COMBINATORIAL METHODS IN TOPOLOGY

BY

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Notes* of Lectures given in Fall 1994

in the

TOPOLOGY SEMINAR

of the

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TOPOLOGY SEMINAR

(1994-95)

The seminar for this year will meet, starting 31st August 1994, on Wednesdays and Fridays, from 3:30 p.m. to 5:30 p.m. in Room No. 17.

As already announced, besides **reviewing Poincaré** again — we went over Poincaré's *Analysis Situs* and its five *Compléments* during the seminar of 1993-94 — we'll focus this year mainly on **simplicial methods** in topology, i.e. the work of **Eilenberg**, Kan, Moore, etc. Our *goal* will be to identify Connes' cyclic cohomology with equivariant loop space cohomology; however the scenery, along the way to this, is so enticing, that we'll also be making *many side-trips* ...

All are welcome to attend the seminar.

K S Sarkaria

(K. S. Sarkaria)

P.S. (23.12.94) -- These are *provisional* notes of the 28 lectures given from 31.8.94 till 23.12.94.

P.P.S. (10.4.95) -- Now these notes look even more provisional than before! A number of corrections – mostly of details but occasionally even of ideas – need to be made. Also we have covered quite a bit of additional material .... However since all this is still not satisfactorily typed up, this'll have to do.....

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Chapter 3. **Poincare' : 3-Manifolds & De Rham Theory**

Chapter 4. **Minimal complexes**

Chapter 5. **Realization**

Chapter 6. **Operator Algebras**

Chapter 1.

POINCARE : HOMOLOGY THEORY

(1.1) THE IMPORTANCE OF POINCARE to twentieth century mathematics is clear e.g. from the rôle which one of his most important discoveries, viz. HOMOLOGY, has played in the last hundred years : in fact, not since Newton's and Leibniz's DERIVATIVE, has a single idea so completely dominated a century of mathematics as this one !

But having said this, it's important also to stress that Poincaré's work too was done on the "shoulders of (other) giants", e.g.

(1.2) **The birth of Euler characteristic theory** of course goes back at least to Euler who knew, that for *any* cell subdivision of the 2-sphere, one has

$$v - e + f = 2,$$

where e, v and f are, respectively, the number of vertices, edges, and faces of the cell subdivision.

REMARK. Thus it all began with "counting" ... and indeed this interaction **topology-combinatorics** is still all-important, e.g. the Euler characteristic theory of today (i.e. Lefschetz Formulae, Zeta Functions, etc.) has given us the celebrated Deligne-Weil Theorem which "counts" the number of points of a variety over a finite field.

We note that Euler's Formula was known even to DESCARTES, and perhaps (?) even to the PLATONISTS, since they (almost) knew of the following corollary of Euler's Formula.

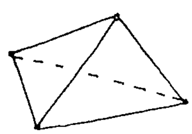
Theorem. *The* 2-*sphere has* precisely *five regular cell subdivisions.*

Here, we are using the word **regular** in a *combinatorial* sense : we want every face to have the same number (say r) of edges, and we want every vertex to be incident to the same number (say s) of edges.

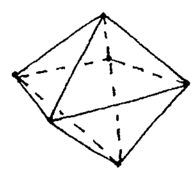
(As against this, the Greeks' "regular" was meant in a *geometric* sense: so their analogous result was partly weaker and partly stronger than the one stated above.)

Proof. Multiply Euler's Formula by rs and substitute $rf = 2e$ and $sv = 2e$ to get $e(2r - rs + 2s) = 2rs$. So $2r - rs + 2s$ has to be positive. Since also — with any reasonable definition of "cell subdivision" — $r \geq 3$ and $s \geq 3$ we see that we must have $\{r,s\} = \{3\}$, $\{3,4\}$ or $\{3,5\}$.

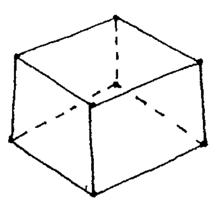
The following pictures show (combinatorially) regular cell subdivisions of S^2 having these values of r *and* s. (The existence of such *geometrically* regular cell subdivisions — which the Greeks showed — is somewhat harder.)



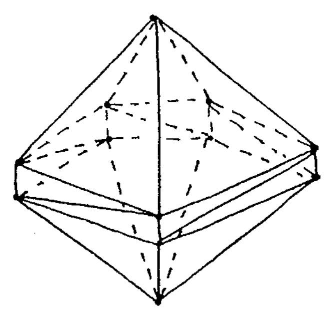
$r = s = 3$



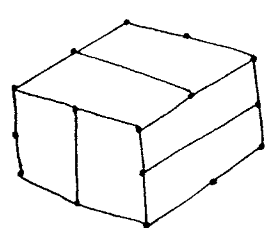
$r = 3 \quad s = 4$



$r = 4 \quad s = 3$



$r = 3 \quad s = 5$



$r = 5 \quad s = 3$

Finally it can be checked that, upto an isomorphism of cell complexes, the above regular cell subdivisions of the 2-sphere are the only ones having these values of r and s. *q.e.d.*

However it was POINCARE who launched Euler characteristic theory in real earnest (in §§ 16-18 of *"Analysis Situs"*, 1895) with the following far-reaching generalization of Euler's Formula.

Theorem. *Given a closed manifold X, the alternating sum*

$$f_0 - f_1 + f_2 - f_3 + \dots$$

of the numbers f_i *of i-dimensional faces of any cell subdivision of X, is* independent *of the cell subdivision, and is thus a topological* **invariant** e(X) *of* X.

In fact Poincaré (improving an earlier definition of Betti) also constructed **finer invariants** $b_i(X)$ of X, these being numbers which determine the aforementioned **Euler characteristic** by

$$e(X) = b_0(X) - b_1(X) + b_2(X) - b_3(X) + \dots .$$

Not only that! Poincaré went on to define **yet finer invariants** of X, which in today's language can be thought of as finetely generated Abelian groups $H_i(X)$ which determine the aforementioned **Betti numbers** by

$$b_i(X) \quad = \quad \dim_{\mathbb{Q}}(H_i(X)).$$

Note. The current jargon of point-set topology and abstract algebra came into vogue only during 1920-30 : so you won't find expressions like "open set" (and perhaps even "set" !), "finitely generated Abelian group", etc., in the *"Analysis Situs"*.

REMARK. For cell subdivisions of manifolds Poincaré also gave additional combinatorial relations (see § 17 of *Analysis Situs* and our notes on it from last year's Seminar) which easily give, for the case of triangulations, the so-called *Dehn-Sommerville equations*.

(1.3) The idea of a manifold too pre-dates Poincaré, and goes back at least to RIEMANN (see p. 10 of Poincaré Seminar notes). However it apparently appeared first in a (more-or-less !) fully developed form only in *Analysis Situs* (see § 3 of last year's notes).

Poincaré made all his definitions (of Euler characteristics, homologies, etc.) *only for manifolds*, and stated (in the third and fourth *Compléments*, which deal with algebraic surfaces) that it was not clear whether the obvious generalizations of these concepts were the "right" ones for singular spaces. Quite obviously he was loathe to renounce POINCARE DUALITY (*"Analysis Situs"* § 9, and the first two *Compléments*) ... it is amusing to note that now we do have the "right" generalizations for singular spaces (e.g. Intersection Homology, for which duality holds in much greater generality).

(1.4) The birth of homology theory. *"Analysis Situs"* contains THREE distinct (and now all equally important) definitions of homology (of which the "second" one we'll consider in 3.6 below) !

SINGULAR HOMOLOGY. Poincaré's "first definition" (which corrected Betti's flawed definition of the numbers named after him) involves considering closed varieties of a manifold X which are "independent" in the sense that no non-trivial integral linear combination is a boundary (Betti only considered combinations with coefficients −1, 0 and +1 : see §§ 5 and 6 of last year's notes). This definition (modulo the fact that Poincaré ignored torsion) is essentially (see Remark in 2.14 below) today's singular homology $H_*(X)$.

HOMOLOGY OF CW COMPLEXES. Though Poincaré never quite *defined* a CW complex (i.e. a space obtained from a point by iteratively adjoining higher-dimensional cells via some, not necessarily bijective, maps of their boundaries) he gave many nice examples of these (see e.g. 3.3) and his "third definition" (see the first and second *Compléments*) gives in fact an algorithm for computing homology (and by this time he was keeping track of torsion too).

More precisely he showed (upto a point only !) that the Betti numbers and torsion coeficients of X can be calculated via any CW complex K homeomorphic to X by computing the **rank** ρ and elementary divisors d_r of its **incidence matrices** M_i (note that because an (i-1)-cell can "occur" more than once in an i-cells some entries of M_i might be integers with absolute values greater than 1). In modern terminology Poincaré's result is the following.

Theorem $H_i(X) = \mathbb{Z}^{f_i(K)-\rho(M_i)-\rho(M_{i-1})} \oplus \{\mathbb{Z}/d_r(M_{i+1})\mathbb{Z}\}$.

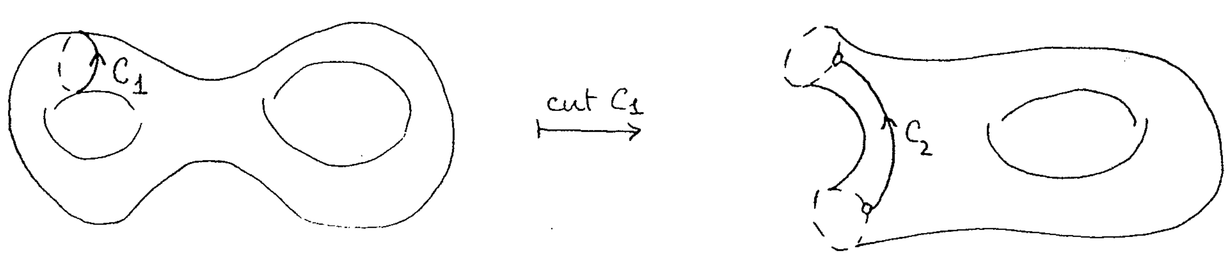
Once again, to keep our perspective historically correct, it is important to remember that, in all of the above, Poincaré was inspired — via Betti — by the ideas of

RIEMANN, who had, beginning with his famous *Inaugural Dissertation* of 1851, visualized the solutions of a non-singular polynomial equation $f(x,y) = 0$ over $\mathbb{C}$ as a closed (real) surface F, and discovered the remarkable fact that, up to a rational substitution of variables, the equation is conversely determined by the **connectivity** $b_1(F) + 1$ of F, which he had defined as follows.

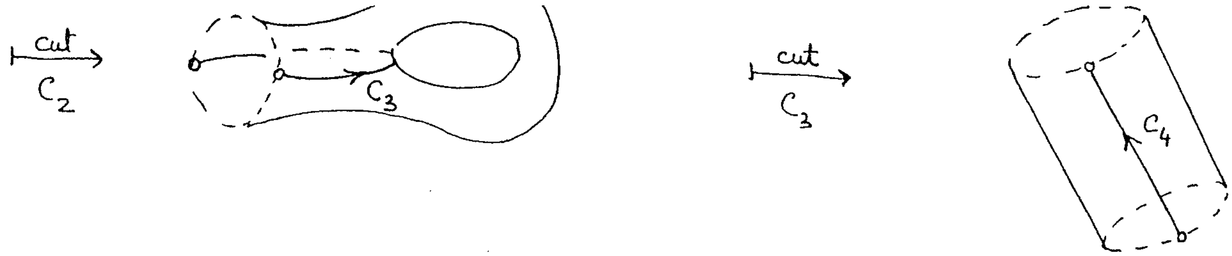
"Wir unter einer nfach zusammenhängenden Fläche eine solche verstehen, die durch n-1 Querschnitte in eine einfach zusammenhängende zerlegbar ist." (*Innaugural Dissertation* of 1851; see p.10 of his Collected Works.)

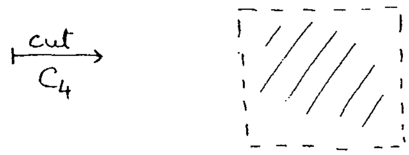
In other words, Riemann deemed the connectivity to be 1 if any closed curve separated the surface into two parts, and then inductively to be of connectivity $n \geq 2$ if, by cutting along a suitable closed curve, one obtained a surface of connectivity $n - 1$.

For example, shown below is a closed surface of connectivity 5.



(note $\partial C_2 = 0$ in mfld)





We remark that in fact almost *all* algebraic topological invariants of X can be interpreted as (sometimes *very* sophisticated !) measures of its "connectivity" in various diverse senses : the simplest such interpretation being that of $b_0(X)$ which is the number of **path components** of X.

(1.5) "Doing sums" (to use J.F. Adams favourite phrase) is what leads to new theories (which in turn enable us to do yet more sums). In particular our "hero", Poincaré, firmly believed in doing *lots and lots* of sums (see e.g. 3.3) : with this as my alibi, I intend to share with you many questions and exercises which I find interesting (or amusing).

Exercise. Classify the following connected subspaces of $\mathbb{R}^2$ upto homeomorphism :

H E N R I P O I N C A R E

Exercise (**Sprouts**). This is a two-person **game** starting with any n points of $\mathbb{R}^2$. The players take turns alternately, and each move consists of first joining two (possibly same) points by an arc not intersecting previously drawn arcs, and then marking a new point in the middle of this arc, taking care that no point ever winds up with more than 3 edges terminating in it. The game ends (with that player losing) when no such move can be made.

(i) Find all possible lengths (in number of moves) of such a game.

*(ii)** Does there exist a winning strategy for one of the players ?

(I remember the rules of Sprouts from an old article in *Scientific American* by Martin Gardner, and that *(ii)* was unsolved at that time.)

Exercise (**Proving**). Again this is a two-person **game** starting with any n points (= "statements") with the players taking turns alternately. But now each move consisting of drawing any directed arc (= "implication") from one of these statements to another, subject only to the restriction that this implication be "new", i.e. not a concatenation of existing implications. The player making the last move wins (= "gets the credit" for having completed the proof of the "theorem" that the n given implications were equivalent).

Answer above questions *(i)* and *(ii)* for this game too.

(*Hint :* The analysis of both the above games is facilitated by some rudimentary topological ideas.)

Poincaré's love for "doing sums" was what made him an "applied" mathematician par excellence, and (also!) one of the great physicists of this century. (To quote Chandrashekar — who himself got his Nobel pretty late — Poincaré was "the greatest physicist of this century not to have received a Nobel Prize".) But these facts about him, and the supposedly "pure" nature of algebraic topology, then prompts the following question.

(1.6) WHY DID THIS PRAGMATIC PROBLEM-SOLVER AND APPLIED MATHEMATICIAN INVENT ALGEBRAIC TOPOLOGY ? For the detailed and very eloquent way in which Poincaré himself provides an answer to this, the reader should at this point re-read the *Introduction* of Poincaré's *"Analysis Situs"* (for this, and our comments on it, see the Notes of last year's Seminar). Briefly we recall that Poincaré realized that

Topology gave us the means (= higher-dimensional analogues of the "badly drawn figures of a good geometer") to appropriately VISUALIZE, and thus perhaps solve, many diverse problems.

In particular he mentions that the solutions of polynomial equations, *resp.* ordinary differential equations, can often be usefully visualized (following Riemann and Picard, *resp.* Poincaré and Dyck) as manifolds, *resp.* manifolds-with-flows. The results which topological methods have bestowed to these fields, i.e. Algebraic Geometry and Global Analysis, are by now just too many to enumerate.

Poincaré hoped that a useful visualization would help also in understanding the structure of the **finite groups** contained in a $GL(n,\mathbb{C})$.

A finite group is just one special kind of **combinatorial configuration** : more generally we can hope that topological methods would shed light on many diverse combinatorial problems. This is indeed so, and now there are many combinatorial results — including some of Quillen, Aschbacher-Segev et al. pertaining to finite groups — which have been discovered or proved by topological methods.

THE EMPHASIS OF THIS YEAR'S SEMINAR will precisely be on this interaction Combinatorics $\longleftrightarrow$ Topology, and the combinatorial configuration to which we'll give pride of place is that of a SIMPLICIAL COMPLEX (see 2.2).

The study of simplicial complexes will inexorably lead us to various generalizations (chain complexes, semi-simplicial complexes, hypersimplices etc.) which have been developed to "visualize" them and/or to get more "elbow-room" (this is analogous to how a number theorist gets more room by enlarging $\mathbb{N}$ to $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, etc. etc.).

We remark that the final generality of these combinatorial methods is incidentally such that they *also* suffice for the other two applications of Topology which Poincaré had in mind, viz. to Algebraic Geometry and Global Analysis.

Chapter 2.
Eilenberg-Steenrod Axioms

(2.1) E-S CATEGORIES. We recall that a **category** $\mathfrak{C}$ is a class — in case $\mathfrak{C}$ is a set we'll call the category **small** — with a partially defined multiplication which is, firstly, **associative** — i.e. whenever fg and gh are defined, then (fg)h and f(gh) are defined and are equal — and, secondly, such that for each $f \in \mathfrak{C}$ there exist **identity elements** D and R of $\mathfrak{C}$ — i.e. elements which leave unchanged all elements of $\mathfrak{C}$ which can be multiplied with them — with fD and Rf defined.

Exercise. Check that $f \in \mathfrak{C}$ has a *unique* such **domain** D and **range** R, and that fg is defined iff range of g coincides with the domain of f.

By a **functor** $T : \mathfrak{C} \longrightarrow \mathfrak{D}$ between two categories we mean a map which preserves multiplication and identity elements.

From now on we'll refer to the members of a category as its **morphisms**, with the invertible ones called **equivalences**, and the identities called **objects**. If a morphism f has domain D and range R then we'll write $f : D \longrightarrow R$, and we'll refer to the partially defined multiplication of $\mathfrak{C}$ as **composition**.

A category will be called an **Eilenberg-Steenrod Category** if it comes equipped with some specified **couples** i.e. ordered pairs (g,h) of morphisms with hg defined, a specified reflexive and symmetric **binary relation** $\simeq$ on each set Morph(A,B) of morphisms from object A to object B, and some distinguished morphisms and objects called **excisions** and **points** respectively.

Two morphisms of Morph(A,B) equivalent under the relation generated by $\simeq$ will be called **homotopic**, and if [f] is an equivalence in the category obtained by replacing each Morph(A,B) by Morph(A,B)/$\simeq$, then f will be called a **homotopy equivalence**. A morphism which is a composition of some homotopy equivalences and excisions will be called a

generalized excision.

By a **morphism of couples** $f : (g,h) \longrightarrow (j,k)$ will be meant a *triple* f of morphisms of $\mathfrak{C}$ yielding the following commutative diagram (thus couples of $\mathfrak{C}$ and their morphisms constitutes another category) :

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & B \arrow[r, "h"] \arrow[d, "f"] & C \arrow[d, "f"] \\ L \arrow[r, "j"] & M \arrow[r, "k"] & N \end{tikzcd}$$

REMARK. Here **commutativity of a diagram** means that any composition of its morphisms is uniquely determined by its domain and range. Imagining each morphism as (*an abstract*) "voltage drop", we see that this is analogous to **Kirchoff's voltage law**, i.e. that the sum of the voltage drops around any closed loop is zero. (See e.g. Bollobas's book on Graph Theory about more on Kirchoff's laws.)

By an **E-S functor** $T : \mathfrak{C} \longrightarrow \mathfrak{D}$ between two E-S categories we'll mean one which preserves couples, homotopies, generalized excisions and points.

Without futher ado, we now take a first look at the (for us) most important category.

(2.2) $\mathfrak{Simp}$. A finite (and unless otherwise specified) *nonempty* set will often be called a **simplex**, its elements being its **vertices**. An (unless otherwise specified) *finite* set of simplices K will be called a **simplicial complex** if it is closed with respect to $\subseteq$.

We will identify the vertices v of simplices of K with the corresponding cardinality one simplices $\{v\}$ of K, and denote this subset of K by $\mathrm{vert}(K)$. A **simplicial map** is a function f from a simplicial complex K to another simplicial complex L, which maps $\mathrm{vert}(K)$ into $\mathrm{vert}(L)$, and is induced by this restriction as follows :

$$f(\sigma) = \{f(v) : v \in \sigma\}.$$

By a simplicial **pair** (K,L) will be meant an ordered pair of simplicial complexes with $K \supseteq L$, and by a simplicial map $f : (K,L) \longrightarrow (S,T)$ *of* pairs we'll mean a simplicial map of K into S which maps L into T. The pair $(K,0)$ — where 0 denotes the empty simplicial complex — will be identified with the simplicial complex K.

We'll associate to each pair (K,L) the **couple** consisting of the two inclusions $L \longrightarrow K \longrightarrow (K,L)$ of pairs, and will associate to each simplicial map f of pairs the obvious morphism f between their couples.

Two simplicial maps f and g between the same pairs will be called **contiguous** iff for each simplex σ of the domain, $f(\sigma) \cup g(\sigma)$ is a simplex of the range.

All inclusions of the type $(K, K \cap L) \longrightarrow (K \cup L, L)$, where K and L are any two simplicial complexes, will be called **simplicial excisions**; and, finally, simplicial complexes having just one vertex will be called **points**.

The class of all simplicial maps between pairs constitutes, under composition, and together with the structure defined above, an E-S Category which we'll denote by $\mathfrak{Simp}$.

(2.3) E-S HOMOLOGY THEORIES. Suppose there are associated to each object A of an E-S category $\mathfrak{C}$ some discrete or compact Abelian **homology groups** $H_q(A)$, $q \in \mathbb{Z}$, and to each morphism $f : A \longrightarrow B$ of $\mathfrak{C}$ are associated **induced homomorphisms** $f_* : H_q(A) \longrightarrow H_q(B)$ of these groups, and to each couple $A \longrightarrow B \longrightarrow C$ are asociated **connecting homomorphisms** $\partial : H_q(C) \longrightarrow H_{q-1}(A)$. Then this constitutes an E-S Homology Theory on $\mathfrak{C}$ provided the following **Eilenberg-Steenrod Axioms** hold.

Axiom 1. $f = \mathrm{id} \Rightarrow f_* = \mathrm{id}$.

Axiom 2. $(fg)_* = f_* \, g_*$.

Axiom 3. $f_* \, \partial = \partial \, f_*$ for any morphism f of couples of $\mathfrak{C}$.

Axiom 4 (EXACTNESS AXIOM). For any couple $A \longrightarrow B \longrightarrow C$ of $\mathfrak{C}$ we have the following **long exact sequence**

$$\ldots \longrightarrow H_q(A) \longrightarrow H_q(B) \longrightarrow H_q(C) \longrightarrow H_{q-1}(A) \longrightarrow \ldots$$

made up of the corresponding induced and connecting homomorphisms.

REMARK. Here, **exactness** means that the kernel of each map coincides with the image of the preceding. We note that this is somewhat remniscent of **Kirchoff's current law**, i.e. that the algebraical sum of the currents entering and leaving any node must be zero. (*Question :* Is there a useful common generalization of electrical network and "diagram-chasing" theories ?) Exactness was recognized explicitly in full generality first by Hurewicz, soon after Kolmogrov had pointed out that Alexander Duality is a corollary of the Poincaré Duality of (S^n, A) on account of the exactness of its long sequence.

Axiom 5 (HOMOTOPY AXIOM). $f \simeq g \Rightarrow f_* = g_*$.
Axiom 6 (EXCISION AXIOM). f excision $\Rightarrow f_*$ isomorphism.
Axiom 7 (DIMENSION AXIOM). A point $\Rightarrow H_q(A) = 0 \ \forall \ q \neq 0$.

In case Axioms 1 through 6 hold, but Axiom 7 does *not* hold, then we say that we have an **extraordinary homology theory** on $\mathfrak{C}$.

REMARK. For all examples of E-S categories $\mathfrak{C}$ to be considered it is easily seen that the above axioms amount to saying that **homology** H_* **is a functor from $\mathfrak{C}$ to the category $\mathfrak{Exact}$ of long exact sequences obeying Axioms 5-7.** So given any E-S functor $T : \mathfrak{D} \longrightarrow \mathfrak{C}$ between such categories, HT is an E-S homology on $\mathfrak{D}$. This observation will be used repeatedly to get many examples of E-S theories starting from one on the category $\mathfrak{Chain}$ (see 2.4) which contains $\mathfrak{Exact}$, and is also "bigger" than $\mathfrak{Simp}$ in the sense of (2.5) below.

REMARK. If a map between categories preserves identities but *reverses* multiplication then it is called a **contravariant functor**. The definition of an **E-S cohomology** on an E-S category is exactly similar to above except one uses * and q's as superscripts and reverses their arrows (also one writes δ in place of ∂). For all examples of E-S categories $\mathfrak{C}$ considered this amounts to saying that cohomology is a contravariant functor $H^* : \mathfrak{C} \longrightarrow \mathfrak{Exact}$ obeying the duals of Axioms 5-7.

Note. It will be understood from now on that "groups" (e.g. co/homology groups) can also be R-**modules** (with R-linear maps between them) where R is *any* fixed commutative ring with unity. However one has now *no simple* (see however the Universal Coefficient Theorem) *relationship between homology and cohomology* unless R happens to be say a field $\mathbb{F}$ (when of course "groups" means vector spaces over $\mathbb{F}$).

(2.4) $\mathfrak{Chain}$. Its objects are **chain complexes** (C), i.e. sequences

$$\cdots \xrightarrow{\partial} C_{q+1} \xrightarrow{\partial} C_q \xrightarrow{\partial} C_{q-1} \xrightarrow{\partial} \cdots$$

with $\partial^2 = 0$, and morphisms are **chain maps** $f : (C) \longrightarrow (D)$, i.e. degree preserving maps commuting with the ∂'s.

Its couples (f,g) are all **short exact sequences of chain complexes** (here (0) is the chain complex with all groups zero) :

$$(0) \longrightarrow (C) \xrightarrow{f} (D) \xrightarrow{g} (E) \longrightarrow (0) .$$

Two chain maps $f,\ g : (C) \longrightarrow (D)$ are called **chain homotopic** iff there exist homomorphisms $h : C_q \longrightarrow D_{q+1}$ such that

$$\partial\, h + h\, \partial \;=\; g - f \;.$$

Exercise. Show that this is an equivalence relation on Morph((C),(D)).

The equivalences of $\mathfrak{Chain}$ will be deemed to be its excisions, and a chain complex (C) will be called **point-like** iff $C_q \xrightarrow{\partial} C_{q-1}$ is an isomorphism whenever q is even and positive or else odd and negative.

Note. We will sometimes consider separately the category $\mathfrak{Cochain}$ of **cochain complexes** obtained by writing q's as superscripts and reversing arrows (and replacing ∂ by δ) in the above definitions, but sometimes we will just identify the two categories by the **"sign-changing trick"** $C^q = C_{-q}$.

REMARK. To see how the terminology "chain *complexes*" came into being we point out that a free Abelian C_* equipped with some specified bases β was essentially Lefschetz's **abstract cell complex**, the members of β being its (abstract) **cells**, and the matrices of ∂ in these bases being its **incidence matrices**. This obvious generalization of Poincaré's **geometrical cell complexes** was soon superseeded by Mayer's chain complexes, whenever the choice of the bases is immaterial. We note however that sometimes — e.g. for computing Whitehead Torsion — the choice of the bases β is all-important.

(2.5) Oriented chain complex. We will now construct a natural embedding $\mathfrak{Simp} \subseteq \mathfrak{Chain}$ of the category of simplicial pairs into that of chain complexes.

By an **orientation of a simplex** of cardinality q we'll understand a total ordering, *upto an even permutation* π, of its vertices, and we'll denote an **oriented simplex** by $[v_0v_1...v_q]$ where $v_0v_1...v_q$ is a total ordering of its vertices representing its orientation.

Now define $C_q(K)$ to be the Abelian group generated by all such oriented simplices subject only to the orientation relations

$$[v_{\pi(0)}v_{\pi(1)}...v_{\pi(q)}] \;=\; (-1)^{\pi}.[v_0v_1...v_q],$$

and we define the homomorphisms $C_q(K) \xrightarrow{\partial} C_{q-1}(K)$ by

$$\partial\,[v_0\;...\;v_q] \;=\; \sum\,(-1)^i\,[v_0\;..\hat{v}_i\;..\;v_q]\;.$$

Exercise. Show that these homomorphisms are well-defined and satisfy $\partial^2 = 0$.

More generally, the chain complex C(K,L) of any pair (K,L) is defined to be the termwise quotient of C(K) by C(L), with ∂ being the induced homomorphisms in these quotient groups.

Furthermore, for each simplicial map f, there is a chain map f between the corresponding chain complexes defined by

$$f[v_0v_1...v_q] = [f(v_0)f(v_1)..f(v)_q]\;,$$

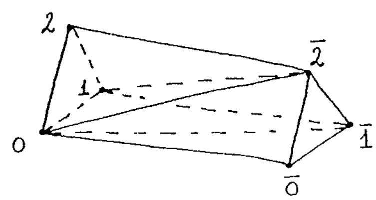
where the right side is to be interpreted as zero in case the f-images of two of the vertices coincide.

This completes our definition of the requisite one-one functor from $\mathfrak{Simp}$ to $\mathfrak{Chain}$.

Theorem. *The oriented chain complex functor is an E-S functor.*

Proof. The only non-trivial part is to check that if f and g are contiguous simplicial maps from K to L, then the corresponding maps f, g : $C_*(K) \longrightarrow C_*(L)$ are chain homotopic.

To see this we define, using the contiguity of f and g and motivated (see also 2.12) by the following picture,



the **"prismatic operator"** $h : C_*(K) \longrightarrow C_{*+1}(L)$ by

$$h[v_0v_1\ldots v_q] = \sum_i (-1)^i\ [f(v_0)f(v_1)\ldots f(v_i)g(v_i)\ldots g(v_q)],$$

and an easy verification (*Exercise*) shows that one has then indeed the requisite formula $\partial h + h\partial = g - f$. *q.e.d.*

(2.6) Homology of chain complexes. This important E-S homology theory, on the E-S category $\mathfrak{Chain}$, is defined as follows :

We associate to each chain complex (C) the groups

$$H_q(C) = \frac{\ker (C_q \xrightarrow{\partial} C_{q-1})}{\mathrm{im}\ (C_{q+1} \xrightarrow{\partial} C_q)},$$

and to each chain map $f : (C) \longrightarrow (D)$ the group homomorphisms $f_* : H_q(C)$

$\longrightarrow H_q(D)$ induced in these quotients. Lastly, to each short exact sequence of chain complexes $(0) \longrightarrow (C) \xrightarrow{f} (D) \xrightarrow{g} (E) \longrightarrow (0)$ we associate the group homomorphisms $\partial : H_q(E) \longrightarrow H_{q-1}(C)$ defined by

$$\partial\, [z] \;=\; [f^{-1}\partial\, g^{-1}\, z].$$

Exercise. Show that the above formula makes sense.

