of its nonzero terms. *q.e.d.*]

Proposition 12. *If* P *has an orientable* q-*skeleton, then* $H_{q-1}(P)$ *is free.*

Proof. This follows at once from the previous two results because $\mathrm{Tor}(H_{q-1}(P)) \cong \oplus_i \mathbb{Z}/d_i(\varepsilon^q(P))$, where $\Pi_i d_i(\varepsilon^q(P))$ is g.c.d. of the biggest sized nonzero minors of $\varepsilon^q(P)$. *q.e.d.*

[The converse is false : one can e.g. extend the triangulation of a Möbius strip embedded in S^3 to a triangulation P of S^3.

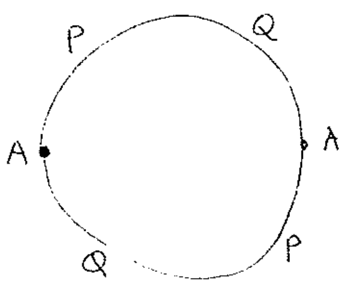
We note that a necessary condition that the manifold admit *some* P all of whose skeleta are orientable is that the **Stiefel-Whitney cohomology classes** of P be all zero. It is possible that this condition is also sufficient ?

For more on these and other **characteristic classes** of a manifold see the book by **Milnor** on this topic.]

For example, the 1-skeleton of any P (for which each edge has two vertices) is clearly orientable, so $H_0(P)$ is always free (of dimension equal to the number of components of P).

Again, the orientability of (the p-skeleton of) the p-manifold P implies that $H_{p-1}(P)$ is free (: a fact which we had verified before in many computations involving orientable closed 3-manifolds).

On the other hand a non-orientable closed p-manifold always has (p-1)-torsion, e.g. the *real projective plane* $\mathbb{R}P^2$ (i.e. **Ex. 7** of *"Analysis Situs"*) has the following cell subdivision P of the second kind, whose incidence matrices give $H_0 \cong \mathbb{Z}$, $H_1 \cong \mathbb{Z}/2\mathbb{Z}$ and $H_2 \cong 0$.



Poincaré ends by announcing that *"if a closed* p-*manifold has the same homology groups as a* p-*sphere, then it is homeomorphic to a* p-*sphere"*. He says that its proof, being long, will be given elsewhere.

Actually this "theorem" is false, as Poincaré himself showed later by giving in the *"Cinquième Complément"* an example of a closed 3-manifold not homeomorphic to S^3 but having the same homology groups as S^3.

[It is amusing to note that, *in this entire series of papers, there are only two results which Poincaré labelled as "Theorems"*, viz. the purely matrix theoretical Propositions 2 and 11 given above ! Perhaps this was done to draw the attention of London mathematicians like Cayley and Sylvester to what was likely to appeal most to them ?]

CHAPTER VI

SUR L'ANALYSIS SITUS

C.R. Acad.Sc. 133 (1901), 707-709.

The most interesting closed 4-dimensional (differentiable) varieties are **complex surfaces**. Of these we will confine our attention here to those given by an equation of the type

$$z^2 = F(x,y),$$

where the polynomial F is such that the **complex curve**

$$F(x,y) = 0$$

has only *ordinary points* or *ordinary double points*. We have calculated the fundamental group of these 4-dimensional varieties.

[Poincaré says that he encountered these surfaces while trying to figure out the *variations of some double integrals* needed to study the (power series) developements of some *perturbation function* : this probably refers to some work on *dynamical systems*, which, together with complex surfaces, was one of his main motivations for writing "*Analysis Situs*".]

We note also that, *if y is constrained to be on a closed curve*, then $z^2 = F(x,y)$ gives the closed 3-manifolds of **Example 6** of "*Analysis Situs*", or else, straightforward generalizations of these.

[One gets the **mapping torus** of a diffeomorphism of some surface M^2 (in Ex. 6, we had $M^2 = T^2$) : so the fundamental group of this 3-dimensional submanifold of the given 4-dimensional variety is an extension of the fundamental group of M^2 by $\mathbb{Z}$.]

We will show that the 4-dimensional varieties given by $z^2 = F(x,y)$ *are simply connected.* Note that here we are considering these varieties *with* their (isolated) singularities, i.e. the **conical points** arising from the (ordinary) double points of the above algebraic curve.

Further, we'll also show that the fundamental group of the smooth part is finite, and its order depends on the number of factors of the polynomial F(x,y) *as follows :* if deg(F) is even (which we can always ensure by a simple transformation) and F decomposes into n factors, which are all of even degree, then $|\pi_1| = 2^{n-1}$, and if these factors are not all of even degree, then $|\pi_1| = 2^{n-2}$.

[*Obviously Poincaré is considering points at infinity also* : we know that in $\mathbb{C}^2$ the smooth part of Heegaard's surface $z^2 = xy$ is not simply connected, but has $\pi_1 = \mathbb{Z}/2$.]

Also, Poincaré mentions again in this note the weaker but older result of his colleague **Picard** : *the first Betti number of a generic complex surface is zero.*

CHAPTER VII

SUR CERTAINES SURFACES ALGEBRIQUES : TROISIEME COMPLEMENT A L'ANALYSIS SITUS

Bull.Soc.Math.France 30 (1902), 49-70.

As the following will show Poincaré's analysis of the variety $V = \{(x,y,z) \in \hat{\mathbb{C}} \times \hat{\mathbb{C}} \times \hat{\mathbb{C}} \mid z^2 = F(x,y)\}$ employs a rich diversity of tools.

(read $\hat{\mathbb{C}} \times \hat{\mathbb{C}} \times \hat{\mathbb{C}}$ as symmetric product i.e. as $\mathbb{C}P^3$?)

Complex polynomial F(x,y):

We will assume throughout that this is such that if $F = 0 = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y}$ at (a,b), then $\frac{\partial^2 F}{\partial x^2} \cdot \frac{\partial^2 F}{\partial y^2} - (\frac{\partial^2 F}{\partial x \partial y})^2$ is nonzero at (a,b) : in other words, if the terms of degree ≤ 1 of $F(x-a, y-b)$ vanish, then those of the second degree should constitute a nondegenerate quadratic form.

From Prop.10 onwards we'll also demand that $\frac{\partial^2 F}{\partial x^2}$ is nonzero at all (a,b) where F and $\frac{\partial F}{\partial x}$ are zero (this condition accounts for the fact that the results are non-symmetrical in x and y).

Proposition 1. *If the degree* $\deg_x(F)$ *of* F *in* x *is* 2p+1 *or* 2p+2, *then for* almost all $y \in \hat{\mathbb{C}}$, *the equation* $F(x,y) = 0$ *in* x *has* 2p+2 *distinct roots* $\{x_0(y), x_1(y), \ldots, x_{2p+1}(y)\}$ *in* $\hat{\mathbb{C}}$.

From now on we'll denote by $Y = \hat{\mathbb{C}} \setminus \{A_1, \ldots, A_q\}$ the set of all y such that $F(x,y) = 0$ has 2p+2 distinct roots in the extended complex plane (there being a root at infinity iff $\deg_x F$ is odd).

Proof. The point to note is that, in the prime decomposition of $F(x,y) \in \mathbb{C}(y)[x]$, no irreducible can repeat : otherwise we would have $F = 0 = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y}$ at infinitely many points $(a,b) \in \hat{\mathbb{C}} \times \hat{\mathbb{C}}$, which is impossible because the second degree terms at any such (a,b) constitute a nondegenerate quadratic form, and so these (a,b)'s are isolated points of the compact space $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$.

So the polynomials $F(x,y)$ and $\frac{\partial F}{\partial x}(x,y)$ are relatively prime in $\mathbb{C}(y)[x]$. Since $\mathbb{C}(y)[x]$ is a principal ideal domain one has $P(x,y).F(x,y) + Q(x,y).\frac{\partial F}{\partial x}(x,y) = 1$ in $\mathbb{C}(y)[x]$ for suitably chosen rational functions P, $Q \in \mathbb{C}(y)[x]$. (Note, on the other hand, that $\mathbb{C}[x,y]$ has non-principal ideals like (x,y).)

So, if y is not one of the finitely many zeros of the denominators of P or Q, then there is no x such that $F(x,y) = 0 = \frac{\partial F}{\partial x}(x,y)$. This implies that if y is also not one of the finitely many common zeros of the coefficients of x^{2p+2} and x^{2p+1} in $F(x,y)$, then $F(x,y) = 0$ has $2p+2$ distinct roots in the extended complex plane. *q.e.d.*

[Poincaré in fact assumes that the degree of F in x is even, and so his x_i's are all in the finite plane $\mathbb{C}$. This can always be ensured by using the homeomorphism $(x,y,z) \longmapsto (x',y,z')$

$$x = a(y) + \frac{1}{x'} \quad , \quad z = \frac{z'}{(x')^{2p+2}} ,$$

where polynomial $a(y)$ is so chosen that $F(a(y),y)$ is not identically zero : then $z^2 = F(x,y)$, $\deg_x(F) = 2p+1$, changes to $(z')^2 = G(x',y)$, where G is a polynomial in x' and y of degree $2p+2$ in x'.]

[We will denote by V_Y the inverse image of Y under the **projection map** $V \to \hat{\mathbb{C}}$. This is contained in the smooth or **non-singular part** $V_{n.s.}$ of V, since at any $(x_0,y_0,z_0) \in V_Y$ there is the well-defined tangent space

$$x.\left(\frac{\partial F}{\partial x}\right)_0 + y.\left(\frac{\partial F}{\partial y}\right)_0 - 2z.(F)_0 = 0.$$

In fact it was checked that we never have $(\partial F/\partial x)_0 = 0 = (F)_0$. So the derivative of the restricted projection map $V_Y \to Y$ is of maximal rank at each $(x_0,y_0,z_0) \in V_Y$, i.e. this map is a smooth **fibration.**

This remark shows that an exact sequence

$$0 \to G_0 \to G_Y \to F^{q-1} \to 0,$$

which we'll meet below is merely a part of the *homotopy sequence of this* *fibration* : G_0, G_Y and F^{q-1} (= free group on q-1 generators) being, respectively, the fundamental groups of its **fibre** V_0, **total space** V_Y and **base** Y.]

By a **complex curve** (resp. **complex surface**) we'll mean a subset of $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ (resp. $\hat{\mathbb{C}} \times \hat{\mathbb{C}} \times \hat{\mathbb{C}}$) satisfying some given complex polynomial in two (resp. three) variables. (This polynomial is of course not uniquely determined by the subset.)

For example $z^2 = F(x,y)$ determines our complex surface V and $F(x,y) = 0$ a complex curve $V_{z=0}$ contained in it. This complex curve has some isolated singularities. We now turn to some *non-singular* curves contained in V.

Hyperelliptic curves V_y :

These are given by $z^2 = F(x,y)$, *where* y *is constrained to have a fixed value in* $\hat{\mathbb{C}} \setminus \{A_1, \dots ,A_q\}$. (If $y = y_t$ then V_{y_t} will be denoted V_t.)

As we saw before (by building it up from two copies of $\hat{\mathbb{C}}$-with-cuts : cf. remark (3) below) the *topology of these curves* is that of a surface with p handles. To analyze the surface V, we however also need the following information about the *geometry of these curves*.

Proposition 2. *Let* U *denote the hyperbolic, parabolic, or elliptic plane respectively for* $p \geq 2$, $p = 1$, *or* $p = 1$. *For each* $y \in Y$ *there exists a complex analytic covering map* $U \longrightarrow V_y$ *whose covering transformations constitute a group* G_y *of motions of* U. *Furthermore there is a* G_y*-invariant tiling of* U *by centrally symmetric convex* 4p*-gons* R_y *such that the covering map images all the vertices to* $x_0(y)$, *all the centres to* $x_{2p+1}(y)$, *and all pairs of mid-points of opposite edges to the remaining roots* $\{x_1(y), \dots ,x_{2p}(y)\}$ *of* $F(x,y) = 0$.

We omit the proof but discuss below some aspects of the fascinating and rich theory (also pioneered by Poincaré !) of Fuchsian groups and automorphic functions to which this result belongs.

Remarks re Prop. 2 :

(1) We first recall the meanings of some words used above :

The **hyperbolic plane** is the open unit complex disk Δ, the "lines" being circular arcs (including its diameters) which are perpendicular to its **horizon** $\partial\Delta$. The angles are the usual ones.

The **parabolic plane** is the complex plane $\mathbb{C}$, with the usual straight lines and angles.

The **elliptic plane** is the extended complex planr $\hat{\mathbb{C}}$ (so it is in fact a 2-sphere, not a plane !), the "lines" being great circle arcs, and the angles being the usual ones.

By an orientation preserving rigid **motion** of U we will mean any complex analytic homeomorphism $U \longrightarrow U$. This makes sense because such a map preserves angles and maps "lines" into "lines"; further we can, in each case, define a **distance** (unique upto a constant multiple) which is preserved by these motions.

(Later we'll also need **affine motions** of U, i.e. those which map each "line" linearly on a "line", but need not preserve angles.)

The notions of **convexity, polygon, central symmetry, mid-point**, etc., are now defined, for each case, in the usual way.

Given a group of motions a complex analytic function on U is called an **automorphic function** of this group if it is invariant under it. Obviously only *discontinuous* or **Fuchsian groups** can have non constant automorphic functions.

(2) The Fuchsian group G_y of Prop.2, which is isomorphic to the **fundamental group of** V_y, is in addition also *fixed point free.*

This implies, for the genus p = 0 case, that $G_y = 1$ and so the tiling of U (a 2-sphere now) has just one 0-gon, viz. all of U ! And, for all $p \geq 1$ this condition implies that the **sum of the angles** of the 4p-gons of the tiling of U is 2π.

The identifications of the opposite pairs of edges of any 4p-gon of the tiling can be effected by 2p motion S_j which **generate** G_y.

However conversely G_y does not determine the tiling uniquely (and this ambiguity will play an important role in the computation of the fundamental group of V) : we will say that two polygons are **equivalent** if they generate the same Fuchsian group.

Turning to the coordinates $(x(u), y_0, z(u))$ of the covering map $U \longrightarrow V_0$ we see that we have $(z(u))^2 \equiv F(x(u), y_0)$ identically in U, i.e. that x(u) and z(u) are **automorphic solutions of the polynomial equation** $z^2 = F(x, y_0)$ with group G_y.

(3) The above generalizes in fact to *all* nonsingular curves $\Phi(x,z) = 0$; also there are generalizations for *non-compact* surfaces : in fact later Poincaré will use one such result (see proof of Prop.6 below).

The special feature of the hyperelliptic case is the central symmetry of the polygons.

Using this we can split (in 2p ways !) any chosen polygon $R = R_y$ into two **(2p+1)-gons** R' and R" which are interchanged by the central symmetry of R.

Accordingly the complex curve V_y splits two parts, viz. the images of R' and R" under the covering map. Each of these is a copy of $\hat{\mathbb{C}}$ with 2p+1 **cuts** running from x_0 to x_j, $1 \le j \le 2p+1$, the two **lips** of the jth cuts being the images of the two halves of the jth edge of R' of R" :

Thus we see again that V_y is obtainable by identifying each lip of a cut of one copy to the other lip of the corresponding cut in the other copy. And, we see also that *the central symmetry of any polygon* R *is a lift to* U *of the* **involution** $(x,y,z) \longleftrightarrow (x,y,-z)$ *of the hyperelliptic curve* V_y.

Consider now the **bigger Fuchsian group** $\overline{G_y}$, obtained from G_y by attaching the central symmetry of the polygon R (and thus of all polygons). So G_y is a subgroup of $\overline{G_y}$ of index 2, and R' and R" are two tiles of a **finer tiling** of U generated by this bigger group.

The group $\overline{G_y}$ is *not* fixed point free, and if we divide U out by its action we obtain the 2-sphere $\hat{\mathbb{C}}$. This corresponds to the fact that *if we divide out the complex curve by the involution* $(x,y,z) \longleftrightarrow (x,y,-z)$ *we obtain a 2-sphere*, viz. the complex curve $z = F(x,y)$, which can be parametrized by the automorphic functions $(z(u))^2$ and $x(u)$ of $\overline{G_y}$.
[(4) The (multiple-valued) inverse $u(x)$ of the Fuchsian function $x(u)$ (and likewise of $z(u)$) equals some **Abelian integral** over the complex curve, i.e. one has $u(x) = \int^x Q(t,\sqrt{F(t,y_0)})dt$, where Q is a suitable rational function of two variables. (The dependence of G_y and R_y on y can be worked out from this formula.)

For p = 1 these are the **elliptic integrals** of **Legendre**, so named because they arose while computing arc length of ellipses. (However the geometry for the case p = 1, i.e. the usual euclidean geometry of $\mathbb{C}$, is *parabolic*, not elliptic !)

We note that each choice of generators of the Fuchsian group determines some **periods** of the above indefinite integral, or **meromorphic differential form** $Q(t,\sqrt{F(t,y_0)})dt$, of the complex curve V_0 and the additive subgroup of $\mathbb{C}$ spanned by these periods is isomorphic to the free part of $H_1(V_0,\mathbb{Z})$.

The so-called **normal periods** correspond to a somewhat different choice (than Poincaré's) of a fundamental polygon, viz. that in which the 4p edges identified as per the defining commutator relation of $\pi_1(V_0)$.

(5) Inverting elliptic integrals **Abel**, **Gauss**, and **Jacobi** innaugurated, in the 1820's, the study of **elliptic functions** $u(x)$. (More generally, for any p, these inverse **Abelian functions** are variously called *automorphic functions*, *Fuchsian functions*, *hyperelliptic functions* etc.)

The same Abel had previously shown that the general fifth degree polynomial equation in one variable is not solvable by radicals : *so maybe Abel was searching for transcendental solutions for degrees* ≥ 5 ?

In fact these functions *do* provide such solutions : e.g. (1) shows that $x(u(z)_{z=0})$ are precisely all the roots of the degree 2p+1 or 2p+2 polynomial equation $F(x, y_0) = 0$ in x. Unfortunately however (the rational functions entering into the integrals giving) *these transcendental solutions have been* explicitly *worked out for very few cases with* $p \geq 2$! (See e.g. **Klein**'s book on the icosahedron for fifth degree equations, and an appendix in **Mumford**'s book on theta functions.)

Of course for $p = 0$ and $p = 1$, when these solutions are rational and elliptic functions respectively, one has explicit formulae. For instance, $z^2 = 1 - x^2$ has the solution $(z(u), x(u)) = (\frac{1}{1 + u^2}, \frac{u}{1 + u^2})$, $u \in \hat{\mathbb{C}}$; while $z^2 = 4x^3 - x - 1$ has the solution $(z(u), x(u)) = (\wp'(u), \wp(u))$, $u \in \mathbb{C}$, where the **Weierstrass function** $\wp(u)$ is the inverse of the elliptic integral

$$\int^x \frac{dt}{\sqrt{4t^3 - t - 1}} .$$

($z^2 = 1 - x^2$ has also the solution $(z(u), x(u)) = (\cos(u), \sin(u))$, $u \in \mathbb{C}$, but this does *not* parametrize the 2 points at infinity.)]

Proposition 3. *The covering map* $U \longrightarrow V_y \subset V_Y$, *the group* G_y, *the set of tilings* (as in Prop.2) *of* U, *and the set of roots of* $F(x,y) = 0$, *are all continuous functions of* $y \in Y$. *Furthermore,* each *of these tilings or roots is* a locally *single-valued continuous function of y which get permuted when y describes a closed curve.*

Proof. The continuity of the roots and of the covering map seem obvious (and the latter can be precised by studying the behaviour of the integrals of remark (4) with respect to the parameter y).

The continuity of G_y follows because these are motions commuting with the continuous covering map, and the continuity of the set of tilings follows because this set determines and is determined by G_y.

To see the local continuity of any our tilings look at any 4p-gon R of this tiling and the roots corresponding to its vertices, centre and edge mid-points. The local continuity of these roots and that of the covering map now uniquely specifies a nearby 4p-gon, and thus a nearby tiling, for any nearby y. *q.e.d.*

Proposition 4. *If* $p = 1$, *and* V_C *consists of all points of* V *for which* y *is on a simple closed curve* $C \subset Y$, *then* V_C *is diffeomorphic to one of the orientable* 3-*manifolds of* Example 6 *of* "Analysis Situs".