Theorem. *The above* H_* *is an E-S homology theory on* $\mathfrak{Chain}$.

Proof. Axioms 1-3 and 5-7 are trivial (*check*). The proof of the Exactness Axiom is long but straightforward using **"diagram chasing"** (i.e. the method you must have used to do the preceding exercise) and so is left as another *Exercise* (cf. Kelley-Pitcher or p.128 of Eilenberg and Steenrod's book.) *q.e.d.*

REMARK. Generalizing the notion of a chain complex we can define a **p-chain complex** to be a sequence $\ldots \xrightarrow{\partial} C_{q+1} \xrightarrow{\partial} C_q \xrightarrow{\partial} C_{q-1} \xrightarrow{\partial} \ldots$ with $\partial^p = 0$. For each $0 < q < p$ we can define its **q-homology** to be the graded group $H_{*,q}(C) = \dfrac{\ker(\partial)^q}{\mathrm{im}(\partial)^{p-q}}$, with the induced and connecting homomorphisms $f_{*,q} : H_{*,q}(C) \longrightarrow H_{*,q}(D)$ and $\partial^q : H_{*,q}(E) \longrightarrow H_{*-q,q-p}(C)$ being defined exactly as above. The same argument shows that, for each q, this gives an E-S homology theory on the category $\mathfrak{pChain}$ of all p-chain complexes.

(2.7) Oriented Simplicial Homology (with coefficients $\mathbb{Z}$) is the E-S homology theory obtained by restricting the homology of chain complexes to $\mathfrak{Simp} \subseteq \mathfrak{Chain}$ (see 2.5).

Coefficients. If instead of the functor $K \longmapsto C_*(K)$ we use the functors $K \longmapsto C_*(K)\otimes G$ or $K \longmapsto \mathrm{Hom}(C_*(K),G)$, then the homology of chain complexes pulls back to oriented simplicial (co)homology $H_*(K;G)$, $H^*(K;G)$ with coefficients G.

Exercise. Show that the functors $\mathfrak{Simp} \longrightarrow \mathfrak{Chain}$ defined by $K \longmapsto C_*(K)\otimes G$ and $K \longmapsto \mathrm{Hom}(C_*(K),G)$ are indeed E-S functors — *Hint :* use that $C_q(K)$ are *free* Abelian groups — and thus the above (co)homologies on $\mathfrak{S}$imp are also E-S.

(2.8) Pontrjagin duality. As a lead-in into what I wish to recall let us look first at the case of **field coefficients** $\mathbb{F}$, when $C_\bullet(K;\mathbb{F}) = C_\bullet(K)\otimes\mathbb{F}$ consists of $\mathbb{F}$-vector spaces and linear maps,

$$\cdots \longleftarrow C_{q-1}(K;\mathbb{F}) \xleftarrow{\partial_q} C_q(K;\mathbb{F}) \xleftarrow{\partial_{q+1}} C_{q+1}(K;\mathbb{F}) \longleftarrow \cdots ,$$

and has the *dual complex*,

$$\cdots \longrightarrow (C_{q-1}(K;\mathbb{F}))^* \xrightarrow{\partial_q^*} (C_q(K;\mathbb{F}))^* \xrightarrow{\partial_{q+1}^*} (C_{q+1}(K;\mathbb{F}))^* \longrightarrow \cdots ,$$

where V^* denotes the vector space of all **linear forms** $\alpha : V \longrightarrow \mathbb{F}$, and $f^* : V^* \to W^*$ denotes the linear map $\alpha \longmapsto \alpha\circ f$.

Exercise. Consider each cochain $a \in C^\bullet(K;\mathbb{F}) = \mathrm{Hom}(C_\bullet(K),\mathbb{F})$ as a function from oriented simplices of K to $\mathbb{F}$ (which changes sign when the orientation of the simplex is reversed) to define its **value** $\langle a,c\rangle$ on any chain $c \in C_\bullet(K;\mathbb{F})$. Check that this bilinear form is non-singular and that one has the **Stokes' formula** $\langle\delta a,c\rangle = \langle a,\partial c\rangle$. (Likewise the dual complex of the cochain complex identifies with the chain complex.)

We will use below this *natural identification* of the cochain complex of K with the above dual complex.

Theorem. $H^q(K;\mathbb{F}) \cong (H_q(K;\mathbb{F}))^*$.

Proof. We define a linear map $\ker(\partial_{q+1}^*) \longrightarrow (H_q(K;\mathbb{F}))^*$ as follows. The elements of $(H_q(K;\mathbb{F}))^*$ are linear forms on $\ker(\partial_q)$ which vanish on $\mathrm{im}(\partial_{q+1})$; but $\partial_{q+1}^*(\alpha) = 0$ iff $\alpha(\partial_{q+1}c)$ for all $c \in C_{q+1}(K;\mathbb{F})$; the required map restricts each α to $\ker(\partial_q)$.

Clearly this linear map is *onto* because any linear form on $\ker(\partial_q)$ can be extended to all of $C_q(K;\mathbb{F})$. Also it is clear that the kernel of this linear map contains $\mathrm{im}(\partial_q^*)$ because if $\alpha = \partial_q^*(\beta)$ then $\alpha(c) = \beta(\partial_q c)$ and thus α vanishes on $\ker(\partial_q)$.

In fact *the kernel is precisely* $\mathrm{im}(\partial_q^*)$. To see this note that ∂_q induces an isomorphism of $(C_q(K;\mathbb{F}))/\ker(\partial_q)$ with $\mathrm{im}(\partial_q)$. So any α vanishing on $\ker(\partial_q)$ is the pull-back under ∂_q of a unique linear form $\bar{\alpha}$ on $\mathrm{im}(\partial_q)$, and thus of any extension β of $\bar{\alpha}$ to all of $C_{q-1}(K;\mathbb{F})$. *q.e.d.*

REMARK. The above arrangement of argument is as in a paper of **Serre** (*Comment. Math. Helv.* of 1955) and shows that one can identify the cohomology of the dual complex (of *continuous linear forms*) of a chain complex of **Hausdorff locally convex vector spaces** with all ∂_q's continuous *and with closed images*, with the dual of the homology of the given chain complex (the hypotheses "$\ker(\partial_q)$ and $\mathrm{im}(\partial_q)$ closed" enables one to use the **Hahn-Banach theorem** to extend continuous linear forms defined on them). Using this he established the well-known *Serre duality* theorem for compact complex manifolds.

We turn now to the case of **discrete Abelian coefficients** G. The the chain complex $C_\bullet(K;G) = C_\bullet(K)\otimes G$ now comprises of Abelian groups and homomorphisms between them :

$$\cdots \longleftarrow C_{q-1}(K;G) \xleftarrow{\partial_q} C_q(K;G) \xleftarrow{\partial_{q+1}} C_{q+1}(K;G) \longleftarrow \cdots .$$

The *dual complex*

$$\cdots \longrightarrow (C_{q-1}(K;G))^* \xrightarrow{\partial_q^*} (C_q(K;G))^* \xrightarrow{\partial_{q+1}^*} (C_{q+1}(K;G))^* \longrightarrow \cdots ,$$

is now defined by using the character groups A^* of these discrete groups and the induced maps $f^* : V^* \longrightarrow W^*$, $\alpha \longmapsto \alpha\circ f$, between them.

Here we recall that by a **character** of A we mean a group homomorphism α from A into the **circle group** $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and the set A^* of all these α's is also made into a group by defining $(\alpha\beta)(a) = \alpha(a)\beta(a)$. Also, following **Pontrjagin** (see *Annals* of 1934) it is useful to equip it with the topology of pointwise convergence which makes it (by virtue of Tychonoff's theorem) into a **compact Abelian group**.

Thanks to this device Pontrjagin was able to prove the very remarkable fact that the original discrete Abelian group is naturally isomorphic to the discrete group of all **continuous characters** of A^* (and likewise, if we start from any compact Abelian group A, then we can recover it as the character group of the discrete group A^* of all its continuous characters).

Thus now the above dual complex consists of compact Abelian group and continuous boundary operators with closed (by virtue of the compactness) images. Thus (cf. above Remark) we can identify its cohomology with the character group of the homology of the original chain complex. (One again uses a "Hahn-Banach" theorem : any continuous character of a closed subgroup of a compact Abelian group extends to a continuous character of the entire grooup.)

Also the dual complex identifies naturally (cf. above Exercise) with the cochain complex of K with **compact Abelian coefficients** provided by the character group G^* of G. So putting all this together we get

Theorem. $H^q(K;G^*) \cong (H_q(K;G)^*$.

We remark that the easier field coefficients duality became clear only *after* the discovery of the above duality theorem by Pontrjagin (perhaps because it is *so easy* to confuse a finite dimensional vector space with its dual).

Some examples of character groups :

(a) For each $z \in \mathbb{T}$ one has the homomorphism $\mathbb{Z} \longrightarrow \mathbb{T}$ given by $n \longmapsto z^n$ and these are the only characters of $\mathbb{Z}$, thus $\mathbb{Z}^* \cong \mathbb{T}$.

Conversely each $n \in \mathbb{Z}$ determines the continuous homomorphism $\mathbb{T} \longrightarrow \mathbb{T}$ given by $z \longmapsto z^n$ and these are the only continuous characters of $\mathbb{T}$, thus $\mathbb{T}^* \cong \mathbb{Z}$.

(b) Each mth **root of unity** $z \in \mathbb{T}$, $z^m = 1$, gives the homomorphism $\mathbb{Z}/m \longrightarrow \mathbb{T}$ defined by $[n] \longmapsto z^n$ and these are the only characters of $\mathbb{Z}/m$, thus $(\mathbb{Z}/m)^* \cong U_m$ the subgroup of $\mathbb{T}$ consisting of all mth roots of unity. In the other direction, the m characters of U_m are $z \longmapsto z^n$, $0 \le n < m$, and so $(U_m)^* \cong \mathbb{Z}/m$.

(c) Now let A be *any finitely generated Abelian group*, so A is isomorphic to the direct product of some cyclic groups. But clearly

$$(G \times H)^* \cong G^* \times H^* ,$$

which shows, on using (a) and (b), that A^* must be a **torus** (of dimension equal to that of $A \otimes \mathbb{Q}$) times the product of some finite cyclic groups.

(d) Let p be any prime number and let $\mathbb{Z}[\frac{1}{p}]$ be the (non-finitely generated) additive subgroup of $\mathbb{Q}$ consisting of all rational numbers which can be written as m/p^n. The images z_1, z_2, ... of the succesive powers of $\frac{1}{p}$, under any group homomorphism

$$\mathbb{Z}[\frac{1}{p}] \longrightarrow \mathbb{T} ,$$

have clearly to obey the condition $(z_{i+1})^p = z_i$, and it is clear that each such sequence

$$\{z_1, z_2, \ldots : z_i \in \mathbb{T}, (z_{i+1})^p = z_i\}$$

determines a different character of $\mathbb{Z}[\frac{1}{p}]$. It can be shown (*Exercise*) that there are no other, thus the character group $(\mathbb{Z}[\frac{1}{p}])^*$ is isomorphic to the **solenoid** $\mathbb{T}_p$, i.e. the compact Abelian group formed by all such sequences of $\mathbb{T}$ under pointwise addition.

Note that $\mathbb{Z}[\frac{1}{p}]$ is isomorphic to the Abelian group (under termwise addition) of all sequences n_1, n_2, ... of integers subject to the relations $pn_i = n_{i+1}$.

In other words $\mathbb{Z}[\frac{1}{p}]$ is the **direct limit** of the sequence of group homomorphisms A_i $(= \mathbb{Z}) \longrightarrow A_{i+1}$ $(= \mathbb{Z})$ given by multiplication by p, while its dual $\mathbb{T}_p$ is the **inverse limit** of the sequence of continuous group homomorphisms B_i $(= \mathbb{T}) \longleftarrow B_{i+1}$ $(= \mathbb{T})$ given by pth powers. (There is a similar relationship between the dual of any direct limit and the inverse limit of the duals.)

REMARK. The reader might wonder if there is any relationship between the above and the additive group of **p-adic integers** $\mathbb{Z}_p$ (i.e. the inverse limit of the sequence of natural epimorphisms $\mathbb{Z}/p^i \longleftarrow \mathbb{Z}/p^{i+1}$) ? There is :

$$\mathrm{Ext}(\mathbb{Z},\mathbb{Z}[\tfrac{1}{p}]) \cong \mathbb{Z}_p/\mathbb{Z} .$$

For this see the paper of **Eilenberg-Maclane** in which Ext was first defined (*Annals* 1942). The problem which led to this work was posed by Eilenberg and (his teacher) **Borsuk** in 1937, viz. to classify homotopy classes of maps from the complement of a solenoid $\mathbb{T}_p$ in the 3-sphere S^3 to the 2-sphere S^2. The above formula of Maclane, and earlier work of Eilenberg (*obstruction theory*) and **Steenrod**, now lets them state the answer as

$$[S^3 \setminus \mathbb{T}_p \ , \ S^2] \cong \mathrm{Ext}(\mathbb{Z},\mathbb{Z}[\tfrac{1}{p}]),$$

(with exactly similar results for the groups $\mathbb{Z}[\frac{1}{a}]$ of (e) below).

Exercise (cf. Eilenberg-Steenrod, pp. 230-231). Give an explicit embedding of $\mathbb{T}_p$ in the 3-sphere and also show that $\mathbb{T}_p$ is a compactification of $\mathbb{R}$ with $T_p/\mathbb{R} \cong \mathbb{Z}_p/\mathbb{Z}$.

(e) It is easily seen (*Exercise*) that **any additive subgroup** A **of** $\mathbb{Q}$ **containing** $\mathbb{Z}$ is uniquely determined by a **generalized natural number**

$$\mathbf{a} = (p_1)^{n_1}(p_2)^{n_2} \ldots (p_i)^{n_i} \ldots$$

of **Steinitz** (is this the mathematician who was also *World Chess Champion* from 1866 to 1894 ?) : let n_i = supremum of the powers of the prime p_i which divides the denominators of the rationals of A (thus n_i is a non-negative integer or infinity). So we'll also write $A = \mathbb{Z}[\frac{1}{a}]$.

Clearly $\mathbb{Z}[\frac{1}{a}]$ is the direct limit of the net of group homomorphisms $G_\alpha = \mathbb{Z} \xrightarrow{\times\, \beta/\alpha} G_\beta = \mathbb{Z}$, $\alpha \le \beta$, where now α and β belong to $\mathcal{P}$, the **divisibility poset** (i.e. $\alpha \le \beta$ iff $\alpha|\beta$) of all natural numbers dividing **a**. So

$$(\mathbb{Z}[\frac{1}{a}])^* \cong \mathbb{T}_a ,$$

with the generalized solenoid $\mathbb{T}_a$ being the inverse limit of the dual net of continuous group homomorphisms of $\mathbb{T}$. (*Question.* Is the topology of the "biggest" solenoid $\mathbb{Q}^*$ of some use vis-à-vis the study of the divisibility poset of all natural numbers ?)

(2.9) $\mathfrak{Top}$**.** The E-S category $\mathfrak{Top}$ of pairs (X,A), $X \supseteq A$, of **topological spaces**, has as morphisms $f : (X,A) \longrightarrow (Y,B)$ continuous maps of X into Y which throw A into B. The pairs $(X,0)$, where 0 denotes the empty topological space, will be denoted X, and, corresponding to each pair, we'll define **couples** and their morphisms just as in $\mathfrak{Simp}$, and **points** will be, well, one-point spaces.

Two maps $f,g : (X,A) \longrightarrow (Y,B)$ will be called **homotopic** if we can find a map $h : (X \times [0,1],\ A \times [0,1]) \longrightarrow (Y,B)$ which composed with the injections $x \longmapsto (x,0)$ and $x \longmapsto (x,1)$ of (X,A) in $(X \times [0,1],\ A \times [0,1])$ yields f and g respectively.

As **excisions** in $\mathfrak{Top}$ we *won't* take (unlike in $\mathfrak{Simp}$) *all* inclusions of the type $(X,X\cap Y) \longrightarrow (X\cup Y,Y)$ but *only* those for which X is closed in $X\cup Y$ and the union of the interiors of X and Y is $X\cup Y$.

(2.10) $\mathfrak{Simp} \subseteq \mathfrak{Top}$. We'll now give a natural way of *visualizing* a simplicial complex (a purely combinatorial object !) as a nice topological space. Much of the interaction between Combinatorics and Topology has been via this **realization functor**, or functors very closely associated (see e.g. 2.11) to it.

We associate to each simplicial complex K the subspace of $\mathbb{R}^{\mathrm{vert}(K)}$ defined by

$$|K| = \{\alpha : \mathrm{vert}(K) \longrightarrow \mathbb{R} \ : \ \alpha(v) \geq 0, \ \sum \alpha(v) = 1, \ \mathrm{supp}(\alpha) \in K\},$$

to each (K,L) the pair $(|K|,|L|)$, and to each simplicial map f the continuous map defined by

$$|f|(\alpha)(w) = \sum \{\alpha(v) : f(v) = w\}.$$

The objects of $\mathfrak{Top}$ equivalent to objects lying in the image of this functor are called **triangulable** pairs of spaces, and we'll denote by $\mathfrak{Triang}$ the full subcategory of $\mathfrak{Top}$ defined by such pairs.

THE IMPORTANCE OF THIS WONDERFUL FUNCTOR $\mathfrak{Simp} \subseteq \mathfrak{Top}$ for topologists arises from the fact that *almost all* (i.e. excluding fractals, foliations, quotients of some group, etc.) spaces which a "normal mathematican" works with are (at least upto homotopy type) triangulable, and so can be studied via *finitistic* methods; and, for combinatorialists, its importance stems from the fact that it often lets them visualize their *finite but shapeless* objects K by means of spaces occuring in "every-day life". We'll be looking at the idea of realization in more depth in Chapters 4 and 5.

Note. We will be sometimes using the above definition of $|K|$ even when K is *not* closed under $\subseteq$.

(2.11) Aff(K) and Proj(K). The above visualization of simplicial complexes is the most common one; so much so that we will mostly just write K and f instead of $|K|$ and $|f|$. However there are some other closely related visualizations which are equally natural.

For example, dropping the inequalities in the definition of $|K|$, we will also associate to K the subspace of $\mathbb{R}^{\mathrm{vert}(K)}$ defined by

$$\mathrm{Aff}(K) = \{\alpha : \mathrm{vert}(K) \longrightarrow \mathbb{R} : \sum \alpha(v) = 1, \ \mathrm{supp}(\alpha) \in K\},$$

to (K,L) the pair (Aff(K), Aff(L)), and to each simplicial map f the continuous map defined by

$$|f|(\alpha)(w) = \sum \{\alpha(v) : f(v) = w\}.$$

Exercise. $|K|$ has the same homotopy type as Aff(K).

For each $v \in \mathrm{vert}(K)$ we'll denote by $v : \mathbb{R}^{\mathrm{vert}(K)} \longrightarrow \mathbb{R}$ the function $v(\alpha) = \alpha(v)$. Under pointwise addition and multiplication these functions generate the **graded ring** $\mathbb{R}[\mathrm{vert}K]$ **of polynomial functions** on $\mathbb{R}^{\mathrm{vert}(K)}$.

Theorem. $\mathrm{Aff}(K) \subseteq \mathbb{R}^{\mathrm{vert}(K)}$ *is an* **affine variety,** *being indeed the subset of the hyperplane* $\sum v = 1$, *defined by the monomial equations* $v^{\sigma} = 0$, $\sigma \notin K$, *where* $v^{\sigma} = \prod \{v \in \sigma\}$.

Proof. The result follows because a point α of this hyperplane is not in $\mathrm{Aff}(K)$ iff $\mathrm{supp}(\alpha) \notin K$, i.e. iff $v^{\mathrm{supp}(\alpha)}(\alpha) = \prod \{\alpha(v) : v \in \mathrm{supp}(\alpha)\}$ is nonzero. *q.e.d.*

Dropping further the sole non-homogenous equation $\sum v = 1$, it is natural also to consider the **projective variety** $\mathrm{Proj}(K) \subseteq \mathbb{RP}^{\mathrm{vert}(K)}$ — here $\mathbb{RP}^{\mathrm{vert}(K)}$ is the space of all one-dimensional vector subspaces of $\mathbb{R}^{\mathrm{vert}(K)}$ — defined by the equations $v^{\sigma} = 0$, $\sigma \notin K$.

Exercise. If U is the subspace of $\mathbb{RP}^{\mathrm{vert}(K)}$ consisting of all one-dimensional subspaces of $\mathbb{R}^{\mathrm{vert}(K)}$ *not* contained in the codimension one subspace $\sum v = 0$, then $U \cap \mathrm{Proj}(K)$ is an open dense subset of $\mathrm{Proj}(K)$ homeomorphic to $\mathrm{Aff}(K)$, and so $\mathrm{Proj}(K)$ is a **compactification** of $\mathrm{Aff}(K)$.

REMARK. We will later on compute (see Chapter 5) the singular (co)homology of Proj(K) and relate it to (the à priori finer) Dolbeaut cohomology of this projective varieties.

Note. The above definitions make sense even if $\mathbb{R}$ is replaced by $\mathbb{Z}$ or any field $\mathbb{F}$. This point will be of importance later when we consider the *Weil zeta function* (this counts points over finite fields) of Proj(K) .

(2.12) More maps for $\mathfrak{Simp}$. For many purposes there is not enough "elbow-room" in $\mathfrak{Simp}$ ... so we will sometimes go the whole hog and work in the full subcategory of $\mathfrak{Top}$ determined by $\mathfrak{Simp}$ (this is essentially equivalent to working in $\mathfrak{Triang}$).

On the other hand we'll at other times just go as far as using all (simplex-wise) **linear maps** $|K| \longrightarrow |L|$, i.e. those which can be extended to a linear maps $\mathbb{R}^{\mathrm{vert}(K)} \longrightarrow \mathbb{R}^{\mathrm{vert}(L)}$. It is easily seen (*Exercise*) that a linear map is determined by its values on the vertices, and that it is (the realization of) a simplicial map iff it maps each vertex to a vertex.

Though compositions of (simplex-wise) linear maps are linear, their inverses (*Exercise*) need not be linear. So it is useful to relent a bit more and admit all **piecewise linear** maps $|K| \longrightarrow |L|$, i.e. those which are of the type gf^{-1} where g and f are linear with f being bijective. Clearly the inverse of a bijective piecewise linear map is also piecewise linear.

We remark that till 1956, when **Milnor** settled it in the negative, the **Hauptvermutung**, i.e. the "big problem", in topology was whether or not homeomorphic simplicial complexes were always piecewise-linearly homeomorphic.

We note also that the notion of being piecewise linearly homeomorphic depends on the field $\mathbb{R}$. Our object in this section is to recall a remarkable theorem from 1926 which tells us that in fact this notion is *purely combinatorial !*

Towards this end, let us (cf. last section) denote our simplicial complex **K as a polynomial in its vertices** with each term (within which no vertex repeats) corresponding to a simplex of K. Dividing this polynomial out by any given $\sigma \in K$ we can write it uniquely as

$$K = \sigma . Q + R ,$$

with none of the terms of the "remainder" R (which is sometimes called the **antistar** of σ in K) divisible by σ (the "quotient" Q is called the **link** of σ in K and is usually denoted by $Lk_K \sigma$).

We now define the **stellar subdivision at σ of K** to be the simplicial complex with one vertex more, denoted $\hat{\sigma}$, defined by

$$sd_{\sigma}(K) = \hat{\sigma}.(\partial\sigma).Q + R ,$$

where $\partial\sigma$ denotes the subcomplex consisting of all proper faces of σ.

Exercise. Imaging $\hat{\sigma}$ to the **barycenter** of $|\sigma|$ — i.e. the point of $\mathbb{R}^{\mathrm{vert}(K)}$ given by $v \longmapsto 0$ unless $v \in \sigma$ when $v \longmapsto \frac{1}{\mathrm{card}(\sigma)}$ — one gets a linear projection $\mathbb{R}^{\mathrm{vert}(\mathrm{sd}_\sigma K)} \longrightarrow \mathbb{R}^{\mathrm{vert}(K)}$. Show that this restricts to a homeomorphism $|\mathrm{sd}_\sigma K| \cong |K|$.

We'll say that two simplicial complexes are **combinatorially homeomorphic** iff they are are related by the equivalence relation *generated* by that of being isomorphic to a stellar subdivision. (In other words we should be able to go from one to the other by a finite sequence of simplicial complexes such that each is either isomorphic to a stellar subdivision of the preceding or else has the preceding isomorphic to a stellar subdivision of itself.)

Newman's Theorem. *Two simplicial complexes are combinatorially homeomorphic iff they are piecewise linearly homeomorphic.*

Exercise*. Is the relation of having a common iterated stellar subdivision transitive ? (If so, this will obviously coincide with the relation of being combinatorially homeomorphic.)

By the **barycentric subdivision sd(K) of K** we'll mean the simplicial complex obtained by performing stellar subdivisions on *all* simplices σ of K in any order such that $\dim(\sigma)$ is *non-decreasing*.

The particular choice of such an order does'nt matter because any such sd(K) will consists precisely of all simplices of the type $\{\hat{\sigma}, \hat{\tau}, \hat{\upsilon}, \ldots\}$, $K \ni \sigma \supset \tau \supset \upsilon \supset \ldots$. To see this assume inductively that the sd of the subcomplex determined by simplices preceding σ is of this type and use $\mathrm{sd}_\sigma(K) = \hat{\sigma}.(\partial\sigma).Q + R$.

More generally (removing the hats ^ from our σ's) we'll also consider the **subdivision functor**

$$\mathrm{sd} : \mathfrak{Posets} \longrightarrow \mathfrak{Simp}$$

on the category of partially ordered sets P by letting sd(P) be the simplicial complex consisting of all totally ordered subsets of P.

Using this one can sometimes even use any $\subseteq$ preserving or reversing, i.e. **monotone maps** $f : K \longrightarrow L$ between simplicial complexes; in fact as the following exercise indicates, such maps can be useful even when K and L are *not* closed under $\subseteq$.

Exercise. Let K be any finite set of simplices *not* closed under $\subseteq$, then $|sd(K)|$ has the same *homotopy type* as the non-compact space $|K|$. (Hint : Assume inductively that there is a deformation of the union of the realizations of the preceding simplices to their sd , and then cone over $\hat{\sigma}$ the part of this homotopy which lies on $\partial\sigma$.)

REMARK. Many natural constructions (e.g. deleting a subcomplex, doing identifications, taking cartesian products, ...) take us out from $\mathfrak{Simp}$ but keep us within $\mathfrak{Posets}$. Provided one takes a little "care" it is usually possible to come back into $\mathfrak{Simp}$, without losing any relevant topological information, by means of the functor.

Exercise. Identifying the two ends A and A' of an edge AA' we get a CW complex P homeomorphic to the circle. However sd(P) of the underlying poset is not. What "care" will you take to corect this ?

(2.13) Products. Given two simplicial complexes $|K|$ and $|L|$ the natural subdivision of the product space $|K| \times |L|$ which presents itself is by means of the **cells** $\sigma \times \theta$, $\sigma \in K$, $\theta \in L$, i.e. the convex hull, in $\mathbb{R}^{vert(K)} \times \mathbb{R}^{vert(L)}$, of the finite set of points (v,w), $v \in \sigma$, $w \in \theta$.

If one wants to stay within $\mathfrak{Simp}$ one way out (see below) is to use the subdivision functor sd on the **poset** $K \times L$, $(\sigma,\theta) \leq (\tau,\upsilon)$ iff $\sigma \subseteq \tau$ and $\theta \subseteq \upsilon$.

Exercise. Check that the linear map which images each vertex $\sigma \times \theta$ of $sd(K \times L)$ to the barycenter (obvious definition) of the cell $\sigma \times \theta$ gives a homeomorphism of $|sd(K \times L)|$ with $|K| \times |L|$.

However since $sd(K \times L)$ has far too many simplices we'll often use instead the **E-S subdivision** $K_t \times L_t$ which depends on the choice of *total orders* t on vert(K) and vert(L) : it consists of all simplices of the type $\{(v_0,w_0),(v_1,w_1),\dots,(v_q,w_q)\}$ with $\{v_0,v_1,\dots v_q\}$ a simplex of K, $\{w_0,w_1,\dots w_q\}$ a simplex of L, and with $v_0v_1\dots v_q$ and $w_0w_1\dots w_q$ *non-decreasing sequences* w.r.t the chosen total orders t.

Exercise. Check that the obvious isomorphism $\mathbb{R}^{\mathrm{vert}(K)\ \times\ \mathrm{vert}(L)} \longrightarrow \mathbb{R}^{\mathrm{vert}(K)} \times \mathbb{R}^{\mathrm{vert}(L)}$ of vector spaces images $|K_t \times L_t|$ onto $|K| \times |L|$. (Hint : see the diagram of 2.5.)

This very economical triangulation of the product only suffers from the aesthetic defect that it depends on the total orders t. So Eilenberg and Steenrod also consider the **simplicial product** $K \Delta L$ which is defined just as above, except that now $v_0v_1\dots v_q$ and $w_0w_1\dots w_q$ can be *any sequences* supported on the simplices of K and L.

Since its dimension is bigger than that of the product we certainly can not expect $K \Delta L$ to triangulate it. However, as the following shows, for most purposes one can work even with this.

Exercise. Show that $|K \Delta L|$ has the same homotopy type as $|K| \times |L|$. (Hint : use the deformation retraction provided by the two projections of each vertex.)

REMARK. Perhaps motivated in part by the above, Eilenberg and Zilber later defined (see 2.18 and § 4) the notion of a *semi-simplicial complex.* In this terminology $K_t \times L_t$ supports the product of the **semi-simplicial complexes of non-decreasing sequences** associated to K and L, while $K \Delta L$ supports the product of the **semi-simplicial complexes of all sequences** associated to K and L. These semi-simplicial complexes of vertex sequences will play a big rôle in the following.

(2.14) Singular chain complex. We fix a Hilbert space V, and by an **(ordered) standard simplex** σ we'll mean the convex hull of a finite sequence of (distinct) affinely independent points $v_0v_1 \dots v_q$: thus there are (q+1)! standard q-simplices having the same set of vertices, and from any standard q-simplex σ, to any other θ, there is a unique *affine homeomorphism* $\theta\sigma$ which preserves the specified vertex orderings.

Now let X be any topological space. By a **singular simplex** $T[\sigma]$ of

X we'll mean any continuous function T imaging a standard simplex σ into X (rather than just its image $T(\sigma) \subseteq X$).

Any linear integral combination of standard simplices will be called a **standard chain**, and likewise one of singular simplices will be called a **singular chain** of X. If T is a continuous function into X defined on the union of some standard simplices σ_i, then we'll denote the singular chain $\sum_i n_i T[\sigma_i]$ by $T[\sum_i n_i \sigma_i]$.

We now equip the free Abelian group of all standard chains with the boundary operator defined by $\partial\sigma = \sum_i (-1)^i \sigma^{(i)}$, where if $\sigma = v_0 v_1 \cdots v_q$ and $0 \le i \le q$, then $\sigma^{(i)}$ denotes the standard simplex $v_0 v_1 \cdot\cdot \hat{v}_i \cdot\cdot v_q$. Likewise we equip the free Abelian group of all singular simplices of X with the boundary operator $\partial\, T[\sigma] = T[\partial\sigma]$. Clearly both operators obey $\partial^2 = 0$.

We now define the **singular chain complex** C(X) to be the quotient of the above determined by the relations

$$T[\sigma] = U[\theta] \text{ iff } T = U \circ \theta\sigma.$$

The **singular chain complex functor** $\mathfrak{Top} \longrightarrow \mathfrak{Chain}$ associates to each pair (X, A) of topological spaces the quotient C(X)/C(A), and to each continuous map f between pairs the chain map defined by $T[\sigma] \longmapsto fT[\sigma]$; the homology of C(X)/C(A) will be denoted $H_*(X, A)$ and called the **singular homology** (with coefficients $\mathbb{Z}$) of the pair (X, A).

The quotient chain complex C(X) *is also free Abelian* : in fact if (following **Eilenberg**) we choose, for each $q \ge 0$, just one standard q-simplex Δ^q, then clearly C(X) is isomorphic to the **free** Abelian group generated by all singular simplices of the type $T[\Delta^q]$, $q \ge 0$.

The above reformulation of Eilenberg's definition seems convenient e.g. we can now represent any element of C(X) by a singular chain of the type T[c], c being some suitable standard chain. (Moreover it can be shown that one gets the same homology even *without* the above relations and this fact might conceivably be of some use ?)

REMARK. Before Eilenberg, there were the **oriented singular chains** of **Lefschetz**, defined just as above, except that he used *oriented* standard simplices σ and his C(X) was given by the relations $T[-\sigma] = -T[\sigma]$ and $T[\sigma] = U[\theta]$ iff $T = U \circ \theta\sigma$, where now $\theta\sigma : \sigma \longrightarrow \theta$ is any orientation preserving affine homeomorphism. It is easily seen (*Exercise*) that this C(X) is *not* free Abelian. However it too gives the same (co)homology. Clearly any element of this C(X) can be represented by T[c], with c being now some oriented standard chain : thus Lefschetz's definition is the same as **Poincare'**'s (see our notes of last year on §§ 4-6 of *Analysis Situs*) except that the latter worked only with smooth manifolds X, and used only smooth T's of maximal rank. As was shown by **Eilenberg** in the *Annals* of 1947 the homology of such **smooth singular chains** is again the ordinary homology of X.

In case $\mathcal{U}$ is an open covering of X we'll say that a singular simplex is **$\mathcal{U}$-small** if its image is contained in some member of $\mathcal{U}$. In the course of the following proof we'll see that the **singular homology $H_*(\mathcal{U})$ of the covering** $\mathcal{U}$, i.e. the homology of the sub chain complex $C_*(\mathcal{U})$ determined by the $\mathcal{U}$-small singular simplices, is the same for all open coverings of X. (This is unlike some other homologies which equal some "limit" as the coverings become smaller and smaller.)

Theorem. *The singular chain complex functor is an E-S functor.*

Proof. Only the verifications re homotopy and excisions are non-trivial. We show first how each homotopy $h : X \times [0,1] \longrightarrow Y$ gives rise to a corresponding chain homotopy $h : C_*(X) \longrightarrow C_{*+1}(Y)$.

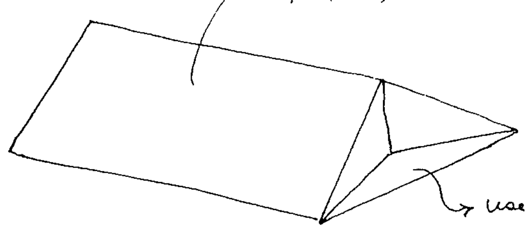
For this we multiply each standard simplex by a perpendicular unit segment and orient (by an increasing induction on dimension) these **standard prisms** $\sigma \times [0,1]$ of V in such a way that the following **boundary formula** holds :

$$\partial(\sigma \times [0,1]) + \partial\sigma \times [0,1] = \sigma \times \{1\} - \sigma \times \{0\}.$$

We now suppose each of these prisms equipped with the E-S subdivision of (2.13). Then for each singular q-simplex $T : \sigma \longrightarrow X$ we have a singular (q+1)-chain $T \times id : \sigma \longrightarrow X \times [0,1]$ and composing this with the given homotopy we get (by virtue of the above boundary formula) the corresponding chain homotopy.

Next we show how, given any open covering $\mathcal{U}$, we can chain homotop singular simplices to $\mathcal{U}$-small singular chains. For this we assume inductively that we have already done this in dimensions less than q, and shall show how it can be now done for each singular q-simplex $T[\sigma]$ imaging each proper face of σ to a member of $\mathcal{U}$.

We subdivide σ so finely that T becomes a $\mathcal{U}$-small singular chain of X. We use a copy of this triangulation on $\sigma \times \{1\}$, leave the rest of the boundary of $\sigma \times [0,1]$ unsubdivided, and extended this triangulation of its boundary to all of $\sigma \times [0,1]$. (For example we can iterate the stellar subdivision shown below.) Using this, and the identity homotopy $h : X \times [0,1] \longrightarrow X$, $h(x,t) = x$, the same construction as above tells us how each T can be chain homotoped to a $\mathcal{U}$-small singular chains.



This implies that *any* inclusion $(X, X\cap Y) \longrightarrow (X\cup Y, Y)$, where the union of the interiors of X and Y in $X\cup Y$ is $X\cup Y$, induces an isomorphism in singular homology. Indeed by above we need to work only with sub chain complexes c generated by singular simplices whose images are contained either in X or in Y, and then $c(X\cup Y) = c(X)+c(Y)$, which gives $c(X)/c(X\cap Y) = c(X)/c(X)\cap c(Y) \cong c(X)+c(Y)/c(Y) = c(X\cup Y)/c(Y)$. *q.e.d.*

More generally we will define **singular (co)homology with coefficients** G other than $\mathbb{Z}$ just as in (2.7). By above result these are all examples of Eilenberg-Steenrod homologies on $\mathfrak{Top}$.

(2.15) Simplicial approximation. Here we give two results re the geometry of simplicial complexes and their implication re E-S homologies.

Theorem. *Every simplicial subcomplex is a strong deformation retract of a closed neighbourhood.*

Proof. In case the subcomplex L is not **full** in K ($\partial\sigma \in L \Rightarrow \sigma \in L$) make it full by doing a subdivision. So it is a retract of its open simplicial neighbourhood. Another subdivision and we can get such a closed neighbourhood too. *q.e.d.*

Corollary. *The restriction to* $\mathfrak{Simp}$ *of any E-S theory on* $\mathfrak{Top}$ *is also an E-S theory,*

Proof. Only the stronger-looking excision axiom of $\mathfrak{Simp}$ was in doubt and now this follows at once by using above result.

Theorem. *Any continuous map between simplicial pairs is homotopic to a simplicial map of some subdivision of the domain.*

We omit the proof of this "simp[licial approximation theorem". Note that this has the following immediate corollary.

Corollary. *If two E-S homology theories on* $\mathfrak{Top}$ *coincide on* $\mathfrak{Simp}$ *then they'll automaticall coincide in* $\mathfrak{Triang}$.

So to prove the uniqueness of E-S theories it suffices to look at the purely combinatorial case of E-S theories on $\mathfrak{Simp}$, and the theorem pertaining to this will be established in (2.17) below. For this we need a new tool.

(2.16) Spectral sequences. In 1946 another applied mathematician called **Jean Leray** (= editor of the Topology section of Poincaré's *Oeuvres*, 1953) added two very original (and as we'll see *very fruitful*) ideas to topology, of which we'll consider one now.

(The other, viz. that of cohomology with coefficients in a "sheaf", we'll consider later. Leray's ideas sprang from a seminar he conducted while a prisoner for five years in a German P.O.W. camp during World War II ! In this context we note that (conscientious objector) André Weil had also, somewhat earlier, made similar use of the "hospitality" of a French jail — in fact such good use that in his *Apprentisage* he strongly recommends a stint in jail to all aspiring mathematicians !!)

The idea is that when a chain complex $C = (C_q, \partial)$ comes (this happens quite often !) with a natural **filtration**, i.e. is a union of a given increasing sequence $C^r = (C_q^r, \partial)$ of sub chain complexes, then it is useful, instead of going *all at once* to $H(C)$, to approach it step-by-step as follows (this simplification of Leray's definition was found later by **Massey**, *Annals*, 1952).

For each r we have

$$\begin{tikzcd}
H(C^r) \arrow[rr] & & H(C^{r+1}) \arrow[dl] \\
& H(C^{r+1}/C^r) \arrow[ul] &
\end{tikzcd} ,$$

the long exact homology sequence (see 2.6) of the pair (C^{r+1}, C^r). On adding (i.e. taking the direct sum of) all these triangles we get an **exact couple**,

$$\begin{tikzcd}
D^1 \arrow[rr, "f"] & & D^1 \arrow[dl, "g"] \\
& E^1 \arrow[ul, "h"] &
\end{tikzcd} .$$

Note now that $(gh)^2 = 0$, and let $E^2 = H(E^1)$ be the homology of E^1 under this **differential** $d^1 = gh$; also let $D^2 = f(D^1)$. Further let $f\colon D^2 \longrightarrow D^2$

be the restriction of the above f; g: $D^2 \longrightarrow E^2$ the quotient of the above g; and define a new h : $E^2 \longrightarrow D^2$ like the connecting homomorphism of (2.6). This gives (*Exercise*) a new exact couple with D^1 replaced by D^2 and E^1 replaced by E^2. Iterating this construction we have thus, for each $r \geq 1$, an exact couple

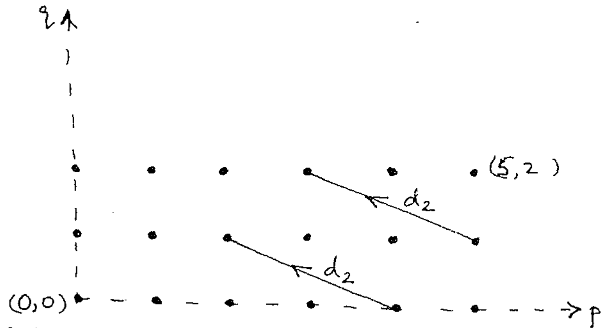
$$\begin{tikzcd} D^r \arrow[rr, "f"] & & D^r \arrow[dl, "g"] \\ & E^r \arrow[ul, "h"] & \end{tikzcd}$$

The sequence of differential groups (E^r, d^r), each of which is the homology of the previous, is called the **spectral sequence** of the filtered chain complex.

In case the filtration is of finite length (i.e. $C^r = C$ for all r sufficiently big) it is easily seen (*Exercise*) that the differentials d^r become zero for r sufficiently big and that one has then $E^r = H(C)$: this fact is denoted briefly by writing $E^\infty = H(C)$, and one says that the spectral sequence **converges** to the homology of C.

While doing "sums" with spectral sequences it is useful to **bigrade** these spectral groups as follows. We start by assigning to the summand $H(C^p_{p+q}/C^{p-1}_{p+q})$ of E^1 the bigrading (p,q). The differential d^1 is homogenous of bidegree (-1,0). So there is an induced bigrading of E^2. Further from the above definition it is easily checked that d^2 is homogenous of bidegree (-2,1) (i.e. is a "knight's move"). More generally d^r has bidegree (-r,r-1).

Using this bigrading it is usual to display the bidegree (p,q) summands of each E^r at the (p,q) point of an integer planar lattice. We note that in most applications the summands are nonzero only in the **first quadrant**, e.g. the second term of a spectral sequence may be



Here the bold dots denote nonzero groups while all other places are zero groups. We note e.g. in the above example the spectral sequence must **collapse** at $r = 4$ (i.e. $E^r = H(C)$) because clearly (just from their length, i.e just from "dimensional considerations") it follows that the the differentials d^r are zero. (It is simply amazing how many interesting results have been proved by playing this — as Bott calls it — **"tic-tac-toe"** with groups !)

(2.17) Eilenberg-Steenrod theorem. Consider any *generalized* (i.e. we insist only on the first six axioms) Eilenberg-Steenrod homology theory h_* defined on $\mathfrak{Simp}$.

We now use the **skeletal filtration** $K^0 \subseteq K^1 \subseteq \ ..$ of a simplicial complex K, and the long exact homology sequences

$$\begin{tikzcd}
h(K^r) \arrow[rr] & & h(K^{r+1}) \arrow[dl] \\
& h(K^{r+1}, K^r) \arrow[ul] &
\end{tikzcd}$$

provided by Axiom 4, to define, exactly as in (2.16), the so-called (*Proc. Sympos. A.M.S.* 1963) **Atiyah-Hirzebruch spectral sequence** $E^r_{p,q}(K)$ of this homology theory h .

Theorem. *This spectral sequence converges to* $h_*(K)$ *and its second term is given by*

$$E^2_{p,q}(K) \cong H_p(K, h_q(pt)),$$

where H_* *denotes the oriented simplicial homology of* K.

Proof. The assertion re convergence is clear. For the rest we'll give a mere sketch. Using the axioms one checks that $E^1_{p,q}(K)$ is isomorphic to the direct sum of the groups $h_q(\sigma, \partial\sigma)$ as σ runs over all p-dimensional simplices of K. But $h_q(\sigma,\partial\sigma) \cong h_q(pt)$. So we see that $E^1_{p,q}(K) \cong C_p(K; h_q(pt))$. Next one turns to d^1 and checks that this is nothing but the oriented boundary operator of the chain complex $C_*(K; h(pt))$. This implies the stated isomorphism. *q.e.d.*

We remark that the recovery of the oriented chain complex from the axioms (mentioned in the above proof) is essentially what is done in Ch.III of the Eilenberg-Steenrod book. In fact the above spectral sequence was "folk-lore" amongst topologists long before Atiyah and Hirzebruch made use of it in the aforementioned 1963 paper (which created topological "K-theory").

Corollary. *An E-S homology theory* h_* *on* $\mathfrak{Simp}$ *is uniquely determined by its coefficients.*

Proof. Since now h_* also satisfies the dimension axiom $h_q(pt)$ is zero for nonzero q. So, using the above and dimensional considerations, it follows that the spectral sequence collapses at the second term. This gives $h_*(K) \cong H_*(K, h_0(pt))$. *q.e.d.*

We remark that if $K \longrightarrow L$ is any simplicial map, then the spectral filtration of L pulls back to a filtration of K. The associated spectral sequence of such a filtration (for the case of ordinary homology) was used by **Serre** in his famous paper on fibrations.

Another very important filtration is that of the singular (semi-simplicial) complex defined by **Eilenberg** using his complexes $S_n(X)$. Its spectral sequence relates the homotopy groups of X to its singular homology groups and was one of the key tools used to unlock some of the musteries of homotopy groups of spheres. We'll look at this more in § 3.

REMARK. A homology theory satisfying Axioms 1-6 is *not* determined by its values on {pt}. However there may be a more involved classification of such homology theories ?

(2.18) Other examples of homologies on $\mathfrak{Simp}$. So far we have considered *just two* homologies for a simplicial complex K, viz. the oriented homology of (2.7), and the singular homology (2.14) of the realization $|K|$ (see 2.11) of K (and miraculously it turned out that these two homologies are in fact isomorphic). We will now remedy this situation by giving a long (but nevertheless still very incomplete) list of many other such homologies. (**Warning.** *Homologies will be coming out of your ears by the time this seminar ends !*)

We'll see, by applying the Eilenberg-Steenrod Theorem (2.17), that some of these definitions also lead to the same homology groups (however it is to be stressed that the novelty of the definition can be very important for combinatorial applications) while for some other of these definitions even the homology groups will be different.

At this point we give the following important definition (although we'll study this notion in more detail only later in Ch. 4).

SEMI-SIMPLICIAL COMPLEX. By this we mean a set S which is partitioned into disjoint subsets $S_{\{q\}}$, $q \geq 0$, with elements of $S_{\{q\}}$ called the q-dimensional *simplices* of S; furthermore, to each q-simplex σ, $q \geq 1$, are associated q+1 (q-1)-dimensional simplices $\partial_i(\sigma)$, $0 \leq i \leq q$, called the *faces* of σ, in such a way that one has

$$\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i \quad \text{if } i < j.$$

There are many natural semi-simplicial complexes which one can associate to a simplicial complex K.

For example we can consider K itself as a semi-simplicial complex *provided* we choose (cf. 2.13) a *total order* t for its vertices : now

$\partial_0(\sigma)$ is obtained by omitting the least vertex of σ, $\partial_1(\sigma)$ by omitting the second least vertex, and so on.

Another very important example is that of 2.14, i.e. all the singular simplices of the realization $|K|$. Now one defines the faces of a standard simplex as above, and the restriction of the singular simplex to these faces gives the faces of the singular complex.

It is to be observed that the two specific homologies we mentioned (oriented and singular) were defined *via* the two semi-simplicial complexes just mentioned. We'll now consider below other semi-simplicial complexes K_{assoc}, K_{comm}, etc., which can be naturally associated to K, and define some other homologies of K *via* them.

(a) Associative homologies.

We will denote by $C(K_{assoc})$ the free graded Abelian group generated by all **associative monomials**, i.e. all finite **sequences of vertices**, which are supported on the simplices of the complex K. So the $C(K)$ of (2.5), which we'll now also denote by $C(K_{orient})$, is the quotient of $C(K_{assoc})$ under the relations

$$v_{\pi(0)}v_{\pi(1)} \cdots v_{\pi(q)} = (-1)^{\pi} v_0 v_1 \cdots v_q .$$

We define the homomorphism $\partial : C_q(K_{assoc}) \longrightarrow C_{q-1}(K_{assoc})$ by

$$\partial(v_0 v_1 \cdots v_q) = \textstyle\sum_i (-1)^i v_0 v_1 \cdot\cdot \hat{v}_i \cdot\cdot v_q .$$

An easy computation shows $\partial^2 = 0$, and we'll denote by $H_*(K_{assoc})$ the homology of this chain complex $(C_*(K_{assoc}),\partial)$.

REMARK. Sometimes (see Ch. 6) we will think of $C(K_{assoc})$ as the quotient of the free **associative algebra** $\mathbb{Z}\langle \mathrm{vert}(K)\rangle$ by the ideal generated by monomials *not* supported on K. When so doing we'll assume that it has **unity**, and accordingly will **augment** the chain complex by $\partial : C_0(K_{assoc}) \longrightarrow \mathbb{Z} = C_{-1}(K_{assoc})$, the homomorphism which assigns to each 0-chain the sum of its coefficients : thus homology gets changed slightly in an obvious way to a so-called **reduced homology**. (For some combinatorial arguments too it is helpful to have the **empty simplex** in K : one can think of C_{-1} as the group corresponding to this "simplex".)

Theorem. *The quotient map* $C_*(K_{assoc}) \longrightarrow C_*(K_{orient})$ *commutes with the boundary operators and induces an isomorphism*

$$H_*(K_{assoc}) \cong H_*(K_{orient}).$$

Proof. It is easily checked (*Exercise*) that the above quotient map is a chain epimorphism, that it induces a natural transformation from the homology $H_*(K_{assoc})$ to the E-S homology $H_*(K_{orient}) = H_*(K)$, and that this last is an isomorphism of homologies for K = pt.

Thus, by (2.17), it'll suffice to check that $H_*(K_{assoc})$ is also an E-S homology, and as far as this is concerned, all axioms are clear (*Exercise*) except the homotopy axiom.

For this we use the two given contiguous simplicial maps f, g : K $\longrightarrow$ L to define (as in 2.5) a homomorphism $h : C_*(K) \longrightarrow C_{*+1}(L)$ by

$$h(v_0v_1\ldots v_q) = \textstyle\sum_i f(v_0)f(v_1)\ldots f(v_i)g(v_i)\ldots g(v_q) ,$$

and an easy calculation (*Exercise*) shows $\partial h - h\partial = g - f$. *q.e.d.*

Warning. Just as we abbreviated $C(K_{orient})$ to $C(K)$ we'll also frequently abbreviate $C(K_{assoc})$, etc., etc., also to $C(K)$ (see 3.1)!

(b) Commutative homologies.

We now turn to $C(K_{comm})$ the free graded Abelian group generated by all **commutative monomials** of vertices supported on the simplices of the complex K, i.e. the quotient of $C(K_{assoc})$ under the relations

$$v_{\pi(0)}v_{\pi(1)} \cdots v_{\pi(q)} = v_0v_1 \cdots v_q .$$

These relations are *not* compatible with the boundary operator of $C(K_{assoc})$: so unlike $C(K_{orient})$ (cf. also (e) below) we *cannot define a quotient boundary operator* in $C(K_{comm})$.

So what we'll do is, we'll choose some **total order** α on vert(K), and transport to $C(K_{comm})$ the boundary operator of the *sub chain complex* of $C(K_{assoc})$ spanned by all **non-decreasing vertex sequences** supported on the simplices of K. Thus *the operator* $\partial = \partial_\alpha$ *for* $C(K_{comm})$ *depends on the total order* α *of* vert(K), however we'll see now that the homology $H_*(K_{comm})$ of $(C(K_{comm}),\partial_\alpha)$ is independent of α.

Theorem. $\alpha : C(K_{comm}) \longrightarrow C(K_{assoc})$ *induces an isomorphism*

$$H_*(K_{comm}) \cong H_*(K_{assoc}).$$

Proof. Exactly as in (a), with the same h. *q.e.d.*

REMARK. Often it is useful (see Ch. 5) to think of $C(K_{comm})$ as the quotient of $\mathbb{Z}[\mathrm{vert}(K)]$ by the ideal generated by monomials *not* supported on K, and while so doing, to augment it and use reduced homology.

(c) Mayer's homology.

We now turn to another definition (due to Mayer from the *Annals* of 1942), also starting from commutative monomials, which has the advantage that the boundary operator does not depend on the choice of a total order on vert(K), but which requires coefficients G of **finite exponent**, i.e. there should exist a (least) $p \in \mathbb{N}$ such that $pG = 0$.

The **Mayer boundary** is obtained by replacing all the -1's of the above order-dependent ∂ by just 1's, thus it is given by

$$\partial(f(v_0,v_1,v_2, \ldots)) = \frac{\partial f}{\partial v_0} + \frac{\partial f}{\partial v_1} + \frac{\partial f}{\partial v_2} + \ldots .$$

This of course means renouncing $\partial^2 = 0$, but we now use $pG = 0$ to see (*Exercise*) that we have at least $\partial^p = 0$, and thus, for each ordered pair (r,s) of positive integers with sum $r + s = p$ we can define

$$H_{*;r,s}(K;G) = \frac{\ker(\partial^r)}{\mathrm{im}(\partial^s)} .$$

The following was observed by **Spanier** (*Bull. A.M.S.* of 1949) and was probably the first public use of (2.17) (being Steenrod's student, Spanier was privy to the still unfinished ms. of the E-S book).

Theorem. *For* p *prime the non-zero Mayer homology groups are :*

$$H_{kp+r-1;r,s}(K;G) \cong H_{kp+s-1;s,r}(K;G) \cong H_{2k}(K;G)$$

$$H_{kp-1;r,s}(K;G) \cong H_{kp-1;s,r}(K;G) \cong H_{2k-1}(K;G)$$

Proof. Clearly it suffices to check, for each (r,s), $r+s = p$, and $0 \le t < r$, that the homology $\mathbf{H}_*(K,L)$ of the chain complex $(\mathbf{C}_*(K,L),\mathbf{d})$,

$$0 \xleftarrow{\partial^r} C_t(K,L) \xleftarrow{\partial^s} C_{t+s}(K,L) \xleftarrow{\partial^r} C_{t+p}(K,L) \xleftarrow{\partial^s} \ \ldots.,$$

is an E-S homology, which is non-trivial, and with coefficients G, only if $t = r-1$.

All the E-S properties are obvious — e.g. use (2.5) for exactness axiom and $\mathbf{C}(K_1)/\mathbf{C}(K_1)\cap\mathbf{C}(K_2) \cong \mathbf{C}(K_1 \cup K_2)/\mathbf{C}(K_2)$ for excision — save the homotopy and dimension axioms.

Dimension axiom. When $K = pt$, then each $C_i = G$, and each Mayer boundary is multiplication by some j, $0 \le j \le p-1$:

$$0 \leftarrow C_0 \xleftarrow{2} C_1 \xleftarrow{3} C_2 \ \ldots \ \xleftarrow{p-1} C_{p-2} \xleftarrow{0} C_{p-1} \xleftarrow{1} C_p \xleftarrow{2} C_{p+1} \xleftarrow{3} C_{p+2} \ \ldots$$

Since p *is a prime*, multiplication by any nonzero j is an isomorphism of

G. Using this we see that $H_*(pt)$ (the homology defined above) is identically zero unless $t = r-1$, when it reads G, 0, 0, ...

We next give another argument which establishes a **strong dimension axiom**, viz. that the homology $H_*(\sigma)$ of *any* closed simplex is zero in all nonzero dimensions (and this strong dimension axiom will then be used to check the homotopy axiom).

To see this let u be the first vertex of σ, and define the **"cone operator"** $h : C_q(\sigma) \longrightarrow C_{q+p-1}(\sigma)$ for $q \geq -1$ (now $C_{-1} = G$) by

$$h(v_0v_1v_2 \dots) = u^{p-1}v_0v_1v_2 \dots .$$

So $[\partial, h](v_0v_1v_2 \dots) = (\partial h - h\partial)(v_0v_1v_2 \dots) = (p-1)(u^{p-1}v_0v_1v_2 \dots)$, and thus a (p-1)-fold iteration of this calculation gives

$$[\partial[\ \dots\ [\partial[\partial,h]]\ \dots] = (p-1)!(id) .$$

Since p is a prime we have $(p-1)! = -1 \bmod p$. So the above formula shows $\ker(\partial)^r = \mathrm{im}(\partial)^s$, i.e. that the reduced homology $H_*(\sigma)$ is zero in all dimensions.

Homotopy axiom. Assume inductively that we have already constructed an $h : C_*(K\backslash\sigma) \longrightarrow C_{*+1}(L)$, on K minus a top-dimensional simplex σ, such that (on $C_*(K\backslash\sigma)$) it satisfies $hd + dh = g - f$, where the chain morphisms $f, g : C_*(K) \longrightarrow C_*(L)$ are induced by the two given contiguous simplicial maps f and g. So $-hd(\sigma) + g(d\sigma) - f(d\sigma)$ is a d-cycle of the closed simplex $f(\sigma)\cup g(\sigma)$ of L. Since $H_*(f(\sigma)\cup g(\sigma)) = 0$ we can find a chain $h(\sigma) \in C_*(f(\sigma)\cup g(\sigma))$ such that $dc(\sigma) = -hd(\sigma) + g(d\sigma) - f(d\sigma)$, and thus extend the chain homotopy h to all of K. *q.e.d.*

We remark that the above iterative way of defining chain homotopies is an example of **the method of acyclic models**.

Exercise. Verify the homotopy axiom directly by defining a degree p-1 "prism operator" in $C_*(K)$ which satisfies a formula analogous to that of the above "cone operator" of $C_*(\sigma)$.

Exercise*. Find a natural map at the chain level which induces the above isomorphisms between Mayer and oriented homologies.

Exercise*. For p non-prime the homologies $H_*(K,L)$ defined in course of the above proof do not satisy the dimension axiom, but do some of them still satisfy the homotopy axiom ?

(d) Bier's homology.

For each $r \geq 1$, let $C(K_{comm,r})$ denote the Abelian group generated by all commutative monomials, in which **the degree of each vertex is at most** r, and which are supported on the simplices of K.

Just as in (b) we use a total order on vert(K) to equip $C(K_{comm,r})$ with a boundary operator ∂, and we shall denote by $H_*(K_{comm,r})$ the homology of $(C_*(K_{comm,r}),\partial)$.

The following striking result of Bier (*M.P.I. preprint* 1992), which is formulated in terms of *reduced* homology, shows that the groups $H_*(K_{comm,r})$ are again independent of the order chosen on vert(K).

Theorem. *For* r odd $H_*(K_{comm,r}) \cong H_*(K)$, *but for* r even *one has*

$$H_*(K_{comm,r}) \cong \oplus_{\sigma\in K} H_{*-r|\sigma|}(Lk_K\sigma).$$

(Here remember, since we are using *reduced* notations, that $\emptyset \in K$, and so one of the summands on the right is $H_*(Lk_K\emptyset) = H_*(K)$.)

Proof. The simple but **key idea** that we'll use is that while calculating ∂ we can "pull out" even powers of vertices (just as one "pulls out" constants while doing differentiation in calculus).

Note that each monomial of $K_{comm,r}$ can be written *uniquely* as

$$x = (\sigma)^r y,\ \sigma \in K,\ y \in (Lk_K\sigma)_{comm,r-1},$$

because such a σ has to consist precisely of all those vertices which occur with degree r in x (this σ may be empty).

If r is *even*, then (by "key idea") for each $\sigma \in K$, all such monomials $(\sigma)^r y$ span a sub chain complex of $C_*(K_{comm,r})$, which is chain isomorphic, under "division" by $(\sigma)^r$, with $C_{*-r|\sigma|}((Lk_K\sigma)_{comm,r-1})$.