Proof. Choose any map $\mathbb{R} \to C$, $\zeta \longmapsto y_\zeta$, with $y_\zeta = y_{\zeta+1}$. Next, using Prop.3, we choose two independent translations ω and ω' of $\mathbb{C}$, which vary continuously with ζ, and generate the group G_ζ of covering transformations of the covering map $\mathbb{C} \to V_\zeta$.

Since $G_\zeta = G_{\zeta+1}$ the pair (ω, ω') of translations should change to an equivalent one when ζ increases by 1, i.e. it should change to $(\alpha\omega + \beta\omega', \gamma\omega + \delta\omega')$ where $T_C = \begin{bmatrix}\alpha & \gamma\\ \beta & \delta\end{bmatrix} \in SL(2,\mathbb{Z})$.

For each (x,y,z) of V_C choose any $\zeta \in \mathbb{R}$ such that $y = y_\zeta$ and let (ω,ω') be the corresponding pair of translations of $\mathbb{C}$. Next choose a $u \in \mathbb{C}$ which is above (x,y,z) with respect to the covering map $\mathbb{C} \to V_\zeta$ and let ξ and η be the real numbers defined by $u = \xi\omega + \eta\omega'$.

It is easy to check that the set of *all* such 3-tuples of numbers (ξ,η,ζ) is an orbit of the discontinuous group of 3-space generated by the transformations :

$$(\xi, \eta, \zeta) \longmapsto (\xi+1, \eta, \zeta),$$
$$(\xi, \eta, \zeta) \longmapsto (\xi, \eta+1, \zeta),$$
$$(\xi, \eta, \zeta) \longmapsto (\delta\xi - \gamma\eta, -\beta\xi + \alpha\eta, \zeta+1).$$

For example to see the last note that $u = \xi\omega + \eta\omega' = (\delta\xi - \gamma\eta).(\alpha\omega + \beta\omega') + (-\beta\xi + \alpha\eta).(\gamma\omega + \delta\omega')$.

So V_C is diffeomorphic to the quotient of 3-space by the group G_C generated by these three transformations. *q.e.d.*

[The fundamental group G_C of V_C is thus the extension of the fundamental group G_0 ($\cong \mathbb{Z}^2$) of V_0 by $\mathbb{Z}$. We recall that in *"Analysis Situs"* Poincaré had classified all such extensions upto group isomorphism and had thus proved that the manifolds of Ex. 6 are diffeomorphic if and only if they have isomorphic fundamental groups.

As noted before Ex.6 is a particular instance of the **mapping torus** construction : given a diffeomorphism ϕ of any manifold M we have a manifold N of one dimension more obtained by dividing out $M \times \mathbb{R}$ by the $\mathbb{Z}$ action $(x, t) \longmapsto (\phi(x), t+1)$.

It is easily seen that a manifold is a mapping torus iff it **fibers over the circle**, and the exact homotopy sequence of this fibration shows that $\pi_1(N)$ is always an extension of $\pi_1(M)$ by $\mathbb{Z}$.]

Monodromy :

By Prop. 3 we can choose, in a unique way, for any curve y_ζ in Y, a continuously varying (4g-gon of a) tiling of U for the group $G_\zeta = G_{y_\zeta}$, starting from a (4g-gon of a) stipulated tiling for $\zeta = 0$. We will denote by M_ζ the corresponding **affine flow** of points of U (i.e M_ζ lies in the same moving 4g-gon for all ζ and has the same convex coordinates with respect to its moving vertices).

That is, we have a a one-parameter group of diffeomorphisms $M_0 \longmapsto M_\zeta$ which map each each "line" of U linearly on a "line" but need not preserve angles.

[This flow of U depends only on the curve y_ζ of Y and is independent of the stipulated tiling for $\zeta = 0$ used in the above definition.]

We note that these affine motions $M_0 \longmapsto M_\zeta$ commutes with the groups G_0 and G_ζ of rigid motions of U (and so if $y_0 = y_\zeta$, these are in the **normalizer** of $G_0 = G_\zeta$). Thus there is also a well-defined flow of points $P_\zeta \in V_\zeta = V_{y_\zeta}$ in V (and if $y_0 = y_\zeta$ a diffeomorphism of $V_0 = V_\zeta$).

Proposition 4'. *For any* p, *and any simple closed curve* $C \subset Y$ *based at* y_0, V_C *is diffeomorphic to the mapping torus of the above monodromy diffeomorphism of* V_0.

Thus the fundamental group of V_C *is isomorphic to an extension of the fundamental group of* V_0 *by* $\mathbb{Z}$.

Proof. Just as in the case p = 1 choose any map $\mathbb{R} \longrightarrow C$, $\zeta \longmapsto y_\zeta$, with $y_\zeta = y_{\zeta+1}$. Next, using Prop. 3, we choose 2p continuously varying generators $S_1, \ldots, S_{2p}$ of the group G_ζ of covering transformations of the covering map $U \longrightarrow V_\zeta$.

For each (x,y,z) of V_C choose any $\zeta \in \mathbb{R}$ such that $y = y_\zeta$ and choose a point $M_\zeta \in U$ which is above (x,y,z) with respect to the covering map $U \longrightarrow V_\zeta$ and let ξ and η be the real and imaginary parts of the initial point $M_0 \in U$ of its flow line.

It is easily seen that the set of *all* such 3-tuples of numbers (ξ,η,ζ) is an orbit of the discontinuous group G_C of $U \times \mathbb{R}$ generated by the 2p+1 transformations :

$$(\xi,\ \eta,\ \zeta) \longmapsto (\phi_k(\xi,\eta),\ \psi_k(\xi,\eta),\ \zeta),\ 1 \le k \le 2p,$$
$$(\xi,\ \eta,\ \zeta) \longmapsto (\theta(\xi,\eta),\ \theta_1(\xi,\eta),\ \zeta+1).$$

Here $\phi_k(\xi,\nu)$ and $\psi_k(\xi,\eta)$ denote the real and imaginary parts of $S_k(\xi+i\eta)$ for $\zeta = 0$; and $\xi+i\eta \longmapsto \theta(\xi,\eta)+i\theta_1(\xi,\eta)$ is the monodromy diffeomorphism $M_0 \longmapsto M_1$ of U.

By definition the quotient of $U \times \mathbb{R}$ by this group is the mapping torus of the induced monodromy diffeomorphism of V_0.

The subgroup of G_C generated by the first 2p transformations is clearly isomorphic to G_0. The fact that it is normal follows from the fact that the diffeomorphism $M_0 \longmapsto M_1$ of U is in the normalizer of G_0. *q.e.d.*

[SOME NOTES ON EXTENSIONS OF GROUPS BY $\mathbb{Z}$ AND POINCARE'S SIXTH EXAMPLE :

We emphasize that for us **an extension of** H **by** $\mathbb{Z}$ always means a *group* G which fits into some exact sequence of the type

$$0 \longrightarrow H \xrightarrow{\subseteq} G \xrightarrow{j} \mathbb{Z} \longrightarrow 0$$

and not, as is sometimes the case, such a short exact sequence j itself. (We note that sometimes such groups G, or else short exact sequences j, are also called " extensions of $\mathbb{Z}$ by H ".)

Thus, for any H, there is the **surjection** $\mathfrak{H}$ which associates to each isomorphism class [j] of such exact sequences the isomorphism class [G] of extensions of H by $\mathbb{Z}$.

PROBLEM. *For which groups* H *is* $\mathfrak{H}$ *a bijection ?*

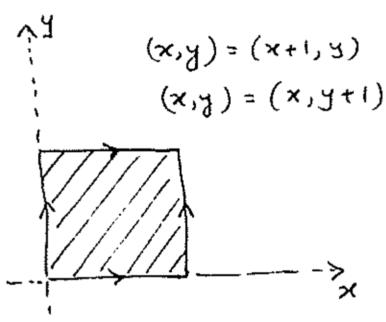
We will reformulate this problem by using the following definition.

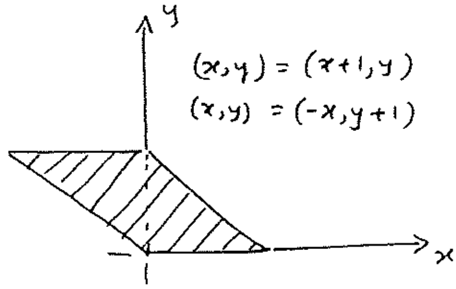
Groups H_J : given an automorphism J of a group H we will denote by H_J the group generated by H and a new element g, subject to the relations of H and the new relations $g + h = J(h) + g \ \forall\, h \in H$.

Example. $H = \mathbb{Z}$ has only two automorphisms, viz. the identity map $I(n) = n$ and its negative $(-I)(n) = -n$, and the two groups one gets are

$$\mathbb{Z}_I = \langle h, g \mid g + h = h + g\rangle \quad \text{and} \quad \mathbb{Z}_{-I} = \langle h, g \mid g + h = -h + g\rangle.$$

These two groups are non-isomorphic since the former is Abelian while the latter is not : more generally note, for any H, that **the group** H_J **is Abelian iff** J **is the identity automorphism** I **of** H. $\mathbb{Z}_I$ and $\mathbb{Z}_{-I}$ are respectively the fundamental groups of the 2-dimensional **torus** and the **Klein Bottle**, viz. the **mapping tori** $(S^1)_I$ and $(S^1)_{-I}$ **of the circle** $S^1 = \mathbb{R}/\mathbb{Z}$ obtained by using the diffeomorphisms $\pm I : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.





We remark that the homology groups of these surfaces are denoted $H_i(\mathbb{Z}_{\pm I})$. So $H_1(\mathbb{Z}_\pm)$ are the Abelianizations $\mathbb{Z}^2$ and $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ of their fundamental groups $\mathbb{Z}_{\pm I}$: more generally, for any G, one defines $H_i(G)$ as follows, and it is true that $H_1(G)$ **is the Abelianization of** G.

Group homology. $H_i(G;\mathbb{Z})$ is defined to be $H_i(X/G,\mathbb{Z})$ where X is any space deformable to a point on which G acts as a discontinuous group of fixed point free transformations.

Of course it needs to be checked that such an X exists and that this definition is independent of the choice of X in a natural way. Following **Milnor** there is also a pleasant *canonical* choice for X, viz. the **infinite join** EG = G * G * ... with the diagonal action of G. Its quotient BG = EG/G is called the **classifying space** of the group G.

It is easily seen that if $G = H_J$, where $H = \pi_1(M)$ and the group automorphism $J : H \longrightarrow H$ is induced by a diffeomorphism $J : M \longrightarrow M$, then we can take X/G = mapping torus M_J : so the **homology of** H_J **coincides with that of the mapping torus** M_J.

We solved above, for the particular case $H = \mathbb{Z}$, THE PROBLEM OF CLASSIFYING ALL GROUPS H_J. This ties up with the problem mentioned before as follows.

For any Abelian H, the exact sequence j determines an automorphism J of H : choose any $g \in G$ such that $j(g) = 1$ and set $J(h) = g + h - g$. Conversely, any $J \in \mathrm{Aut}(H)$, determines such an exact sequence j : let $G = H_J$ and set $j(g) = 1$.

It is easily verified that this gives a **bijection** $[j] \longleftrightarrow [J, J^{-1}]$ **between isomorphism classes of exact sequences** j **and conjugacy classes of all pairs** $\{J, J^{-1}\} \subseteq \mathrm{Aut}(H)$.

(These remarks generalize also to non-Abelian groups H, provided we replace Aut(H) by $\frac{\mathrm{Aut(H)}}{\mathrm{Inn(H)}}$: this because the above J is now well-defined only upto an inner automorphism, i.e. $J \in \frac{\mathrm{Aut(H)}}{\mathrm{Inn(H)}}$.)

So e.g. the statement " $\mathfrak{H}$ *is a bijection when* $H = \mathbb{Z}^2$ " is equivalent to the following.

Proposition 5. *The group* $G = (\mathbb{Z}^2)_J$ *is isomorphic to* $G' = (\mathbb{Z}^2)_{J'}$, *if and only if J is conjugate to J' or its inverse in* GL(2,$\mathbb{Z}$).

This strengthens Prop. 17a of "Analysis Situs", moreover the argument given below is more conceptual than Poincaré's.

Proof of "if" . If AJ = J'A then the map $H_J \longrightarrow H_{J'}$ (where $H = \mathbb{Z}^2$) given by $h \longmapsto A(h)$ and $g \longmapsto g$ is well-defined because $g + h \longmapsto g + A(h) = J'A(h) + g = AJ(h) + g$ and $J(h) + g \longmapsto AJ(h) + g$. This can be checked to be an isomorphism $H_J \cong H_{J'}$.

If $J' = J^{-1}$ then the map $H_J \longrightarrow H_{J'}$ given by $h \longmapsto h$ and $g \longmapsto -g$ is well-defined : for this note on the one hand that $g + h \longmapsto -g + h = (J')^{-1}(h) - g$ (since putting $h = J'(k)$ here we get $-g + J'(k) = k - g$, i.e. $g + k = J'(k) + g$, a relation of $H_{J'}$) $= J(h) - g$ and on the other that $J(h) + g \longmapsto J(h) - g$. This gives an isomorphism $H_J \cong H_{J'}$.

Proof of "only if". Let $\phi : G' \cong G$ be any isomorphism. We'll use the following properties of $\phi(\mathbb{Z}^2)$:

(a) $\phi(\mathbb{Z}^2)$ is, like $\mathbb{Z}^2$, a free Abelian group of rank 2.

(b) $\phi(\mathbb{Z}^2)$ is normal in G because $\mathbb{Z}^2$ is normal in G'.

(c) If a nonzero multiple of some element of G is in $\phi(\mathbb{Z}^2)$ then that element is in $\phi(\mathbb{Z}^2)$: this follows from $G'/\mathbb{Z}^2 \cong \mathbb{Z}$ which that the subgroup $\mathbb{Z}^2$ of G' has the analogous property.

It follows from (b) and (c) that if $\phi(\mathbb{Z}^2) \subseteq \mathbb{Z}^2$ then $\phi(\mathbb{Z}^2) = \mathbb{Z}^2$ and so, in this case, ϕ itself provides the required isomorphism between the exact sequences j and j' or, equivalently, the required conjugation from J to J' or its inverse.

So lets assume from here on that $\phi(\mathbb{Z}^2)$ *is not contained in* $\mathbb{Z}^2$. Then we have in addition the following :

(d) $\phi(\mathbb{Z}^2) \cap \mathbb{Z}^2$ is free Abelian of rank 1 : to see this use (a) and

$$\frac{\phi(\mathbb{Z}^2)}{\phi(\mathbb{Z}^2) \cap \mathbb{Z}^2} \cong \mathbb{Z},$$

which follows since $j : \phi(\mathbb{Z}^2) \longrightarrow \mathbb{Z}$ is a nonzero (but possibly non-surjective) map with kernel $\phi(\mathbb{Z}^2) \cap \mathbb{Z}^2$.

(e) $\dfrac{\mathbb{Z}^2}{\phi(\mathbb{Z}^2) \cap \mathbb{Z}^2} \cong \mathbb{Z}$, since by (c) this quotient is free.

(f) J, being an inner automorphism of G, maps the normal subgroup $\phi(\mathbb{Z}^2) \cap \mathbb{Z}^2$ of G onto itself.

It follows from (e) and (f) that we can choose a basis β of $\mathbb{Z}^2$ such that with respect to it J has a matrix of the type

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & t \\ 0 & -1 \end{bmatrix}$$

if det(J) = 1, or of the type

$$\begin{bmatrix} -1 & t \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & t \\ 0 & -1 \end{bmatrix}$$

if det(J) = -1.

(So unless tr(J) = ±2 and det(J) = 1, or tr(J) = 0 and det(J) = -1, $\phi(\mathbb{Z}^2)$ is necessarily equal to $\mathbb{Z}^2$: thus **generically one has** $\phi(\mathbb{Z}^2) = \mathbb{Z}^2$ and it is only a few **exceptional cases** which make the proof hard.)

Moreover in the det(J) = -1 case we can modify β so that the matrix of J is one of the following two matrices :

$$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} \text{ or } \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}.$$

To do this we use the conjugations,

$$\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & t\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & -b\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & t-2b\\ 0 & -1\end{bmatrix},$$

and their analogues for the case when the (1,1) element is -1, followed by the conjugations

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix} \text{ and}$$

$$\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}.$$

In the det(J) = +1 case we can only change the sign of t by means of a similar simple conjugation (which we omit). Thus we have the following.

(h) We can choose an integer **basis** β of $\mathbb{Z}^2$ with respect to which J has one of the following matrices, where $t \geq 0$:

$$(1)\ \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix},\ (2)\ \begin{bmatrix}-1 & t\\ 0 & -1\end{bmatrix}\ (3)\ \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} \text{ or } (4)\ \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}.$$

Likewise it follows, by using ϕ^{-1} instead of ϕ, that there is an integer basis β' with respect to which the matrix of J' is also from one of the above listed possibilities.

We note that for a matrix J of the above four types the Smith invariants of $J - I$ are (t,0), (2,2) or (1,4), (2,0) and (1,2) respectively; and also that, in case (2), the Smith invariants of $J^2 - I$ are (2t,0). These facts will be used to prove the following.

(i) **The matrix of J with respect to β is identical with that of J' with respect to β'.**

This will complete the proof, for then $B \in GL(2,\mathbb{Z})$, where $B(\beta) = \beta'$, provides a conjugation between J and J'.

Since $g + h - g - h = J(h) + g - g - h = J(h) - h$ we start by noting that the subgroup $H_J \# H_J$ of H_J generated by all its **commutators** coincides with $\mathrm{Im}(J - I)$. (More generally the lower **central series** G_i, $i \geq 0$, of the group $G = (\mathbb{Z}^2)_J$ is given by $G_i = \mathrm{Im}(J - I)^i$: here $G_0 = G$ and $G_{i+1} = G \# G_i$, the subgroup generated by all commutators $g + h - g + h$ with $g \in G$ and $h \in G_i$.)

So the first homology group of H_J , i.e. its Abelianization, is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}^2/\mathrm{Im}(J - I))$, and we have a similar expression for the first homology of $H_{J'}$. Since $H_J \cong H_{J'}$ their homologies are isomorphic, and so **the matrices** $J - I$ **and** $J' - I$ **must have the same Smith invariants.**

Also the isomorphism of homologies implies that J **and** J' **have the same determinant** : to see this note that $H_3((\mathbb{Z}^2)_J) = H_3((T^2)_J) \cong \mathbb{Z}$ iff the mapping torus $(T^2)_J$ is **orientable**, and this happens iff $\det(J) = 1$.

These necessary conditions show that the matrices of J and J' (w.r.t. β and β') are same except possibly when they are both of type (2).

For this we use the fact that $H_J \cong H_{J'}$ implies $2H_J \cong 2H_{J'}$ where 2G denotes the group generated by **doubles** of all elements of G, and hence the Abelianizations of these doubled groups are also isomorphic. So using $2g + 2h - 2g - 2h = J^2(2h) + 2g - 2g + 2h = J^2(2h) - 2h$ it follows as above that **the matrices** $(J)^2 - I$ **and** $(J')^2 - I$ **also have the same Smith invariants** (and likewsise $(J)^N - I$ and $(J')^N - I$ have the same Smith invariants for all $N \geq 1$). This new condition shows that the matrices of J and J' are same when they are both of type (2). *q.e.d.*

Examples. $J = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} \in GL(2,\mathbb{Z})$ is not conjugate to its inverse $J^{-1} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix}$ (even in $GL(2,\mathbb{Q})$!) because their **characteristic polynomials** $|J-xI| = -1-4x+x^2$ and $|J^{-1}-xI| = -1+4x+x^2$ are distinct. For the same reason a matrix $A \in GL(n,\mathbb{Z})$ and its inverse A^{-1} are *usually* not similar.