So in this case we have a direct sum decomposition of chain complexes,

$$C_*(K_{comm,r}) \cong \oplus_{\sigma\in K} C_{*-r|\sigma|}((Lk_K\sigma)_{comm,r-1}) ,$$

which shows that **case r even follows from the case r odd.**

So by (2.17) it will suffice to check that **for r odd $H_*(K_{comm,r})$ is an E-S homology.** Once again only the homotopy axiom is non-trivial. For this we employ the *quotient* of the h used in (a), i.e. from the right side of that definition we omit all terms involving monomials in which some vertex occurs with degree r+1. Since r+1 is even the "key idea" tells us that these omitted terms commute with ∂, and so this quotient h satisfies the required formula $\partial h - h\partial = g - f$. *q.e.d.*

Exercise*. Let $\mathbf{r} : \mathrm{vert}(K) \longrightarrow \mathbb{N} \cup \{\infty\}$ be any "Steinitz" monomial on the vertices, and let $C(K_{comm,\mathbf{r}})$ be determined by all commutative vertex monomials dividing $\mathbf{r}$ and supported on K. With $C(K_{comm,\mathbf{r}})$ equipped again with a similar ∂ calculate $H_*(K_{comm,\mathbf{r}})$.

Exercise*. Calculate the homology of $C_*(K_{assoc,r})$, the subcomplex of $C_*(K_{assoc})$ determined by all vertex sequences in which no vertex repeats more than r times. (Even the case r = 1 is unusual now.)

Exercise. Do Bier-type modifications of Mayer's definition also.

(e) Cyclic cohomology.

Note. The following homely example is probably known to CONNES (and others) but seems to have been totally ignored !

We recall that *oriented* cochains $c \in C^*(K_{assoc})$ are those which satisfy

$$c(v_{\pi(1)}v_{\pi(2)} \ldots) = (-1)^{\pi}c(v_1v_2 \ldots),$$

for *all* permutations π of the sequence. We'll now look at the bigger subgroup $C^*(K_{cycl})$ consisting of all c's which satisfy the above requirement just for all **rotations** π of the vertex sequence.

Theorem. *The coboundary operator δ maps $C^*(K_{cycl})$ into itself.*

Proof. Let us say that a set of vertex sequences is an **orbit** if it is generated by any one of its members by taking all its rotations, e.g. {abc bca cab} is an orbit. Also, for any set S of vertex sequences, we'll denote by $\delta(S)$ the set of all vertex sequences having some member of S as a codimension one face.

Note now that a cochain c belongs to $C^*(K_{cycl})$ iff it is constant, upto the factor $(-1)^{\pi}$, on each orbit. Thus the result follows from the fact that **if S is an orbit then $\delta(S)$ is a union of orbits**, e.g.

$$\delta\{\text{abc} \quad \text{bca} \quad \text{cab}\} = \left\{\begin{array}{lll} \text{vabc}* & \text{vbca} & \text{vcab} \\ \text{avbc} & \text{bvca} & \text{cvab}* \\ \text{abvc} & \text{bcva}* & \text{cavb} \\ \text{abcv}* & \text{bcav} & \text{cabv} \end{array}\right\},$$

is the union of three orbits of which one is indicated by * . *q.e.d.*

We postpone (see Chapter 6) the computation of the **cyclic cohomology** $H^*_{cycl}(K)$ of this cochain complex $(C^*(K_{cycl}),\delta)$ but note that it certainly is not the ordinary cohomology : for example an easy computation shows that $H^q_{cycl}(pt) \cong \mathbb{Z}$ for q even, and zero otherwise.

We have thus seen that the two sequences, (i) all permutations π of the n+1 vertices, and (ii) rotations π of the n+1 vertices, both have the property that if we only consider cochains of $C^*(K_{assoc})$ which are constant upto factor $(-1)^\pi$ when such a permutation is made, then one gets a sub cochain complex. So a natural question is : *are there other such sequences of permutation groups ?* Indeed there are :

Exercise. The sequences consisting of (iii) the *reversals* of the n+1 vertices and (iv) rotations or reversals π of the n+1 letters also have the above property. Thus one can define (using these two sub cochain complexes respectively) cohomologies $H^*_{rev}(K)$ and $H^*_{dih}(K)$.

Exercise*. Show that the above four sequences of permutation groups are the *only* such sequences. (See FIEDOROWICZ-LODAY, *T.A.M.S.* 1991, for a fancy generalization of this.)

Exercise. Show that $H^*_{rev}(K) \cong H^*(K)$. On the other hand the **dihedral cohomology** is different : for a point $H^{4k}_{dih}(pt) \cong \mathbb{F}$ with other groups being zero. (A full computation of $H^*_{dih}(K)$ is given later.)

REMARK. The dual **cyclic homology** $H^{cycl}_*(K)$ is of course defined as the homology of *quotient* chain complex $(C_*(K_{cycl}),\partial)$ of $(C_*(K_{assoc}),\partial)$ dual to the sub cochain complex $(C^*(K_{cycl}),\delta)$ of $(C^*(K_{assoc}),\delta)$. Likewise for **dihedral homology** of K.

(f) Cyclotomic homology.

Note. Somewhat amazingly the following natural definition was apparently first considered only in our seminar of 1993-94 (see notes of the first *Complément* where an even more general definition is given). It gives a useful *lifting of Mayer's definition to characteristic zero* (however I learnt of Mayer's paper, via the references on pp. 182-183 of Eilenberg-Steenrod, only later, viz., while preparing for this talk).

We will work with commutative monomials K_{comm} and will pay the usual price for this, i.e. we will use a total order α on vert(K) to define our boundary operator. (Alternatively one can dispense with a total order and work with K_{assoc}). Now $C_*(K_{comm})$ will denote the free module generated by these monomials over the ring $\mathbb{Z}[U_p]$ of **cyclotomic integers**, and we'll equip it with the **cyclotomic boundary** $\partial : C_*(K_{comm}) \longrightarrow C_{*-1}(K_{comm})$ defined by

$$\partial(v_0 v_1 \ \ldots\ ) \ = \ \textstyle\sum_r \ (\omega)^r \ (v_0 v_1 \ .. \ \hat{v}_r \ .. \ ) \ ,$$

where ω denotes the pth root of unity $\omega_p = \exp(2\pi i/p)$, $p \geq 2$. (In other words we simply replace the -1's of the usual definition, which corresponds to the case $p = 2$, by ω's.)

Just as for the Mayer boundary, we again have

$$\partial^p \ = \ 0.$$

Proof. Let $i_1 < i_2 < \ldots < i_p$ and consider the $(p-1)!$ terms of $\partial^p(v_0 v_1 \ \ldots\ )$ involving omission of vertices having these subscripts. Each time the factor coming from the omission of the vertices indexed by $i_1 < i_2 < \ldots < i_{p-1}$ is the same. In case the vertex indexed i_p is omitted first this factor gets multiplied by the i_p th power of ω ; in case this gets omitted second, it gets multiplied by the (i_p-1)th power of ω ; and so on. This gives us a factor $(1 + \omega + \omega^2 + \ldots + \omega^{p-1})$ which of course is zero. *q.e.d.*

So for each ordered pair (r,s) of positive integers with sum $r + s = p$ we can define the **cyclotomic homology groups**

$$H_{*;r,s}(K;\mathbb{Z}[U_p]) \ = \ \frac{\ker(\partial^r)}{\mathrm{im}(\partial^s)} \ .$$

Once again, as in (c), we have for each such (r,s) and $0 \leq t < r$, the homology $\mathbf{H_*(K,L)}$ of the chain complex $(\mathbf{C_*(K,L),d})$,

$$0 \ \xleftarrow{\partial^r} \ C_t(K,L) \ \xleftarrow{\partial^s} \ C_{t+s}(K,L) \ \xleftarrow{\partial^r} \ C_{t+p}(K,L) \ \xleftarrow{\partial^s} \ \ \ldots..$$

Even for p prime this homology H_* need not obey the dimension axiom. For example for $p = 3$ the cyclotomic boundaries for $K = pt$ (now each $C_i = R = \mathbb{Z}[U_p]$) are given by

$$0 \leftarrow C_0 \xleftarrow{1+\omega} C_1 \xleftarrow{0} C_2 \xleftarrow{1} C_3 \xleftarrow{1+\omega} C_4 \xleftarrow{0} C_5 \xleftarrow{1} C_6 \xleftarrow{1+\omega} C_7 \xleftarrow{0} C_8 \xleftarrow{1} \ \cdots$$

which shows (since $R/(1+\omega)R \cong \mathbb{Z}/2$) that for $(r,s,t) = (1,s,0)$ we have

$$H_*(pt) = (R,\ 0,\ \mathbb{Z}/2,\ 0,\ 0,\ \mathbb{Z}/2,\ 0,\ 0,\ \mathbb{Z}/2,\ 0,\ 0,\ \mathbb{Z}/2,\ \ldots\ ).$$

However we see that the dimension axiom is satisfied (for any p) *mod torsion*. Indeed cyclotomic cohomology $H_{*;r,s}(K;\mathbb{Q}[U_p])$ with coefficients in the quotient field of **cyclotomic numbers** identifies with usual homology as follows.

Theorem. *For any* p *the cyclotomic homology with field coefficients is given by :*

$$H_{kp+r-1;r,s}(K;\mathbb{Q}[U_p]) \cong H_{kp+s-1;s,r}(K;\mathbb{Q}[U_p]) \cong H_{2k}(K;\mathbb{Q}[U_p])$$

$$H_{kp-1;r,s}(K;\mathbb{Q}[U_p]) \cong H_{kp-1;s,r}(K;\mathbb{Q}[U_p]) \cong H_{2k-1}(K;\mathbb{Q}[U_p])$$

Proof. As in (c) all the E-S properties of $H_*(K,L)$ are obvious except the homotopy property which will follow once we have verified the strong dimension axiom.

For this we again let u be the first vertex of simplex σ and define $h : C_q(\sigma) \longrightarrow C_{q+p-1}(\sigma)$ for $q \geq -1$ (now $C_{-1} = \mathbb{Q}[U_p]$) by

$$h(v_0v_1v_2\ \ldots\ ) \ = \ u^{p-1}v_0v_1v_2\ \ldots\ .$$

Then the following formula holds :

$$\partial^{p-1}h + \partial^{p-2}h\partial + \partial^{p-3}h\partial^2 + \ldots + \partial^2h\partial^{p-3} + \partial h\partial^{p-2} + h\partial^{p-1} \ =$$

$$(1+\omega)(1+\omega)(1+\omega+\omega^2)\ \ldots\ (1+\omega+\omega^2+\ \ldots\ +\omega^{p-2})\ (\mathrm{id}).$$

To check this we define, intermediate between the above $h = h_{p-1}$ and $h_0 = \text{id}$, the degree r homomorphisms $h_r : C_q(\sigma) \longrightarrow C_{q+r}(\sigma)$ by $h(v_0v_1v_2 \dots) = u^r v_0v_1v_2 \dots$. Clearly

$$\partial h_r = \omega^r . h_r \partial + (1 + \omega + \dots + \omega^{r-1}) . h_{r-1}, \quad 1 \le r \le p-1.$$

So on eliminating these intermediate h_r's — using the fact that the elementary symmetric functions in the roots $\{\omega, \omega^2, \dots, \omega^{p-2}\}$ of $1 + x + \dots + x^{p-1}$ are all ± 1 — we get the stated formula.

Each of the factors $1 + \omega + \omega^2 + \dots + \omega^r$, $1 \le r \le p-2$, on the right side of our formula is nonzero. So $\ker\partial^r = \mathrm{im}\partial^s$, i.e. the reduced homology $\mathbf{H}_*(\sigma)$ of the closed simplex σ is identically zero. *q.e.d.*

Exercise*. Prove above result by constructing, for any two contiguous maps f and g, an explicit degree p-1 "prism operator" h which satisfies a formula analogous to that of the above "cone operator".

Exercise*. Compute $H_{*;r,s}(K;\mathbb{Z}[U_p])$, or at least find all cases when the homotopy axiom holds for the corresponding homology $\mathbf{H}_*(K)$.

Exercise. Give the precise sense in which cyclotomic homology is a "lifting" of Mayer's definition to characteristic zero.

Other examples. Later we'll also meet **(g) Cohomology with compact supports** (and, dually, homology with infinite chains) of infinite simplicial complexes, **(h) Equivariant homologies** (see Ch. 4) of simplicial complexes equipped with group actions, **(i)** Singular homology $H_*(\mathrm{Proj}(K))$ (see Ch. 5) **and many, many others ...**

REMARK. Note that if we take any natural construction (within $\mathfrak{Simp}$ or from it into $\mathfrak{Top}$) and compose it with any of the above homologies (or of 2.19) we get another homology ... e.g. (for the case of $\dim K = 1$) **Lovasz** once used the homology (of the sd) of the **poset of all acyclic subcomplexes** of K ... . However despite this we feel that the E-S theorem is only a begining, and large classes of non E-S simplicial homologies can also be characterized axiomatically.

Exercise. Show that the homology of the poset of *all* subcomplexes of K is trivial.

(2.19) Other examples of homologies on $\mathfrak{Top}$. So far we have defined only the singular homology on $\mathfrak{Top}$. Let us start with some

(a) Variants of singular homology.

1. Firstly we can define **algebraical variations** mimicking (2.19). For example we can put a total order (!) on our space X and use only **commutative singular simplices** i.e. those which restriced to the set of vertices of the standard simplex are non-decreasing. Likewise it makes sense, if one is working with coefficients mod p, to use a signless boundary and use **Mayer singular homology**. Or, using a primitive pth root of unity use a **cyclotomic singular homology**, or define a **cyclic singular homology** by going to the cyclic quotient, or use the oriented or (Alexander-Veblen) **Lefschetz singular homology** by going to the oriented quotient.

2. When X is a differentiable manifold we can use **differentiable singular simplices** (of maximal rank) only with any of the above variants, e.g. with the Lefschetz variant : cf. (1.4).

Exercise. Prove that this **Poincaré singular homology** of a differentiable manifold coincides with its singular homology.

3. Another variant is obtained by replacing standard simplices by standard cubes : this **cubical singular homology** was used by **Serre** in his famous paper (*Annals* of 1955) on the spectral sequence of a fibration. More precisely the definition goes as under.

An n-dimensional **singular cube** of X is a continuous function $T(u_1, u_2, \ldots, u_n)$ of n real variables $0 \le u_i \le 1$ which takes its values in X. It has 2n (n-1)-dimensional singular cubes as its faces, viz. the n

front faces $F_i T$ given by

$$(F_i T)(u_1, u_2, \ldots, u_{n-1}) = T(u_1, \ldots u_{i-1}, 0, u_i, \ldots, u_n),$$

and the n **back faces** $B_i T$ given by

$$(B_i T)(u_1, u_2, \ldots, u_{n-1}) = T(u_1, \ldots u_{i-1}, 1, u_i, \ldots, u_n).$$

The obvious cubical boundary operator to use is of course

$$\partial T = \sum_i (-1)^i (F_i T - B_i T),$$

and one has, as an easy calculation shows, $\partial^2 = 0$. But now there is a surprise in store for us : the **non-normalized cubical singular homology** $\ker\partial/\operatorname{im}\partial$ thus defined *does not obey the dimension axiom.*

More precisely for a point this homology is $\mathbb{Z}$ in all dimensions : this follows because the cubical singular homology of a point is $\mathbb{Z}$ in all dimensions and the above boundary operator is identically zero in all dimensions.

Theorem. *The non-normalized cubical singular homology obeys all E-S axioms other than the dimension axiom.*

Proof. We imitate the argument of 2.14. The proof of the homotopy axiom is now in fact simpler since a cube times [0,1] is also a cube and no subdivision is necessary to get the required chain homotopy h. Next we note the following cubical subdivision of a cube times [0,1] which can be iterated to makes cubes arbitrarily small :

Using this the same argument as before shows again that this homology does not change if we use only $\mathcal{U}$-small singular cubes ($\mathcal{U}$ being any given open covering of X) and the excision property follows at once from this fact. *q.e.d.*

Exercise*. Compute the above generalized E-S homology. What is its "spectrum" ?

The **cubical singular homology** which Serre used was obtained by quotienting out the above complex by the subcomplex generated by the **degenerate singular cubes** i.e. those which factor through a projection of the standard cube to one of its proper faces.

Corollary. *This (normalized) cubical singular homology coincides with the singular homology.*

Proof. The dimension axiom follows because the positive dimensional singular cubical simplices of a point are degenerate. The remaining axioms follow from the above result. *q.e.d.*

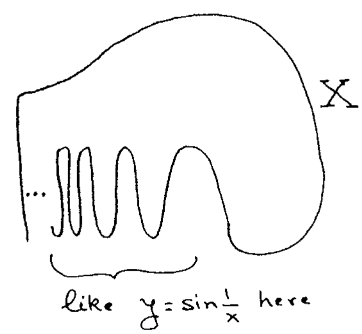
Exercise. Since we appealed to the E-S Theorem (2.17) the above "coincidence" has been shown only on triangulable spaces. Show that in fact it holds for *all* topological spaces.

Exercise*. Show that any closed 3-manifold has a **cubation**, i.e. can be subdivided into cubes intersecting in common faces only. Likewise show that it also admits a **dodecation**, a subdivision into non-overlapping dodecahedra which meet in common pentagonal faces.

The aforementioned suggests that a variant of singular homology using dodecahedra might be useful in dimensions 3 and 4 ?

(b) Cech cohomology.

It was Eilenberg who resurrected singular homology by showing its convenience (see Ch. 4) when dealing with homotopy-theoretic questions. In the 1930's however the emphasis was on trying to find generalizations of Alexander duality to subsets of the sphere which are *not* triangulable (e.g. the solenoids of 2.8). From this point of view singular homology is *not* the right homology to use, as becomes clear from the following simple example of **Alexandrov**.



Exercise. Show that the first singular homology group of the above one-dimensional compact space is trivial even though its complement in the plane has two path components.

The "correct" definition from the above point of view was given by Cech. We give below this definition noting that we are just giving the mainstream : one again has various algebraical variations defined using the ideas of (2.19).

We begin by noting that the set of all open coverings $\mathcal{U}$ of our space X is a directed poset under **refinement**, i.e. any two open coverings have a common finer open covering. We now associate to each $\mathcal{U}$ a simplicial complex $K_{\mathcal{U}}$ called its **nerve** : its simplices are all those finite subsets of $\mathcal{U}$ which have a nonempty common intersection. (We note that here we are using *infinite* simplicial complexes.) We can take say the oriented simplicial (co)homology of these simplicial complexes.

Now note that if $\mathcal{V}$ is finer than $\mathcal{U}$ then there is an obvious simplicial map $K_{\mathcal{U}} \longrightarrow K_{\mathcal{V}}$. So we can take the inverse limit of these simplicial homologies, or the direct limit of these simplicial cohomologies : these are the required Cech groups of X.

Theorem. *Cech cohomology obeys the E-S axioms, however, for non-compact spaces* X*, Cech homology may not obey the exactness axiom.*

Proof. We leave this as a challenging *Exercise* (cf. Eilenberg-Steenrod) contenting ourselves by pointing out that the peculiarity regarding Axiom 4 arises because exactness commutes with direct limits but not necessarily with inverse limits. For compact spaces one can limit oneself to *finite* coverings and, using this, it turns out that now even this axiom is true. *q.e.d.*

Thus for (finite) simplicial complexes the Cech approach also gives the usual homology and cohomology groups, the difference (from singular homology) is for non-triangulable spaces.

Exercise. Calculate the Cech (co)homology groups of Alexandrov's example (see above).

Exercise*. Calculate the Cech (co)homology of the solenoids of (2.14).

(c) Alexander-Kolmogorov homology.

etc., etc. ..... .

(2.20) Acyclic models theorem. We have already used the method of proof called "the acyclic models method" in (2.19). Here we'll first formulate the general theorem of Eilenberg and Maclane (*Amer. Jour. Math.* of 1953) pertaining to this, and then show how it connects to the ideas of *resolutions* and *derived functors* which were developed later in the mid-1955's in Cartan-Eilenberg, etc.

The main point here is the definition of a **representable functor**. Informally

Chapter 3

POINCARE : 3-MANIFOLDS & DE RHAM THEORY

(3.1) On "rattling". Before I start I would like to share with you the following quotation of THURSTON (from p.165 of *"On proof and progress in mathematics"*, Bull. A.M.S. 30 (1994), 161-177) :

" Personally, I put a lot of effort into "listening" to my intuitions and associations, and building them into metaphors and connections. This involves a kind of simultaneous quieting and focusing of my mind. Words, logic and detailed pictures rattling around can inhibit intuitions and associations."

To reduce "rattling" I have, and will, continue to minimize or even drop indices, tildes, etc. ... this of course carries some danger of ambiguity and over-simplification, but a willingness to run this risk seems necessary if one wants to see the big picture ....

(I would like to emphasize that "rattle-free thinking" and a fondness for "doing sums" don't contradict each other : Thurston is one of the great problem-solvers of this century.)

(3.2) The birth of Homotopy Theory, i.e. the part of topology concerned mostly with *homotopy groups* $\pi_i(X)$, also arose from Poincaré's *"Analysis Situs"* in whose § 12 is defined the **fundamental group** $\pi_1(X)$. Starting from this, the higher homotopy groups were later defined inductively by HUREWICZ in 1935 as follows :

$$\pi_{i+1}(X) = \pi_i(\Omega X),$$

where ΩX denotes the *loop space* of the space X. (Of course this induction can start from one step lower — $\pi_0(X)$ being the set of path components of X — but then the group structure is not clear.)

Poincare's first definition of π_1 is based on looking at a system of *linear partial differential equations* of the type

$$\frac{\partial y_\alpha}{\partial x_i} = \mathcal{F}_{\alpha,i}(x_1, \ldots, x_n ; y_1, \ldots, y_\lambda).$$

Here x runs over a domain D of n-space, and y runs over all of λ-space. So one can imagine the right sides as n slopes of an n-dimensional **plane field** on $D \times \mathbb{R}^\lambda$ transverse to the fibers of the projection $D \times \mathbb{R}^\lambda \longrightarrow D$, and *solving the above differential equations* amounts to partitioning, or **foliating**, all of $D \times \mathbb{R}^\lambda$ into n-dimensional smooth **leaves** which are, at each point, tangent to the plane field at that point.

There are obvious **integrability conditions**, stemming from

$$\frac{\partial^2 y_\alpha}{\partial x_i \partial x_j} = \frac{\partial^2 y_\alpha}{\partial x_j \partial x_i},$$

which the slopes $\mathcal{F}_{\alpha,i}$ of our plane field must obviously satisfy, for the differential equations to have a solution. It was known to Poincaré (this is now called **Frobenius' Theorem** but actually goes way back to **Deahna**) that these integrability conditions are also sufficient.

At this point Poincaré apparently confines himself to the case when the leaves of our foliation **cover** the base space D **evenly**. Using this he now "follows" any solution as its projection describes a **loop** on D. This yields a *group of substitutions* of $\mathbb{R}^\lambda$, which nowadays is called the **monodromy** of the above integrable system of differential equations.

Poincaré notes that this group has the property that the substitution corresponding to any **lacet**, i.e. a very narrow loop, is the identity substitution. He now *defines* the fundamental group of D to be the monodromy of any system of differential equations $\mathcal{F}$ for which the converse is also true, i.e. we should obtain the identity substitution *only if* the loop is a concatenation of lacets, i.e. only if, in today's language, the loop is **homotopically trivial**.

REMARK. Poincaré omits to prove that such an integrable system $\mathcal{F}$ of PDEs exists. Much later, **Sullivan,** had need for a Poincaré-like definition of π_1, and he pointed out that such an $\mathcal{F}$ certainly exists, at least if λ is *infinite*, and then we can ensure also that each of our substitutions is a *linear* substitution of the Hilbert space $\mathbb{R}^{\lambda}$. Note on the other hand, that there are closed manifolds having fundamental groups which do not sit inside any $GL(\lambda,\mathbb{R})$ for $\lambda < \infty$.

We note however that Poincaré (and Schwarz etc.) had shown the existence (for the case of surfaces) of a **universal cover**, so the above existence question did not seem much in doubt. And, anyway, Poincaré was interested in *calculating* his groups, and for this purpose he switched to the following, which also simultaneously inaugurated (as TIETZE, who analyzed Poincaré's definition very closely thereafter showed) the *combinatorial theory of groups*.

The second definition of π_1 which Poincaré gave was couched as follows. He set each lacet C equal to zero thus : $C \equiv 0$. Then he defined (in analogy with his earlier notion of an *homology*) an **equivalence** to be any expression of the type $A \equiv B$ (where A and B are linear combinations of loops) obtainable from these by working "just as if these were equations" with the sole difference that we don't demand commutativity.

Clearly this second definition is equivalent to the one found in text books nowadays : π_1 is the group, under concatenation, of homotopy classes of loops at the given **base point**. (Poincaré was very clear about the importance of base point vis-à-vis this definition.)

Noticing that one obtains all one-dimensional homologies of the (connected) manifold from its equivalences if one allows commutativity, Poincaré also saw that **H_1 is π_1 made Abelian**.

REMARK. While Poincaré's "second definition" of π_1 is topological, we'll see in (3.6) that his "first definition" was essentially de Rham-theoretic. For the case of CW complexes obtained by pairwise identifications of facets of a 3-polyhedron Poincaré also gave a "third definition" (see the "cyclic equivalences" of 3.3 below) which is purely combinatorial . We note that purely combinatorial definitions of the higher homotopy groups are somewhat harder (because of the infinite-dimensionality of the loop space : a fact which also makes these Abelian groups much harder to compute than the possibly non-Abelian π_1).

(3.3) The birth of 3-manifold theory. We'll now go back in time "and do some sums" with Poincaré ....

All closed 2-manifolds were "known" at that time, so it was natural to turn to the next dimension and look at closed 3-manifolds.

For 2-manifolds, the starting point for the classification had been their **triangulability** (those days surfaces were always complex analytic and then this followed e.g. from Poincaré's work on Fuchsian groups) or equivalently (*Exercise*) the fact that they could be built from some 2k-gon by a pairwise identification of sides.

Likewise, the triangulability of a closed 3-manifold (this was established much, much later by **Moise**) is equivalent to saying that it is obtainable by **pairwise identification of the facets of a 3-dimensional polyhedron.**

So Poincaré turned to one of the simplest polyhedrons one can think of, viz. the **cube**, and examined what happens when we identify opposite facets of the cube. (But note also, since *any* 3-polyhedron is a *subdivided cube*, that the undivided cube is an obvious starting point for a search of all 3-manifolds !)

Poincare's smoothness criterion. Unlike the case of polygons, the (in today's language) CW COMPLEX one gets by pairwise facet identification of a polyhedron has some **singularities**. However Poincaré noted that these singularities are **very mild** : the only doubtful points are the finitely many vertices of this CW complex, and each of these will be non-singular iff the Euler characteristic of its **link** is 2. This follows of course by using the classification theorem of surfaces.

Later (see p. 216 of the 1935 text book of **Seifert-Threlfall** : I know of no earlier reference) it was observed that Poincaré's criterion could be reformulated much more elegantly as follows (clearly one does *not* have a similar elegant version in higher dimensions).

Theorem. *A pairwise identification of all the facets of a 3-dimensional polyhedron yields a closed 3-manifold if and only if the Euler characteristic of the resulting CW complex is zero.*

(However, to work out the topology in the singular case, there is, in general, no getting around examining, as Poincaré does, the vertex-links one by one !)

Proof. "Only if" is trivial by Poincaré duality. For "if" we can, without loss of generality, replace the CW complex by a simplicial subdivision K. Using Poincaré's criterion (see above) we see thus the inequality

$$2f_0 \geq \sum_v e(Lk_K v) = 2f_1 - 3f_2 + 4f_3 ,$$

with equality holding iff K is a 3-manifold; the result follows because (by virtue of $4f_3 = 2f_2$) the above is same as saying $e(K) \geq 0$. (Note also that one *always* has the equality $\sum_v (2 - e(Lk_K v)) = e(K)$.) *q.e.d.*

Cyclic equivalences for π_1. Note that each edge of the CW complex arises as the result of a cyclic sequence of identifications of the sort

$$\bar{A}B = \bar{C}D = \bar{D}E = \ldots = \bar{Z}A ,$$

where $\{A, \bar{A}\} = \mathcal{A}$ are the identified **facet-pairs** (so $\bar{\bar{A}} = A$), and $A\bar{B}$ etc. denotes the intersection of the facets $\bar{A}$ and B.

Theorem. *The fundamental group of the CW complex is isomorphic to the group generated by the facet pairs subject to the cyclic relations*

$$\mathcal{B}\mathcal{D}\mathcal{E} \ldots \mathcal{A} = 1,$$

one for each edge of the CW complex.

(See last year's Seminar notes to § 13 of *Analysis Situs.*)

Poincaré now has all the weapons to do "sums" : the fundamental group of the CW complex and thus its Abelianization, the first homology group, can be calculated using these cyclic equivalences. Then, in case the smoothness criterion has certified the CW complex to be an orientable manifold, duality gives the rest of the homology. (Even in the singular case using "duality" gives something meaningful, namely the **intersection homology** of these orientable 3-pseudomanifolds !)

(In the second *Complément* Poincaré' gave another calculation using the incidence matrices of these CW complexes : see 1.4)

Theorem. *Pairwise identifications of the opposite facets of a cube yields exactly seven orientable* 3*-manifolds.*

Proof (with Keerti Vardhan). Lets orient the boundary of our cube, and denote by A, B, and C the oriented facets incident to some chosen vertex v. We want the resulting CW complex P to be **orientable** i.e. we want the boundary of its sole 3-cell to be zero: so we must identify A to $\bar{A}$ (etc.) with the *opposite* orientation, i.e. we must *rotate* A through $e^{ia\pi/2}$, where $a \in \{0,1,2,3\}$, and then identify with $\bar{A}$ by a translation. Accordingly we'll denote P by **abc**, $0 \le a,b,c \le 3$.

So à priori there are $4^3 = 64$ possibilities. But the symmetries involving interchanges of the 3 edges incident to v show that **abc** is homeomorphic to the complex denoted by any permutation of its symbols: this cuts down the possibilities to 20 (the number of degree 3 monomials in 4 variables). These possibilities reduce further to 13 because the antipodal symmetry ($v \longleftrightarrow \bar{v}$ etc.) of the cube shows that the complex (abc) is always homeomorphic to $(a^{-1}b^{-1}c^{-1})$.