If $A \in SL(2,\mathbb{Z})$, then it has the same characteristic polynomial as its inverse and the above argument does not apply. Nevertheless it is easy to see that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SL(2,\mathbb{Z})$ is not conjugate to its inverse $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *in* $SL(2,\mathbb{Z})$ (or even in $GL_+(2,\mathbb{R})$!) because

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ i.e. } \begin{bmatrix} b & -a \\ d & -c \end{bmatrix} = \begin{bmatrix} c & d \\ -a & -b \end{bmatrix},$$

gives b = c and a = -d, and so ad-bc = $-a^2-b^2$ can't be positive.

Further we note that a conjugation with $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ changes any A $\in$ SL(2,$\mathbb{Z}$) to $(A^{-1})'$, so we ask : *is an integer matrix* A *always similar over* $\mathbb{Z}$ *to its* **transpose** A' ? As we'll see later the answer to this question is "No", and **there exist matrices** A $\in$ SL(2,$\mathbb{Z}$) **not conjugate to their inverses in** GL(2,$\mathbb{Z}$).

(On the other hand recall that A **is similar to** B **over** $\mathbb{Q}$ **iff** A − xI **and** B − xI **have the same Smith invariants over the p.i.d.** $\mathbb{Q}$[x], so it follows that A and A' are similar *over* $\mathbb{Q}$.)

However note e.g. that the matrix $\begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix} \in$ SL(2,$\mathbb{Z}$) is conjugate to its inverse $\begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$ in GL(2,$\mathbb{Z}$) by virtue of the next result which applies because the common characteristic polynomial $1+x^2$ of these 2 matrices is irreducible over $\mathbb{Q}$ and the **Gaussian integers** $\mathbb{Z}$[i] constitute a p.i.d..

Before stating this we recall that ideals $\mathfrak{A}$ and $\mathfrak{B}$ of the ring $\mathbb{Z}[\theta]$ are said to belong to the same **ideal class** iff $a\mathfrak{A} = b\mathfrak{B}$ for some nonzero a,b $\in \mathbb{Z}[[\theta]$, and that for any algebraic number θ, **there are only finitely many ideal classes**, and their number is the **class number** h of $\mathbb{Z}[\theta]$.

This basic finiteness theorem of **Kummer** is a by-product of his careful analysis of an insufficient argument which he (and independently **Lame'**) had found for **Fermat's Last Theorem** $p \geq 3 \Rightarrow x^p + y^p \neq z^p$. Kummer showed that this argument established FLT for **regular primes** p, i.e. those which do not divide the class number of $\mathbb{Z}[\exp(2\pi i/p)]$. (For more regarding this see **Edwards** book on FLT.)

Proposition 6 (LATIMER-MACDUFFEE THEOREM). *The class number of the ring* $\mathbb{Z}[\theta]$, *where* θ *is a root of a monic degree* n *polynomial* $f(x) \in \mathbb{Z}[x]$ *irreducible over* $\mathbb{Q}$, *is equal to the number of similarity classes of* n×n *integer matrices* A *such that* f(A) = 0.

This they deduce from a natural 1-1 onto correspondence $[A] \longleftrightarrow [\mathfrak{A}]$ between similarity classes over $\mathbb{Z}$ of matrices having f(x) as characteristic polynomial and ideal classes of $\mathbb{Z}[\theta]$: we remark that (for n = 2) this correspondence was studied by **Poincare'** himself, *Jour. de l'Ecole Polytech.* 47 (1880), 177-245 !

We note that while proving Prop.5 we saw that the infinitely many integer matrices $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, $n \geq 0$, are not similar to each other, yet they all satisfy $f(x) = (1-x)^2$. So above result is not true if f(x) has multiple roots; however we note that L-M also give a version of the above result for all f(x) having distinct roots.

Prop. 6 shows that **the full classification of the groups** $(\mathbb{Z}^2)_J$ **is quite deep**, e.g. just to enumerate such groups with tr(T) = t we need class numbers of some quadratic number fields *and more*, but what we have is already enough for the following.

Proposition 7. *Let* $\mathfrak{G}(t)$ *be the number of diffeomorphism classes of 3-manifolds defined as in* Example 6 *of* "Analysis Situs" *with* a + d = t. *Then, for* $t \neq \pm 2$, $\mathfrak{G}(t)$ *is finite, and we have*

$$h(t)/2 \leq \mathfrak{G}(t) \leq h(t),$$

where h(t) *is the class number of* $\mathbb{Z}[(t^2-4)^{1/2}]$.

For t = 2 or −2 the number of manifolds is countably infinite (cf. proof of Prop. 5).

Proof. $\mathfrak{G}(t)$ is the number of isomorphism classes of groups $(\mathbb{Z}^2)_J$ with det(J) = 1 and tr(J) = t. So, by Prop.5, this is the number of conjugacy classes in $GL(2,\mathbb{Z})$ of pairs $\{J,J^{-1}\}$ of 2x2 integer matrices having characteristic polynomial $x^2 - tx + 1$. Its discriminant t^2-4 is the square of a rational, only if it is the square r^2 of an integer r. So $|t|-|r|$, $|t|+|r|$ must be equal to 1, 4 or 2, 2; and only the latter can happen, and then $t = \pm 2$ and r = 0. Otherwise, the polynomial is irreducible over $\mathbb{Q}$, and the result follows by using Prop.6. *q.e.d.*

The finite set of ideal classes $[\mathfrak{A}]$ of $\mathbb{Z}[\theta]$, equipped with the operation, $[\mathfrak{A}] + [\mathfrak{B}] = [\mathfrak{A}\mathfrak{B}]$ where $\mathfrak{A}\mathfrak{B}$ denotes the ideal generated by all products ab, $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, becomes an Abelian group which is called the **class group** of $\mathbb{Z}[\theta]$.

(Thus each polynomial $f(x) \in \mathbb{Z}[x]$ has two important finite groups : the usually non-Abelian **Galois group** and the Abelian class group. In some ways these are, respectively, the algebraical analogues of the fundamental and homology groups of a space.)

Under the **Poincaré-Latimer-MacDuffee correspondence** $[A] \longleftrightarrow [\mathfrak{A}]$ (which has been extensively studied by **Taussky**) the **zero ideal class** is represented by the **companion matrix**

$$\begin{bmatrix} a_1 & a_2 & & a_n \\ 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & & 1 & 0 \end{bmatrix}$$

of the polynomial $f(x) = x^n - a_1x^{n-1} - \dots - a_n$, and **negatives of ideal classes correspond to transposition of matrices** (*: thus the class group of* $\mathbb{Z}[\theta]$ *contains elements of order* ≥ 3 *if and only if there are matrices* A *satisfing* $f(x) = 0$ *for which* A *is not similar over* $\mathbb{Z}$ *to* A').

(If $f(x) = |A - xI|$ is reducible, then the direct sum of the companion matrices of its **elementary divisors**, i.e. prime powers occuring in the Smith invariants over $\mathbb{Q}(x)$ of $A - xI$, is the **Frobenius normal form** of the similarity class over $\mathbb{Q}$ of A. Taking all possible "Smith factorizations" of a given monic $f(x) \in \mathbb{Z}[x]$, this process gives, upto similarity over $\mathbb{Q}$, all possible nxn **matrix solutions** of $f(x) = 0$.)

We can now improve Prop.7 to the following

Proposition 7'. *For* $t \neq \pm 2$, *the number* $\sigma(t)$ *of diffeomorphism classes of 3-manifolds defined as in* Example 6 *of* "Analysis Situs" *with* $a + d = t$ *is given by*

$$\sigma(t) = \frac{h(t) + n_2(t) + 1}{2} ,$$

where $h(t)$ *is the order of, and* $n_2(t)$ *the number of elements of order 2 in, the class group of* $\mathbb{Z}[(t^2 - 4)^{1/2}]$.

Proof. This follows because, by the above remarks, $\sigma(t)$ = number of pairs $\{[\mathfrak{U}],-[\mathfrak{U}]\}$ in the class group, and this equals $1 + n_2(t)$ plus half of the remaining $h - n_2(t) - 1$ elements. *q.e.d.*

We conjecture that there is, but have so far not found, an H for which $\mathfrak{H}$ is not bijective. This amounts to finding a group G with two isomorphic normal subgroups H and H' having $G/H \cong \mathbb{Z} \cong G/H'$ and such that there is no automorphism of G which maps H onto H'. On the other hand we can improve Prop.5 as follows.

Proposition 5'. *$\mathfrak{H}$ is bijective whenever* H *is a finite group, simple group, or a surface group.*

Here by a **surface group** we mean the fundamental group of a surface.

Proof. If j(x) is not zero then x cannot be of finite order in G. So, *if* H *is finite*, H = Tor(G). Thus any isomorphism $G \longrightarrow G'$ between two extensions of H by $\mathbb{Z}$ preserves H, and so determines an isomorphism of the two exact sequences.

(The same remark shows that Tor(H) = Tor(G) for any H. So for an Abelian H, when Tor(H) is a subgroup, there is an induced exact sequence $0 \longrightarrow H/Tor(H) \xrightarrow{\subseteq} G/Tor(G) \xrightarrow{j} \mathbb{Z} \longrightarrow 0$. Thus G is an extension of H by $\mathbb{Z}$ iff G/Tor(G) is an extension of H/Tor(H) by $\mathbb{Z}$.

For a finitely generated Abelian H, $H/Tor(H) \cong \mathbb{Z}^n$ for some n, and any isomorphism $G \longrightarrow G'$ between two extensions of H by $\mathbb{Z}$ preserves Tor(H) : so *the result will follow for all finitely generated Abelian groups* H *also if it were known for* $H = \mathbb{Z}^n$.)

If H *is a simple group* then the result follows because (with same notation as in proof of Prop.5) now either $\phi(H)\cap H = 1$ or $\phi(H)\cap H = H$, with the former possible only if $H \cong \mathbb{Z}$.

Prop.5 dealt with the (hard) case of the fundamental group $\mathbb{Z}^2$ of the 2-torus. *If* H *is isomorphic to the fundamental group of a surface of* genus $p \geq 2$ then again we have $\phi(H) = H$ since H has only one non-trivial normal subgroup which is isomorphic to $\mathbb{Z}$, with quotient $\mathbb{Z}^{2p}$. *q.e.d.*

One has parallel generalizations of the topological Prop. 17 of "*Analysis Situs*".

Proposition 5". *Let* M *be a manifold whose fundamental group* H *has* $\mathfrak{H}$ *one-one, and is such that each automorphism of* H *is induced by some diffeomorphism. Then two mapping tori of* M *are diffeomorphic iff their fundamental groups are isomorphic.*

Proof. The fundamental groups of the two tori are extensions G and G' of H by $\mathbb{Z}$. Let $G \cong H_J$ and $G' \cong H_{J'}$ and let $\phi \in \mathrm{Aut}(H)$ be the conjugation from J to J' or its inverse. Choose an $F \in \mathrm{Diff}(M)$ which induces ϕ. Then $[x,t] \longmapsto [F(x),t]$ gives the required diffeomorphism of the two mapping tori. *q.e.d.*

We remark that the harder problem of classifying *all* mapping tori (rather than of a specific manifold) would entail tackling the following

Question. *If two diffeomorphic* (n+1)*-dimensional manifolds are mapping toruses of* n*-manifolds* M *and* M', *then is it true that* M *is diffeomorphic to* M' ? (There is a similar question for homeomorphisms.)

We note that the hypothesis of the above question implies that $M \times \mathbb{R}$ is diffeomorphic to $M' \times \mathbb{R}$, thus an affirmative answer would follow if the the following **cancellation property** were valid.

Q. $M \times \mathbb{R} \cong M' \times \mathbb{R} \Rightarrow M \cong M'$?

However the answer to this second question (which has been extensively studied) is in general NO !

For example, a theorem of **Mazur** says that in dimensions $n \geq 5$ one has $M \times \mathbb{R}^{n+2} \cong M' \times \mathbb{R}^{n+2}$ if and only if the n-manifolds M and M' have the same **simple homotopy type** (a notion much weaker than being diffeomorphic).

Again, **Kirby** and **Siebenmann** have shown that, for any simply connected *topological* 4-manifolds M, either there is no other, or exactly one other such manifold M' with $M \times \mathbb{R} \cong M' \times \mathbb{R}$, and that both cases occur.

The cancellation property is known to hold for many classes of smooth manifolds, e.g. all simply connected ones.

For groups $(\mathbb{Z}^n)_J$, $n \geq 3$, we give only a preliminary result which can be much improved (however we do not know if N can be reduced to 1).

Proposition 8. *There exists an* N *depending only on* n *such that we have* $(\mathbb{Z}^n)_J \cong (\mathbb{Z}^n)_{J'}$ *only if* $(J)^N$ *is conjugate to* $(J')^{\pm N}$ *in* $GL(n,\mathbb{Z})$.

Proof. As in the proof of Prop. 5 we check that if the isomorphism ϕ : $(\mathbb{Z}^n)_{J'} \longrightarrow (\mathbb{Z}^n)_{J'}$ maps $\mathbb{Z}^n$ into itself then $\phi(\mathbb{Z}^n) = \mathbb{Z}^n$ and ϕ itself provides the required isomorphism between the exact sequences j and j' or, equivalently, a conjugation from J to J' or its inverse.

So, in this **generic case** (we'll justify this terminology below), we can in fact take N = 1. It is the remaining **exceptional case,** $\phi(\mathbb{Z}^n)$ *not contained in* $\mathbb{Z}^n$, which will concern us from now on.

Exactly as before we see that the quotient of $\mathbb{Z}^n$ by $\phi(\mathbb{Z}^n) \cap \mathbb{Z}^n$ is $\mathbb{Z}$, that J preserves this subgroup, and so $\mathbb{Z}^n$ has an integer basis β w.r.t. which the matrix of J is

$$\begin{bmatrix} \pm 1 & T_{1,n-1} \\ 0_{n-1,1} & \psi_{n-1,n-1} \end{bmatrix}$$

where $\psi \in GL(n-1,\mathbb{Z})$. (Likewise a similar basis β' for J'.)

(A) **ψ is of finite order in** $GL(n-1,\mathbb{Z})$. To see this note that the restriction $j : \phi(\mathbb{Z}^n) \longrightarrow \mathbb{Z}$ of the surjection $j : (\mathbb{Z}^n)_J \longrightarrow \mathbb{Z}$ has image $r\mathbb{Z}$, for some $r > 0$, and kernel $\phi(\mathbb{Z}^n) \cap \mathbb{Z}^n$.

Thus $\phi(\mathbb{Z}^n) \cong K_L$, where $K = \phi(\mathbb{Z}^n) \cap \mathbb{Z}^n$, and $L \in Aut(K)$ is the restriction of either J^r or J^{-r} to K. Since $\phi(\mathbb{Z}^n)$ is Abelian, this restriction must be the identity automorphism of $\phi(\mathbb{Z}^n) \cap \mathbb{Z}^n$.

(B) **There exists an** N **depending only on** n **such that for any element** ψ **of finite order in** $GL(n-1,\mathbb{Z})$ **one has** $(\psi)^N = I$.

For this, note that if ψ is of order r, then the **minimal polynomial** (= highest Smith invariant of $\psi - xI$ = least degree polynomial satisfied by ψ) has *non-repeated* irreducible factors because $x^r = 1$ has distinct roots, and each of these factors is a dth **cyclotomic polynomial** for some $d|r$, with the l.c.m. of the d's being r : so r must be bounded. (For a more precise result see below.)

We will take such an even N, so w.r.t. our basis β the matrix of $(J)^N$ is of the type

$$\begin{bmatrix} 1 & t_{1,n-1} \\ 0_{n-1,1} & I_{n-1,n-1} \end{bmatrix},$$

and $(J')^N$ has a similar basis with respect to β'.

(Thus $\phi(\mathbb{Z}^n) = \mathbb{Z}^n$ unless the eigenvalues of J are Nth roots of unity.)

(C) **We can choose β in such a way that the sub-matrix $t_{1,n-1}$ of $(J)^N$ has at most one nonzero entry.** This follows because if the first row has element a in the pth column and a nonzero b in the qth column, $2 \le p < q$, then we can make q zero, replace p by (p,q), and leave all other elements unchanged by using the conjugation $M^{-1}(..)M$, where the nxn matrix M is like I, except for 4 spots (p,p), (p,q), (q,p), and (q,q), at which it has, respectively, c, $\frac{b}{(a,b)}$, d, and $-\frac{a}{(a,b)}$, with $c.\frac{a}{(a,b)} - d.\frac{b}{(a,b)} = 1$ (cf. **Newman**, *"Integral Matrices"*, p.42).

Using these all the off-diagonal elements of the matrix of $(J)^N$ can be made zero except possibly that at the (1,2) spot which we'll denote by t. Likewise for the matrix of $(J')^N$, for which the element at the (1,2) spot will be denoted t'.

As explained in the proof of Prop.5, the Smith invariants $(|t|, 0, \ldots, 0)$ of $(J)^N - I$ must coincide with the Smith invariants $(|t'|, 0, \ldots, 0)$ of $(J')^N - I$. So $t = \pm t'$, and thus the element of $GL(n,\mathbb{Z})$ relating the integer bases β and β' of $\mathbb{Z}^n$, conjugates $(J)^N$ to $(J')^{\pm N}$. *q.e.d.*

We note that (B) of the above proof is part of the argument by which

Vaidyanathaswamy proved that if an element of GL(n,ℤ) has finite order r, then the sum of the **totients** $\phi(p^a) = p^{a-1}(p-1)$ of all maximal prime powers p^a, other than 2, which occur in r, must be no more than n. Also, conversely (by using a method mentioned before) he showed that, for each such r, $x^r = 1$ has a matrix solution in GL(n,ℤ).

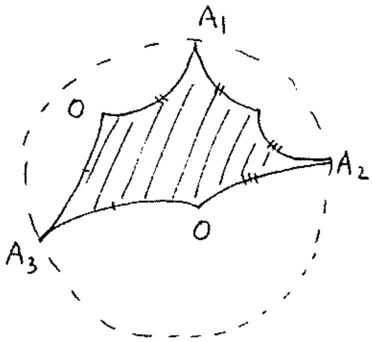
It is known also that there is an N depending only on n such that any **finite group of** GL(n,ℤ) has order less than N.]

RETURNING TO THE PAPER IN HAND we now find Poincaré beginning his analysis of the smooth 4-manifold $V_Y \subseteq V_{n.s} \subseteq V$.

Proposition 9. *The smooth 4-manifold* $V_Y = \{(x,y,z) \in V : y \in Y\}$ *is the orbit space of a fixed point free discontinuous group* G_Y *of 4-space which is isomorphic to an extension of the fundamental group* G_0 *of any of its complex curves* $V_0 = \{(x,y,z) \in V : y = y_0 \in Y\}$ *by the free group* F^{q-1} *on* q-1 *generators.*

Proof. The *non-compact* surface $Y = \hat{\mathbb{C}} \setminus \{A_1, \ldots, A_q\}$, a 2-sphere minus q points has fundamental group isomorphic to F^{q-1}.

In fact Y is the quotient of U by a **Fuchsian group of the second kind,** i.e. one whose tiles, which can be chosen as shown below, have some **cusps** on the horizon. We note that the sum of the tile's angles is again 2π with the angle at each cusp being zero, and the group ($\cong F^{q-1}$) is generated by q motions T_i which identify pairs of edges incident to the same cusp, and one has the relation $T_1 T_2 \ldots T_q = 1$.



The corresponding covering map $\zeta \longmapsto y_\zeta$ obeys $y_{T_i(\zeta)} = y_\zeta \ \forall i$ maps each tile onto Y with the non-cuspidal vertices going to a single point O, and pairs of identified edges determining disjoint **cuts** OA_i .

Next, using Prop. 3, we choose 2p continuously varying generators $S_1, \ldots, S_{2p}$ of the group G_ζ of covering transformations of the covering maps $U \to V_\zeta$ $(= V_{y_\zeta})$.

For each (x,y,z) of V_Y choose any $\zeta \in U$ such that $y = y_\zeta$ and then a point $M_\zeta \in U$ which is above (x,y,z) with respect to the covering map $U \to V_\zeta$. Next, choose any path in Y from y_0 to y_ζ, and let ξ and η be the real and imaginary parts of the initial point $M_0 \in U$ of the corresponding monodromy flow line through M_ζ.

It can be checked that the set of *all* 4-tuples of numbers $(\xi,\eta,\zeta',\zeta'')$, $\zeta'+i\zeta'' = \zeta$, thus associated to each point (x,y,z) of V_Y, is an orbit of the discontinuous group G_Y of $U \times U$ generated by the 2p+q transformations :

$$(\xi, \eta, \zeta', \zeta'') \longmapsto (\phi_k(\xi,\eta), \psi_k(\xi,\eta), \zeta', \zeta''), \ 1 \le k \le 2p,$$
$$(\xi, \eta, \zeta', \zeta'') \longmapsto (\theta_i(\xi,\eta), \theta_i'(\xi,\eta), \kappa_i(\zeta',\zeta''), \kappa_i'(\zeta',\zeta'')), \ 1 \le i \le q.$$

Here $\phi_k(\xi,\nu)$ and $\psi_k(\xi,\eta)$ denote the real and imaginary parts of $S_k(\xi+i\eta)$ for $\zeta = 0$; $\xi+i\eta \longmapsto \theta_i(\xi,\eta)+i\theta_i'(\xi,\eta)$ is the monodromy diffeomorphism $M_0 \longmapsto M_{1i}$ of U determined by any closed curve C_i of Y obtained by projecting an arc in U from some $\zeta = \zeta'+i\zeta''$ to its conjugate $T_i(\zeta) = (\kappa_i(\zeta',\zeta''), \kappa_i'(\zeta',\zeta''))$.

To see this note that the complement of the cuts is simply connected, so the monodromy $M_0 \longmapsto M_\zeta$ is unchanged as long as we deform the path from y_0 to y_ζ in this complement, but if we add C_i (i.e. a closed path around one exceptional point A_i) it changes by the amount $M_0 \longmapsto M_{1i}$.

On the other hand, the first 2p transformations account of course for the fact that the choice of M_0 (resp. M_ζ) was ambiguous upto action of G_0 (resp. G_ζ). Since the monodromies $M_0 \longmapsto M_{1i}$ commute with G_0 it follows also that the isomorph of G_0 generated by these transformations is a *normal subgroup* of G_Y.

(**Notation** *:* We'll use the notations S_k, $1 \le k \le 2p$, and T_i, $1 \le i \le q$, also for the defining transformations of G_Y. So in particular G_0 will be identified with the forementioned normal subgroup of G_Y.)

Finally, since we have exhibited G_Y as the group of covering transformations of the simply connected covering $U \times U$ of V_Y it follows that it is isomorphic to the fundamental group of V_Y (more details, some special to this case, of this isomorphism, are given below). *q.e.d.*

The isomorphism $G_Y \longrightarrow \pi_1(V_Y)$: this associates to each $g \in G_Y$ the element of $\pi_1(V_Y)$ consisting of the homotopic closed curves of Y based at y_0 which occur as the projections of arcs from any point $N_0 \in U \times U$ above y_0 to the conjugate point $g(N_0)$.

Its *surjectivity* is immediate since each closed curve at y_0 lifts to such an arc. For *injectivity*, Poincaré's argument (whose refinements will be used later to prove Prop. 11) runs as follows.

If a closed curve of V_Y at y_0 can be continuously made **arbitrary small** (i.e. is homotopically trivial) in V_Y, then it can also be made arbitrarily small in a **deformation** $(V_Y)^{\#}$ of V_Y determined by constraining y to lie outside some **guardian circles**, i.e. small disjoint 2-disks D_i, one around each exceptional point A_i.

So the y-projection of this closed curve, being also arbitrarily small, encloses none of these disks. So the given closed curve of V_Y can be deformed to one having constant y-projection y_0, i.e. to a curve in V_0.

Turning to the x-projection, we now note that, being also arbitrarily small, it cannot obviously enclose *more than one* of the roots $x_0, \dots, x_{2p+2}$ of $F(x,y_0) = 0$. In fact, it cannot enclose *any*, because otherwise z, which is a square root of $F(x,y_0)$, undergoes a sign change, which is not possible because this is projection of a *closed* curve of V_0.

But this implies that our curve lifts under $U \longrightarrow V_0$ to a closed curve. So only $1 \in G_Y$ images to $1 \in \pi_1(V_Y)$.

Proposition 10. *Any pair* $\{x_a(y), x_b(y)\}$, $y \in Y$, *of roots of* $F(x,y) = 0$ *approaches a common value* x_{ab} *at some* A_i, *and conversely, for any* i, *the equation* $F(x,A_i) = 0$ *has some* **double roots** x_{ab} *in* $\hat{\mathbb{C}}$, *but none of higher order. Furthermore, the variety* V *has only finitely many singularities, viz. the* **conical points** $(x_{ab}, A_i, 0)$ *with* $\frac{\partial F}{\partial y}$ *zero, and at these the links are real projective 3-spaces.*

In the following we'll refer to the non-singular points $(x_{ab}, A_i, 0)$ with $\frac{\partial F}{\partial y}$ *nonzero* as the **removable singularities** of V.

Proof. We remark that the first assertion is not explicit anywhere in the paper but Poincaré does seem to need something like it later. It seems to be true "in general", for otherwise $\frac{1}{x_a(y) - x_b(y)}$ would be the germ of a bounded analytic function, which is possible by Liouville's theorem only if it is a constant.

Conversely, since $F(x, A_i) = 0$ has less than 2p+2 distinct roots, it follows that some pair of roots approaches a common value ξ, and ξ cannot be the limiting value of more than two roots, because then $\frac{\partial^2 F}{\partial x^2}$ would be zero at (ξ, A_i), contradicting the second hypothesis on F.

Using the Jacobian criterion, i.e. that a point of V is non-singular if one of the three partial derivatives of $z^2 - F(x,y)$ is nonzero, it follows that the *possible* singularities are at the points mentioned.

To see that these are *indeed* singularities we use the first hypothesis on F which says that its Taylor expansion at these points starts with a non-degenerate second degree form. So, by using the discussion in the *Complément* re **Heegaard**'s example, we see that the links at these points are not 3-spheres, but real projective 3-spaces. *q.e.d.*

We remark that in the paper Poincaré makes the connection with Heegard's example only after giving a geometric argument (see proof of Prop. 12) which shows that $\pi_1(V_{n.s.})$ might have elements of order two.

[And, just before this reference, Poincaré repeats a mistake of *"Analysis Situs"* by stating that such loops don't bound 2-*manifolds* ! This curious slip, at this stage of the game, suggests that parts of this paper might have been written before the first *Complément* ?]

We recall that the group $\overline{G}_0$ has as 2p+2 generators the **central symmetries** s_a of U with respect to u_a, where $u_a \in U$ is a chosen preimage in of the root x_a; besides $(s_a)^2 = 1$ these are subject to the **Fuchsian relation** that a certain product of all of them is 1. The subgroup G_0 of $\overline{G}_0$ consists of products s of an *even* number of these s_a's.

Proposition 11. *The fundamental group* $\pi_1(V_{n.s.}$ or V) *is isomorphic to the quotient of* G_0 *by the relations* $\{T_i s T_i^{-1} = s,\ s_a s_b = 1\}$, *where* (a,b) *runs over all pairs for which there is a removable singularity, resp. removable singularity or conical point,* $(x_{ab}, A_i, 0)$ *of* V.

Proof. The above is equivalent to saying that $\pi_1(V_{n.s.}$ or V) is isomorphic to the quotient of G_Y obtained by dividing out by the normal subgroup generated by the T_i's and the stipulated $s_a s_b$'s.

To see this we start by noting that the isomorphism $G_Y \longrightarrow \pi_1(V_Y)$, followed by the map induced by the inclusion $V_Y \subseteq (V_{n.s}$ or V), gives an epimorphism $G_Y \longrightarrow \pi_1(V_{n.s.}$ or V), because clearly any loop of V can be deformed slightly so that its y-projection misses the q points A_i.

The kernel of this epimorphism keeps an account of the new deformations involving the points A_i of the y-plane.

For example T_i is in this kernel because $V_{n.s.}$ has arbitrarily small loops whose x-projections don't enclose any roots and whose y-projections are loops around A_i shrinking to this point.

Likewise $s_a s_b$ is this kernel because within $V_{n.s.}$ or V one has an arbitrarily small closed curve whose y-projections are constants approaching A_i and whose x-projections are small loops around the nearby roots $x_a(y)$ and $x_b(y)$ shrinking to the double root $x_{ab}(A_i)$.

Since any other deformation involving the A_i's is a composition of such deformations it follows that the kernel is the aforementioned normal subgroup of G_Y. *q.e.d.*

Poincaré notes that **Picard** had already proved for these (and other) surfaces that $b_1(V_{n.s.}$ or V) = 0. But, for p = 1, when G_0 = the fundamental group of the torus, $\pi_1(V_{n.s.}$ or V) is Abelian by above result. So we see that $\pi_1(V_{n.s.}$ or V) = $H_1(V_{n.s.}$ or V) is a finite group.

[From Prop. 12 below it will follow that $\pi_1(V_{n.s.})$ is actually the quotient of G_0 just by the relations $T_i s T_i^{-1} = s$. We remark that an analogous result for first homology had also been proved by Picard, and was the inspiration for Poincaré's *"Quatrième Complément"*.]

We'll see below that $\pi_1(V_{n.s.}$ or $V)$ is a finite group for all p and we'll compute this group. For this Poincaré's key argument is the geometric one given below (the figures below being essentially as in the paper).

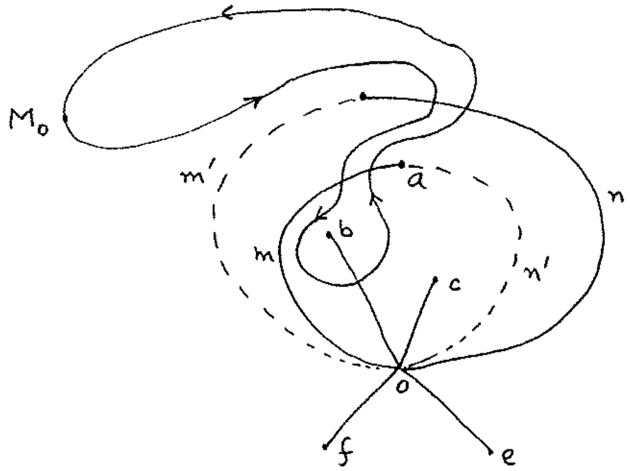
Proposition 12. *If* $(x_{ad}, A_i, 0)$ *is the only removable singularity, resp. conical point, with* $y = A_i$, *then the relations* $T_i s T_i^{-1} = s$ *in* $\bar{G}_0$ *are equivalent to* $s_a s_d = 1$, *resp.* $(s_a s_d)^2 = 1$.

(If there are many such points with $y = A_i$ then $T_i s T_i^{-1} = s$ is equivalent to the totality $\{s_a s_d = 1$ or $(s_a s_d)^2 = 1\}$ of such relations.)

Proof. With Poincaré we'll look at the case p = 2 only though it will be clear that the argument is quite general.

The six roots x_a, as well as the generating central symmetries $s_a \in \bar{G}_0$, will be *denoted just by their indices* a, b, c, d, e and f. We join these roots, in clockwise order, to another point 0 of the x-plane. Thinking of this cut x-plane as a G_0 tile of U we now examine the monodromy of these cuts as y describes a small loop about A_i.

If (ad, A_i, 0) **is a removable singularity** the (cuts to the) roots a and d interchange, i.e. Oma becomes Om'd and Ond becomes On'a :



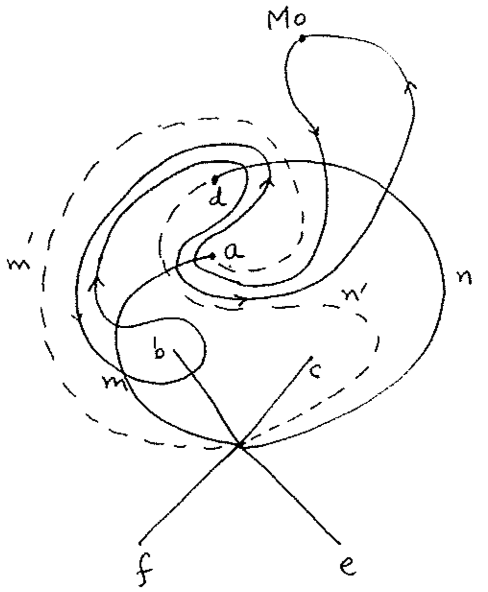
Poincaré gives the following **method for computing the conjugation** $T_i.s.T_i^{-1}$ **of** $\bar{G}_0$:

A base point M_0 is chosen, say outside Om'dna. The generating central symmetries are represented by loops, starting and ending at this base point, which intersect just one of the final cuts. To see the effect of the conjugation on it one notes in order the initial cuts which this loop intersects.

For example in the above picture we show a loop which intersects only Ob out of the final cuts, and we note that it intersects in order the initial cuts Ond, Oma, Ob, Oma and Ond. This shows that $T_i.b.T_i^{-1}$ = dabad. Likewise one checks that a $\mapsto$ d, c $\mapsto$ dacad, d $\mapsto$ dad, e $\mapsto$ e and f $\mapsto$ f under this conjugation.

The conjugating relations a = d, b = dabad, c = dacad, d = dad, e = e and f = f are clearly equivalent to the single relation ad = 1.

If (ad, A_i, 0) **is a conical point** the (cuts to the) roots a and d remain distinct during the monodromy, with Oma changing to Om'a and Ond to On'd as shown below :



Computing the conjugation by above method — e.g. fig. shows why $T_i.b.T_i^{-1}$ = dadabadad — the result follows because the conjugation relations a = dad, b = dadabadad, c = dadacadad, d = dadad, e = e and f = f are equivalent to the single relation $(ad)^2 = 1$. *q.e.d.*

Despite Poincaré's assertion that "nothing is easier now than determining the fundamental group" there certainly seem to be some loose ends left in the argument given below.

Proposition 13. *The variety* V *is simply connected, i.e.* $\pi_1(V) = 1$. *Moreover* $\pi_1(V_{n.s.}) \cong (\mathbb{Z}/2\mathbb{Z})^{n-1}$ *if the degree* 2p+2 *polynomial* $F(x,y) \in \mathbb{C}(y)[x]$ *has* n *factors all of even degree, while* $\pi_1(V_{n.s.}) \cong (\mathbb{Z}/2\mathbb{Z})^{n-2}$ *if some of these factors are of odd degree.*

Proof. As y describes a loop the 2p+2 roots of our polynomial undergo a permutation. Using this we obtain a homomorphism from F^{q-1}, the fundamental group of Y, to **the Galois group** of the polynomial F.

If the polynomial F *is irreducible* ($V = V_{n.s.}$ now) the Galois group of course interchanges any two roots x_a and x_b. However (though it seems likely) it is not clear why the image of the above homomorphism also (as Poincaré asserts) acts transitively on the roots of F.

Granting this, we see from Prop. 11 that our group is G_0 divided out by *all* relations $s_a = s_b$, so it is = 1.

In case F *is not irreducible* (now $V_{n.s.}$ is smaller) the asserted $\pi_1(V) = 1$ seems to require even more : if roots x_a and x_b belong to different prime factors we need an A_i at which they approach a common value (see Prop. 10). Using this we are still dividing out G_0 by all relations $s_a = s_b$ and so $\pi_1(V) = 1$ always.

For $\pi_1(V_{n.s.})$, using Prop. 12, we still have $s_a = s_b$ if x_a and x_b are roots of the same prime factor of the polynomial (these $(x_{ab}, A_i, 0)$ are the removable singularities) but if they belong to different prime factors we have only $(s_a s_b)^2 = 1$. Thus $\pi_1(V_{n.s.})$ is generated by n (= number of factors of F) commuting elements of order two.

We finally note that these conditions, in conjunction with the Fuchsian relation of G_0 , also give $\prod (a)^{n_i} = 1$, where a runs over representative roots of distinct irreducible factors of F, and n_i denotes the degrees of these factors. When all these n_i's are even this just reads 1 = 1, but otherwise is a nontrivial relation. *q.e.d.*

[More re fundamental group of complex varieties :

The theorem of **Picard** (i.e. that a smooth complex surface of $\mathbb{C}P^3$ is simply connected) is now a particular case of the result that smooth projective varieties obtainable from some smooth hypersurfaces as their (transversal) **complete intersections** *are simply connected.*

Shafarevich gives a proof of the above using a homotopy version of the **"weak Lefschetz theorem"** (provable via **Morse theory** as e.g. in **Milnor**).

On the other hand Shafarevich's book also contains examples which show that **any finite group can occur as the fundamental group of a smooth complex surface** (not of course in $\mathbb{C}P^3$)! The starting point for this is the *branched covering*

$$(\hat{\mathbb{C}} \times \ldots \times \hat{\mathbb{C}})/ \textstyle\sum_n = \mathbb{C}P^n$$

(to which we referred before also) which has, for n large, a linear section which is an unbranched covering over a 2-dimensional smooth variety, etc.

However on the whole "very little is known" (in Shafarevich's words) about the universal covering spaces and fundamental groups of varieties.

We note also that when $\pi_1(V) = 1$ and the singularities are points with known links (as was the case in this paper) $\pi_1(V_{n.s.})$ can be calculated by using **van Kampen**'s theorem re fundamental groups of two spaces and their intersection and union.]

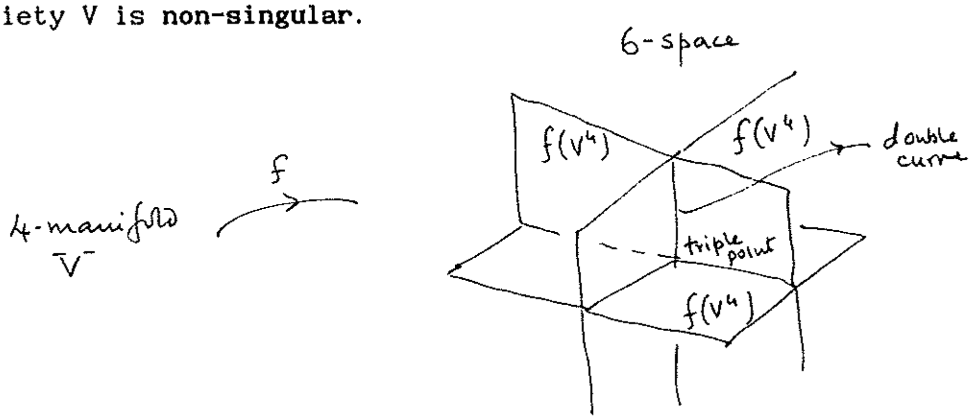
CHAPTER VIII

SUR LA CONNEXION DES SURFACES ALGEBRIQUES

(*Comptes Rendu de l'Acad. Sci.* 133 (1901) 969-973.)

In this note Poincaré announces some improvements to results which **Picard** had obtained very recently re connectivity of surfaces.

Let f(x,y,z) = 0 be a complex **surface** [in 6-dimensional space $(\hat{\mathbb{C}})^3$] to which corresponds a four-dimensional **variety** V [viewed intrinsically]. We will assume that the surface can have only **ordinary singularities**, i.e. complex **double curves** or **triplanar points**, and that the associated variety V is **non-singular**.



[Note in this context that *two* general position 4-dimensional (affine) subspaces of 6-space have as common intersection a subspace of dimension 4 + 4 - 6 = 2, while *three* such subspaces have as common intersection a subspace of dimension 4 + 4 + 4 - 6 - 6 = 0, i.e. a point; and that *more than three* such subspaces have an empty common intersection.]

As partial justification for his assumption Poincaré recalls an (older) **theorem of Picard on resolution of singularities of surfaces:** *in complex 3-space, any surface is birationally equivalent to one having only ordinary singularities, while in complex 5-space, any surface is birationally equivalent to one having no singularities.*

Also Poincaré remarks that the assumption is necessary because otherwise even the underlying variety V might have singularities, and then the definitions of "Analysis Situs" become ambiguous, unless one completes them by adopting some new conventions. (See also below.)