For each of these 13 CW complexes we calculated the number of cells and thus their Euler characteristic. It turned out that e = 0 only in seven cases, and then a fundamental group calculation showed that these 3-manifolds are all topologically distinct from each other.

All this information (and more) is shown in the following table :

CW Complex P	Number of cells and e(P) f_0	f_1	e	Fundamental group	First homology (One has $H_2 = \mathbb{Z}^{b_1}$.)	Geometrical description
000	1	3	0	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{R}^3/\mathbb{Z}^3$
001	1	3	0	$gh = hg$, $k^{-1}gk = h^{-1}$, $k^{-1}hk = g$.	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{R}^3/(0,0,\pi/2)$
002	1	3	0	$gh = hg$, $k^{-1}gk = g^{-1}$, $k^{-1}hk = h^{-1}$.	$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{R}^3/(0,0,\pi)$
011	1	1	2			
012	1	2	1			
013	1	1	2			
022	2	4	0	$g^2 = 1$, $h^2 = 1$.	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	?
111	2	4	0	Hamilton's order 8 group.	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$S^3/\langle i,j,k\rangle$
112	1	1	2			
113	2	2	2			
122	1	3	0	$g^2 = 1$, $h^4 = 1$.	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$	?
123	1	2	1			
222	4	6	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$S^3/\langle \pm 1\rangle = \mathbb{RP}^3$.

Some sample calculations for 022 (the remaining calculations are left as *Exercises* for the tireless reader !): The three faces of our CW complex P being

$$(X =) \quad ABCD = A'B'C'D' \quad (= \overline{X})$$

$$(Y =) \quad BAA'B' = D'C'CD \quad (= \overline{Y})$$

$$(Z =) \quad CBB'C' = A'D'DA \quad (= \overline{Z})$$

we obtain four edges, viz.,

$$AB = A'B' = CD = C'D' \quad , \quad BC = B'C' = DA = D'A' \ ,$$

$$AA' = C'C \quad , \quad BB' = D'D \ ,$$

and the two vertices

$$A = A' = C' = C \quad , \quad B = B' = D' = D \ .$$

So $e(P) = f_0 - f_1 + f_2 - f_3 = 2 - 4 + 3 - 1 = 0$. Further since the edge identifications can be written cyclically as

$$YX = \overline{X}Y = \overline{Y}X = \overline{XY} \quad , \quad ZX = \overline{X}Z = \overline{Z}X = \overline{XZ} \ ,$$

$$\overline{Z}Y = \overline{Y}Z \quad , \quad YZ = \overline{ZY} \ ,$$

we see that (in multiplicative notation) the fundamental group is generated by X, Y, and Z, subject to the four relations

$$XYXY^{-1} = 1 \quad , \quad XZXZ^{-1} = 1 \ ,$$

$$YZ = 1 \quad , \quad ZY^{-1} = 1 \ ;$$

i.e. it is the (infinite non-Abelian) group generated by the 2 elements $g = Z = Y$ and $h = XY$ subject to the sole relations $g^2 = 1 = h^2$. *q.e.d.*

Exercise. Complete the calculations for all the 13 CW complexes listed. (It seems that either all 13 or at least 12 of them are pairwise non-homeomorphic.) Also do some homology calculations via the chain complex of the CW complex. In case of singularities calculate also the genus of the link of each singularity.

With the above examples Poincaré *started* 3-manifold theory, but he went *much, much, further*.

For example, he saw that three of the above manifolds, to wit **000**, **001**, and **002**, belong to an infinite series — the so-called **twisted 3-tori 00T**, $T \in SL(2,\mathbb{Z})$ — which he classified by means of a remarkable **"rigidity theorem"** (see § 14 of *Analysis Situs* : this result is analogous to the later "rigidity theorems" of Bieberbach and Mostow).

Note that each of these **00T**'s can be made into a Riemannian manifold whose metric is locally the left-invariant metric of a three-dimensional solvable Lie group. It is known (see **Fried-Goldman**, *Advances*, 1983) that these and their finite covers are the only closed 3-manifolds having such geometries. We note also that these geometries are three of the eight geometries of **Thurston** (Lectures, Princeton, 1980). Thurston has conjectured (and proved under one additional condition) that any closed "irreducible" (we omit the definition) admits one of these eight "geometries". It is known that this conjecture would quickly lead to a classification of all closed 3-manifolds. (This is analogous to how the classification of closed 2-manifolds follows from the **uniformization theorem** of Klein, Poincaré, and Köbe.)

Exercise*. Show that **022** and **122** also belong to an infinite series **T22** of closed 3-manifolds. Classify these manifolds (these may be called **"projective" lens spaces** since they can be obtained by a construction similar to that of 3.4 but starting from projective 3-space minus a disk) and give a number theoretical "enumeration" of this infinite series analogous to the one we gave (see p.113 of last year's notes, first ed.) for Poincaré's series **00T**. (Hint: equip these manifolds with a hyperbolic geometry.)

Exercise*. Given T, U, $V \in SL(2,\mathbb{Z})$, one can define a CW complex **TUV** by identifying the three opposite face-pairs of the cube after "twisting" one member of each pair by these matrices respectively. Find necessary and sufficient conditions that **TUV** has no singularities.

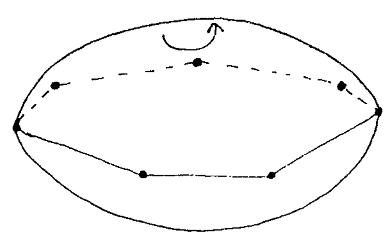
Exercise*. Starting from the (undivided) n-cube find a formula for

M(n) the number of orientable closed n-manifolds that can be obtained by identifying its opposite facets.

Poincaré's most enchanting contribution to 3-manifold theory is undoubtedly the ubiquitous **Poincare's Manifold** (i.e. the main character of his long *Cinquième Complément* !) We note that this EXOTIC HOMOLOGY SPHERE lies at the next level of complexity compared to the seven manifolds of the table above : it is a 3-manifold obtainable by suitably identifying opposite facets of a dodechedron, that is to say (see 1.2) a very simple *subdivided* cube.

Lastly Poincaré left to us the celebrated **Poincare' Conjecture** (or problem) — viz. that S^3 is the only HOMOTOPY SPHERE amongst closed 3-manifolds — which is still (?) open. We note in this context that *Analysis Situs* does not contain *any* example of non-homeomorphic 3-manifolds having isomorphic fundamental groups. These (see below) were found a little later in 1908 by TIETZE who was one of the first mathematicians to have read *Analysis Situs* thoroughly.

(3.4) Linsenräumen. To define these start with the **lens** (see below) obtained by "fattening" a p-gon, and consider the CW complex obtained by rotating its top facet by $2\pi/q$ (p and q are relatively prime to each other) and then identifying with the bottom facet.



Clearly this complex is orientable and has $f_0 = f_1 = f_2 = f_3 = 1$ and so its Euler characteristic is zero. Thus it is an orientable 3-manifold, called the **lens space** L(p,q). Using Poincaré's (sole) cyclic equivalence we see that its fundamental group is $\mathbb{Z}/p$. However in 1919 **Alexander** showed (this was conjectured by Tietze) that for different values of q these lens spaces need not be homeomorphic to each other, and subsequently in 1935, **Reidemeister**, using a new invariant called **torsion** (we will look at this, and its off-spring, Algebraic K-Theory, later) was able to give a *complete* homeomorphism classification of the lens spaces.

It turns out that (unlike the case of 3-manifolds whose universal cover is contractible) even the *homotopy type* of the lens spaces is *not* determined by its fundamental group : the classification of lens spaces upto homotopy type (which is *not* the same as upto homeomorphism) was given in 1950 by **Whitehead**.

Exercise. Triangulate the top and the bottom facets of the lens above by coning their boundary over two new vertices N and S. Let now $p = 2k$, so the bounding simplicial 2-sphere has an antipodal simplicial involution ν. We now identify each triangle σ of the top facet of the lens with $\nu(\sigma)$ rotated through $2\pi/3$. Find all possible 3-manifolds of this type. Also give classifications of all these 3-complexes upto homeomorphism and homotopy type.

Exercise. Show that $L(p,q)$ is homeomorphic to the quotient space of the unit sphere $S^3 \subset \mathbb{C}^2$ obtained by dividing out by the free $\mathbb{Z}/p$ - action $(z_1, z_2) \longmapsto (e^{2\pi i/p} z_1, e^{2\pi iq/p} z_2)$.

Note that this again shows, since S^3 is simply connected, that the fundamental group of $L(p,q)$ is $\mathbb{Z}/p$. We recall, from last year's seminar, that Poincaré was aware of, and frequently used this method — of finding a universal cover — to calculate fundamental groups. For example he knew that **111** (of the table in 3.4) is S^3 mod the order eight group generated by the quaternions **i**, **j** and **k**, and that **222** $\cong \mathbb{R}P^3$.

(3.5) On the combinatorics of the 2-sphere. This section is somewhat of an aside in which I want to emphasize the fact that Poincaré's conjecture — or even the classification problem of 3-manifolds — can be reformulated entirely within $\mathfrak{Simp}$ and indeed are questions pertaining to the combinatorics of the ordinary commonplace simplicial 2-sphere !

Simplicial 2-sphere. *This means a connected simplicial complex* K *having* $e(K) = 2$ *for which each vertex-link* $Lk_K v$ *is a polygon.*

The above purely combinatorial definition is valid because, by virtue of **the classification theorem of surfaces,** $|K|$ is homeomorphic to a 2-sphere iff K is as above. (*Exercise.* Is "connected" necessary in the above ?)

To indicate how challenging such K's can be let me remind you of the following of which a somewhat controversial proof was found 15 years ago by **Appell-Haken** (a conceptual proof is still lacking).

Four Colour Theorem. *For any simplicial 2-sphere* K *we can find a function* $f : \mathrm{vert}K \longrightarrow \mathbb{F}_4$ *with* $f(v) \neq f(w)$ *for all edges* $\{v,w\} \in K$.

Our choice of the field $\mathbb{F}_4$ as "colouring box" was dictated by the elegance of the following argument (which seems — to use Erdös's favourite phrase — to come right out from **"The Book"** of proofs).

Tait's Theorem. *The above theorem is equivalent to saying that the edges of* K *can be assigned three colors in such a way that each triangle of* K *has all its edges colored differently.*

Proof (**M. Brown**). Consider any f of the above kind as a 0-cochain of K with values in $\mathbb{F}_4$. Then its coboundary $z = \delta f$ is a 1-cocycle with values in $\mathbb{F}_4^*$. Now note, on the one hand, that saying it is a cocycle is same as saying that for any triangle $\{u,v,w\}$ of K we have $z(v,w) + z(u,w) + z(u,v) = 0$; and, on the other hand, that *sum of any three elements of* $\mathbb{F}_4^*$ *is zero iff they are distinct.* So z gives us the required edge coloring. Conversely any such edge coloring amounts to an $\mathbb{F}_4$-valued 1-cocycle z of K with values in $\mathbb{F}_4^*$. *Since* $H^1(K;\mathbb{F}_4) = 0$ we have $z = \delta f$ and this f gives the required vertex coloring. *q.e.d.*

Now start from a 3-ball with center c, bounded by a simplicial 2-sphere K, and let ν be any pairwise identification of its (necessarily *even* number of) triangles. The resulting CW complex $c.K/\nu$ is (*Exercise*) homeomorphic to the simplicial complex $sd(c.K/\nu)$.

Theorem. *Upto homeomorphism any closed 3-manifold is a* $sd(c.K/\nu)$, $e(sd(c.K/\nu)) = 0$, *where* K *is some simplicial 2-sphere equipped with a pairwise identification* ν *of its triangles.*

Proof. This follows from the preliminary remarks of (3.3), the essential points being **Moise**'s theorem, and that for such identification spaces $e = 0$ ensures absence of singularities. *q.e.d.*

Thus the classification theorem of 3-manifolds can be reformulated as the purely combinatorial question, of classifying all objects of $\mathfrak{Simp}$ of the above type, under the equivalence relation generated by elementary stellar subdivisions (here of course we are making use of the basic theorem of **M.H.A.Newman**).

Turning now to the cyclic equivalences of $c.K/\nu$ (see 3.3) we note that their definition was purely combinatorial in terms of (K,ν), and saying that the fundamental group is trivial is the purely combinatorial assertion that, in the equivalence relation generated by the **Tietze transformations**, the equivalence class of these relations is that of the empty set.

Poincaré's conjecture is thus the purely combinatorial assertion that this combinatorial hypothesis implies the combinatorial conclusion that, in the equivalence relation generated by elementary stellar subdivisions, $sd(c.K/\nu)$ is equivalent to the boundary of a 4-simplex.

Note however that the remarks of (3.4) show that there is no chance of always "lifting" the Tietze relation step-by-step to the Newman relation. Thus the above reformulations do not, as such, tell us how to solve these recalcitrant problems. However it would be of course very fascinating if there were to exist some non-trivial connection between 3-manifold theory and say the 4CT ???

(3.6) The birth of "de Rham" theory. We'll first outline a familiar situation in (a) and then show how Poincaré generalized it in (b). In (c) we see the birth of the de Rham *homotopy* theory.

(a) If f(z) is *analytic* in a domain D of the complex plane, then its **indefinite integral** $\int f(z)dz$ does not depend just on the initial and final points, but also on the *path* between them over which integration is done : indeed **the ambiguity of the generic indefinite integral over** D **measures the homology of** D.

By this we mean that, for each f analytic on D, one can find some complex numbers — called the **residues** or **periods** of f — and the aforementioned ambiguity is an integral linear combination of these periods; moreover these periods are given by integration over paths going just once around just one "hole" of D, and so their number is bounded by the first Betti number of D; finally this bound is the best possible in the sense that one can construct an f analytic over D having precisely $b_1(D)$ distinct residues over D.

(b) The above was known to Riemann, and indeed had been his motivation in defining the "connectivity" of closed surfaces, and Betti had generalized all of the above to higher dimensional manifolds, but only for one-dimensional or codimension one indefinite integrals.

Exercise. Check that analyticity of f(z), i.e. the *Cauchy-Riemann equations*, is equivalent to demanding that the real and imaginary parts of the $\mathbb{C}$-valued 1-form f(z)dz are *closed* (in the sense defined below).

In § 7 of *Analysis Situs*, Poincaré extended this to indefinite integrals of *all* dimensions on a manifold. Before we go further it needs to be pointed out that the **differential forms** of today are essentially synonymous with the indefinite integrals of yore.

REMARK. Imitating **Abhyankar** who starts with a polynomial $a_0 + a_1z + a_2 + \dots + a_nz^n$ and then erases the last part to "define" power series $a_0 + a_1z + a_2 + \dots$, one can likewise start with an indefinite integral, say $\int f(x,y)dxdy$, and then erase the integration sign to "define" the differential form f(x,y)dxdy. The anticommutativity dxdy = − dydx follows as a consequence of this definition because the corresponmding indefinite integralds are same.

In the aforementioned § 7 of *Analysis Situs* the manifold in question is in some n-space and D denotes a neighbourhood of it. One works with indefinite integrals $\int\omega$ which are (in place of being *analytic*) now **closed** in the modern sense, i.e. the **exterior derivative** $d\omega$ of the differential form ω is zero throughout D. Only **in Poincare's paper the equation** $d\omega = 0$ **is written in a curious cyclic notation** (see 7.2 of last year's seminar notes)!

REMARK. Though de Rham cohomology is à priori **homology via calculus** we remark that this cyclic viewpoint of d has led now, in the hands of **Connes**, to a complete combinatorialization of de Rham cohomology called **cyclic cohomology** (a more obvious kind of combinatorialization had been done before by Thom).

Poincaré notes that the ambiguity of the indefinite integral $\int\omega$ over M is again given by an integral linear combination of at most $b_{|\omega|}(M)$ *periods* obtained by integrating over so many homologically independent closed sub varieties of M, ands with the **assertion** that this bound is the best possible.

Theorem. *This assertion of Poincaré is equivalent to de Rham's theorem, i.e. that* $H^{*}(M;\mathbb{R})$ *is isomorphic to* (kerd)/(imd).

Proof. Integration lets us identify each closed form as a

Chapter 4.
MINIMAL COMPLEXES

(4.1) Eilenberg-Steenrod properties of π_*. The higher homotopy groups π_q , $q \geq 2$, were first studied by HUREWICZ in 1934, who defined them in terms of POINCARE's fundamental group π_1 by

$$\pi_q(X,*) = \pi_1(\Omega^{q-1}(X),*),$$

where $\Omega^{q-1}(X)$ is the (q-1)th iterated **loop space** of X at the **base point** * of X (with the constant loop * considered as its base point).

So as a set **(a)** $\pi_q(X,*)$ consists of homotopy classes $[\alpha]$ of loops of the iterated loop space $(\Omega^{q-1}(X),*)$. Since a loop is a map of the unit interval I which images ∂I to *, we can reformulate this as saying that **(b)** $\pi_q(X,*)$ consists of homotopy classes [f] of maps

$$f : (I^q,\partial I^q) \longrightarrow (X,*)$$

from the pair (q-cube, its bdry) into (X,*), and this in turn amounts to saying that **(c)** $\pi_q(X,*)$ consists of homotopy classes [g] of maps

$$g : (S^q,*) \longrightarrow (X,*),$$

from a q-sphere with base point into (X,*).

Exercise. Carefully carry out these reformulations (b) and (c).

The base point is (as Poincaré emphasized) *very important* : it is because we consider only loops based at a point that concatenation supplies the quotient set of their homotopy classes with *a group structure*. It is easy to interpret this structure directly in the above reformulations, e.g. if we use q-cubes, $0 \leq x_1, \ldots ,x_q \leq 1$, then [f] + [g] is the homotopy class of the map f+g defined by

$$(f+g)(x_1, x_2, \ldots, x_q) = f(2x_1,x_2, \ldots ,x_q) \quad \text{if } x_1 \leq 1/2, \text{ and}$$

$= g(2x_1-1, x_2, \ldots, x_q)$ if $x_1 \geq 1/2$.

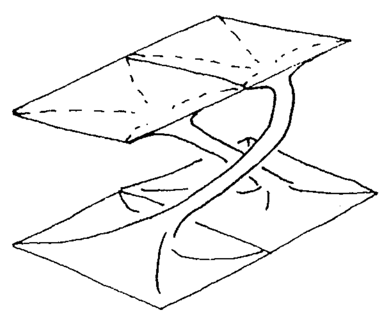
Here we have preferred additive notation (which incidentally Poincaré used also for the non-Abelian π_1 !) because of the following.

Theorem. *For all* $q \geq 2$ *the groups* $\pi_q(X,*)$ *are Abelian.*

In this context we remark that the higher homotopy groups had in fact been defined earlier by CECH (he used (c) : see *ICM Zûrich* 1932, vol. 2, p.203) who mentions that even DEHN had been aware of this definition (around 1910) ! However the study of these groups was not pursued because of the above "discouraging" property : it was generally felt that, like π_1, one should really be looking for an interesting *non-Abelian generalization of the homology groups* H_q *for* $q \geq 2$.

(Later work of MOORE and KAN will show that, *in a certain sense*, the homotopy groups π_i's, despite being Abelian, are nevertheless the natural "non-Abelian analogues" of the homology groups H_i !)

Proof (I found this argument in 1969 during my first encounter with π_i's). Choose two disjoint arcs in the (q+1)-cube from a point with $x_1 < 1/2$ (*resp.* $x_1 > 1/2$) in the bottom to a point with $x_1 > 1/2$ (*resp.* $x_1 < 1/2$) in the top, and "fatten" these (see fig.) into disjoint tubes having these portions of the bottom and top as their ends.

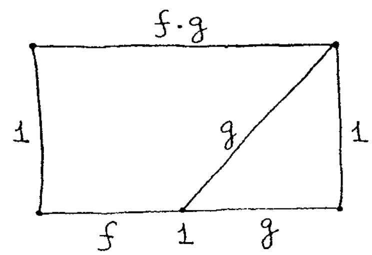


The required homotopy from f+g to g+f can now be defined by mapping each horizantal section of these two tubes as per f and g respectively, and by mapping the complement of these tubes to the base point. *q.e.d.*

It is usual also to denote *the set* of path components of X by $\pi_0(X,*)$, with the path component of * to be denoted by 0. Next we "relativize" all these definitions.

Relative homotopy groups. Given a pair (X, A) of nonempty spaces with a base point $*$ specified in A we consider rel homotopy classes of maps of the q-cube I^q into X, which image its boundary ∂I^q into A, with all of ∂I^q but possibly the front face $x_1 = 0$ being mapped into $*$. The group operation is defined again by concatenation (i.e. by the equations above). These groups $\pi_q(X, A, *)$ can be checked to be Abelian for $q \geq 3$, for $q = 2$ they may be non-Abelian, and for $q = 1$ we only have a set with a distinguished element 0, we don't define them for $q = 0$.

Exercise. Show that π_1 of a topological group is Abelian. *Hint.* Look at the following picture.



Exercise. Show that $\pi_q(X, A)$ too can be defined as a fundamental group, viz. of the (q-2)th iterated loop space of $\Omega(X, A)$, the space of all paths of X starting from $*$ and ending at some point of A.

Eilenberg-Steenrod properties. Axioms 1 and 2 are obvious : each map ϕ of spaces which preserves their base points gives rise to the **induced map** ϕ_* , $[g] \longmapsto [\phi \circ g]$, and clearly one has $(\mathrm{id})_* = \mathrm{id}$ and $(\phi\xi)_* = \phi_*\xi_*$. Again if the **connecting map** $\partial : \pi_q(X, A) \longrightarrow \pi_{q-1}(A)$ is defined by imaging each $[g]$, where $g : (I^q, \partial I^q) \longrightarrow (X, A, *)$ to $[g|\partial I^q]$, then Axiom 3 also holds. There is also the following analogue of the Axiom 4 which too we leave as another straightforward *Exercise.*

Theorem (EXACTNESS). *To each $(X, A.*)$ is associated a functorial long exact homotopy sequence*

$$\cdots \longrightarrow \pi_q(A) \longrightarrow \pi_q(X) \longrightarrow \pi_q(X, A) \xrightarrow{\partial} \pi_{q-1}(A) \longrightarrow \quad \cdots .$$

Note here that *exactness* makes sense for any sequence of maps of *sets with distinguished elements* which preserve these elements : the subset imaging to the next distinguished element should coincide with the image of the preceding map.

Thus if we interpret $\mathfrak{Exact}$ as above Axioms 1-4 together amount to saying that we have a functor from the category of pairs of spaces with base points to $\mathfrak{Exact}$.

Continuing his *Exercise* the reader will have no problem in checking easily that the homotopy Axiom 5 is also true. So only the excision property remains, for which of course everything hinges on *what* we'd like to call "excisions". *If* we adopt (for the new category of pointed pairs of topological spaces in which we are working now) a definition close to that in $\mathfrak{Top}$ or $\mathfrak{Simp}$ then Axiom 6 will *not* be true.

Example. We'll see below that $\pi_3(S^2,*) \cong \mathbb{Z}$. But $(S^2,*)$ has the homotopy type of $(S^2,S^2_+,*)$ where S^2_+ denotes the hemisphere with center $*$. Now if we "excise out" a small open disk of S^2_+ not containing $*$ then we'll be left with the homotopy type of $(D^2,\partial D^2,*)$. The circle ∂D^2 has higher homotopy groups trivial (*Why ?*) while the disk D^2 is contractible, so by above exact sequence $\pi_3(D^2,\partial D^2,*) = 0$.

REMARK. Despite being Abelian, the higher homotopy groups are much more difficult to compute than π_1, a difficuly which stems no doubt from *the infinite dimensionality* of the loop spaces used in their definitions, and results in a failure of the "usual" excision property.

Exercise*. Find conditions necessary and sufficient for an open subset U of 3-space to be such that any identically nonzero continuous vector field $\overrightarrow{P}$ on U can be expressed as the **vector product** $\overrightarrow{E} \times \overrightarrow{H}$ of two other such vector fields on U. (This problem was communicated to me in 1969 by Prof. N.E. Steenrod, while solving it I made my own personal rediscovery of the following important concept.)

As **"excisions"** for our category we'll now use **fibrations**, viz.

maps $\begin{array}{c}(E,F,*)\\ \big\downarrow \\ (B,*,*)\end{array}$ having the **covering homotopy property** : i.e. any homotopy f_t into $(B,*,*)$ for which f_0 is liftable into $(E.F,*)$ should be liftable into $(E.F,*)$ (with "lifting" having the obvious meaning). For such maps the required Axiom 6 is verified easily.

Theorem ("EXCISION"). *If* $(E.F,*) \longrightarrow (B,*,*)$ *has the covering homotopy property then it induces an isomorphism in homotopy groups.*

Corollary. *We have the following* **exact sequence of fibration**

$$\ldots \longrightarrow \pi_q(F) \longrightarrow \pi_q(E) \longrightarrow \pi_q(B) \longrightarrow \pi_{q-1}(F) \longrightarrow \ldots .$$

This follows immediately if we replace the $\pi_q(E,F)$'s in the exact sequence of the pair (E,F) by their isomorphs $\pi_q(B)$.

Exercise. Consider S^3 as the unit sphere of $\mathbb{C}^2$ and S^2 as the projective space $\mathbb{C}P^1$ of all one-dimensional complex vector subspaces of $\mathbb{C}^2$. Show that the restriction $S^3 \longrightarrow S^2$ of the projection map $\mathbb{C}^2 \longrightarrow \mathbb{C}P^1$, $(z_1,z_2) \longmapsto [z_1,z_2]$, has the covering homotopy property and each fiber is homeomorphic to S^1. Use the exact homotopy sequence of this **Hopf fibration** (and $\pi_3(S^3) \cong \mathbb{Z}$ to see that $\pi_3(S^2) \cong \mathbb{Z}$.

REMARK. The above simple yet remarkable example of Hopf, 1930, probably motivated Hurewicz to start studying the higher homotopy groups in 1934 : clearly one had a *new* phenomena, unlike homology these groups could be nonzero in dimensions bigger than that of the space ! Inspired in turn by Hurewicz's work Hopf later studied higher dimensional analogues of his example and formulated his famous "Hopf invariant one" problem. We remark that all these examples have played a vital rôle (cf. Milnor's *exotic spheres*) in later developements in topology.

Exercise. Let T_1S^2, the **unit sphere bundle** of the 2-sphere, be the space consisting of all orthonormal pairs $(\overrightarrow{E},\overrightarrow{H})$ of vectors of $\mathbb{R}^3$. Show that the map $T_1S^2 \longrightarrow S^2$ given by projecting onto the first coordinate has the covering homotopy property (this is the point required in an exercise given above).

Exercise. Show that S^3 is a double cover of T_1S^2 and that the latter space is homeomorphic to $\mathbb{R}P^3$ (i.e. Poincaré's *Example* 5 or **222** of 3.4) or to the continuous group SO(3) of all 3x3 orthogonal matrices having determinant 1. One can also think of S^3 as the group of all unit quaternions and then T_1S^2 identifies with the coset space $S^3/\{\pm 1\}$.

Only Axiom 7 is now left and that of course holds because in fact $\pi_q(pt) = 0$ *for all* q (this is more like saying that the homology of the *empty space* is trivial, rather than that that of a point is trivial in all positive dimensions).

Thus the homotopy "groups" π_q give an example of an E-S homology theory in the sense of (2.3). Here the quotes are meant to remind us that for small values of q our groups π_q are not Abelian and for the lowest values indeed just sets-with-distinguished-elements.

Action of fundamental groupoid on homotopy groups. Given any path α from x to y there is an induced isomorphism

$$\alpha_{\#} : \pi_*(X,y) \longrightarrow \pi_*(X,x)$$

obtained by "dragging back" the defining maps via α.

Exercise. Make this "dragging back" precise. (*Hint.* Use the fact that $S^q\times\{0\} \cup \{*\}\times[0,1]$ is a deformation retract of $S^q\times[0,1]$.)

Clearly $\alpha_{\#}$ depends only on the homotopy class of the path, and concatenation of paths results in a composition of these isomorphisms. So we see that **the fundamental group $\pi_1(X,*)$ acts on all the homotopy groups $\pi_q(X,*)$** via "drag back".

The action of π_1 on itself is conjugation, so will be trivial iff π_1 is Abelian. However even in this case its action on the higher homotopy groups need not be trivial. One says that the space X is **simple** iff this action is trivial.

Note. Without always mentioning it we will often assume our spaces simple and π_1's Abelian : the changes required to generalize the formulae obtained are almost always very straightforward.

REMARK. The homotopy groups of a space are probably best considered as **the natural coefficient groups of the space**. To each point x there are associated these groups $\pi_*(X,x)$ and if we let x vary we get a space which covers X : thus each space X *comes with its canonical covering space* or **coefficient sheaf** π_X.

An analogy. This coefficient sheaf is analogous to the **structure sheaf** $\mathcal{O}_V$ of a variety V. Just like $H^*(V;\mathcal{O}_V)$ the cohomology $H^*(X;\pi_X)$ of X with coefficients in π_X is very important but hard to compute (e.g. the zeroth cohomology of a simple space is its homotopy groups). So one uses a spectral sequence, this time provided by the **Eilenberg filtration** of (4.3) below — for varieties one has the **Dolbeaut filtration** — to get some partial information (results of Serre, Cartan, etc.) about these hard-to-compute "cohomology" groups.

(4.2) Milnor's Uniqueness Theorem. We saw in (4.1) that the π_q's give us an example of another E-S homology (in the sense of 2.3 slightly generalized). So in analogy with (2.17) it is natural to enquire (this question was posed by Eilenberg-Steenrod in their book) whether it is the only such E-S homology ? The answer was given by MILNOR in the *Annals* of 1956.

Theorem. *Any functor from the category of pairs of pointed spaces to sequences of sets with distinguished elements which satisfies* Axioms 1 - 7 *of* (4.1) *must be isomorphic to the functor* π_* *of* (4.1).

Moreover Milnor shows that *there are exactly two ways of making these sets into groups* (for all but the least values of q and in such a way that the distinguished elements become the identity elements and all induced maps and connecting maps become homomorphisms) the second structure being of course the *opposite* of the group structure of (4.1).

Proof. Assume inductively that we have already identified the lesser dimensional given sets with the π_q's in a natural way.