If we keep y constant on our complex surface we obtain a complex curve $f(x,z) = 0$, which corresponds to a Riemann surface S (= S_y). [It should be absolutely clear by now that Poincaré thinks of varieties (= "manifolds") and Riemann surfaces as *intrinsic* objects, while complex curves and complex surfaces are *within* some ambient space.] For almost all values of y this complex curve has a constant genus p — and for the exceptional values $A_1, \dots, A_q$, i.e. for the **singular points** of the y-projection map, the genus is lesser — and we'll denote by $\omega_1, \dots, \omega_{2p}$ any choice of 2p (homologically) distinct 1-cycles of S.

Poincaré now recalls that as y traces a small loop around A_i the Riemann surface S varies (this is the **monodromy** of the *Troisième Complément*) and comes back to its original position S with its points having undergone some transformation. The induced automorphism of $H_1(S)$ ($\cong \mathbb{Z}^{2p}$) is denoted T_i, and Poincaré christens the subgroup of $\mathrm{Aut}(H_1(S))$ ($\cong GL(2p,\mathbb{Z})$) generated by these T_i's as the **Picard group.**

Picard has shown that $H_1(V)$ *is generated by* $\omega \in H_1(S)$ *subject to the relations* $\omega = T_i(\omega)$ [i.e. $H_1(V) \cong \frac{H_1(S)}{\sum_i \mathrm{Im}(I-T_i)}$] : so the first Betti number of V is equal to the number of distinct solutions of the system of 2pq equations in 2p unknowns

$$\textbf{(A)} \quad x_t = \textstyle\sum_j c^i_{tj}\, x_j \,,$$

where $T_i(\omega_t) = \sum_j c^i_{tj}\, \omega_j$.

At first sight it seems that the number of distinct 1-cycles of V can be still smaller. For example we can imagine a surface with one A, which occurs as the limiting position of a surface with two nearby singular points A_1 and A_2 around which the monodromy transformations are inverses of each other. For such a surface there are no new relations, but some of the 1-cycles of S do become zero as $y \longrightarrow A$, because the genus of the curve $f(x,A,z) = 0$ is less than p .

However such a case never arises for the surfaces with ordinary singularities that we are considering !

Poincaré goes on to make a very persipicuous observation : *"If this does arise for other surfaces, one might wonder if these cycles ought to be regarded as equivalent to zero; one would be now in a situation where the ordinary definitions are ambiguous, unless supplemented by new conventions, and the answer would depend on the conventions one adopts"*.

[Quite clearly Poincaré is, unlike his inheritors, not keen to hastily generalize his definitions to varieties-with-singularities in the "obvious way" : the recent discovery of **intersection (co)homology** — which, and *not* the obvious singular (co)homology, is the "correct" definition for such varieties — shows that his caution was justified.]

Now Poincaré recalls the definition of **Picard's 2- tori** $\Omega \times C$, where 1-cycle Ω of S is invariant under monodromy as y varies over the closed curve C. It was unknown if these 2-cycles are distinct or if there are other 2-cycles besides these; the following are the results to which Poincaré was led while looking at this question.

$H_2(V)$ *is generated by the 2-cycles listed below :*
Two 2-cycles of the first type, namely the surfaces S_x and S_y obtained by keeping x or y constant, and

2-cycles of the second type, $W_1 + \dots + W_q$, $W_i = \Omega_i \times C_i$, where the Ω_i's are 1-cycles of S obeying

$$\Omega_1 + \dots + \Omega_q = T_1(\Omega_1') + \dots + T_q(\Omega_q'),$$

and each C_i is a closed curve in the y-plane around A_i.

The only relations amongst these 2-cycles are the following :

$W_1 + \dots + W_q \simeq 0$ iff for some 1-cycles V_i of S such that $T_i(V_i) = V_i$ the sequence of 1-cycles

$$\Omega_1 - V_1 \ , \ \dots \ , \ \Omega_q - V_q \ , \ C$$

is **cyclic**, i.e. one has $T_1(\Omega_1 - V_1) = \Omega_2 - V_2, \ \dots \ , \ T_q(\Omega_q - V_q) = \Omega_1 - V_1$.

Poincaré was unable to decide if Picard's tori sufficed to generate all 2-cycles of the second kind.

Proceeding now to $H_3(V)$ we will show that it has as basis the distinct 3-cycles $\Omega \times S^2$ where Ω is an **invariant 1-cycle** of S (i.e. preserved by all transformations of the Picard group) and $S^2 = \hat{\mathbb{C}}$ is the y-plane.

Since the number of such invariant 1-cycles equals the number of distinct solutions of (A) we verify that $b_1(V) = b_3(V)$.

Thus the consideration of the Picard group suffices to determine all the Betti numbers of V. Poincaré asserts that it likewise suffices for the calculation of the **torsion coefficients** of V.

CHAPTER IX

SUR LES CYCLES DES SURFACES ALGEBRIQUES; QUATRIEME COMPLEMENT A L'ANALYSIS SITUS

(*Jour. de Math.* 8 (1902) 169-214.)

§ 1. **Introduction.** We will use the principles of *"Analysis Situs"* and its first two complements to improve some results re cycles on algebraic surfaces which have been obtained by **Picard** in the course of his *"beaux travaux"* on algebraic surfaces.

Notations. Let V be a given closed variety (= smooth manifold) of dimension p, and let W_q denote varieties (closed or not) of dimension q "traced" on it.

Then Poincaré recalls that a **congruence** $\sum W_q \equiv \sum W_{q-1}$ signifies that the terms of the lower dimension (written here on the right side) constitute the boundary of the remaining terms : thus this is equivalent to the modern $\partial(\sum W_q) = \sum W_{q-1}$.

Supressing the higher dimensional terms Poincaré gets a **homology** $\sum W_{q-1} \simeq 0$: thus this signifies that the (q-1)-chain $\sum W_{q-1}$ is a boundary of some q-chain.

For Poincaré a "**cycle**" of dimension q means a $\sum W_q$ such that $\sum W_q \equiv 0$ but not $\sum nW_q \simeq 0$ for any nonzero integer n : these days of course q-cycles are q-chains satisfying just the first condition (for which Poincaré uses the word "**closed**") and Poincaré's "cycles" are those which are homologically non-trivial over $\mathbb{Q}$.

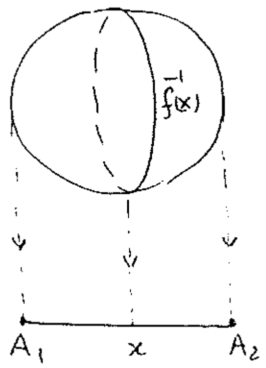
To avoid confusion we will sometimes change Poincaré's notations so as to conform with modern usage.

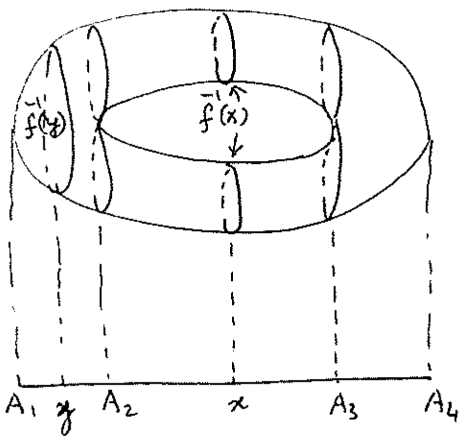
OUR OBJECT is to calculate the (rational) homology of a closed smooth 4-manifold V which occurs in complex 3-space as a complex surface f(x,y,z) = 0 having only ordinary singularities. For all but finitely many values $y = A_1, \ldots, A_q$ ($\neq 0$ or ∞) the complex curve defined by setting y equal to a constant will be assumed to be of constant **genus** p.

[We emphasize that our complex variety V is itself smooth, and the A_i's are merely the **singular points of the map** $(x,y,z) \longmapsto y$ from V to the complex line S^2 $(= \hat{\mathbb{C}})$.

Picard and Poincaré's method of *studying the complex variety* V *via the singularities of a map to the complex line*, as well as their results, were generalized to higher dimensions by **Lefschetz**. The partitioning of an algebraic variety into codimension-one varieties obtained by keeping one variable constant is called a **Lefschetz pencil**.

The analogous method of *studying a real variety via a map to the real line* was initiated by **Lyusternik-Schnirelman** and **Morse** and is called **Morse theory**. Singularities of some such maps are shown below :





In fact **Thom** observed that one of Lefschetz's results — the **weak Lefschetz theorem** — is best proved via Morse theory. On the other hand the best proof of the deeper **hard Lefschetz theorem** is still via **Hodge theory** which involves complex analysis on the smooth complex variety.]

Polyhedron P **and its subdivisions** P', P". As y $(\neq A_i)$ varies, the genus p Riemann surface S above y varies. *If we constrain* y *to be in the complement of some chosen* **cuts** $OA_1, \ldots, OA_q$, then this variation gives a unique homeomorphism between any two of these Riemann surfaces, and so, the choice of a cell subdivision P of *any one* of these S's fixes a *unique* homeomorphic cell subdivision on *all* these S's.

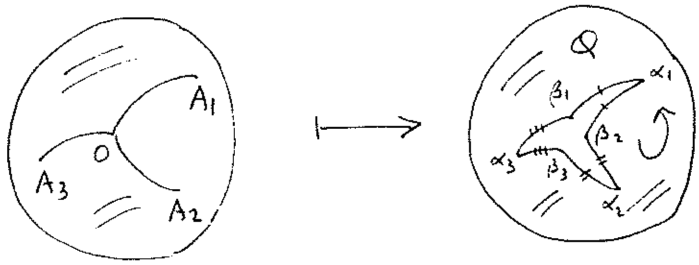
On the other hand, a y ($\neq A_i$ and $\neq 0$) *lying on a cut* OA_i can be approached from two sides, so we obtain by this process, for its S, *two* homoeomorphic subdivisions, related to each other by the **monodromy transformation** T_i. The *superposition* (in the obvious sense) of these two subdivisions of S will be denoted P'_i or just P'.

As will be justified later in § 5, we can assume that **the variation of** y **between** O **and** A_i **on the cut** OA_i **preserves** P'_i.

Finally, since the point O can be approached from within any of the q sectors, we obtain, for its S, q homeomorphic subdivisions. The superposition of all of these q subdivisions will be denoted by P".

Poincaré denotes the faces, edges and vertices of P (resp. P', resp. P") by F, B and C (resp. F', B' and C', resp. F", B" and C").

Polygon Q. Just as in the *"Troisième Complément"* we now "open up" the q cuts OA_i and consider the complex line $\hat{\mathbb{C}} \cong S^2$ as a polygon Q with 2q sides, with identifications of pairs of edges done in the manner shown :



Thus, there is a unique vertex α_i of the polygon Q which corresponds to A_i, but the q vertices β_i all correspond to O. Note also that, with the indicated orientations, the boundaries are given by $\partial(\alpha\beta) = \alpha - \beta$ and $\partial Q = \sum(\alpha_i\beta_i - \alpha_i\beta_{i+1})$, the summation being over all i mod q.

Cell subdivision H **of** V. We note that we can obtain V from the cartesian product of Q and P by identifying each of its **fibers** $\{y\} \times P$, $y \neq \alpha_i$, with the subdivided Riemann surface S above the corresponding point y of $\hat{\mathbb{C}}$, and by making, in the q fibers $\alpha_i \times P$, the identifications governed by the nature of the singularity at A_i.

However these identifications do not identify cells with cells. To make them cellular, Poincaré now modifies the above construction slightly, and defines H by making the above identifications on the **subdivided cartesian product** Q×P which uses, for each fiber $\{y\} \times P$ with y lying in the open intervals $\beta_i\alpha_i\beta_{i+1}$ of ∂Q the subdivision $\{y\} \times P'_i$, and for the q fibers $\{\beta\}_i \times P$ the still finer subdivision $\{\beta_i\} \times P''$.

Proposition 1. *The cell complex* H *is obtained from the subdivided cartesian product* Q × P *by making the cellular identifications*

$$\alpha_i\beta_i \times \sigma = \alpha_i\beta_{i+1} \times \sigma, \ \sigma \in P'_i,$$
$$\beta_i \times \theta = \beta_j \times \theta, \ \theta \in P'',$$

and, for each i, *some identifications amongst cells of the type* $\alpha_i \times \phi$, $\phi \in P'_i$ *governed by the nature of the singularity at* A_i (which too will be considered cellular, for we can always, if need be, replace P'_i by a finer subdivision).

Furthermore, each chain of H *is homologous to one which does not contain such cells* $\alpha_i \times \phi$, $\phi \in P'_i$.

We remark that in the paper the verification of the last assertion, which uses the **product formula** for boundaries,

$$\partial(\sigma \times \theta) = \partial(\sigma) \times \theta + (-1)^{\dim \sigma} \sigma \times \partial(\theta),$$

is completed only in the beginning of § 2.

Proof. Let us with Poincaré partition the cells of Q×P into **four categories** depending on whether their first factor is Q, an $\alpha\beta$, an α, or a β. Then a little reflection shows that two cells can get identified only if they are in the same category other than the first (which has no identifications) and that the identifications above the points of ∂Q are as stated.

To see the second part note that $\partial(\alpha_i\beta_i \times \phi) = \partial(\alpha_i\beta_i)\times\phi - \alpha_i\beta_i\times\partial(\phi)$

implies $\alpha_i \times \phi \simeq (\beta_i \times \phi) - \alpha_i \beta_i \times \partial(\phi)$. *q.e.d.*

[REMARKS REGARDING ABOVE IDENTIFICATIONS :

Most of the arguments of this paper apply to any 4-dimensional polyhedron V constructed as follows : start with

(i) a 2q-gon Q with identifications $x = t_i(x)$ of edge pairs as above,
(ii) a surface S of genus p, and
(iii) q diffeomorphisms $T_i : S \longrightarrow S$ whose product is the identity,

and let V be obtained from Q×S by making the identifications $(x,y) = (t_i(x), T_i(y))$ whenever $x = t_i(x)$.

We note that such a V is necessarily smooth except possibly over the vertices α_i of Q : these **singular fibres are** S/T_i. (Some arguments of this paper need a condition on these fibers e.g. that $H_2(S/T_i) \simeq \mathbb{Z}$.)

If these singular fibers are also smooth surfaces (with projection $S \longrightarrow S/T_i$ being in general **branched**) then the 4-manifold V is **foliated** (i.e. partitioned smoothly) into **compact leaves** (of which only a finite number have genus less than p).

However this nice situation can prevail only if the tangent bundle of V splits off a smooth 2-dimensional plane bundle, i.e. when the second **Stiefel-Whitney class** of V is 0.

"Quand le point M *vient en* A_i*, les deux modes de décomposition de la surface* S *en polyèdre* P *se confondent; d'autre part, ce polyédre dégénere ..."*

We will interpret these words of Poincaré as meaning that all complex surfaces $f(x,y,z) = 0$ are *quotients* of V's of the above (i-ii-iii) kind, the extra identifications being all only over the points A_i.

For "ordinary singularities" the geometry of these identifications had been worked out in **Picard**'s book and Poincaré will later cite many of these facts e.g. that in this case each singular fiber has just one fundamental 2-cycle : $H_2(S/T_i) \cong \mathbb{Z}$. Also § 5 of this paper will elaborate further on the identifications above the points A_i.

We note that because of the identifications at α_i the two copies of P at a point M approaching A_i along the cut OA_i do approach coincidence in the topology of H, *but not in the topology of* Q×P :

If they did, the two copies contained in them, of *any* loop of P, would be deformable to arbitrarily near, and thus homotopic, loops : i.e. the action of the monodromy transformation T_i on $\pi_1(P)$ (and so on $\pi_1(P)$) would be trivial (cf. proof of Prop. 4). Thus *the discontinuity* (in the topology of Q×P) *between the two copies of* P *as* M $\longrightarrow$ A_i *is in fact our measure of the non-triviality of the monodromy action.*

Lastly we remark that it would be interesting to see how far *any smooth 4-manifold* can be treated by these methods : e.g. is it always of (i-ii-iii) type ? In the light of the fact that the (co)homology of closed *smooth* 4-manifolds is subject to *severe* restrictions, it seems one can hope that many techniques used only to investigate the homology of complex surfaces actually apply to all smooth 4-manifolds ?]

Notation. Poincaré omits the product sign × : so e.g. his $\alpha_i\beta_{i+1}F_k$ corresponds to our $\alpha_i\beta_{i+1} \times F_k$. Moreover, he omits the first factor of a cell of Q×P if it is Q : so e.g. his F_k denotes, depending upon the context, a face of P, as well as the 4-cell (or **hyperbox**) $Q\times F_k$ of Q×P.

We note also that the first factor in $\beta_i\times\phi \in H$, $\phi \in P''$, is essentially redundant because $\beta_i\times\phi = \beta_j\times\phi$ in H (see Prop. 1).

§ 2. **Three dimensional cycles.**

Proposition 2. *For any face* F *of* P *we have*

$$Q \times \partial(F) \simeq \sum_i(\alpha_i\beta_{i+1} \times F) - \sum_i(\alpha_i\beta_i \times F)$$

in H. *So, if* z *is a homologically trivial* 1*-cycle of* P, *then the* 3*-chain* $Q \times z$ *of* H *is homologous to one which only contains* 3*-cells* (or **boxes**) *of type* $\alpha_i\beta_i \times F'_k$.

In the above, and in some other formulae, it is understood that some terms are to be chain-subdivided as per the definition of H given above.

Proof. For this note that the boundary of any hyperbox is given by $\partial(Q \times F) = \partial(Q) \times F + Q \times \partial(F) = \sum_i(\alpha_i\beta_i - \alpha_i\beta_{i+1}) \times F + Q \times \partial(F)$. *q.e.d.*

Notation. Poincaré denotes by S(M) the Riemann surface above a point M of the cut OA_i between O and A_i, and its two P's, obtained by thinking of M as being respectively on the **lip** $\alpha_i\beta_i$ or $\alpha_i\beta_{i+1}$, are denoted (rather whimsically !) by MP and (MP) with cells being {MF, MB, MC} and {(MF), (MB), (MC)}. The superposition of MP and (MP) is denoted MP' with cells {MF', MB', MC'}.

[This time we'll stick to Poincaré's notations, though perhaps we should have changed S(M) and MP' to M × S and M × P', and MP and (MP) to M × P and (M) × P, where M and (M) denote the two copies of the point M on the two lips of the cut.]

Proposition 3. *The map* $\sum_q\zeta_q.(Q\times B_q) + \sum_{k,i}\theta'_{ki}.(\alpha_i\beta_i\times F'_k) \longmapsto \sum_q\zeta_q.B_q$ *associates to each* 3*-cycle of* H, *a* 1*-cycle of* P *which is* **invariant**, *i.e. one whose homology class is preserved by the Picard group.*

We note that we have used θ'_{ki} instead of Poincaré's θ'_k because these integral coefficients can depend on both k and i.

Proof. Since the boundary $\partial(\sum_q\zeta_q.(Q \times B_q) + \sum_{k,i}\theta'_{ki}.(\alpha_i\beta_i \times F'_k))$ is zero, its sub-chain consisting of all cells of category Q, i.e. Q ×

$\partial(\sum_q \zeta_q B_q)$, must also be zero : so $\partial(\sum_q \zeta_q B_q) = 0$ and

$$\partial(\sum_{k,i} \theta'_{ki}.(\alpha_i\beta_i \times F'_k)) = -\sum_q \zeta_q.\partial(Q) \times B_q \text{ , i.e.}$$

$$\partial(\sum_{k,i} \theta'_{ki}.(\alpha_i\beta_i \times F'_k)) = \sum_{q,i} \zeta_q.(\alpha_i\beta_{i+1} \times B_q) - \sum_{q,i} \zeta_q.(\alpha_i\beta_i \times B_q).$$

The **intersections** with $S(M)$, $M \in OA_i$, $1 \le i \le q$, of (the cells of) the last equation, gives us the equations

$$\partial(\sum_k \theta'_{ki}.MF'_k) = \sum_q \zeta_q.(MB_q) - \sum_{q,i} \zeta_q.MB_q.$$

As y departs from M, from the side $\alpha_i\beta_i$ of the cut OA_i, makes one complete turn around A_i, and comes back to M from the side $\alpha_i\beta_{i+1}$ of this cut, the cycle $\sum_q \zeta_q.B_q$ deforms continuously from its initial position $\Omega_i = \sum_q \zeta_q.MB_q$ to its final position $\Omega'_i = \sum_q \zeta_q.(MB_q)$: thus the above equations show that the homology class of our 1-cycle is invariant under the monodromy transformations T_i, $1 \le i \le q$. *q.e.d.*

Notation. As in the above proof we denote by $\Omega_i = \sum_q \zeta_q.MB_q$ and $\Omega'_i = \sum_q \zeta_q.(MB_q)$ the 1-cycles of S(M) corresponding to a given 1-cycle $\sum_q \zeta_q.B_q$ of P. The Riemann surface above O will be denoted S_0, and U_i and U'_i will denote the cycles of S_0 obtained from the cycles Ω_i and Ω'_i of S(M) by moving M to O olong OA_i. We note that U_i and U'_i depend on M, and that as M approaches O these cycles approach $\Omega^0_i = \sum_q \zeta_q.\beta_i \times B_q$ and $\Omega^0_{i+1} = \sum_q \zeta_q.\beta_{i+1} \times B_q$ respectively.

Proposition 4. *The image of the map of Proposition* 3 *consists of* all *invariant* 1-*cycles of* P.

As we'll see below, Poincaré's proof is somewhat incorrect, however it does give a weaker result, and probably can be repaired completely for the case of "ordinary singularities".

Proof. If $\sum \zeta.B$ is an invariant 1-cycle of P we can choose, for each $1 \le i \le q$, an integral 2-chain $\sum_k \theta'_{ki}.F'_k$ of P'_i, such that $\partial(\sum_k \theta'_{ki}.MF'_k) = \sum_q \zeta_q.(MB_q) - \sum_{q,i} \zeta_q.MB_q$ for any point M between O and A_i on the cut OA_i.

As M approaches A_i along this cut *the cycles* Ω_i *and* Ω'_i *approach* *coincidence in* H, and so the above equation gives $\partial(\sum_k\theta'_{ki}.\alpha_i\times F'_k) = 0$, i.e. that $\sum_k\theta'_{ki}.\alpha_i\times F'_k$ is a 2-cycle.

We can in fact choose the above 2-chain $\sum_k\theta'_{ki}.F'_k$ in such a way that, not only is $\sum_k\theta'_{ki}.\alpha_i\times F'_k$ a 2-cycle, but that *it is identically zero* :

Assume first that singular complex curve above A_i has only one non-trivial 2-cycle z (this happens when only "ordinary singularities" are allowed) : choose as above an initial 2-chain $\sum_k\theta'_{ki}.F'_k$ for which $\sum_k\theta'_{ki}.\alpha_i\times F'_k$ is a 2-cycle = nz for some n, and if n is nonzero replace it by the 2-chain $\sum_k\theta'_{ki}.F'_k - nz$.

[However in the general case below Poincaré's argument seems to use that

(*) the cycles Ω_i and Ω'_i approach coincidence in Q×P.

As was pointed out before also (*) can happen only if

(**) Ω and $T_i(\Omega)$ represent the same element of $\pi_1(P)$ $\forall$ i,

a hypothesis which apparently is stronger (is it really ?) than the given hypothesis that they represent the same element of the Abelianization $H_1(P)$ of $\pi_1(M)$. Conversely, it is very likely (cf. Prop.6) that any element of $\pi_1(P)$ which is invariant under all the T_i's can be *represented* by an Ω satisfying (*).]

For the general case Poincaré gives the following argument :

Let R be the region of S_0 **swept out** by the cycles U_i and U'_i of S_0 as M travels from M to A_i. (This is suspect, however if one has (*) one only needs to come *close enough* to A_i and bridge the small gap by a homology to define this R.) This region R (which depends on M) is bounded by U_i and U'_i because the coincidence of the two cycles Ω_i and Ω'_i as M approaches A_i implies that the corresponding cycles U_i and U'_i also coincide as M approaches A_i. Transporting R along OA_i to M we get a region of S(M) bounded by Ω_i and Ω'_i and we will choose the 2-chain $\sum_k\theta'_{ki}.MF'_k$ so that it covers this region. The above assertion follows because R becomes zero as M approaches A_i.

[Since this "swept out" argument will be used again also (even for the "ordinary singularities" case) it seems that as such **Poincaré's argument will only show that Ω's obeying (*) are in the image of the map in question.** However it is fairly certain that, at least for the "ordinary singularities" case, the necessary repair work is not hard.]

With the 2-chains chosen as above, we have for each $1 \le i \le q$,

$$\partial(\textstyle\sum_k \theta'_{ki}\cdot\alpha_i\beta_i\times F'_k) = \sum_q(\zeta_q\cdot\alpha_i\beta_{i+1}\times B_q) - \sum_q(\zeta_q\cdot\alpha_i\beta_i\times B_q) - \sum_k \theta'_{ki}\cdot\beta_i\times F'_k ,$$

and adding these q equations we get

$$\partial(\textstyle\sum_q\zeta_q\cdot Q\times B_q + \sum_{k,i}\theta'_{ki}\cdot\alpha_i\beta_i\times F'_k) = -\sum_{k,i} \theta'_{ki}\cdot\beta_i\times F'_k .$$

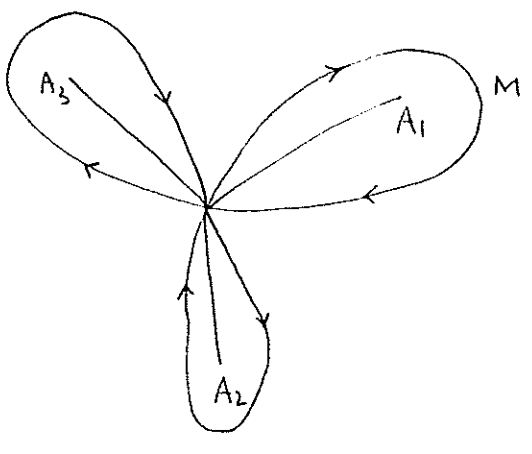
To complete our proof we'll now show that the right side of the last equation is zero :

When M goes to O along OA_i the 2-chain $\sum_k\theta'_{ki}\cdot MF'_k$ becomes $\sum_k\theta'_{ki}\cdot\beta_i\times F'_k$, i.e. the 2-chain of S_0, bounded by the values Ω_i^0 and Ω_{i+1}^0 which U_i and U'_i take as M goes to O, which is swept out by U_i and U'_i as M traverses the complete cut OA_i. So

$$\partial(\textstyle\sum_k\theta'_{ki}\cdot\beta_i\times F'_k) = \Omega_{i+1}^0 - \Omega_i^0 ,$$

and summing these q equations (here i is mod q) we see that $\partial(\sum_{k,i} \theta'_{ki}\cdot\beta_i\times F'_k) = 0$, i.e. that *the "right side" in question is a 2-cycle.* So $\sum_{k,i} \theta'_{ki}\cdot\beta_i\times F'_k$ is some multiple $n.S_0$ of the fundamental 2-cycle of the Riemann surface above O. *We want to show that* $n = 0$.

For this, Poincaré considers a point M describing a **contour** of the following type which is **infinitely close to** (but away from !) the cuts :



We will think of $\sum_{k,i} \theta'_{ki}.\beta_i \times F'_k$ as the region of S_0 chalked out by a **moving** 1-cycle which initially is Ω_1^0 , moves like U_1 as M moves from 0 to (infinitely close to) A_1, then like U'_1 as M moves back to 0, then like U_2 as M moves from 0 to (infinitely close to) A_2, etc.

[This is okay : the "small jump" which the moving 1-cycle actually makes at some contour point M close enough to A_1 when motion "like U_1" is changed to that "like U'_1 " can be filled in by a small homology of S_0.]

The movements of our 1-cycle stem from monodromy w.r.t. y, which in turn (see *"Troisième Complément"*) is due to the motion w.r.t. y of the branch points of $f(x,y,z) = 0$ considered as a (multivalued) function $z = F(x,y)$. These values of x are the branch points of a moving **branched covering** $S^0 \longrightarrow S^2$ (initially given by $(x,0,z) \longmapsto x$). If the projection under it of our 1-cycle misses these branch points initially, then it will continue to so miss, or as Poincaré puts it, **flee from** these moving branch points, for all time.

We note now that, since $\sum_{k,i} \theta'_{ki}.\beta_i \times F'_k$ is a *2-cycle* of S_0, it is unchanged if we replace its generating moving 1-cycle by a nearby one : so we can, without loss of generality, assume that our moving 1-cycle does flee from the above singularities as above. Note now that if such a family of moving 1-cycles is deformed in any way through families fleeing from branch points, then the 2-cycle $\sum_{k,i} \theta'_{ki}.\beta_i \times F'_k$ still remains unchanged.

Since our contour was away from the cuts (despite being infinitely close to them !) each branch point retains its identity, i.e. they describe disjoint arcs on the x-plane S^2. So we can deform our moving family in such a way that it misses a chosen **fixed point** in the complement of these arcs. This shows that the branched projection of our 2-cycle misses a small disk of S^2 around this fixed point. Thus $n = 0$ and our 2-cycle must vanish. *q.e.d.*

Proposition 5. *If a 3-cycle* $\sum_q \zeta_q.(Q\times B_q) + \sum_{k,i}\theta'_{ki}.(\alpha_i\beta_i\times F'_k)$ *of* H *is such that* $\sum_q \zeta_q.B_q$ *bounds in* P, *then the* 3-*cycle must bound in* H.

In conjunction with Prop. 3 and of Prop. 4 this shows (modulo a little doubt about the proof of the latter !) that $H_3(V)$ *is isomorphic to the subgroup of* $H_1(V)$ *invariant under the action of the Picard group.*

Proof. Using the hypothesis and Prop. 2 we see that our 3-cycle is homologous to one which has only cells of the category $\alpha\beta$. So to complete the proof it suffices to check that

H *has no nonzero* 3-*cycle containing only cells of category* $\alpha\beta$:

If $\sum_{k,i}\theta'_{ki}.(\alpha_i\beta_i\times F'_k)$ were a nonzero 3-cycle of H, then its intersection with S(M), i.e. $\sum_k\theta'_{ki}.MF'_k$ would be a nonzero 2-cycle of MP'_i. So when M approaches A_i its limit $\sum_k\theta'_{ki}.(\alpha_i \times F'_k)$ should be a nonzero multiple of the 2-cycle covering the entire singular complex curve above A_i. This contradicts the fact that if in $\partial(\sum_{k,i}\theta'_{ki}.(\alpha_i\beta_i\times F'_k)) = 0$ we consider only cells of category α we get $\sum_k\theta'_{ki}.(\alpha_i \times F'_k) = 0 \ \forall$ i. *q.e.d.*

Proposition 6. *Each* element of $H_1(P)$ *which is invariant under all* T_i *can be represented by a* 1-*cycle* Ω *for which* $\Omega_i = \Omega'_i$ *for all* i.

For **totally invariant cycles** of this kind one gets a 3-cycle of V just by multiplying by the complex line $\hat{\mathbb{C}}$: so the surjectivity of Prop. 4 follows from this result, and a perhaps more natural approach towards $H_3(V)$ is to establish Prop. 6 first ?

Poincaré however gives no proof of Prop. 6 but simply writes that this is clear because of "the arbitrary fashion in which we can make our Riemann surfaces correspond with each other".

By this he might be meaning that we have the liberty of deforming the fibers above one of the two lips of any cut before matching it to the pre-image of the other lip. This argument does appear to show the above but only under the apparently stronger hypothesis that our homology class is the image of an element of the fundamental group $\pi_1(P)$ which is invariant under all the T_i's.

§ 3. **Two dimensional cycles.** Using Prop. 1 we need to consider only 2-cycles *without cells of category* α. Then, re cells of category Q, we observe the following.

Proposition 7. *Any two 2-cycles of* H *which contain 2-cells of category* Q *are homologous to each other mod cells of other categories.*

Proof. If a 0-chain $\sum\varepsilon_k C_k$ of the surface P is the boundary of $\sum\zeta_q B_q$ then $\partial(\sum_q\zeta_q(Q\times B_q)) = \sum_k\varepsilon_k(Q\times C_k)$ + cells of other categories.

So any 2-cycles of H whose category Q part is $\sum\varepsilon_k(Q\times C_k)$, with $\sum\varepsilon_k = 0$, is homologous to a 2-cycle having no cells of category Q : this follows from the above because *a 0-chain* $\sum\varepsilon_k C_k$ *of* P *bounds iff the sum* $\sum\varepsilon_k$ *of its coefficients is zero.*

The result now follows because, given any two 2-cycles with category Q parts $\sum\varepsilon_k(Q\times C_k)$ and $\sum\varepsilon'_k(Q\times C_k)$, we can certainly find integers n and n' such that $n(\sum\varepsilon_k) + n'(\sum\varepsilon'_k) = 0$. *q.e.d.*

A 1-cycle $K_i = \sum_k\theta'_k.B'_k$ of P'_i is said to be a **vanishing cycle at** A_i iff $\sum_k\theta'_k.(\alpha_i\times B'_k) = 0$ in H.

Proposition 8. *There is a 1-1 correspondence between 2-cycles* $\sum_{k,i}\theta'_{ki}(\alpha_i\beta_i\times B'_k) + \sum_k\theta''_k.(\beta_i\times F''_k)$ *of* H *and sequences* $K_i = \sum_k\theta'_{ki}.B'_k$ *of 1-cycles of* P'_i, *whose sum* $\sum_i K_i$ *bounds* $\sum_k\theta''_k.F''_k$ *in* P", *and each* K_i *vanishes at* A_i.

We note that, in the above, the coefficients θ" depend only on the index k because we have $\beta_i\times F''_k = \beta_j\times F''_k$ in H.

We note also that the **exceptional cycle** $S_0 = \sum_k \beta_i\times F''_k$ — i.e. the Riemann surface $f(x,0,z) = 0$ — is one of the above 2-cycles, and that whenever all θ' are zero, the above 2-cycle must be a multiple $n.S_0$, i.e. then all θ" must be equal to each other.

Proof. That $\sum_i K_i$ bounds $\partial(\sum_k\theta''_k.F''_k)$ in P is clearly equivalent to saying that we have $\sum_{k,i}\theta'_{ki}(\beta_i\times B'_k) = \partial(\sum_k\theta''_k.\beta_i\times F''_k)$ in H.

Further, if each K_i vanishes at A_i, i.e. $\sum_k\theta'_{ki}(\alpha_i\times B'_k) = 0$ in H, we have $-\sum_k\theta'_{ki}\cdot\beta_i\times B'_k = \partial(\sum_k\theta'_{ki}(\alpha_i\beta_i\times B'_k))$. Adding these q equations to the one above we get $\partial(\sum_{k,i}\theta'_{ki}(\alpha_i\beta_i\times B'_k) + \sum_k\theta''_k\cdot\beta_i\times F''_k) = 0$.

Conversely, intersecting the equation $\partial(\sum_{k,i}\theta'_{ki}(\alpha_i\beta_i\times B'_k) + \sum_k\theta''_k\cdot\beta_i\times F''_k) = 0$ with S(M), where M is on OA_i, we get $\partial(\sum_k\theta'_{ki}(MB'_k) = 0$, which shows that $K_i = \sum_k\theta'_{ki}\cdot B'_k$ is a cycle of P for each i.

Further, considering category α terms of $\partial(\sum_{k,i}\theta'_{ki}(\alpha_i\beta_i\times B'_k) + \sum_k\theta''_k\cdot\beta_i\times F''_k) = 0$ we get $\sum_{k,i}\theta'_{ki}(\alpha_i\times B'_k) = 0$ which happens iff $\sum_k\theta'_{ki}(\alpha_i\times B'_k) = 0$ for each i. So each K_i vanishes at A_i and we have $\sum_{k,i}\theta'_{ki}(\beta_i\times B'_k) = \partial(\sum_k\theta''_k\cdot\beta_i\times F''_k)$. *q.e.d.*

These vanishing 1-cycles also explain the lowering of the generic genus p at the singularities, e.g. if the fiber is a torus with a double curve C as vanishing cycle as shown below, then genus decreases from 1 to 0 :

We note that the property that a 1-cycle $\sum_k\theta'_k\cdot MB'_k$ becomes zero (when M approaches A_i) *in the cartesian product* Q×P is much stronger than that it is a vanishing cycle at A_i (i.e. that it becomes zero *in* H when M approaches A_i) : e.g. in contrast with the following assertion, we saw before that a 1-cycle of the type $\omega - T_i(\omega)$ satisfies this stronger property only if it is homologically (and even homotopically) trivial.

Proposition 9. *If* ω *is any 1-cycle of* P'_i, *then the cycle* $\omega - T_i(\omega)$

vanishes at P'_i. *Conversely, any vanishing cycle of* P'_i *is of this type.*

Proof. The direct part is immediate since we have the identifications $\alpha_i \times \phi = \alpha_i \times T_i(\phi)$, $\phi \in P'_i$, at each singularity A_i.

For the converse, Poincaré only gives references to **Picard's** book (vol. I, pp. 82 and 95) which gives explicit description of the **ordinary singularities** and their monodromies T_i. *q.e.d.*

The notion of vanishing cycle of course makes sense even if the singularities are not ordinary. However then (Poincaré notes) all vanishing cycles are not necessarily of the above **first kind**.

Proposition 10. *A 2-cycle* $\sum_{k,i}\theta'_{ki}(\alpha_i\beta_i \times B'_k) + \sum_k \theta''_k.\beta_i \times F''_k$ *of* H *bounds iff there is a 1-cycle* K *of* P *such that for any* M_i *lying on any cut* OA_i *we have* $\sum_k \theta'_{ki}.M_iB'_k \simeq M_iK - (M_iK)$. *Furthermore, the 2-cycle* S_0 *does not bound in* H.

Proof. If the 2-cycle bounds we have some equation of the type

$$\partial(\sum_k \varepsilon_k.Q \times B_k) + \sum_{k,i}\zeta'_k.\alpha_i\beta_i \times F'_k) = \sum_{k,i}\theta'_{ki}(\alpha_i\beta_i \times B'_k) + \sum_k \theta''_k.\beta_i \times F''_k \qquad (0).$$

On intersecting this with the Riemann surface S above a point $y \in \mathrm{int}(Q)$ we get $\partial(\sum_k \varepsilon_k.B_k) = 0$ in the cell subdivision P of S. We assert that *this* $\sum_k \varepsilon_k.B_k$ *is our required 1-cycle* K *of* P.

To see this we note that (0) is equivalent to

$$\partial(\sum_{k,i}\zeta'_k.\alpha_i\beta_i \times F'_k) = \sum_{k,i}\theta'_{ki}(\alpha_i\beta_i \times B'_k) + \sum_{k,i}\varepsilon_k.(\alpha_i\beta_{i+1} \times B_k)$$

$$- \sum_{k,i}\varepsilon_k.(\alpha_i\beta_i \times B_k) + \sum_{k,i}\theta''_k.\beta_i \times F''_k , \qquad (1)$$

and on intersecting this with the Riemann surface $S(M_i)$ we get

$$\partial(\sum_{k,i}\zeta'_k.M_iF'_k) = \sum_k \theta'_{ki}.M_iB'_k + (M_iK) - M_iK , \qquad (2)$$

which shows the required $\sum_k \theta'_{ki}.M_iB'_k \simeq M_iK - (M_iK)$.