Now consider the **Serre fibration**,

$$\Omega X \longrightarrow PX \xrightarrow{f} X,$$

whose **total space** PX consists of all paths starting from the base point, and the **projection** f maps each path to its end point. So the total space is contractible, and the **fiber** is the loop space ΩX of X. Now use the **exact sequence of the fibration** (an immediate consequence of Axioms 4 and 6) to identify the given qth set with distinguished element of X with the (q-1)th set of ΩX which by the inductive hypothesis we already know is $\pi_{q-1}(\Omega X)$ i.e. $\pi_q(X)$.

Now let o be any any natural group multiplication in the sets $\pi_1(X,*)$. *When* X *is a bouquet of two circles* we know that the usual group structure makes this $\pi_1(X,*)$ into a free group with two generators a and b. Let the new product aob equal the **reduced word** w(a,b), then (because new operation also makes the set into a group) we have

$$w(a,1) = 1 = w(1,b) \quad \text{and} \quad w(w(a,b),c) = w(a,w(b,c)).$$

Exercise. The only words of $\langle a\rangle * \langle b\rangle$ having the above properties are w(a,b) = ab and w(a,b) = ba.

Using this and the naturality of o it now follows that *for any* X the product of any two loop classes $[\alpha]$ and $[\beta]$ is either $[\alpha\beta]$ or $[\beta\alpha]$ (i.e. concatenation in the usual or opposite order. The assertion re the multiplications of the higher groups follows because of the isomorphism between the qth group of X and the (q-1)th of ΩX. *q.e.d.*

The above argument seems to suggest that the group structure of the π_q's is perhaps not too important. We remark also that even the **local family** $x \longmapsto \pi_*(X,x)$ (i.e. the groups together with the action of the fundamental groupoid) can be characterized axiomatically.

Exercise*. Note that Milnor's uniqueness theorem was proved quite differently and more simply than the Eilenberg-Steenrod uniqueness theorem of (2.17). Still, there is even now (same definition as in 2.16) an Atiyah-Hirzebruch spectral sequence for π_* (one needs only Axioms 1-4 for its definition so it is defined also for a "generalized homotopy theory") : what can one say about its second term E^2 ?

(4.3) Obstructions. In the second of his two paper in the *Annals* of 1940 (see also a 1938 paper in *Fund. Math.* motivated by Hopf's classification of maps of an n-complex into the n-sphere) Eilenberg introduced the very fruitful idea of obstruction cocycles. We recall some basic facts re these here.

First lets start with the two **homotopy addition theorems**. Here U will denote a q-**sphere with holes** i.e. an S^q minus disjoint disks D_1, D_2, ... , D_t. Our q-sphere's base point will be in U, and we'll assume it joined in U by some paths α_1, α_2, ... , α_t to the base points of the bounding (q-1)-spheres S_1, S_2, ... , S_t of the disjoint disks.

Theorem. **(i)** *If f be any map from the above q-sphere with holes* $(U,*)$ *into* $(X,*)$, *then we have in* $\pi_{q-1}(X,*)$

$$\sum (f\alpha_i)_\# [f|S_i] = 0.$$

(ii) *If f and g be two maps from* $(S^r,*)$ *to* $(X,*)$ *which coincide on* U, *then we have in* $\pi_q(X,*)$

$$[g] - [f] = \sum (f\alpha_i)_\# [g|D_i - f|D_i].$$

Here by $g|D_i - f|D_i$ we mean the map defined on the r-sphere obtained by identifying two copies of D_i by using g on the copy having the right orientation and f on the other copy.

Proof. The result is intuitively obvious, and we'll leave as an *Exercise* the straightforward job of formalizing this intuition. *q.e.d.*

To reduce "rattling" (see 3.1) we will often simplify (cf. Note above) such addition formulae to just $\sum_i [f|S_i] = 0$ and $[g] - [f] = \sum_i [g|D_i - f|D_i]$ respectively.

Instead of strait-jacketing Eilenberg's iterative *method of obstructions* into formalized statements we'll illustrate it by looking at some important cases.

EXTENSION PROBLEMS. Suppose a map f is given, from the n-skeleton K^n of a simplicial complex K, to a space X (which we'll assume simple to cut down on notation !) and we want to extend f to the (n+1)-skeleton K^{n+1} of K.

We now define an (n+1)-cochain o of K with values in $\pi_n(X)$ by $o(\sigma) = [f|\partial\sigma]$: clearly *the map* f *extends iff this cochain* o *is trivial*. Note that $\delta o = 0$ i.e. that o *is a cocycle* : this follows because given any (n+2)-simplex θ the map f is given on its n-skeleton U, so by using the above homotopy addition lemma (i) it follows that $o(\partial\theta) = 0$. Because of these properties we'll refer to o as the **obstruction cocycle** defined by $f : K^n \longrightarrow X$.

We now look at the **obstruction class** $\mathbf{o} = [o] \in H^{n+1}(K;\pi_n(X))$. We defined it using the map $f : K^n \longrightarrow \pi_n(X)$. But suppose we change f on just one n-simplex ξ. Clearly this only adds the elementary coboundary $\delta\xi$ to o. Thus we see that *the obstruction class is unaffected if we change* f *only on the* n-*simplices of* K. So the vanishing $\mathbf{o} = 0$ of the obstruction class is equivalent to saying that f *can be extended possibly after altering it on some* n-*somplices of* K.

FINDING A HOMOTOPY. We are given two maps f and g from a simplicial complex K to a (still simple !) space X, and we know of a homotopy F defined on the n-skeleton of K, between these two maps. We want to extend it to the (n+1)-skeleton of K.

Take any (n+1)-simplex σ and look at $\sigma \times [0,1]$. The map F is given already on the boundary of this (n+2)-cell. Thus we have an (n+1)-dimensional cochain d of K with values in $\pi_{n+1}(X)$, viz. $d(\sigma) = [F|\partial(\sigma\times[0,1])]$ and clearly the *homotopy extends iff this cochain is zero*. Note also that in case F is identity on K^n (we can obviously always reduce to this case) then $d(\sigma) = [g|\sigma - f|\sigma]$ and we therefore refer to it as the **difference cochain** defined by f and g.

FINDING A SECTION. Let $p : E \longrightarrow K$ be a fibration and suppose we already know a section (i.e. a right inverse) f of p defined over the n-skeleton and we want to extend f to the (n+1)-skeleton of K.

Using f the boundary of each (n+1)-simplex (a contractible thing over which E is just $\sigma \times F$, where F is the **fiber** of p) is getting mapped to the fiber. Thus we have an (n+1)-cochain o of K with values in $\pi_n(F)$. Clearly the section extends iff o is zero. And as above we can easily verify that o is a cocycle, that changing the section only on the n-simplices only adds coboundaries to it. Thus the **characteristic class** o = [o] is independent of the section upto such changes and one has **o** = 0 iff after possibly changing o on the n-simplices of K we can extend to a section of p defined over the (n+1)-skeleton of K.

(4.4) Eilenberg's filtration and Hurewicz's theorem. Singular homology theory goes back to Poincaré, or as Eilenberg puts it in his *Annals* 1944 paper it is **"as old as topology itself"**. As we mentioned in (2.19) this method was neglected in the 1920s and 1930s because then the focus was on generalizing the duality theorems of Poincaré for which the dual method of Cech was more appropriate.

The revival of singular homology theory began in 1940 with the following striking observation of EILENBERG (the proof however appeared later in the above cited paper of 1944).

Theorem. *For* $q \geq 2$ *the group* $\pi_q(X)$ *is naturally isomorphic to the* q*th homology of the sub chain complex of the singular complex determined by all singular simplices which are such that their faces of dimensions less than* q *are constant* = * , *the chosen base point of* X.

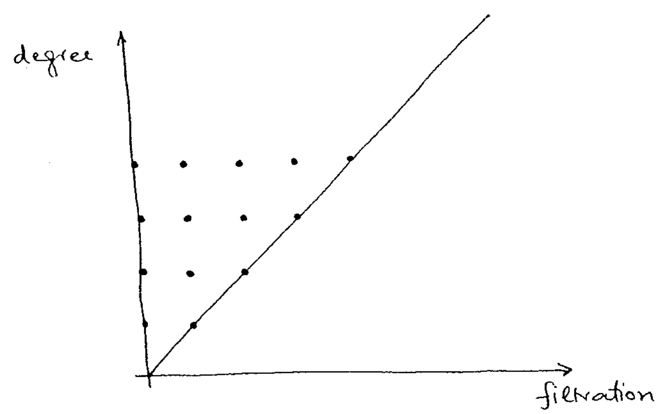
Proof. Let $S_q(X,*)$ denote the set of all singular simplices of the indicated kind. Interpreting each of these as a map of the q-cell into X which images its boundary to * we get a group homomorphism from the group $C_q(S_q(X,*))$ of all integral combinations of such simplices *onto* $\pi_q(X,*)$. Further from the very definition of the group operation in $\pi_q(X,*)$ it is clear that the subgroup of boundaries $B_q(S_q(X,*))$ is contained in the kernel of this homomorphism. Thus we have a surjective homomorphism $H_q(S_q(X,*)) \longrightarrow \pi_q(X,*)$.

We now construct an inverse map $\pi_q(X,*) \longrightarrow H_q(S_q(X,*))$. For this we note that each homotopy class $[g]$ of maps from the q-sphere S^q to X induces a chain map class $S(S^q) \longrightarrow S(X)$. Since S^q is (q-1)-connected we can chain homotope this into $S_q(X)$. This gives a homomorphism of $\mathbb{Z} = H_q(S^q)$ into $H_q(S_q(X)$, and the image of 1 under this map is defined to be the image of $[g]$. *q.e.d.*

Eilenberg's spectral sequence. By this we'll mean the spectral sequence of the following decreasing *filtration*

$$S(X) = S_1(X) \supseteq \ldots \supseteq S_q(X) \supseteq S_{q+1}(X) \supseteq \ldots$$

of the singular chain complex, the $S_q(X)$ being as defined in the above proof. We note that though the filtration is infinite, it is a first quadrant sequence, so at each point (p,q) the groups will stabilize after finitely many times. So, in this sense, it converges to the singular homology of the space X.



Usually (as before) p+q will denote the dimension, but note that in case q denotes the dimension, this spectral sequence becomes confined to the *upper half of the first quadrant* (see above fig.) : $S_q(X)$ has only the constant simplices in dimensions less than q, and so $H_*(S_q/S_{q+1})$ is zero in all these dimensions. As an example of the use of this spectral sequence let us now give Eilenberg's 1944 proof (couched in the later spectral sequence language) of a 1935 theorem of HUREWICZ.

Theorem. *If* q *is bigger than* 1 *and* $\pi_q(X)$ *is the first nonzero homotopy group of* X *then it is isomorphic to* $H_q(X)$.

Proof. From the hypothesis it follows that for $i \leq q-1$ each $S_i(X)$ can be deformation retracted onto $S_{i+1}(X)$. So the first q-1 columns of the first term of our spectral sequence are zero. Thus the qth homology (which corresponds to the lattice points (i,j) with $i+j = q$) comes only from the (q,q) point of the final term of the spectral. But this point stablizes at the first term, and, by above result, the group here is $\pi_q(X)$. *q.e.d.*

Here we remark that the later results of Serre, Cartan, etc. re calculations of homotopy groups all use (the spectral sequence of) the Eilenberg filtration, or else the dual sequence of **Postnikov fibrations**

$$\cdots \longleftarrow S/S_q \longleftarrow S/S_{q+1} \longleftarrow \cdots .$$

REMARK. There is also a **relative Hurewicz theorem,** also a relative version of the Eilenberg filtration. Now we are given a subspace A of X and a singular simplex is deemed to be of filtration $\geq n$ iff its $(n-1)$-skeleton is imaged into A. The proof of the relative Hurewicz theorem now follows as above by using this new spectral sequence. In this context we remark also that closely related to the relative Hurewicz theorem is a theorem of WHITEHEAD which say that a map between two triangulable spaces induces an isomorphism in homology iff it induces an isomorphism of the homotopy groups. The relation is brought about by the important trick of the **mapping cylinder** of a map $f\colon X \longrightarrow Y$ i.e. the identification space M_f obtained from the disjoint union of $X \times [0,1]$ and Y by identifying $(x_1,1)$ and $(x_2,1)$ iff $f(x_1) = f(x_2)$. This construction converts statements about the map into corresponding statements about the inclusion $X = X\times\{0\} \subseteq M_f$ e.g. *f is a homotopy equivalence iff* $X \subseteq M_f$ *is a deformation* and f *induces an isomorphism in homology iff* $H_*(M_f,X) = 0$, etc.

(4.5) Semisimp. The 1950 *Annals* paper of EILENBERG-ZILBER introduced the notion (cf. 2.18) of a **semi-simplicial complex** as a set of objects called *simplices* σ, each assigned a *dimension* $q \geq 0$, and (for $q > 0$) $q+1$ $(q-1)$-dimensional simplices $\sigma^{(i)}$, $0 \leq i \leq q$, called the *principal faces* of σ, such that

$$(\sigma^{(j)})^{(i)} = (\sigma^{(i)})^{(j-1)}$$

whenever i is less than j. (We'll also write $\sigma^{(i)} = \partial_i(\sigma)$.)

SOME EXAMPLES OF SEMI-SIMPLICIAL COMPLEXES

(A) Singular complexes. The motivating example for Eilenberg-Zilber was of course the set of singular simplices (see 2.14) of a space X : we'll denote this **singular complex** by X_{sing} or S(X). Then we have various subcomplexes of it which we have already considered e.g. the **Eilenberg subcomplexes** of (4.4) above.

Theorem. *Any semi-simplicial complexes is a subcomplex of some singular complex.*

Proof. To see this one can form the **Giever realization** of K as follows. For each simplex σ of K take a disjoint standard simplex Δ_σ of the same dimension as σ. In the disjoint union of spaces $\amalg_{\sigma\in K}\ \Delta_\sigma$ identify the ith face $\partial_i(\Delta_\sigma)$ of the standard simplex with the standard simplex $\Delta_{\partial_i(\sigma)}$ attached to the ith face of σ. This gives us a topological space $|K|$. Corresponding to each σ we have the singular simplex $\Delta_\sigma \subseteq \amalg_{\sigma\in K}\ \Delta_\sigma \longrightarrow |K|$ and thus K identifies with a subcomplex of the singular complex $S(|K|)$. *q.e.d.*

Exercise. Check that for a *finite* simplicial complex (with vertK totally ordered) the Giever realization is homeomorphic to that of 2.11.

Exercise. In this context note that we could have defined $|K|$ as in (2.11) even for an infinite K by using the canonical basis of a Hilbert space as our vertices. Show that if a vertex of K has infinitely many neighbours then this metric space is *not* homeomorphic to the Giever realization defined above.

REMARK. Thinking of a semi-simplicial complex visually (via the above Giever realization or else, when it is complete, via the more economical Milnor realization which we will consider below) renders many ponderous constructions very vivid, so we will be frequently resorting to it as motivation or short-cut for many definitions.

(B) Finitistic examples. These are essentially simple generalizations of 2.18 but definition (d) is new (of course all these too can be viewed as suitable singular subcomplexes).

(a) Simplicial complexes (possibly infinite) : to make an s.c. into an s.s.c. put a total order on vert(K) and define the ith principal face of any σ to be that obtained by omitting its ith vertex.

Exercise. Show that a semi-simplicial complex is of this kind *iff* each simplex is uniquely determined by the *set* of its principal faces *iff* given any two distinct simplices we can find a vertex which is incident to one of them but not the other.

(b) Complexes of simplicial type. By this we'll mean any semi-simplicial complex consisting of some (finite) **vertex sequences without repetitions**, the ith face being the subsequence obtained by omitting the ith member (the vertices come out from some specified set).

For example all vertex sequences without repetitions which are supported on a given simplicial complex K (i.e. $K_{\text{assoc},1}$ in the notation of 2.18) are of this kind (but there are other examples too since we need not take *all* vertex sequences of the supporting simplices).

Exercise. Show that an s.s.c. is of this kind iff the principal faces of any simplex are distinct and their *sequence* determines the simplex uniquely.

(c) Associative complexes. By this we'll mean any s.s.c. consisting of some (finite) **vertex sequences** (possibly with repetitions) with ith principal face again defined by omission of ith vertex (e.g. the K_{assoc} of 2.18).

Exercise. Show that an s.s.c. is of this kind iff each simplex is uniquely determined by the *sequence* of its principal faces.

REMARK. A combinatorial description of the **face vectors** (f_0, f_1, ...), f_i = number of i-dimensional simplices, of the class of all associative complexes on N vertices, is apparently unknown. A theorem of MACAULAY gives such a characterization for the face vectors of the smaller class having simplices which are *non-decreasing* with respect to a fixed total order on the vertices (these we'll also call **commutative complexes** or **order ideals of monomials**); further a theorem of KRUSKAL-KATONA similarly characterizes the face vectors belonging to the still smaller subclass whose simplices are *strictly increasing*, i.e. the class of all simplicial complexes on N vertices. (This face vector problem makes sense for many other classes of s.s.c.'s also.)

(d) Hyperassociative complexes and bicomplexes. Now the *r-simplices* σ will be functions *not* from just $[r] = \{0,1, \ldots ,r\}$, but from $2^{[r]}$, the *set of all subsets* of $[r] = \{0,1, \ldots ,r\}$ (the values of the functions can be in any set of *vertices*, or in the bigger set of all finite *vertex sequences*, or say the values can be in a group).

The *"skip* i" monotonic injection $\mathbf{i} : [r-1] \longrightarrow [r]$ induces the injection $\mathbf{i}_1 : 2^{[r-1]} \longrightarrow 2^{[r]}$, $\mathbf{i}_1(A) = \mathbf{i}(A)$, and we define the **principal ith face of the first kind** $\sigma_1^{(i)}$ to be the composite $\sigma\circ\mathbf{i}_1$. If our set of simplices σ is closed with respect to such principal faces we shall say that we have an **hyperassociative complex of the first kind**.

The "skip i" map $\mathbf{i}$ also induces the injection $\mathbf{i}_2 : 2^{[r-1]} \longrightarrow 2^{[r]}$, $\mathbf{i}_2(A) = \mathbf{i}(A)\cup\{i\}$, and we define the **principal ith face of the second kind** $\sigma_2^{(i)}$ to be the composite $\sigma\circ\mathbf{i}_2$. If our set of simplices σ is closed with respect to such principal faces we shall say that we have an **hyperassociative complex of the second kind**.

In case the set of simplices σ is closed with respect to *both* kinds of principal faces then we'll call it a **hyperassociative bicomplex**.

Note. We'll see later that some minimal complexes (4.7) are hyperassociative complexes (4.8) of the first kind and using these $K(\pi,n)$'s we'll work out the general structure of *any* minimal complex (4.10). However since this corresponds to only the "columns" of the Eilenberg bigrading (of the minimal complex) it might be useful to consider also the above combinatorial *bicomplex*, i.e. to consider also the notion of faces of the second kind.

REMARK. The above immediately brings to mind the work (being still) done by GELFAND and others (see e.g. the paper of **Gelfand-Macpherson** in the *Advances* of 1982) re the **combinatorics of the Pontrjagin characteristic classes** of a smooth manifold : they use **hypersimplices** Δ_p^r, i.e. the convex hull of the barycentres of the p-dimensional faces, $0 \le p \le r-1$, of an r-simplex. Except in the extreme cases $p = 0$ or $r-1$, this r-dimensional convex polytope has $2(r+1)$ facets, of which $r+1$ are affinely isomorphic to Δ_p^{r-1} (of the first kind) and the remaining $r+1$ are affinely isomorphic to Δ_{p-1}^r (of the second). Thus what we are suggesting (see *Exercises* below) is that **hypersimplices may also give a combinatorial definition of homotopy groups and of the Eilenberg-Maclane-Postnikov characteristic classes** ?

Exercise*. Let $K_{\text{hyperassoc}}$ be the bicomplex whose r-simplices are functions from $2^{[r]}$ (to vertex sequences) whose images are supported on simplices of a given simplicial complex K. Equip $K_{\text{hyperassoc}}$ with the sum of the boundary operators (see 4.6) of its two semi-simplicial structures. Identify this with a subcomplex of the singular complex of $|K|$ and show that it has the same homology over $\mathbb{Z}[\frac{1}{2}]$ coefficients.

Exercise*. Say that a function from $2^{[r]}$ is of **filtration** $\ge$ p if it vanishes on all subsets of cardinality p+1 or more. Show that this gives a subcomplex of $K_{\text{hyperassoc}}$. Compute the spectral sequence of this Eilenberg filtration, in particular the (n,n) term, i.e. "the nth homotopy group".

(4.6) Completion. Let us say that a singular subcomplex of S(X) is complete iff it is closed under composition with any non-decreasing map $[j] \longrightarrow [k]$. We saw above that any semi-simplicial complex is a singular subcomplex, so we can always complete it by adding singular simplices obtained by doing all these compositions.

Exercise. Formulate the above *completion process* combinatorially.

REMARK. The notion of completeness is also from the Eilenberg-Zilber paper in the *Annals* of 1950. Note that *an* **s.s.c.** *is a contravariant functor from the category of order preserving* injections *amongst sets* $[r] = \{0,1, \dots ,r\}$, while *a* **c.s.s.c.** — also called a

SIMPLICIAL OBJECT — *is a contravariant functor from the bigger category* $\mathfrak{Nat}$ *of all order preserving maps* amongst such sets [r]. Still more generally (after Connes' work) these days one also uses CYCLIC OBJECTS which are contravariant functors from the yet bigger category of all functions between such sets [r].

For example K_{assoc} is complete (and even a cyclic object) while $K_{assoc,1}$ is not. Note also that in a c.s.s.c. one has the notion of a **degenerate** simplex, i.e. those lying in the image of map induced by some "repeat t" surjection $s_t : [j+1] \longrightarrow [j]$, $0 \le t \le j$. So note that a non-degenerate simplex of K_{assoc} need not be in $K_{assoc,1}$: they comprise of all vertex sequences of K_{assoc} for which no two *neighbouring* vertices repeat. Note also that a face of a degenerate (resp. non-degenerate) simplex can be non-degenerate (resp. degenerate), so *neither degenerate nor non-degenerate simplices may form a subcomplex.*

CARTESIAN PRODUCT OF TWO SEMI-SIMPLICIAL COMPLEXES. As remarked in (2.13) the Eilenberg-Steenrod subdivision of the product of two simplicial complexes might have been a motivation for introducing *complete* semi-simplicial complexes : product $K \times L$ of any two s.s.c.'s is the s.s.c. whose r-simplices are all pairs (σ,θ), where σ is an r-simplex of K and θ of L, with the ith face $(\sigma,\theta)^{(i)}$ being the pair $(\sigma^{(i)},\theta^{(i)})$. In case we take K and L to be just ordered simplicial complexes $K \times L$ does not triangulate the cartesian product $|K|\times|L|$. However if we look at $K_{comm} \times L_{comm}$ then its Milnor realization (see below) is precisely the cartesian product we want.

In this context see also the Eilenberg-Zilber Theorem. We remark also that in (2.10) below we'll also use a "twisted cartesian product" (or "fiber bundle") of two s.s.c.'s.

Milnor realization of a complete semi-simplicial complex K is defined thus. We start with the disjoint union $\amalg(\Delta\times\sigma)$, where Δ is any standard simplex and $\sigma \in K$, and for any non-decreasing map ϕ we make the identification $\phi_*\Delta\times\sigma = \Delta\times\phi^*\sigma$. Thus we now also use the **degeneracies** (i.e. non-injections ϕ) to make more identifications, so we get a much smaller $|K|$ (having however the same homotopy type as the Giever realization). In fact it can be checked that the ensuing thing is a CW complex (see below) whose cells are in one-one correspondence with the non-degenerate simplices of K. So in particular, as asserted above, the Milnor realization of $K_{comm} \times L_{comm}$ is the E-S subdivision of the cartesian product that we considered in (2.13).

Exercise. Show that $|K_{assoc}|$ has the same homotopy type as $|K|$. Also formulate, for cyclic objects, another definition of realization involving further identifications corresponding to permutations. With this new notion the realization will have still fewer cells, in particular that of K_{assoc} will be precisely $|K|$.

Very briefly now we'll digress to give some facts about

CW COMPLEXES. These are **cell complexes** which, *in the finite case*, are essentialy those called "of the third kind" in **Poincare''**s *Analysis Situs*. More precisely a space K partitioned into open cells in such a way that the boundary of each cell is contained in (may not equal, this is the only difference from Poincaré) a union of lower-dimensional cells, and closure of each cell is continuous image of the closed cell of that dimension. In his *B.A.M.S.* 1949 paper, **Whitehead** gave the "right" (see Exercises below for justification) conditions which one should put *in case* K *is infinite* : (C) we demand that it is **closure finite**, i.e. that the closure of each cell is contained in a finite union of cells, and (W) we demand the **weak topology** on K, i.e. a set should be open iff its intersection with all cells is open. Cell complexes satisfying (C) and (W) are CW complexes.

Exercise. Assume (C) and the condition (L) : that K is **locally finite**, i.e. that each cell has a neighbourhood covered by finitely many cells. Show that then K is a CW complex.

Exercise. In the following examples check which of the conditions (C), (L), and (W), holds, and which fails :

(i) $\mathbb{R}$ partitioned into its points (so only zero cells).

(ii) The boundary of a triangle partitioned into its points.

(iii) The triangle partitioned into its interior (open 2-cell) and points (0-cells) of the boundary.

(iv) A metric realization (using canonical basis of a Hilbert space as vertices) of an infinite simplicial complex.

Exercise*. Show that the cartesian product of two CW complexes need not be a CW complex, but that in case one of them is locally finite then this product is a CW complex.

Exercise. Show that if the CW complex L is a subcomplex of the CW complex K, then $L \subseteq K$ is a **cofibration** (i.e. that the pair (K,L) has the *homotopy extension property* of 4.1).

Exercise. Show that any continuous map between CW complexes can be homotoped to a **cellular map**, i.e. one which maps cells into cells. Likewise show that any homotopy between two cellular maps is homotopic rel its "ends" to a cellular homotopy.

Exercise* **(Whitehead's first theorem).** Show that if a map between CW complexes is a **weak homotopy equivalence**, i.e. if it induces an isomorphism in all homotopy groups, then it is in fact a homotopy equivalence. (In other words one gets an induced isomorphism of the entire Eilenberg E_1 term — i.e. the "homotopy type" or "minimal complex" of the spaces — as soon as one gets such an induced isomorphism on the x-axis.)

Exercise. Show that two CW complexes can have all homotopy groups isomorphic without being of the same homotopy type. (Thus we can not dispense with "a map" in the above theorem.)

Exercise **(Whitehead's second theorem)***. Show that if a map between two simply connected CW complexes induces an isomorphism in all singular homology groups then it is a homotopy equivalence. (*Hint.* Use the "mapping cylinder" construction, and the relative Hurewicz theorem of 4.1, to check that this is a weak homotopy equivalence. The best reference for all these "Exercises" is still the 1949 paper of Whitehead even though now most text books — e.g. Spanier's — also cover these.)

(Co)homology of semi-simplicial complexes. The definition of the homology of an s.s.c. is the obvious one. We define $C_r(K)$ to be the free Abelian group generated by the r-simplices of K and the *boundary* *operator* $\partial : C_r(K) \longrightarrow C_{r-1}(K)$ is then defined by

$$\partial(\sigma) = \sum_i (-1)^i \sigma^{(i)}.$$

Because of the defining property of a s.s.c. (this was the motivation for it of course !) we have $\partial^2 = 0$ and so we define $H_*(K) = \ker\partial/\mathrm{im}\partial$ as before. The definition of cohomology is dual of this.

We'll also need cohomology with **local coefficients**, so lets define this now (the motivation of this comes from the homotopy groups $\pi_*(X,x)$ of a space). We are now given for each edge of the s.s.c. a group isomorphism from a group attached to the second vertex of the edge to a group attached to the first vertex of the edge. (Note the two vertices, or maybe even *all* the vertices of the s.s.c., may coincide : then of course the groups attached to the vertices coincide but these group isomorphisms need not be the identity automorphism.)

Each cochain will now assign to a simplex *coefficients from the group attached to its first vertex*. Now the usual definition of coboundary does'nt quite make sense : on omitting the first vertex we get a principal face whose first vertex can be different. So what one does is one changes the definition of δ slightly : one drags back the coefficient of this principal face to the original first vertex also.

Note. These local coefficients become necessary to take into account the action of the fundamental group. On the first time around it is wise to just ignore this finesse and assume all spaces simple.

REMARK. Note that we did *not* use degeneracies in the above definitions of (co)homology and indeed for purposes of *computing homology* completeness is somewhat of a luxury because *even though degenerate simplices of* K *do not form a subcomplex of* K, *the cochains vanishing on them do form a sub cochain complex of* $C^*(K)$, *and this has the same cohomology.* Even though it would be quite easy to give a proof of this here (*Exercise !*) we'll postpone this **normalization theorem** of Eilenberg-Zilber to (4.14), because we want to look at it in conjunction with a later analogous result of Moore. In contrast to homology, if we want to *compute homotopy* (i.e. initial terms of the Eilenberg sequence) then we'll see that completeness is necessary (that *some* such notion must intervene is clear e.g. from the fact that homotopy groups can be nonzero for dimensions higher than that of the simplicial complex).

REMARK. The various homologies considered in (2.18) make sense for arbitrary semi-simplicial complexes, thus one can talk e.g. of the Bier or Mayer homology of a singular complex. One can thus define some topological invariants which are not invariants of the homotopy type.

(4.7) Minimal complexes. The same 1950 paper of Eilenberg-Zilber also introduced this very important idea.

Given a path connected space X and a point * of X by a minimal subcomplex M(X,*) of S(X) is meant a sub semi-simplicial complex containing all *constant* = * singular simplices, which is such that if any singular simplex σ has all its faces in M(X,*), then M(X,*) contains a unique singular simplex θ *homotopic* to σ rel boundary.

Theorem. *Any path connected space* X *has upto a semi-simplicial isomorphism a unique minimal complexes* M(X).

Proof. The existence is straightforward, because we can inductively build up the skeletons by choosing from each homotopy class of dimension q simplices having all their faces in M(X) choose any one representative.

We assert that any other minimal subcomplexes at * must be semi-simplicially isomorphic to M(X). To see this construct (cf. the prism operators of 2.14) a deformation retraction of S(X) onto M(X). Restricted to the second minimal complex this will furnish (*Exercise*) the required isomorphism with M(X).