Conversely note that the right side of (2) is a 1-cycle vanishing at A_i: so this equation can hold only if $\partial(\sum_{k,i}\zeta'_k.\alpha_i\times F'_k) = 0$. In case the singular fiber above A_i has a single fundamental 2-cycle z — *Poincaré asserts that this case alone can happen for the case of ordinary singularities under consideration* — this 2-cycle $\sum_{k,i}\zeta'_k.\alpha_i\times F'_k$ must be n.z for some integer n. We can in fact assume n = 0 for otherwise we can always replace $\sum_{k,i}\zeta'_k.\alpha_i\times F'_k$ by $\sum_{k,i}\zeta'_k.\alpha_i\times F'_k - nz$.

[Poincaré here asserts that we can *always* ensure $\sum_{k,i}\zeta'_k.\alpha_i\times F'_k = 0$ in H, however this seems moot as the "swept out" argument *of* § 2 is in doubt.]

Using (2) and $\sum_{k,i}\zeta'_k.\alpha_i\times F'_k = 0$ we now get (1), and so (0), if we set the subdivided $\sum_{k,i}\zeta'_k.\beta_i\times F'_k$ equal to $\sum_{k,i}\theta''_k.\beta_i\times F''_k$.

For the last part assume that the θ' are all zero and (0) holds. We note that K is now an invariant 1-cycle, and (2) is the homology between it and its transform under T_i. So the corresponding 3-cycle (given by the slightly cloudy Prop. 4 of § 2 !) of H is the chain on the left side of (0) on which ∂ acts. The left side of (0) is thus zero, which shows that all θ" are also zero. *q.e.d.*

Proposition 7'. *Besides the 2-cycles of* Props. 8 *and* 10 *there is exactly one more homologically distinct 2-cycle.*

Proof. We know already (Props. 8 and 10) that this additional 2-cycle must have some cells of category Q, and (Prop. 7) that there is *at most one* such 2-cycle, and that in it the sum of the coefficients of all the 2-cells of category Q must be nonzero.

Also it is clear that if a 2-cycle bounds, then the sum of the coefficients of its category Q cells is zero. Thus it only remains to exhibit a 2-cycle such that the sum of the coefficients of its category Q cells is nonzero.

For this we'll assume that for some $x = x_0$, all the m points of our P (above the chosen base point $y \in \mathrm{int}(Q)$) which have $x = x_0$ are vertices of P. (In § 5 we'll check that this is permissible.) Then if $C_1, \dots, C_m$ denote these vertices we must have $\partial(Q\times C_1 + \dots + Q\times C_m) = 0$. To see this we note that as y turns around an A_i the C_i's, being the m roots of the equation $f(x_0,y,z)$, just permute with each other : so the sum $\sum_k\alpha_i\beta_i\times C_k$ equals the sum $\sum_k\alpha_i\beta_{i+1}\times C_k$.

Alternatively, $Q\times C_1 + \ldots + Q\times C_m$ is a 2-cycle (and will be called *exceptional* alongwith the cycle S_0 or $f(x,0,y) = 0$) because it obviously coincides with the Riemann surface $f(x_0,y,z) = 0$. *q.e.d.*

Proposition 11. *The second Betti number of* V *equals* $2 + (\mu - \rho) - (2p - n)$, *where* $\mu = \sum k_i$,
k_i = *number of homologically independent 1-cycles which vanish at* A_i,
ρ = *rank of the* $\mu\times 2p$ *matrix formed by the coefficients of these* μ *cycles with respect to a basis of* $H_1(P)$, *and*
n = *number of homologically distinct invariant 1-cycles of* P.

[In the paper one has $\rho = 2p - r - 1$. We note that though Poincaré proceeds exactly as follows, some misprints at the very end make his final formula wrong.]

Proof. Using the preceding results we have the following **generating 2-cycles** of H :

Besides S_0 and $f(x_0,y,z) = 0$ (which are homologically distinct and independent of the others) the remaining 2-cycles are determined by length q sequences $U_1, \ldots, U_q$ of 1-cycles of S_0, whose sum bounds in S_0, and which vanish as we transport them, respectively, to $A_1, \ldots, A_q$. Furthermore, we know that there is a homology between any two of these 2-cycles iff the difference of the two sequences is of the type $\Omega_2 - \Omega_1, \ldots, \Omega_1 - \Omega_q$, where Ω_i, $1 \le i \le q$, are the copies in S_0 of the same 1-cycle Ω of P.

The requirement that the sum of the U_i's bounds means that the number of non-exceptional cycles equals the number of distinct dependencies amongst the rows of our $\mu\times 2p$ matrix, i.e. $\mu - \rho$. Also, since $\Omega_2 - \Omega_1, \ldots, \Omega_1 - \Omega_q$ is a zero sequence iff Ω is invariant, we see that the number of distinct homologies is $2p - n$. *q.e.d.*

We remark that Poincaré claims the above even when there are **vanishing**

cycles of the second kind, i.e. not necessarily of the type $\omega - T_i(\omega)$. However, for the case of ordinary singularities, he now points out the following **explicit construction of the generating 2-cycles** of H (cf. the preceding *C. R. Note*) :

Let the vanishing cycle $U_1 = T_1(\omega_1) - \omega_1$, then transport ω_1 along one loop ("petal") of the flower-shaped contour we used in § 2 to get a surface with boundary $T_1(\omega_1) - \omega_1$. Do likewise with U_2, etc. Since sum of the U's bounds, $\sum \omega_i$ is homologous to the sum $\sum T_i(\omega_i)$, and the sum of all these surfaces has zero boundary. This is the required 2-cycle.

§ 4. **One dimensional cycles.** Once again we start by noting (see Prop. 1) that we only need to consider cycles not having any category α cells.

Proposition 12 (Picard). *Any 1-cycle* $\sum_{k,i}\theta'_{ki}.(\alpha_i\beta_i\times C'_k) + \sum_k\theta''_k.(\beta_i\times B''_k)$ *of* H *is homologous to one having all* θ' *zero, i.e. to a 1-cycle of* S_0 .

We will use below the assumption that *the maps induced in 1-cycles at each* α_i *by the defining identifications of* H *are surjective* : presumably Picard had checked this for "ordinary singularities" in his book.

Proof. Considering terms of category α in $\partial(\sum_{k,i}\theta'_{ki}.(\alpha_i\beta_i\times C'_k) + \sum_k\theta''_k.(\beta_i\times B''_k)) = 0$ we get $\sum_{k,i}\theta'_{ki}.(\alpha_i\times C'_k) = 0$ which happens iff $\sum_k\theta'_{ki}.(\alpha_i\times C'_k) = 0$ in H for each i. So the sum of all the θ'_{ki} for which $\alpha_i\times C'_k$ is a fixed vertex of H is zero, and this implies $\sum_k\theta'_{ki} = 0 \ \forall\ i$.

So the 0-chain $\sum_k\theta'_{ki}.C'_k$ of P is the boundary of some 1-chain $\sum_k\zeta'_{ki}.B'_k$. In fact *this 1-chain can be so chosen that* $\sum_k\zeta'_{ki}.(\alpha_i\times B'_k) = 0$ *in* H.

To see this note first that when M approaches A_i along OA_i the equation $\partial(\sum_k\zeta'_{ki}.MB'_k) = \sum_k\theta'_{ki}.MC'_k$ becomes $\partial(\sum_k\zeta'_{ki}.\alpha_i\times B'_k) = \sum_k\theta'_{ki}.\alpha_i\times C'_k = 0$, so $\sum_k\zeta'_{ki}.(\alpha_i\times B'_k)$ is certainly a 1-cycle of the singular fiber above A_i. In case this is nonzero, we use the aforementioned surjectivity to choose a 1-cycle $\sum_k\varepsilon'_{ki}.B'_k$ of P'_i such that $\sum_k\varepsilon'_{ki}.(\alpha_i\times B'_k) = \sum_k\zeta'_{ki}.(\alpha_i\times B'_k)$ and replace $\sum_k\zeta'_{ki}.B'_k$ by $\sum_k\zeta'_{ki}.B'_k - \sum_k\varepsilon'_{ki}.B'_k$.

Having chosen such a 1-chain we now have $\partial(\sum_k\zeta'_{ki}.(\alpha_i\beta_i\times B'_k) = -\sum_k\zeta'_{ki}.(\beta_i\times B'_k) + \sum_k\theta'_{ki}.(\alpha_i\beta_i\times C'_k)$ which shows that the given 1-cycle $\sum_{k,i}\theta'_{ki}.(\alpha_i\beta_i\times C'_k) + \sum_k\theta''_k.(\beta_i\times B''_k)$ is homologous to one having only category β edges. *q.e.d.*

Proposition 13 (Picard). $H_1(V)$ *is isomorphic to the quotient of* $H_1(S)$ *obtained by dividing out by all classes of the type* $\omega - T_i(\omega)$.

Proof. We saw above that the 1-cycles ω of S_0 generate $H_1(V)$. Besides the relations given by homologies within S_0 we have the relations $\omega - T_i(\omega) = 0$ amongst these 1-cycles. This follows since $\omega - T_i(\omega) = 0$ can be transported to A_i where it vanishes. Also we know that for the "ordinary singularities" these are the only 1-cycles vanishing at A_i (see Prop. 9). This implies (? *this is not clear* ?) that there are no further relations amongst our generators for $H_1(V)$. *q.e.d.*

We shall call a 1-cycle of P" a **surviving 1-cycle** if it is not a linear combination of the vanishing cycles of P'_i, $1 \le i \le q$. Using the Poincaré duality theorem , § 2 and the above result give the following.

Proposition 14. *The number of homologically distinct surviving 1-cycles of* P" *is equal to the number of homologically distinct invariant 1-cycles of* P.

Poincaré also gives an argument independent of his duality theorem :

Proof. With respect to a choice of a basis ω_j of $H_1(S)$ we will think of each T_i as a (2p)x(2p) matrix. The number of subsisting cycles is equal to the number of distinct solutions of the system

$$\textbf{(A)} \quad \mathbf{x} = T_i(\mathbf{x}),\ 1 \le i \le q,\ C$$

of 2pq equations in 2p unknowns.

On the other hand we can also assume that our basis is such that the bilinear form $(y_2x_1 - y_1x_2) + (y_4x_3 - y_3x_4) + \dots$ is invariant when both **x** and **y** undergo the same transformation T_i , $1 \le i \le q$.

[Though for our purpose any non degenerate skew-symmetric form invariant under the finite group generated by the T_i's will do, Poincaré seems to be using a geometrically interesting one, viz. the **intersection form** of $H_1(V)$ (see § 9 of *"Analysis Situs"*) which is obviously preserved by the monodromy : thus the Picard group sits within the subgroup Sym(2p,$\mathbb{Z}$) of GL(2p,$\mathbb{Z}$) determined by this skew-symmetric form.]

On comparing with this form, we see that a cycle $m_1\omega_1 + m_2\omega_2 + m_3\omega_3 + m_4\omega_4 + \dots$ of S is invariant under all T_i iff the 2p-tuple $(-m_2, m_1, -m_4, m_3, \dots)$ is a solution of the system of equations (A). This shows that the two numbers in question are the same. *q.e.d.*

Poincaré concludes this section with a discussion of what transpires if we allow singularities of other kind. Now V may not be a 4-manifold, so the definitions of "*Analysis Situs*" become debatable : should one e.g. consider a 1-cycle which only bounds a 2-manifold with a singularity (say a conical point) as a boundary or not ? In sum he feels it is better to postpone this study and consider only the case when V is a 4-manifold.

§ 5. **Miscellaneous remarks.** Poincaré now wants to examine further the following points which had come up in the course of the above arguments:

1° That the polyhedron P' (= P'_i) covering S(M) remains homeomorphic to itself as M varies between 0 and A_i on OA_i.
2° That there is a natural 1-1 correspondence between the Riemann surface S_0 at O and each of the above Riemann surfaces S(M).
3° That there is indeed a region R of S_0 as in the proof of Prop. 4.
4° That a region of S_0 defined in the proof of Prop. 4 (when M described that flower-shaped contour) does not cover all of S_0.
5° That a vanishing cycle at A_i is necessarily of type $\omega - T_i(\omega)$.
6° That all the m points of S_0 taking a certain constant value of $x = x_0$ are vertices of P.

He says that the legitimacy of these hypotheses being almost evident he did'nt want to interrupt the above arguments to give an "explicit proof" of these, especially since it would have entailed choosing a particular P, but that *now* he will give such a proof.

[Actually, in our opinion, Poincaré still won't give an "explicit proof" of these hypotheses (of which all but 3°-5° *do* seem evident anyway) but rather some **examples** — since quite a few algebro-geometrical assumptions will be made below — which do however serve to bolster one's belief in the above hypotheses, and more important, **which serve to elaborate further the nature of the basic identifications of § 1.**]

A canonical polyhedron P. We start with a generic **base point** y and cover its Riemann surface S with a P having $m = \deg_z(f)$ faces F_k, these being 2n-gons given by the copies of **the x-plane with cuts** OB_1, ... , OB_n — the n **branch points** B_j are all the solutions of the equations $f = 0 = \frac{\partial f}{\partial z}$ — which are to be joined to each other (as per the nature of the multi-valued function z of x defined by f = 0) by gluing a lip of a cut in one copy to the other lip of the same cut in some other copy.

[Instead of choosing these cuts as straight lines of the x-plane it is better to choose them to be projections of (usually non-euclidean) "straight lines" in the u.c.s. of S : see *"Troisième Complément"*.]

We will assume that when y varies, the branch points $B_1, \dots, B_n$ remain distinct from each other and from the origin 0 of the x-plane, except for finitely many **singular points** $A_1, \dots, A_q$ ($\neq 0$) of the y-plane, at which

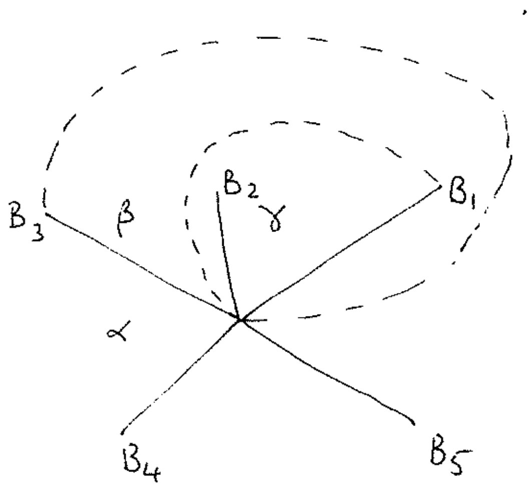
(1) some two of the branch points interchange, or else

(2) one of the branch points comes to the origin 0 of the x-plane.

[Poincaré asserts that this is so for the case of "ordinary singularities", and similar assertions are made later re other assumptions : our policy will be to just record the assumptions, without worrying about the extent of their validity, and thus simply treat what happens as an example of a V !]

As before we now cut the y-plane along $OA_1, \dots, OA_q$ and denote by Q the resulting 2q-gon, and we transport our P by monodromy (over arcs of intQ) from the base point to all points excepting the q-singular points. This gives, above points M_i on the cuts OA_i, two polyhedra M_iP and (M_iP), and above the origin 0 of the y-plane, q polyhedra, whose superpositions are P'_i and P''. We now examine these superpositions separately for the above cases (1) and (2).

EXAMPLE (1). Let us suppose that when y makes one full circuit around A_i the cut OB_1 becomes the dotted line OB'_3 from 0 to B_3 and the cut OB_3 becomes the dotted line OB'_1 from 0 to B_1, but all other cuts return to their initial values :



For the sake of simplicity let us assume that if we go around the branch point B_1 in the x-plane we simply permute two of the m sheets (= faces of P) of our Riemann surface : we'll call these the **first leaf** and **second leaf** of S. Then going around B_3 will also simply permute these same two sheets of S, and P'_i will coincide with P except for these two sheets which will each get subdivided into three cells $\{\alpha_1, \beta_1, \gamma_1\}$ and $\{\alpha_2, \beta_2, \gamma_2\}$ as shown.

Our P' will now be same as P except that these two faces will each get subdivided into three faces α, β, and γ, as shown above.

Verification of 1°. This is clear because, as M varies between 0 and A_i, the branch points remain distinct from each other and from 0, and so the subdivision P' retains this shape.

Verification of 2°. The above isomorphism between the polyhedra P' above various points M of OA_i gives the required diffeomorphism between their underlying spaces S(M)'s.

The identifications at A_i *will be assumed to be as under :*

The cuts OB_3 and OB_1 get identified respectively with the dotted cuts OB'_1 and OB'_3 and the region β disappears. More precisely what we mean by this is the following.

We first define GB_i or GB'_i — resp. DB_i or DB'_i — to be the edges of P' above OB_i or OB'_i such that we have

$$\partial(\beta_1) = B'_1D - B_1D + B'_3D - B_3D \text{ and}$$

$$\partial(\beta_2) = B'_1G - B_1G + B'_3G - B_3G.$$

[In the paper GB_i or GB'_i — resp. DB_i or DB'_i — is defined to be the edge of P' above OB_i or OB'_i which has the first leaf to its left (G is for *"gauche"*) and the second leaf on its right — resp. the first to

its right (D is for "*droite*") and the second to its left — as we move out on it starting from G, resp. D. However this is incorrect for then we do not have the above formulae which we will use below.]
Then as we go around A_i the pairs of edges $\{GB_1, GB'_3\}$, $\{DB_1, DB'_3\}$, $\{GB_3, GB'_1\}$ and $\{DB_3, DB'_1\}$ permute with each other, and at A_i these pairs of edges get identified with each other; furthermore the 2-cells β_1 and β_2 of P' disappear at A_i.

Verification of 5°. Clearly the 1-chain

$$\omega = B_3D - B'_1D - B_3G + B'_1G$$

of P' is a cycle. Moreover it does not bound, and since $T_i(\omega) = -\omega$, it (or rather its double 2ω) is a vanishing cycle at A_i of the type $\omega - T_i(\omega)$. It can be checked that there is no other vanishing cycle at A_i.

Verification of 3°. Let Ω be any 1-chain of P, having at a point M_i on OA_i the copy

$$\Omega_i = \zeta_1 . B_1D + \zeta_2 . B_1G + \zeta_3 . B_2D + \zeta_4 . B_2G + H \text{ (other edges)}$$

in M_iP, and thus the copy

$$\Omega'_i = \zeta_1 . B'_3D + \zeta_2 . B_3G + \zeta_3 . B'_1D + \zeta_4 . B'_1G + H$$

in (M_iP). If Ω is a 1-cycle we'll have here $\zeta_1 = -\zeta_2$ and $\zeta_3 = -\zeta_4$. An easy calculation now gives the equation

$$\Omega_i - \Omega'_i + (\zeta_1 - \zeta_3).\omega = \zeta_1(B'_1G - B_1G + B'_3G - B_3G - B'_1D + B_1D - B'_3D + B_3D)$$

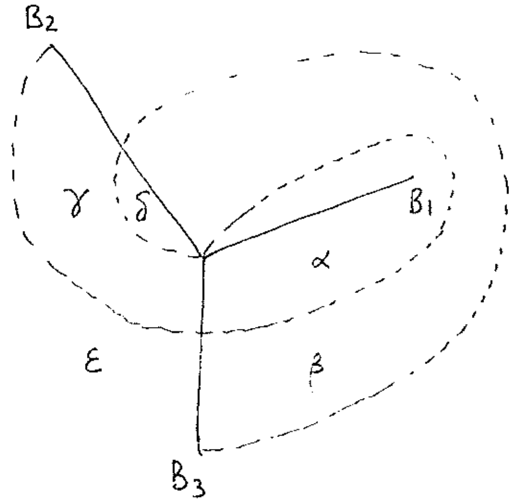
whose right hand side is $\zeta_1 . \partial(\beta_2 - \beta_1)$.

If Ω *is an invariant cycle* (Poincaré had failed to use this information in the "swept out" argument of Prop. 4 !) we must have $\zeta_1 = \zeta_3$: this because now $\Omega_i - \Omega'_i$ is homologically trivial while the vanishing cycle ω is not. So now the above equation reads

$$\zeta_1 . \partial(\beta_2 - \beta_1) = \Omega_i - \Omega'_i$$

and we can choose our 2-chain to be $\zeta_1.(\beta_2 - \beta_1)$ which vanishes at A_i as desired because the regions β_1 and β_2 disappear due to the identifications at A_i.