The dependence of M(X,x) on the base point x can easily (*Exercise*) be seen to be like those of the homotopy groups : i.e. each homotopy class of paths α from x to y will induce a "drag-back" isomorphism $\alpha_{\#}$: $M(X,y) \longrightarrow M(X,x)$ with concatenation corresponding to composition of these isomorphisms (in particular the fundamental group at * acts on the minimal complex at * via these isomorphisms). *q.e.d.*

Theorem. *If* X *is homotopy equivalent to* Y *then* M(X) *is isomorphic with* M(Y). *The converse is also true for triangulable spaces.*

Proof.

q.e.d.

The importance of the minimal complex is now clear : it is a combinatorial model of the homotopy type of X ! It follows that all the homotopy invariants should be "readable" from M(X). For this we need to understand the **structure of the minimal complex** (see §§ 4.8-4.9 below), for starters the following are obvious.

o M(X) *is built up from some bunches of* $|\pi_q(X)|$ *comparable* q-*simplices,* $q \geq 0$, where **comparable** means having the same faces.

o *If all homotopy groups are trivial then* M(X) *contains only the constant simplices.*

Exercise. Show that the Alexandrov space (see 2.20) is not contractible even though all its homotopy groups are trivial.

This shows that the converse assertion of the last theorem is not true for arbitrary topological spaces.

(4.8) Eilenberg-Maclane Complexes, Hopf's Theorem. We turn now to another paper, also from the *Annals* of 1950, by EILENBERG-MACLANE.

DEFINITION OF $\mathcal{K}(\pi,n)$. This will denote the semi-simplicial complex whose r-simplices are all functions ϕ, from the set $\begin{bmatrix} r \\ n \end{bmatrix}$ of all cardinality (n+1)-subsets of $[r] = \{0,1, \ldots ,r\}$, to π; the faces being defined by composing ϕ with the r+1 injections $\begin{bmatrix} r-1 \\ n \end{bmatrix} \longrightarrow \begin{bmatrix} r \\ n \end{bmatrix}$ induced by the r+1 strictly increasing injections $[r-1] \longrightarrow [r]$.

DEFINITION OF $K(\pi,n)$. Now suppose further that π is a group. Thinking of each function $\phi : \begin{bmatrix} r \\ n \end{bmatrix} \longrightarrow \pi$ as an n-cochain of the standard simplex [r] with coefficients π, it has a coboundary $\delta\phi : \begin{bmatrix} r \\ n+1 \end{bmatrix} \longrightarrow \pi$. (This δ should not be confused with the coboundary of the semi-simplicial complex $\mathcal{K}(\pi,n)$ which of course runs $C^r(\mathcal{K}(\pi,n)) \longrightarrow C^{r+1}(\mathcal{K}(\pi,n))$.) We will denote by $K(\pi,n)$ the sub semi-simplicial complex of $\mathcal{K}(\pi,n)$ consisting of all ϕ such that $\delta\phi = 0$.

These 1950 papers had been inspired by an older result of HOPF (inspired in turn by a 1935-36 result of HUREWICZ, viz. that if all the higher homotopy groups are trivial, then the homotopy type depends only on the fundamental group) which now can be reformulated as follows.

(We note that this same result of Hurewicz also inspired WHITEHEAD to showing that if any map between triangulable spaces induces an isomorphism of all their homotopy groups then it is a homotopy equivalence.)

Theorem. *Let* $M(X) \longrightarrow \mathcal{K}(\pi_1(X),1)$ *be the semi-simplicial map which associates to any singular r-simplex* σ *of the minimal complex the function* $\phi_\sigma : \begin{bmatrix} r \\ 1 \end{bmatrix} \longrightarrow \pi_1(X)$ *determined by the 1-skeleton* σ_1 *of* σ. *Then the kernel of this map is the Eilenberg subcomplex* $M_2(X)$ *of* $M(X)$, *while its image is* $K(\pi_1(X),1)$.

So, if all higher homotopy groups are trivial, one has $M(X) \cong K(\pi_1(X),1)$, i.e. one has a complete combinatorial description of the homotopy type of X in terms only of the fundamental group $\pi_1(X)$.

Proof. If σ is in $M_2(X)$ it is constant = * on each 1-simplex of the standard simplex [r], so $\phi_\sigma = 0$. Conversely if $\phi_\sigma = 0$ then the restriction of σ to each 1-simplex of [r] is homotopically trivial, and so, by the minimality of M(X), this restriction must be constant = *.

If $r \geq 2$, and $\{i,j,k\}$, $i < j < k$, is any 2-simplex of [r], then $\phi_\sigma(j,k).\phi_\sigma(i,k)^{-1}.\phi_\sigma(i,j) = 1 \in \pi_1(X)$ follows because the concatenation $\sigma|_{jk}\ \sigma|_{ki}\ \sigma|_{ij}$, i.e. the restriction of σ to the boundary of this 2-simplex, is homotopically trivial. Conversely if $\phi : \begin{bmatrix} r \\ 1 \end{bmatrix} \longrightarrow \pi_1(X)$ is given such that $\phi(j,k).\phi(i,k)^{-1}.\phi(i,j) = 1$, $i < j < k$, then by replacing each class $\phi(i,j)$ by the 1-simplex of M(X) contained in it , we are given a continuous map from the 1-skelton of [r] (this has the homotopy type of a bouquet of 1-spheres) into X which is homotopically trivial. So it extends to a continuous map of [r] into X. By minimality of M(X) this extension can be chosen to be an r-simplex σ of M(X). *q.e.d.*

Theorem. *If* X *has only one nonzero homotopy group, say* $\pi_n(X)$, *then its minimal complex* M(X) *is isomorphic to* $K(\pi_n(X),n)$; *and, somewhat more generally, if* $\pi_n(X)$ *is the only nonzero homotopy group in dimensions less than* j, *then the* j*-skeleton of* M(X) *is isomorphic to that of* $K(\pi_n(X),n)$.

Proof. An obvious generalization of the above argument. *q.e.d.*

REMARK. For an X as above the homology groups in dimensions less than j thus depend only on the group $\pi_n(X)$. Hopf raised the question whether the next, i.e. the jth, homology group of X depended only on $\pi_n(X)$ and $\pi_j(X)$? We'll see in (4.10) that the answer is "no" and that to determine this homology group, one needs to know, besides these two homotopy groups, a certain characteritic class (a "k-invariant") of X.

(4.9) Equivariant homology. Even though we have *explicit complexes* K(G,n) to calculate the (co)homology of spaces having only one nonzero homotopy group, the *actual computation* is not that easy, so (for the moment) we'll only make some remarks about

The case n = 1. (Later we'll take up the case n ≥ 2, and also tie-up the cohomology of the $K(\pi,n)$'s with *cohomology operations.*) Consider the (possibly infinite dimensional) *closed simplex* G whose vertices are all the group elements of G. Thus the **words** (i.e. finite sequences of elements) of G identify with the semi-simplicial complex G_{assoc}. Note now that the left translations L_g of the group G are simplicial maps of the simpicial complex G. We'll say that a chain c of $C_*(G_{assoc})$ is **left-invariant** if $(L_g)_*(c) = c\ \forall\ g \in G$. We'll denote the sub chain complex of left-invariant chains by $C_*^G(G_{assoc})$.

Theorem. $C_*^G(G_{assoc}) \cong C_*(K(G,1))$.

Proof. Each class of left equivalent words of length r+1 has a unique representative of the type $1g_1g_2\dots g_r$. We associate to it the function $\phi : \begin{bmatrix} r \\ 1 \end{bmatrix} \longrightarrow G$ which obeys $\phi(i-1,i) = g_1$ and $\delta\phi = 1$. Note that such a function is uniquely determined because of the latter condition which reads $\phi(a_1,a_2)(\phi(a_0,a_2))^{-1}\phi(a_0,a_1) = 1$ i.e. $\phi(a_0,a_2) = \phi(a_0,a_1)\ \phi(a_1,a_2)$ for all $0 \le a_0 < a_1 < a_2 \le r$, and obviously any such function is associated to such a word. It is easy to check that it also commutes with the boundary maps. *q.e.d.*

On setting $\Phi_{ij} = \phi(i,j)$ we can identify $K(G,1)$ with the **matric complex** of G (this is the original 1943 definition of Eilenberg-Maclane) i.e. the semi-simplicial complex whose r-simplices are $r\times r$ matrices Φ over G obeying $\Phi_{ij}\Phi_{jk} = \Phi_{ik}$, with the sth face being the $(r-1)\times(r-1)$ matrix obtained by deleting the sth row and column. On the other hand the complex on the left side of the above theorem is called the **homogenous complex** of G.

The bijection used in the above proof shows also that this complex is isomorphic to the **non-homogenous complex** of G, i.e. one whose r-simplices are all words $g_1g_2\dots g_r$ of length r (*not* r+1) of G, with the r+1 faces of such an r-simplex being $g_2\dots g_r$, $(g_1g_2)g_3\dots g_r$, ... $,g_1..(g_ig_{i+1})..g_r$, ... , $g_1\dots g_{r-1}g_r$, and $g_1\dots g_{r-1}$.

REMARK. As we'll see in Chapter 5 these *Hochschild faces* arise naturally when one considers the *semi-simplicial complex of a category*. Also we'll check that for any associative algebra its *Hochschild homology,* and so in particular the **homology of the group** G (i.e. the homology of any of the above complexes), can be interpreted as a *derived functor* (2.20). This yields many computational tricks (see e.g. the book by K.S.BROWN), however here we want to go in another direction.

Exercise. Give the non-homogenous description of the cochain complex of $K(\pi,1)$ with local coefficients (in some Abelian group G).

Exercise*. Give a non-homogenous description of $K(\pi,n)$, $n \ge 2$, by considering as r-simplices all functions from n-simplices of T_r to G; with T_r being now a suitable sequence of n-**trees** T_r on r+1 vertices.

We now consider the obvious generalization of the homogenous complex to *any* simplicial complex K equipped with *any* G-**action** : we define its **equivariant homology** to be that of the subcomplex $C^G_*(K_{assoc}) \subseteq C_*(K_{assoc})$ of chains c such that $g_*(c) = c \; \forall \; g \in G$.

Furthermore we'll also sometimes consider equivariance with respect to a non-trivial *representation of* G *on the coefficients*. For example in the simple case of $G = \mathbb{Z}/2$ and coefficients $\mathbb{Z}$, one should consider *both* the subcomplexes $C^{\pm}_*(K_{assoc})$ of $C_*(K_{assoc})$ defined respectively by $g_*(c) = \pm\, c \; \forall \; g \in G$, and thus consider not one but *two* equivariant homologies $H^{\pm}_*(K)$.

If we assume further that this $\mathbb{Z}/2$-action is **free**, i.e. that $\nu \neq 1$ implies $\nu(\sigma) \neq \sigma \; \forall \; \sigma \in K$, then clearly id $\mp \nu_*$ are surjections of C(K) onto $C_{\mp}(K)$, and so we have the two short exact sequences of chain complexes,

$$0 \longrightarrow C_{\pm}(K) \longrightarrow C(K) \longrightarrow C_{\mp}(K) \longrightarrow 0,$$

whose long exact sequences are called the two **Smith-Richardson sequences** of K. We'll now use these to prove the following, which was conjectured by ULAM, and proved by Eilenberg's teacher, BORSUK, in 1933.

Borsuk-Ulam Theorem. *There exists no continuous map from a sphere to a lower dimensional sphere which commutes with their antipodal actions.*

Proof. Concatenating alternately the connecting homomorphisms of the two Smith-Richardson sequences of a free $\mathbb{Z}/2$-space (defined just as for a $\mathbb{Z}/2$-complex) one gets its **characteristic classes** o, one in each dimension.

If there were a continuous $\mathbb{Z}/2$-map from S^n to S^m, then the nth characteristic class of S^m must pull back to the nth characteristic class of S^n. But if m is smaller than n the former class is zero, and so can not possibly image onto the latter which (because of the vanishing of the groups $H_*(S^n)$ occuring in the Smith Richardson sequence of S^n) is nonzero. *q.e.d.*

The above proof illustrates that, just as some results of ordinary ($G = 1$) homotopy theory (e.g. the Brouwer fixed point theorem) are proved using ordinary homology theory, likewise, equivariant homology theory serves as a tool in equivariant homotopy theory (4.11).

Exercise. Interpret the "characteristic classes" occuring in the above argument as characteristic classes (see 4.3) of the $\mathbb{Z}/2$-covering space $X \longrightarrow X/ /2$.

The Borsuk-Ulam Theorem was probably the first *really striking and basic* fact of which there is (apparently !) no hint in Poincaré's magnum opus, *Analysis Situs.* This result innaugurated equivariant homotopy theory, which supplies us with many delicate invariants of the *homeomorphism type* (as against homotopy type) of the space; also it has numerous applications to many diverse parts of mathematics (see 4.11). Like all basic theorems, the Borsuk-Ulam Theorem comes in numerous guises, e.g. the ones given in the next two exercises.

Exercise. Show that the Borsuk-Ulam Theorem is equuivalent to saying that, given any continuous map f from S^n to $\mathbb{R}^n$, there must exist an antipodal pair $\pm x$ of points of S^n such that $f(x) = f(-x)$.

Exercise. Another reformulation : "Given any covering of S^n by $n+1$ open subsets U_i there must be an antipodal pair $\pm x$ which is contained in the same U_i".

The next exercise shows that for many existence and classification questions, there is no loss of generality in considering (as we'll do in 4.11) only free actions.

Exercise. Show that there is a 1-1 correspondence between the G-homotopy classes of G-maps $X \longrightarrow Y$ and G-homotopy classes of G-maps $X \times G \longrightarrow Y \times G$. Here the products are assumed equipped with the **diagonal action** $g(p,h) = (gp,gh)$.

Exercise. Show that for any free $\mathbb{Z}/2$-complex K there is a natural spectral sequence going from the direct sum of the two equivariant homologies, to the ordinary homology of K, and that this spectral sequence is equivalent to the two S-R sequences of K. (The spectral sequence degenerates if one takes field coefficients with characteristic $p \neq 2$.)

Exercise*. Is there a natural spectral sequence (at least for finite G), with an initial term given by suitably fitting together all (as one runs over all representations) equivariant homologies, and converging to the ordinary homology of K ? (Looking at *Brauer Theory* literature should help.)

We turn now back to the unfinished task of getting a structure theorem for M(X) in full generality.

(4.10) Self-obstruction classes, Postnikov's theorem. In general homotopy type is *not* determined just by the homotopy groups even if one takes into account the action of π_1 on them.

We saw e.g. in (4.9) that if the only nonzero homotopy groups are π_i and π_j, $i < j$, then all homology upto dimension $j-1$ is determined by just π_i; however the jth homology is not determined by π_i and π_j .

Example (EILENBERG-MACLANE, *T.A.M.S.*).

Continuing the study of this case in their 1950 *Annals* paper EILENBERG-MACLANE showed that the jth homology is determined by π_i and π_j and a certain *characteristic class.* In fact they almost worked out (see § 4 of their paper) the entire combinatorial structure of the minimal complex (and so the homotopy type of X) for the case of two non-trivial homotopy groups.

Working to a large extent independently of them, POSTNIKOV soon (announcement 1951, paper 1955 = *A.M.S.Translations*, no. 7, 1957) thereafter obtained the full structure theorem of the minimal complex in terms of the homotopy groups (together with the action of π_1 on them) and some characteristic classes of the same type. (This same result was also obtained by ZILBER, unpublished.) Before stating his result we need some definitions.

POSTNIKOV QUOTIENTS $P_n(X)$. These are nothing but the quotients of $M(X)$ by the Eilenberg subcomplexes $M_n(X)$, i.e. the r-simplices of $P_n(X) = M(X)/M_n(X)$ are obtained by identifying r-simplices σ and θ of $M(X)$ whenever their (n-1)-skeletons σ_{n-1} and θ_{n-1} coincide. We note that below dimension n the simplices of $P_n(X)$ are the same as that of $M(X)$ (so knowing $M(X)$ is equivalent to knowing $P_n(X)$ for all n).

More formally the quotient map $p : M(X) \longrightarrow P_n(X)$ sets up a simplicial isomorphism of the (n-1)-skeleton of $M(X)$ with that of $P_n(X)$. Inverting this, and further *choosing*, above each n-simplex of $P_n(X)$ an n-simplex of $M(X)$, we get a **partial section** s of p above the n-skeleton of $P_n(X)$ (i.e. $p \circ s = id$ on the n-skeleton of $P_n(X)$).

The required characteristic classes measure the obstruction (cf. 4.3) to extending s to a section of p over the (n+1)-skeleton of $P_n(X)$.

DEFINITION OF SELF-OBSTRUCTION CLASSES. Consider the (n+1)-cochain of $P_n(X)$ with local coefficients $\pi_n(X)$ which associates to each (n+1)-simplex $p\theta$ of $P_n(X)$, the element of $\pi_n(X)$ determined by the map from the boundary of the standard simplex [n+1] into X, corresponding to the section $s((p\theta)_n)$ of the n-skeleton of $p\theta$.

This cochain $k^{n+1} \in C^{n+1}(P_n(X),\pi_n(X))$ *is a cocycle and its cohomology class is independent of the partial section* s *of the map* $p : M(X) \longrightarrow P_n(X)$ *used in its definition. Its cohomology class* $\mathbf{k}^{n+1} \in H^{n+1}(P_n(X),\pi_n(X))$ *is zero iff after modifying* s *on the* n-*simplices we can extend to a section of the* (n+1)-*skeleton on* $P_n(X)$.

We omit the verification of the above since it is entirely similar to the arguments of (4.3).

We now turn to showing how these **Eilenberg-Maclane Postnikov characteristic classes** (together with the homotopy groups, and the action of π_1 on these groups) give the complete structure of $M(X)$ and thus characterize the homotopy type. The key point is the following generalization of Hopf's Theorem (4.8).

Theorem. *Let* $k \in C^{n+1}(P_n(X), \pi_n(X))$ *be the obstruction to extending a chosen section* s *of* $p : M(X) \longrightarrow P_n(X)$ *defined on the n-skeleton of* $P_n(X)$ *to the (n+1)-skeleton of* $P_n(X)$. *Then the kernel of the map* $M(X) \longrightarrow P_n(X) \times \mathcal{K}(\pi_n(X), n)$ *defined by*

$$\sigma \longmapsto (p\sigma \ , \ [\sigma_n - s(p\sigma)_n])$$

is $M_{n+1}(X)$, *while its image consists of all pairs* $(p\sigma, \phi)$ *such that for any* $a = \{a_o, \ .. \ , a_{n+1}\} \subseteq \{0, \ 1, \ \ldots \ , \dim\sigma\}$, *one has*

$$k(p\sigma)_a + \delta_{p\sigma}\phi(a) = 0.$$

Here the notation "$\delta_{p\sigma}$" (instead of just δ as in 4.8) means that $\pi_n(X)$ are *local* coefficients assigned to the vertices i of the standard simplex [r], with these coefficients being "dragged-back" along each directed edge {i,j}, as per the action of $[p\sigma|_{ij}] \in \pi_1(X)$ on $\pi_n(X)$.

Proof. The verification that the kernel is M_{n+1} is again trivial (cf. 4.8).

Turning next to the image of any (n+1)-simplex σ, we note that $k(p\sigma)$ added to the value of $[\sigma_n - s(p\sigma)_n]$ on the n-skeleton of [n+1] must be zero because $[s(p\sigma)_n] + [\sigma_n - s(p\sigma)_n] = [\sigma_n]$, i.e. the homotopy class determined by restricting σ to the boundary of [n+1]. Likewise the required condition holds for images of σ's of dimension $\geq$ n+2.

Conversely let $\phi : \begin{bmatrix} r \\ n \end{bmatrix} \longrightarrow \pi_n(X)$ satisfy the given condition $\delta_{p\sigma}\phi = 0$ with respect to some r-simplex $p\sigma \in P_n(X)$. Just as in 4.8 we replace each class occuring in the image of ϕ by the n-simplex of M(X) contained in it, to get a a continuous map from the n-skeleton of [r] (this has the homotopy type of a bouquet of n-spheres) into X. The required condition now says that if we *add to it* the map corresponding to the section $s(p\sigma)_n$ of the n-skeleton of $p\sigma$, then we get a homotopically trivial map of this n-skeleton into X. We extend this trivial map to a map of the entire standard simplex [r] into X, and further, by minimality, ensure that this extension is in M(X). Then $(p\sigma, \phi)$ is the image of this r-simplex of M(X). *q.e.d.*

TWISTED CARTESIAN PRODUCT (cf. the "augmentations" of Postnikov, p. 70). The subsemi-simplicial complex of $P_n(X) \times \mathcal{K}(\pi_n(X),n)$ determined by the condition of the above theorem will be denoted by $P_n(X) \times_{\mathbf{k}} K(\pi_n(X),n)$. This notation is justified by the following.

Exercise. Show that upto semi-simplicial isomorphism this is independent of the cocycle k used, i.e. that it depends only on π_n, the action of π_1 on π_n, and the characteristic class **k**.

REMARK. Considered as a continuous map the projection $P_n(X) \times \mathcal{K}(\pi_n(X),n) \longrightarrow P_n(X)$ onto the first factor is a fibration with fiber $K(\pi_n(X),n)$ having only the nth homotopy group nonzero. Thus above theorem says that the constant map $X \longrightarrow *$ has a **Postnikov factorization** into a sequence of such simple fibrations. Recall also that if the fiber has only the nth homotopy group nonzero then the fibration is characterized by an (n+1)-dimensional cohomology class of the base with coefficients in this group : these are precisely the classes **k**.

COMBINATORIAL STRUCTURE OF THE MINIMAL COMPLEX. This now follows at once as an immediate corollary of the above result.

$$M(X) \cong K(\pi_1(X),1) \times_{\mathbf{k}^3} K(\pi_2(X),2) \times_{\mathbf{k}^4} K(\pi_3(X),3) \times_{\mathbf{k}^5} \ldots .$$

That is, M(X) is the product of the Eilenberg-Maclane complexes of its homotopy groups π_i, with products *twisted* as per the action of π_1 on the π_i's, and the self-obstruction classes **k**.

REMARK. For the case which Eilenberg-Maclane considered in their 1950 paper M(X) is thus the twisted product of two Eilenberg-Maclane complexes. The groups "E" which occur in their computation of the homology of such spaces form the E_2 term of the Serre spectral sequence of this twisted product. We remark also that the minimal complex is essentially $\ker d_0$ where $d_0 \colon E_0 \longrightarrow E_0$ is the zeroth differential of the Eilenberg spectral sequence (4.4) determined by the chosen base point.

(4.11) Equivariant homotopy. Given a (unless otherwise specified, *finite*) group G consider the category of spaces (or complexes etc.)

equipped with a **free G-action** and all maps between them which commute with their actions. The usual notions (e.g. homotopy) have now their G-counterparts, and one can pose the G-analogues of the usual (i.e. G = 1) problems, say one can ask for a definition of the minimal G-complex of a G-space and questions re its structure ...

The object of this section is to emphasize that these are not idle generalizations and lead to very powerful topological invariants.

The point is that ordinary homology and homotopy only provide us with invariants of the *homotopy type* of a space, so even to distinguish between say $\mathbb{R}^2$ and $\mathbb{R}^3$ and Y and a pt (which are all contractible spaces) one has to resort to some more or less *ad hoc tricks* (e.g. we can distinguish between the homotopy types of $\mathbb{R}^2 \setminus$ pt and $\mathbb{R}^3 \setminus$ pt and this obviously implies that $\mathbb{R}^2$ cannot be homeomorphic to $\mathbb{R}^3$). (Actually this particular "trick", of using say *local (co)homology*, gives some very general results also : see Chapter 5.)

What we clearly need is *a method* for defining lots (enough hopefully to eventually characterize the p.l. type of K analogously to the characterization in 4.10 of its homotopy type) of finer, and if possible, *computable* topological invariants. It was shown by WU that equivariant homotopy theory gives such a method.

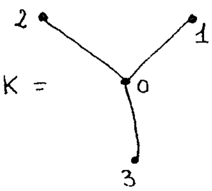
EXAMPLE. To understand Wu's general result (see below) we'll first look at a perfectly trivial problem, viz. that **the letter Y does not embed in** $\mathbb{R}$, but deliberately do it in the following non-trivial, but perfectly general — *that's the point !* — way :

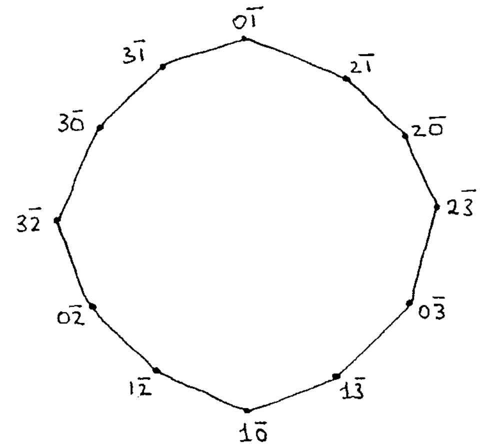
(1) If there were a continuous one-one map from Y into $\mathbb{R}$ then there would be one from $Y \times Y \setminus$ diag to $\mathbb{R}\times\mathbb{R} \setminus$ diag *which commutes with the* **switching action** $(x,y) \longrightarrow (y,x)$ *of* $\mathbb{Z}/2$ *on these spaces*. (2) Next we note that $\mathbb{R}\times\mathbb{R} \setminus$ diag has the $\mathbb{Z}/2$-homotopy type of S^0.

Exercise. More generally show that $\mathbb{R}^n\times\mathbb{R}^n \setminus$ diag has the $\mathbb{Z}/2$-homotopy type of the antopodal (n-1)-sphere.

(3) Triangulate Y using four vertices and three edges. Consider

now the **deleted product** of this simplicial complex K, i.e. the subcomplex K_* of K×K consisting of all cells $\sigma\times\theta$ with σ and θ disjoint simplices of K. A straightforward computation (see fig. below) shows that $(Y)_*$ *is a* 12-*vertex polygon* equipped with the antipodal action.

K = 



(4) We can now obtain the desired contradiction because (by Borsuk-Ulam, no less !) there is no $\mathbb{Z}/2$-map from S^1 (the above polygon) to S^0. So Y does not embed in $\mathbb{R}$.

Now it seems that we were "lucky" (!) in finding just the right thing, viz. the small and computable K_* in the much bigger space Y×Y\diag. However the fact is that K_* and its big brother $|K|\times|K|$\diag must always sink or swim together by virtue of the following.

Wu's Theorem. *For any simplicial complex* K *the* $\mathbb{Z}/2$-*homotopy type of its deleted product* K_* *is the same as that of the* $\mathbb{Z}/2$-*space* $|K|\times|K|$\diag.

We'll give a very elegant argument for this later, for the moment the reader should at least check the following.

Exercise. Let K be the 4-vertex triangulation of Y, then there is a deformation retraction of Y×Y\diag onto K_*.

Corollary. *The* $\mathbb{Z}/2$-*homotopy type of* K_* *is a* topological invariant *of* K.

However, as the contractible Y shows, this $\mathbb{Z}/2$-homotopy type is *not* a homotopy invariant of K. Thus we have found a general way of defining computable and finer topological invariants. Moreover this method **generalizes to all finite groups** G (we leave to the reader the task of formulating the definition of G-fold deleted products of K and formulating analogues of the above theorem). Thus we have a whole host of topological invariants of K, so many in fact that it may be that they suffice to characterize the p.l.type of K ?

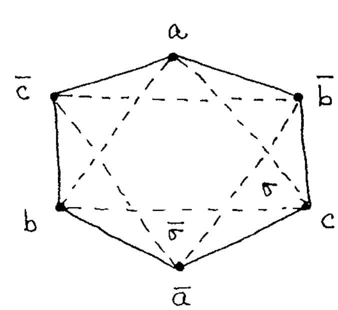
As our second example of the above methodology we'll now prove something more substantial than the non-embeddability of Y in $\mathbb{R}$.

Van Kampen-Flores Theorem. *The* n*-skeleton of a* (2n+2)*-dimensional simplex does not embed in* (2n)*-dimensional space.*

We remark that VAN KAMPEN (see *Abhand. Math. Sem.* of 1932) proved this result *before* Borsuk's paper, and in fact the Borsuk-Ulam theorem follows easily from the methods of Van Kampen's paper. (Indeed the argument which we gave in 4.9 can be considered a modern slick version of this, Borsuk's proof was different.) On the other hand the proof given below (of FLORES, *Abhand.* 1933) uses Bursuk's theorem.

Proof. For this we consider the **deleted join** $K_{\#}$ of a simplicial complex K, i.e. the largest subcomplex of the join K.K of two disjoint copies of K on which the switching $\mathbb{Z}/2$-action is free.

Now let $K = \sigma_n^{2n+2}$, the n-skeleton of a (2n+2)-simplex σ. We leave it as an *Exercise* (see picture below for the case n = 0) that $K_{\#}$ consists of the faces of $\text{conv}(\sigma \cup \bar{\sigma}) \subset \mathbb{R}^{2n+2}$, where σ is a geometrically symmetric (2n+2)-simplex, and $\bar{\sigma}$ its reflection throught its centroid, and that the switching action now coincides with the antipodal action.



If a K embeds in $\mathbb{R}^{2n}$, then by taking the join of the embedding with itself it follows that there will be a $\mathbb{Z}/2$-map from $K_\#$ to the $\mathbb{Z}/2$-subspace $\mathbb{R}^{2n}\#\mathbb{R}^{2n}\setminus$diag (here "diagonal" means all points $\frac{1}{2}.x + \frac{1}{2}.x$, $x \in K$) which can be easily checked (cf. a similar *Exercise* above for deleted products) to be the antipodal 2n-sphere. So by Borsuk-Ulam σ_n^{2n+2} can not embed in $\mathbb{R}^{2n}$. *q.e.d.*

REMARK. More generally one can ask for optimal conditions on r, s, t, and p, which guarantee that all maps of σ_s^r in $\mathbb{R}^t$ have a p-tuple point: for some such results see my paper in the *Proc. A.M.S.* of 1991. Most of the properties of the deleted product (e.g. Wu's theorem) carry over to deleted joins, and from the *combinatorial* viewpoint the $K_\#$'s are more natural (while from the *topological* viewpoint the K_*'s are more convenient). We'll see later in this chapter that the K's for which $K_\#$ is a sphere (e.g. the skeleton σ_n^{2n+2} considered above) are of great importance in equivariant homotopy theory. Similarly in Chapters 5 and 6 we'll make use of some "deleted functors".