EXAMPLE (2). If at A_i the branch point B_1 goes to 0, then the other cuts can spiral around 0 (see below) with their final dotted positions possibly cutting their initial positions, but not cutting the small (and vanishing as we approach A_i) cut OB_1 :



We now obtain P' from P by subdividing each of its faces into five faces α, β, γ, δ, and ε as shown above.

Verification of 1∘. As we approach A_i the above shape remains same, only the cut OB_i becomes smaller and smaller, so P' remains same.

Verification of 2∘. The diffeomorphisms $S(M) \longleftrightarrow S(M')$ between the surfaces above pairs M, M' of points of OA_i between O and A_i are again immediate from 1∘. We note further (this is equally true in Example (1) and will be used later) that the points at infinity correspond to each other under these diffeomorphisms.

The identifications at A_i *will be assumed to be as under :*

The cut OB_1 will disappear. Moreover all intercepts of all (solid or dotted) cuts will diasappear, excepting the **final intercepts** (containing

B_i) : the final intercept of the solid cut OB_i will get identified with the final intercept of the dotted cut OB'_i. Moreover the faces α, β, γ, and δ will disappear.

Verification of 5°. This time there is no (homologically nontrivial) vanishing cycle. (Poincaré says that these singular points A_i are not "essential" and that in fact they can be avoid altogether.)

Verification of 3°. Let the two copies of a 1-cycle at $M_i \in OA_i$ be

$$\Omega_i = \sum \zeta_1 . OB_1 + \sum \zeta_2 . OB_2 + \sum \zeta_3 . OB_3 \text{ and}$$

$$\Omega'_i = \sum \zeta_1 . OB_1 + \sum \zeta_2 . OB'_2 + \sum \zeta_3 . OB'_3$$

(the summation being over various edges of P above indicated cuts). This gives

$$\Omega_i - \Omega'_i = \sum \zeta_2 . \partial(\alpha + \gamma + \delta) + \sum \zeta_3 . \partial(\alpha + \beta + \delta)$$

because of

$$\partial(\alpha + \gamma + \delta) = OB_2 - OB'_2 \text{ and}$$

$$\partial(\alpha + \beta + \delta) = OB_3 - OB'_3.$$

So we can take $\sum \zeta_2 . (\alpha + \gamma + \delta) + \sum \zeta_3 . (\alpha + \beta + \delta)$ to be the required region since it approaches zero as M_i approaches A_i. (Note that in this "inessential" case no use was made of the hypothesis that Ω is also invariant : so any cycle Ω is unchanged by the T_i's corresponding to these inessential A_i's.)

The remaining remarks apply to both examples.

Verification of 4°. As y describes the flower-shaped contour, the arcs described by the branch points remain in the finite x-plane. So the moving cycle Ω_i of S_0, being a combination of the edges above the cuts OB_i, will also remain away from the point at infinity. (Use is being made here of the fact that the points at infinity correspond to each other under $S_0 \longleftrightarrow S(M)$.) So we don't sweep out all of S_0.

Verification of 6∘. This is clear because we had chosen all points of our Riemann surface having $x = 0$ as vertices of P.

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APPENDIX

A LOOK BACK AT POINCARÉ'S "ANALYSIS SITUS"

by

K. S. Sarkaria

(Talk given in "Congrès International Henri Poincaré" at Nancy on 17/5/94)

§ 1. THIS IS A VERY BRIEF REPORT on a year-long Topology Seminar which I ran during 1993-94 at Panjab University. The detailed lecture notes of this seminar will be published elsewhere.[1]

Our object in this seminar was to get an over-all picture of what had been happening in this century's Topology, and with this in mind we had adopted the following strategy.

(i) To understand the mathematics of Poincaré's *"Analysis Situs"* and its five *Compléments* as clearly as possible, and

(ii) to understand the threads connecting Poincaré's ideas to future developements as clearly as possible.

In the course of doing (i) and (ii) we also got

(iii) some new results.

§ 2. THERE IS NO DOUBT that Poincaré's *"Analysis Situs"* and its five *Compléments* [P], 1892-1904, constitute a breathtaking, epic, monumental (almost any superlative seems inadequate !) work.

In fact if I were merely to make A LIST of the *big* ideas which occur one after another in it, I would over-step my time ! Nevertheless, let me at least *start* making such a list :—

- Boundary operator, **Betti numbers, homologies** (using smooth and

[1] A first edition of most of these notes (about 150 pp) is available.

oriented **"singular chains"** of a **differentiable manifold** : §§ 1-6). (However we note that Poincaré became aware of torsion only later, in the first *Complément*, while giving another definition of homology via incidence matrices of cell complexes.)

(The extent to which Poincaré's ideas have overshadowed this century's mathematics can perhaps be gauged from this simple little fact:

Out of all the Fields Medallists, with the exception of perhaps three or four, everyone of them — irrespective of his domain : number theory, algebra, analysis, ... — has used some *homology* in his work !!

Perhaps not since the invention of the *calculus* has a single tool so strongly influenced mathematics as *homology*.)

- Periods of indefinite integrals (= differential forms) and (implicitly) **de Rham cohomology** (§ 7).

- **Intersection matrices** and **Poincare' Duality** in orientable closed manifolds (§ 9, with a correct proof only later in § IX of the first *Complément*).

(Again it is remarkable how many fantastic results of this century — going back from Freedman and Donaldson, through Rochlin and Whitehead, to this beautiful duality of Poincaré — are at heart really assertions about the intersection matrix of an M^{4k} !)

- **Triangulability of differentiable manifolds** (assumed in § 10, with attempts at proof later in § 16 via *quadrillages*, and in § XI of the first *Complément* via a method of *rays*.)

- **Monodromy of integrable linear PDEs** (= flat connection) **on a manifold** and definition of the **fundamental group** $\pi_1(M)$ as the "most general" such group of M (§ 12).

(This definition of π_1 was later put on a firm footing, and used in his **de Rham homotopy theory**, by SULLIVAN [8], 1977.)

- Also the now standard (via **homotopy classes of loops** based at a point) definition of $\pi_1(M)$, and again a third "combinatorial" definition which gives relations for $\pi_1(M)$ if M is a **CW complex** obtained by pairwise identification of facets of a polyhedron (§§ 12-13).

- Many **computations** of fundamental groups and homologies, and **a classification theorem for some affine 3-manifolds** (which is perhaps the "deepest" result of the main paper : § 10-11, 13-14).

(In Dennis Sullivan's words this result is **"1/8 th of Thurston's theorem"** : the latter says roughly that any "irreducible" 3-manifold can be equipped with one of 8 specified "geometries".)

The above is *only the beginning* (based on §§ 1-14 of the main paper), but let me just stop here, and now tell you some more about the first and last items of the above partial list.

§ **3.** POINCARÉ'S FIRST DEFINITION OF HOMOLOGY. He starts off *Analysis Situs* by defining what we would now perhaps call a "differentiable quasi-affine non-singular complete intersection" $V^n \subset \mathbb{R}^N$, i.e. a clean intersection of N-n smooth hypersurfaces of an open set of $\mathbb{R}^N$ defined by so many smooth equations.

The aforementioned open set is assumed defined by some inequalities. He now starts replacing, one-by-one, these inequalities by equations, and by adding these, one-by-one, to the N-n defining equations of V, gets the **complete boundary** of V. Then the **boundary** ∂V of V is defined by dropping further the singularities : so e.g.

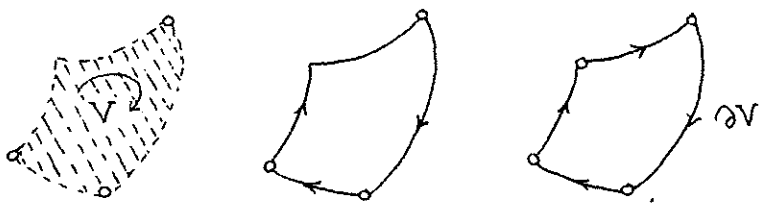

fig.1

Each V is (transversely) **oriented** by ordering its defining N-n equations (so a transposition of 2 of these equations gives not V but −V), and the boundary components are oriented by placing the new equation in the end (this is what gives the arrows in the above picture). Poincaré realizes (unlike Betti before perhaps ?) that "varieties" can repeat in ∂V, e.g.

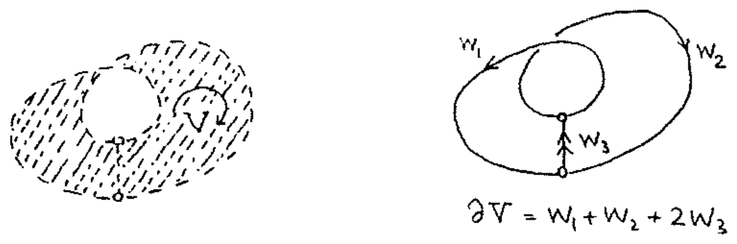

fig. 2

Whenever an integral linear combination of r-dimensional "varieties" equals a boundary ∂V he writes

$$c_1W_1 + c_2W_2 + \dots \quad \simeq \; 0.$$

Starting with these primitive relations (with all $W_i \subseteq M$) he now generates all **homologies of a manifold** M (see below for his definition of manifold) by "treating them just like equations" : i.e. by allowing such relations to be added, and terms taken to the other side if one changes sign, or multiplied by integers (and occasionally — and this of course makes a big difference ! — even division by nonzero integers).

WE BREAK HERE FOR POINCARÉ'S DEFINITION OF A MANIFOLD M : First he sort of retreats and considers more restrictive "**parametrized varieties**" i.e. $\nu^n \subset \mathbb{R}^N$ with a 1-1 onto θ from an open subset of $\mathbb{R}^n$ to ν^n given. But next he generalizes enormously via the idea of **continuation** : ν_1 and ν_2 are called continuations of each other if $\nu_1 \cap \nu_2$ is nonempty *and* is also a parametrized n-variety, e.g.

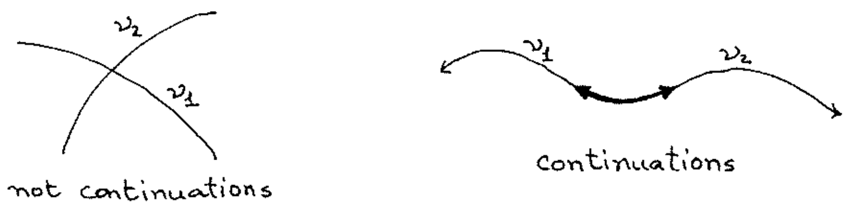

fig. 3

He defines M as a graph (= *résaux connexe*) whose vertices i are parametrized varieties ν_i(of $\mathbb{R}^N$) with edges {i,j} corresponding to pairs of varieties which are continuations of each other. In modern terms he has defined the notion of an abstract manifold M together with an immersion into some $\mathbb{R}^N$ (but for him the latter is always "extra baggage", and this is quite explicit as one reads on the paper).

RETURNING TO HOMOLOGIES, we now see Poincaré defining **Betti**

numbers $b_r(M)$ as the cardinality of a maximal set of linearly independent (i.e. no non-trivial homology between them) and closed (i.e. with $\partial c = 0$) combinations of r-dimensional subvarieties of M.

(**Betti's numbers** on the other hand had been defined restricting the coefficients c_i to be always $\{-1, 0, +1\}$: so Betti was in fact talking of the **least number of generators required to generate** $H_r(M)$ — see below.)

IN MODERN TERMS POINCARÉ'S DEFINITION RE-INTERPRETS AS FOLLOWS : Let $C_r(M)$ be integral combinations of (oriented) "r-varieties" of M, generalize ∂ by linearity to all these to get $\partial : C_r(M) \longrightarrow C_{r-1}(M)$ and since $\partial \circ \partial = 0$ define $H_*(M) = \ker\partial/\mathrm{im}\partial$. Then Poincaré's $b_r(M)$ is the $\mathbb{Z}$-dimension of $H_r(M)$ mod torsion. (As mentioned before Poincaré *did* become aware of torsion too, but later.)

RELATIONSHIP OF THIS DEFINITION WITH SINGULAR HOMOLOGY. There are essentially 2 differences. If we use all **continuous** (instead of just smooth) oriented ν's we get the definition of singular homology as given by LEFSCHETZ [5], 1933. If we further use **ordered** (instead of oriented) ν's, we get the current definition of EILENBERG [2], 1944. We note finally that standard techniques — cf. EILENBERG [3], 1947 — show that, for the case of smooth manifolds M, the aforementioned Poincaré homology groups $H_*(M)$ coincide with the singular homology groups of M.

§ **4.** POINCARÉ AND 3-MANIFOLD THEORY. Extrapolating from the case of 2-manifolds (also from his experience with fundamental domains of some Kleinian groups) Poincaré assumes the triangulability of closed 3-manifolds, i.e. that they can be obtained from a 3-polyhedron by a pairwise identification of its facets.

Since analysis of similar identifications had led to a classification of 2-manifolds, Poincaré now quite naturally wants to make lists of the 3-dimensional ones the same way.

LIKE ALWAYS HE STARTS OFF FROM SOMETHING VERY SIMPLE. He points out that the 2-torus is a **square**, with opposite sides identified, and it is the only orientable 2-manifold obtained this way. So what can we say about the parallel 3-dimensional case of the **cube** ? (Note also that any 3-polyhedron is a *subdivided cube*, so the undivided cube can serve as a starting point as one scans for *all* closed 3-manifolds : e.g. the famous homology sphere P^3 of Poincaré would be encountered at the "next" level in such a scan because the dodecahedron is combinatorially a very simple subdivided cube.)

fig.4

THE CUBE. Even if we only allow its *opposite* facets to be identified, we have much more leeway now than for a square : we are allowed (see fig. above) to first *rotate* a facet (through 0, $\pi/2$, π, or $3\pi/2$) and *then* identify with the opposite one. (We disallow *reflections* because we want *orientable* 3-manifolds only.)

Accordingly let us adopt the notation **abc**, $0 \le a,b,c \le 3$, to denote the cell complex obtained by rotating three compatibly oriented adjacent facets of the cube through these multiples of $\pi/2$, and then identifying with the opposite facets.

POINCARÉ'S FIRST FIVE EXAMPLES are (in the above notation) **000** (the 3-torus), **113** (a non-manifold), **111** (quaternionic space), **001** (a twisted 3-torus), and **222** (projective space, which he defines a little differently by using an octahedron).

In each case he tests for non-singularity by computing the Euler characteristic of the **links** at the vertices of the cell complex (the other points are obviously non-singular) and computes (for the four manifolds he gets) the fundamental group and the Betti numbers to show that they are topologically distinct.

During our seminar we checked that there are exactly *three more manifolds* of this kind : **002**, **022**, and **122** (Poincaré was certainly aware of at least the first of these because it belongs to the series below).

POINCARÉ'S SERIES **00T**, $T \in SL(2,\mathbb{Z})$. These manifolds are defined *combinatorially* as follows. Each integral matrix T with det(T) = 1 determines in a natural way (see fig. below) a **T-subdivision** of the unit square. Use this to subdivide the top of the cube, and analogously subdivide the bottom using the inverse matrix, and leave the vertical faces of the cube unsubdivided. Then **00T** is obtained by identifying opposite pairs of vertical faces without doing any preliminary rotation, and by identifying each piece of the subdivided bottom with the corresponding piece of the subdivided top.

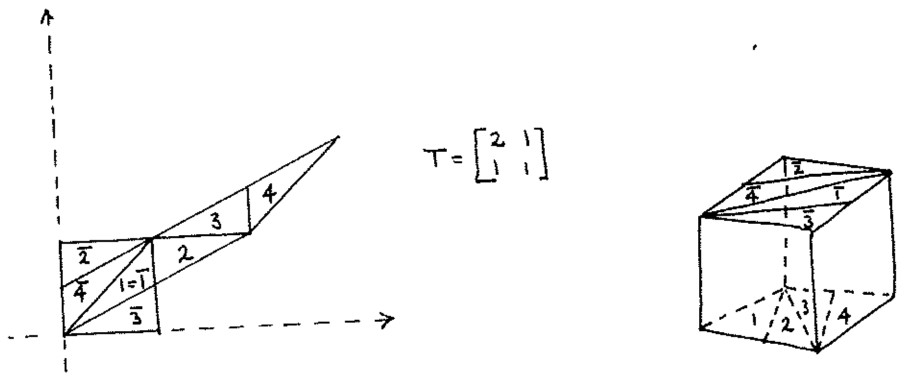

fig. 5

Alternatively OOT is also defined *group-theoretically* (Poincaré uses this for all his computations) as the quotient of 3-space by the **discontinuous group** generated by the three **affine motions**

$$(x,y,z) \longmapsto (x+1,y,z), \quad (x,y,z) \longmapsto (x,y+1,z), \quad \text{and}$$
$$((x,y),z) \longmapsto (T(x,y),z+1).$$

(These manifolds play a big rôle in the third and fourth *Compléments* — which deal with monodromy, etc., of algebraic surfaces — where Poincaré thinks of OOT as a **torus bundle** over the circle, viz. the **mapping torus** of the toral automorphism $T : \mathbb{R}^2/\mathbb{Z}^2 \rightarrow \mathbb{R}^2/\mathbb{Z}^2$: more generally he also considers surface bundles over the circle.)

Using his group theoretic definition of these manifolds Poincaré now proves the following (which he had announced in his 1892 *Comptes Rendus* note).

POINCARÉ'S RIGIDITY THEOREM. OOT *is homeomorphic to* OOU *if and only if* T *is conjugate to* U *or its inverse in* $GL(2,\mathbb{Z})$.

We remark that the above is in fact a *corrected* version of the result stated in the paper (the "or its inverse" is necessary, also Poincaré seems to conjugate within $SL(2,\mathbb{Z})$ which won't do). In our seminar we checked that the above is true even if $\det(T) = -1 = \det(U)$ (when of course these manifolds are non-orientable) and we also obtained the following arithmetical addendum to Poincaré's result.

ENUMERATION OF POINCARÉ'S SERIES. *There are infinitely many topologically distinct manifolds* OOT *with* $tr(T) = \pm 2$. *However for all* t *other than* ± 2, *the number* P(t) *of such manifolds with* $tr(T) = t$ *is finite, and is given by*

$$P(t) = \frac{h(t) + n_2(t) + 1}{2},$$

where h(t) = *number of ideal classes of* $\mathbb{Z}[(t^2-4)^{1/2}]$ *and* $n_2(t)$ = *number of elements of order* 2 *in this class group.*

A similar rigidity and enumeration result can most probably be established for another infinite series **22T** containing the manifolds **222**, **221**, and **220**. Since these manifolds 22T are definable "like" **lens spaces** (starting from $\mathbb{R}P^3$ minus a disk instead of a disk) we see that Poincaré's result is close (in spirit at least !) to the classification of lens spaces given in REIDEMEISTER [7], 1935.

More obviously Poincaré's rigidity theorem is akin to the later rigidity theorems of BIEBERBACH [1], 1911, and of MOSTOW [6], 1966. The former deals with conjugacy, by means of affine motions, of discontinuous groups of *Euclidean motions*, and thus is especially close to Poincaré's result, which deals with a similar problem for some discontinuous groups of *affine motions*.

We remark also that some definitive general results on the conjugacy of discontinuous groups of affine motions of 3-space have been proved by FRIED-GOLDMAN [4], 1983. For example they show that a closed 3-manifold is affinely flat if and only if it is finitely covered by a torus bundle over the circle (i.e. an **00T**). These authors also show that these are all the closed 3-manifolds which admit three (viz. the ones modelled by the left-invariant metric of a solvable 3-dimensional Lie group) of the eight "geometries" of THURSTON [9], 1982.

From the above I think it is amply clear that Poincaré's impressive contributions to 3-manifold theory are by no means limited to the very famous problem about closed 3-manifolds which he left to us (IS ONE OF THEM AN EXOTIC HOMOTOPY SPHERE ?) or to the enchanting (and ubiquitous !) EXOTIC HOMOLOGY SPHERE P^3 which he discovered in the fifth *Complément* of this paper.

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