(4.12) Pigeon-hole theorems. The following result was conjectured by M.KNESER in 1955 and proved by LOVASZ in 1978.

Theorem. *If* $n > m + 2s$ *then any coloring of the* s-*dimensional faces of an* n-*simplex by* m+1 *colors must result in two disjoint* s-*faces getting the same color.*

Proof (from my paper in *Jour.Combin.Theory (B)* of 1990). Let

$$f : \{\text{s-faces } \alpha \text{ of } \sigma^n\} \longrightarrow \theta^m$$

be such that $f(\alpha_1) \neq f(\alpha_2)$ whenever $\alpha_1 \cap \alpha_2 = \emptyset$. Then we have the $\mathbb{Z}/2$-monotone (*not* simplicial) map

$$f_\# : (\sigma_n^n)_\# \longrightarrow (\theta_m^m)_\# , \text{ where}$$

$$f_\#(\lambda,\mu) = (\{\text{colors of all } \alpha\text{'s in } \lambda\},\{\text{colors of all } \alpha\text{'s in } \mu\}).$$

We note that $f_{\#}(\lambda,\mu)$ is nonempty iff (λ,μ) belongs to the $\mathbb{Z}/2$-subposet $(\sigma_n^n)_{\#} \setminus (\sigma_{s-1}^n)_{\#}$ of $(\sigma_n^n)_{\#}$. So applying the subdivision functor we get a $\mathbb{Z}/2$-simplicial map $sd(f_{\#})$ from the $\mathbb{Z}/2$-simplicial complex $sd((\sigma_n^n)_{\#} \setminus (\sigma_{s-1}^n)_{\#})$ to the $\mathbb{Z}/2$-simplicial complex $sd((\theta_m^m)_{\#})$.

Since $sd((\sigma_n^n)_{\#})$ is contained in the join of $sd((\sigma_n^n)_{\#} \setminus (\sigma_{s-1}^n)_{\#})$ and $sd((\sigma_{s-1}^n)_{\#})$ we can thus obtain, by taking the join of $sd(f_{\#})$ with the identity map of $sd((\sigma_{s-1}^n)_{\#})$, a $\mathbb{Z}/2$-simplicial map from $sd((\sigma_n^n)_{\#})$, the derived of the n-dimensional **octahedral sphere**, to the join of $(\theta_m^m)_{\#}$ and $sd((\sigma_{s-1}^n)_{\#})$. This latter $\mathbb{Z}/2$-complex is at most (m+2s)-dimensional. So, by working symmetrically up its skeletons, we can find a $\mathbb{Z}/2$-map from it to the antipodal (m+2s)-sphere.

Thus, starting from our coloring f, we have now got a $\mathbb{Z}/2$-continuous map from the antipodal n-sphere to the antipodal (m+2s)-sphere. So, by Borsuk-Ulam, we must have $n \leq m+2s$. *q.e.d.*

Exercise. Show that the above bound is the best possible.

REMARK. Note that for $s = 0$ the above result gives us the well-known **pigeon hole principle** (of the great DIRICHLET, no less !), viz. that if N (= n+1) pigeons must live in M (= m+1) holes, where N > M, then some two pigeons must share the same hole. (I am grateful to Oged Schramm for pointing out that we have thus obtained a proof of this *deep* result of Dirichlet by *only* using the Borsuk-Ulam theorem !) We remark that Lovasz's proof of the above theorem also used Borsuk-Ulam but was much more complicated. For some other applications of the above methodology (including a more involved pigeon-hole conjecture of ERDOS) see my paper cited above. Also see my paper in the *Illinois J.* of 1989, for a generalization for colorings of s-simplices of *any* simplicial complex K^s which is non-embeddable in a given Euclidean space.

By a **circuit** of a simplicial complex K is meant a simplex $\sigma \subseteq$ vert(K) which is itself *not* in K, but all of whose proper faces are in K. This notion is very useful for the case when K is a **matroid**, i.e. when, for each $S \subseteq$ vert(K), the subcomplex K_S consisting of all simplices of K contained in S, is *dimensionally homogenous*.

Exercise. Show that if m+1 colors are assigned to the circuits of a matroid on N vertices in such a way that no two disjoint circuits have the same color then we must have $N-1 \le \dim(K_{\#})+m+1$.

We now turn to a quite different (inasmuch as it is essentially *infinitistic*) generalization of the pigeon-hole principle. This was found in 1935 by RAMSEY (and since then has been re-discovered repeatedly, *e.g. by yours truly* : see 5.1.1 of my paper in the *T.A.M.S.* of 1983). (The 1980 book by GRAHAM-ROTHSCHILD-SPENCER is a good introduction to the subject spawned by Ramsey's discovery.)

Theorem. *For any coloring of the* s*-faces of an infinite simplex by* m+1 *colors there is an infinite face whose* s*-faces have the same color.*

Proof (cf. G-R-S, p.147). This time we'll use induction on s, the case s = 0 being obvious (infinitistic pigeon-hole principle). For the inductive step we'll associate, to our coloring

$$f : \{\text{s-faces } \alpha \text{ of } \mathbb{N}\} \longrightarrow \theta^{m}$$

the coloring

$$f_{\mathcal{U}} : \{\text{(s-1)-faces } \alpha \text{ of } \mathbb{N}\} \longrightarrow \theta^{m} ,$$

uniquely determined by the requirement that the set of all s-faces θ which contain an (s-1)-face σ and have color $f_{\mathcal{U}}(\sigma)$ belongs to a (chosen) *non-trivial ultrafilter* $\mathcal{U}$ on the set of all s-faces of $\mathbb{N}$.

(We recall that a **filter** $\mathcal{U}$ on a set is a class of nonempty subsets closed with respect to intersections and supersets. It is called an **ultrafilter** if given any subset, either it or its complement (but not both) is in $\mathcal{U}$. If further no finite subset is in $\mathcal{U}$ then it is called **non-trivial**. These definitions were invented by H.CARTAN for use in BOURBAKI's general topology text. *Exercise.* Show that the Axiom of Choice implies the existence of a principal ultrafilter $\mathcal{U}$ on any set.)

We now use the inductive hypothesis that there is a face A *in* $\mathcal{U}$ such that for all (s-1)-faces $\sigma \subset A$ one has the same $f_{\mathcal{U}}(\sigma)$ = a say. Now let $B \subseteq A$ consist of all vertices v such that $f(\sigma.v) = a$ for all $(s-1)$-faces on vertices less than v. Then B is in $\mathcal{U}$ and such that for all s-faces $\theta \subset B$ one has $f(\theta) = a$. *q.e.d.*

Addendum. *For each* r *we can find, within each sufficiently big finite subset of* $\mathbb{N}$, *an* r-*face all of whose* s-*faces have the same color.*

This corollary can be obtained by a compactness argument (cf. pp. 13-15 of G-R-S) akin to that which you might use to do the following.

Exercise. Deduce, from the four color theorem (for finite planar graphs) the four color theorem for infinite planar graphs.

REMARK. It should be borne in mind that the above is only a neat reformulation, using topological language, of the original inductive arguments of RAMSEY (G-R-S, pp.19-20, 16, 7-9). However this reformulation suggests that it has probably even something in common with the Borsuk-Ulam argument we used for the Lovasz-Kneser theorem ? Indeed later, while analyzing minimal $\mathbb{Z}/2$-complexes in (4.16), we'll use the notion of a **self-dual simplicial complex**, which is somewhat like that of (the class of sets *not* lying in) an ultrafilter.

Exercise (from 1994 Hong Kong Math Olympiad). Find a set A of numbers such that in any infinite set S of primes, one can find for some $k \geq 2$, k primes whose product is in A, and k other primes whose product is not in A.

Optional exercises : do also the other five 1994 IMO questions ! (I won't let modesty get in the way of my telling you that *I got them all*, though admittedly at somewhat less than gold-medal-winning speed.)

To see the relation of the exercise stated above to what we have been doing it is best to regard $\mathbb{N}$ **as a semi-simplicial complex** : namely, using the fundamental theorem of arithmetic, I'll identify $\mathbb{N}$ with K_{comm}, where $K = \overline{\mathbb{P}}$, the (infinite) simplicial complex whose simplices are all finite nonempty sets of distinct primes. (It is possible that this construction — e.g. the homologies of some subcomplexes of K — might be of independent number-theoretical interest ?)

Reformulation of above IMO exercise. There exists a 2-coloring of $\overline{\mathbb{P}}$, i.e. of *all* the finite subsets of $\mathbb{P}$, such that any infinite set $S \subseteq \mathbb{P}$ contains two simplices of the same size having different colors. (So the IMO problem roughly says that *Ramsey's theorem is "best possible"*.)

AN AMUSING STORY (and what led to this lecture of 21.11.94). Two weeks ago, when I wrote down this IMO problem for Dharam Bir (at that time I had not connected it with Ramsey theory, and so had no idea that it is on p. 162 of the G-R-S book) I inadvertently replaced "for some $k \geq 2$" by "for all $k \geq 2$"! I noticed this slip only yesterday when Dharam Bir asked me to show him my solution. Still, even the "new problem" (i.e. whether there was an A having this stronger property) appeared interesting : *and sure it was*, because I realized, after reformulating it on $\overline{\mathbb{P}}$, that I had once again, and somewhat serendipitiously this time, rediscovered Ramsey's theorem !

REMARK. Starting with RAMSEY himself, the most striking applications of Ramsey theory are probably in **logic** (i.e. the branch of mathematics inspired by the *form* of mathematical arguments). For example, PARIS and HARRINGTON found a statement (see p.150 of G-R-S) in **Arithmetic** (the formal language of PEANO et al. in which all number theoretical proofs can be formalized) which is a corollary of Ramsey's theorem (this proof can be formalized in "Set Theory") and thus **true**, but which cannot be **proved** formally within Arithmetic. (The previous examples of GODEL are not the kind which one meets in street maths.)

Equally, the fact that Ramsey's theorem is "best possible" is very important in logic. A set P for which there is *no* 2-coloring of the kind mentioned in the reformulated IMO exercise is called an **inaccesible set**. The name is well-deserved, because no known construction can produce such a set, but on the other hand logicians hope to show one day (all power to them !) that the existence of inaccesible sets can be neither proved nor disproved within **Set Theory** (the bigger formal language of VON NEUMANN et al. in which apparently all of what is commonly called mathematics can be formally written).

REMARK. To a topologist Godel's theorem brings to mind the picture of a **space "Arithmetic"**, whose topology reflects somehow the permissible formal deductions of the language, with the theorem in question being : *this space "Arithmetic" is disconnected*, i.e. that π_0(Arith) is not trivial. Working with this in mind I defined one such natural topology (see the *Zeit. f. Math. Logik* of 1984) and showed using it that

$$\pi_i(\text{Arith}) = 0 \ \forall\ i \geq 1.$$

Here the logical interpretation of π_1(Arith) = 0 is of course that *"all formal proofs of an assertion are equivalent"* (in the homotopy theoretical sense chosen). It seems that **a homology and homotopy theory of languages** should be very interesting and useful (e.g. the homology of a language *mod* a sub-language can be non-trivial). However, to the best of my knowledge, no contribution other than the one just mentioned, has so far been made in this direction.

(4.13) Extension. Given any (simplicial, semi-simplicial, or complete semi-simplicial) complex K it is natural to refer to the homotopy groups of its (either Giever or Milnor) realization as the homotopy groups of K. This definition is topological, so

Question. *How can we define these homotopy groups directly in terms of the combinatorics of K ?*

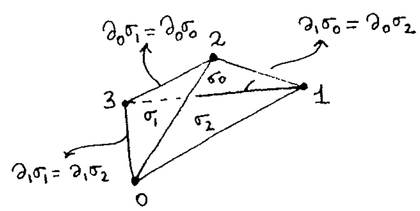
An obvious way of doing this (and in fact of even combinatorially defining the entire "homotopy type" = minimal complex of $|K|$) is indicated at once by *the simplicial approximation theorem* (2.15). It tells us that we don't really need all of the singular complex $S(|K|)$, it is enough to consider the sub c.s.s.c. K_{pl} consisting of all **piecewise linear singular simplices** $|\Delta^q| \to |K|$, i.e. realizations of simplicial maps $sd^n(\Delta^q) \longrightarrow K$, $n \geq 0$. Thanks to this approximation theorem we can now (after having chosen a base vertex *) construct (just as before) the **minimal semi-simplicial complex** M(K) of K within this purely combinatorial object K_{pl}. (More details of the required definitions, esp. that of "homotopic simplices", are given later.)

Note, in the context of the above, that K_{assoc} consists of all simplicial maps $\Delta^q \to K$, $q \geq 0$, and, when vertK is equipped with a total order, K_{comm} consists of all order-preserving simplicial maps $\Delta^q \longrightarrow K$, $q \geq 0$, and K itself can be identified with all order-preserving injective simplicial maps. Thus we have $K \subset K_{comm} \subset K_{assoc} \subset K_{pl} \subset S(|K|)$.

The singular complex K_{pl} is obviously the smallest complex containing K and having the property that it is closed with respect to composition with continuous maps $\Delta^m \longrightarrow \Delta^n$ which are realizations of simplicial maps $sd^r(\Delta^m) \longrightarrow sd^s(\Delta^n)$ for some r and s (these form a category $\mathcal{N}_{pl}$). It is clear how we can similarly enlarge *any* semi-simplicial complex K to the smallest complex K_{pl} satisfying the analogous closure property (i.e. a contravariant functor from $\mathcal{N}_{pl}$).

Exercise. Carefully write down the details of the above purely combinatorial *subdivision process* $K \longmapsto K_{pl}$.

We have thus a combinatorial method $K \longmapsto K_{pl}$ (as against the topological way $K \longmapsto S(|K|)$) of enlarging (without changing the homotopy type of the realization) any s.s.c. K to one which is **extended** in the sense of **Kan**, *Proc.Nat.Acad.U.S.A.*, 1955 : i.e. that if we have q+1 singular q-simplices which fit compatibly (see fig. below) to give a map from the boundary minus one q-simplex of a standard (q+1)-simplex, then this map can be extended to a singular (q+1)-simplex.



Kan showed (see definitions below) how to define homotopy groups of any K obeying his extension condition. (Initially he used the *cubical analogues* of s.s.c.'s; it was **Moore** who first translated his work into s.s.c.'s, we will only sketch these definitions below.) In case K is not extended we will replace it by its combinatorially defined enlargement K_{pl}, which obeys this condition. (For an explicit description of K_{ext}, a smallest subcomplex of K_{pl} containing K and obeying the extension condition, see Kan, *Amer. J.*, 1957.)

Exercise*. Define a subcategory $\mathcal{N}_{\text{ext}}$ of $\mathcal{N}_{\text{pl}}$ such that extended complexes can be identified with contravariant functors $\mathcal{N}_{\text{ext}} \longrightarrow \mathcal{S}ets$.

Before giving the Kan-Moore definitions (which won't be mysterious at all) we want to first look at a non-obvious example of an extended complex. (However note from above figure that any extended complex is sort of "group like"; so this example is not totally unexpected.)

By a **simplicial group** we mean a contravariant functor $\mathcal{N} \longrightarrow$ Groups, i.e. it is a c.s.s.c. with, for each $q \geq 0$, a specified group structure on the set of all q-simplices, and such that all face and degeneracy operators are group homomorphism.

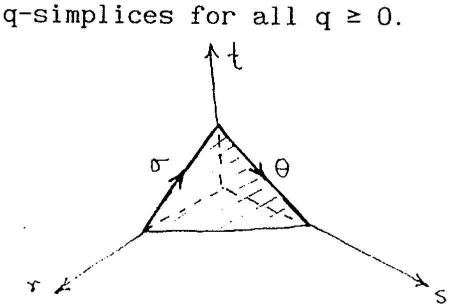
For example let G be a topological group and let K be a sub c.s.s.c. of the singular complex S(G) which is *closed* with respect to the obvious group operations induced on the sets of all q-simplices, $q \geq 0$; then clearly K is a simplicial group.

Conversely if K is a simplicial group its Milnor realization $|K|$ is usually (a slightly technical condition is required to ensure that $|K \times K|$ is homeomorphic to $|K| \times |K|$) a topological group. Thus the example we have given is in fact typical : all simplicial groups are such !

Theorem. *Any simplicial group is extended.*

This was observed by **Moore** in the *Sém. Cartan* of 1955. Since all formal proofs of this (see e.g. May pp. 67-68) tend to be (easy but) non-intuitive, we prefer an informal but vivid argument which is remniscent of how one checks that π_1 of a topological group is Abelian.

Proof. By remarks above it is enough to consider a singular subcomplex K of a topological group which is closed under the induced group operations on q-simplices for all $q \geq 0$.



Suppose we are given two (see above figure) 1-simplices σ and θ with $\sigma(0,1) = \theta(1,0)$ (the "compatibility" condition). Now define a singular 2-simplex F (on the shaded standard 2-simplex) by

$$F(r,s,t) = \sigma(r,t+s).\sigma(0,1)^{-1}.\theta(t+r,s) .$$

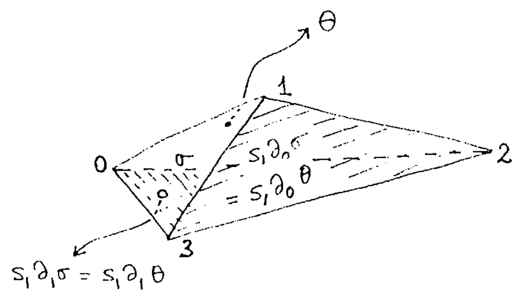
An easy verification shows that F extends the map defined on two arms of our triangle by σ and θ. Similar explicit formulae can be given (*Exercise*) to check Kan's condition for compatible length q+1 sequences of q-simplices for all $q \geq 2$ also. *q.e.d.*

REMARK. In *T.A.M.S.* of 1991, **Loday-Fiedorowicz** look at a generalization of the notion of a simplicial group, called a *crossed simplicial group*. For any such K too the realization $|K|$ is a topological group. It turns out that, upto a small and classifiable ambiguity, crossed simplicial groups are essentially simplicial groups. However this "small ambiguity" is just the thing which is responsible for the cyclic, and some other similar, (co)homologies which have sprung up ever since the famous 1983 *I.H.E.S.* paper of Connes.

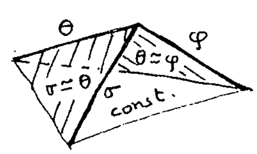
We now give (*toujours* informally !) the aforementioned detailed

DEFINITIONS. To define $\pi_i(K)$ we just mimic *the geometric way* (cf. 4.1) of defining homotopy groups of a space. The key definition is that of **homotopic simplices**. For the singular case this meant (see 4.5) that our q-simplices σ and θ have all faces same and are homotopic *rel* boundary. To mimic this identify each $\sigma^q \in K$ with *the* simplicial map from the c.s.s.c. $\overline{(\Delta^q)}_{comm} = [[q]]$ of all increasing sequences in $\{0,1, \ldots , q\}$ to K, which takes the sequence $01\ldots q$ to σ. (The realization of this $\sigma : [[q]] \longrightarrow K$ is the associated singular simplex $\sigma : \Delta^q \longrightarrow |K|$.) Then $\sigma^q \simeq \theta^q$ will mean that there is a simplicial map $\omega : [[q]] \times [[1]] \longrightarrow K$ which "does'nt move bdry" and on the "ends" equals σ and θ.

To make this more combinatorial-looking we now use the following figure, where the domains of σ and θ are the last two q-faces of a standard (q+1)-simplex, and degenerate singular q-simplices are defined on the other q-1 q-faces in the indicated way (these are constant on the hatched lines). What we want is that this map of the boundary should extend to a singular (q+1)-simplex ω. Then we'll say that ω is a **homotopy** from σ to θ, and write $\omega : \sigma \simeq \theta$. (*Exercise.* Show that this is equivalent to the definition of the last para. Cf. May, § 5.)



The above did *not* use that K is extended. We use this *now* (cf. May, pp. 5-6) to check that *this* $\simeq$ *is indeed a* **transitive** (and so an equivalence) *relation*. Once again we'll be content with a suggestive picture proof leaving the writing of a formal proof as an *Exercise* :



$$\sigma \simeq \theta,\ \theta \simeq \varphi \Rightarrow \sigma \simeq \varphi$$

Now by a **base point** * we'll understand the *subcomplex* generated by some chosen vertex of K. And, as expected, we'll denote by $\pi_q(K)$ the set of homotopy classes $[\sigma]$ of q-simplices σ having all faces in * . (Using other subcomplexes one can likewise define $\pi_q(K,L)$.)

The combinatorial definition of the group operation (cf. May, p.9) on this set is suggested by the homotopy addition theorem. Map all the q-faces of the standard (q+1)-simplex on * , excepting the last one which is mapped as per θ, and the third last which is mapped as per σ. By the extension condition this map extends to a singular (q+1)-simplex, whose value on the second last q-face is deemed to determine $[\sigma].[\theta]$.

Theorem. *For any extended complex* K *the above purely combinatorially defined groups* $\pi_q(K)$ *coincide with the homotopy groups of the realization of* K.

Proof. Using realization we can view K as a singular subcomplex. The fact that K is extended now implies that if a singular simplex of K_{pl} has all its faces in K, then we can find a homotopic simplex which is in K. So by the simplicial approximation theorem each element of any homotopy group of $|K|$ can be represented by a singular simplex of K having all its faces at the base point. The rest is obvious because the definitions above mimicked the topological ones. *q.e.d.*

We note that one also has a combinatorial Milnor uniqueness theorem for "ES" functors on the category of extended complexes. The seven properties of (4.1) and others follow from above result or can be directly checked combinatorially (see book of May).

REMARK. For spaces we also had given a *homological way* of defining homotopy groups via the Eilenberg filtration determined by the chosen base point (see 4.4). This theorem of Eilenberg, i.e. that $E^*_{p,0}(K) \cong \pi_p(K)$ for $p \geq 2$ (for $p = 1$ one gets only the Abelianization of π_1) is still true for any Kan complex. Following Moore, Kan, and Milnor, we'll now give in the next two sections a different but related homological way which yields another, and much more striking, combinatorial definition of homotopy groups (see 4.15). From the computational viewpoint however there is as yet no truly efficient definition. For example it is known that there is an algorithm for computing $\pi_i(S^2)$ for all $i \geq 3$, and also that all these groups are non-trivial, however no one has a clue as to their actual computation.

(4.14) Moore homology. In all of the following K will be a *complete* semi-simplicial complex. So far while defining any of the homologies our starting point has been always the free *Abelian* groups $C_q(K)$ generated by the q-simplices of K. Even though the induced face and degeneracy operators make C_* into an *Abelian simplicial group*, for defining our homologies we have always used only a part of this rich structure. For example the ordinary $H_*(C)$ is the homology of the associated *chain complex* (C_*, ∂) where ∂ is the alternating sum ∂ of the face operators. However, on account of the degeneracies, we can **normalize** our defining chain complex, i.e. make it much smaller.

Theorem. *The homology of the chain complex* (C_*, ∂) *coincides with that of the chain complex whose qth group is* $C_q \cap (\ker \partial_0) \cap \ldots \cap (\ker \partial_{q-1})$ *and whose boundary maps* ∂ *are induced by the last face operators* ∂_q .

We remark that sometimes it is more convenient to replace this **Moore chain complex** with the isomorphic (*Exercise* : check this !) chain complex whose qth group is $C_q \cap (\ker \partial_1) \cap \ldots \cap (\ker \partial_q)$ and whose boundary maps ∂ are induced by the *first* face operators ∂_0 .

Proof (cf. **May**, pp. 94-95). For each $p \geq 0$ let (C_*^p, ∂) be the sub chain complex of (C_*, ∂) consisting of all x such that $\partial_i(x) = 0$ whenever $0 \leq i < \min(p, \dim x)$. The Moore chain complex is clearly isomorphic (upto signs $\pm$ for ∂) to the intersection $\bigcap_p C_*^p$ of this decreasing filtration.

So it suffices to check that each inclusion $C_*^{p+1} \subseteq C_*^p$ induces an isomorphism in homology. This follows because the chain map $f^p : C_*^p \longrightarrow C_*^{p+1}$, which maps x to x unless $\dim x \geq p+1$ when x is mapped to $x - s_p \partial_p x$, is a right inverse, and furthermore, we have $\mathrm{Id} - f^p = \partial t + t \partial$ where t takes x to 0 unless $\dim x \geq p$ when it goes to $(-1)^p s_p x$. *q.e.d.*

REMARK. The above proof shows that the E^1 term of the filtered chain complex (C_*^p, ∂) is zero off the x-axis, and that the Moore chain complex is nothing but the **basic chain complex** $(E^1_{p,0}, d_1)$ of this filtration. As remarked before another way of normalizing was given in the initial 1950 *Annals* paper of **Eilenberg-Zilber**, and consists simply of modding out linear combinations of degenerate simplices (these form a sub chain complex even though degenerate simplices may not form a sub semi-simplicial complex). Are the two normalizations related ? Yes, indeed : **the Moore chain complex complements the sub chain complex of** (C_*, ∂) **generated by the degenerate simplices**. In fact if we denote by F the iterated composition of the chain projections f^p used in the above proof then it is easy to see that C_* is the direct sum of the sub chain complexes $(\mathrm{im} F, \partial)$ and $(\ker F, \partial)$: the former is the Moore subcomplex and the latter that generated by the degenerate simplices.

DEFINITION (Moore, *Sém. Cartan*, 1954-55). The merit of Moore's normalization is that his chain complex *makes sense even for non-Abelian simplicial groups* F_* !! To see this we have to check that in it each $\mathrm{im}\partial$ is a *normal* subgroup of the next $\ker\partial$. We leave this verification (it uses completeness, i.e. degeneracy operators) as an easy *Exercise* (cf. May, p. 69). Thus there are defined the **Moore homology groups** $(\ker\partial)/(\mathrm{im}\partial)$ of F, which we will denote by $H_*^{\mathrm{Moore}}(F)$.

We know from the above theorem that for any Abelian simplicial group C, Moore homology coincides with the homology of the associated chain complex C_* . (*Warning* : the homology *of* C, i.e. of the chain complex $C_*(C)$, is bigger : cf. May, p. 97.) We now turn to the computation of the Moore homology of an arbitrary simplicial group F.

What could $H_*^{\mathrm{Moore}}(F)$ be ? Dare we hope that it is related to the homotopy groups $\pi_*(F)$ (which are defined since, being a simplicial group, F is extended) ? *Magic happens* : the two are the same !!

Theorem. *For any simplicial group* F *we have* $H_*^{\mathrm{Moore}}(F) \cong \pi_*(F)$.

Proof (cf. May, pp. 68-69). Here of course the right side makes sense because we saw in 4.13 that F is extended. The base point * will be the subcomplex consisting of the identity elements.

We note that Moore q-cycles x of F are precisely the same thing as singular q-simplices x of F with all faces in * . We can define a surjection $\pi_q(F) \longrightarrow H_q^{\mathrm{Moore}}(F)$ by $[x] \longmapsto \langle x\rangle$ because if $z : x \simeq y$ then $\partial_{q+1}(s_q x^{-1} z) = x^{-1}y$. To see that this is a group homomorphism it suffices to check $[x][y] = [xy]$ which follows because all the faces of the (q+1)-simplex $(s_{q-1}x)(s_q y)$ are on * excepting the last three which are, respectively, x, xy, and y. Finally this map is injective because $\partial_{q+1}z = x$ implies $z : 1 \simeq x$. *q.e.d.*

REMARK. Thus Moore is giving a homological definition of homotopy groups, but so did **Eilenberg** in his foundational *Annals* 1944 paper (see 4.4) ! Are these two definition related ? We'll see below that indeed they are, which of course is very satisfying, even though it does dim the above "magic" a bit ! However it is not clear if Moore himself was aware of this connection. His immediate inspiration was rather **Thom,** who had guessed (as always correctly !) that the *ith homotopy group of the infinite symmetric product* $S_*(X)$ *is isomorphic to the ith homology group of* X (the proof appeared in his joint *Annals* 1958 paper with **Dold**). Note that the algebraical version of the **Dold-Thom theorem,** i.e. $\pi_i(C(K)) \cong H_i(K)$, follows at once from the two theorems given above. We remark that Dold-Thom proved their topological version by using the Eilenberg-Steenrod uniqueness theorem (2.17) and the fact that the functor S_* converts excisions into quasi-fibrations.

Moore's filtration. We can filter the chain complex $C_*(F)$ of any simplicial group (cf. the proof of the first theorem) thus : ... . We now check easily that $H_*^{\text{Moore}}(F)$ is the x-axis of the E_1 term of the spectral sequence associated to this filtration. On the other hand in 4.4 we had used the **Eilenberg filtration** of $C_*(F)$ which is coarser being defined thus : ... . We saw in 4.4 that the homotopy groups correspond also to the x-axis of the E_1 term of the spectral sequence of this filtration too. Thus the above theorem amounts to saying that the inclusion map induces an isomorphism on the x-axes of the E_1 terms of these two spectral sequences. In fact it is probable that the two spectral sequences have exactly the same spectral sequence from E_1 on.

(4.15) The free group of a simplicial complex. We now have the tool, i.e. Moore homology, to make a *basic* change in our homology defining strategy : we will start, instead of the groups $C_q(K)$, with the *free non-Abelian group* $F_q(K)$ *generated by the* q-*simplices of* K !!

In the following we'll abbreviate $H_*^{\text{Moore}}(FK)$ to $H_*^{\text{Moore}}(K)$; likewise given a subcomplex L of K, *the relative Moore homology* (obvious definition) $H_*^{\text{Moore}}(FK,FL)$ will be abbreviated to $H_*^{\text{Moore}}(K,L)$.

The following result was proved by **Milnor** in a very popular 1956 paper which he did'nt publish ! (It is fortunately now available in **Adam**'s, *A student's guide.*)

Theorem. *For any c.s.s.c.* K *one has* $H_q^{\mathrm{Moore}}(K, pt) \cong \pi_{q+1}(SK)$ *where* SK *denotes the suspension of* K.

Proof. . *q.e.d.*

REMARK. Thus we are now going beyond K_{assoc} (words in vertK supported on K) and looking at all words in vertK *and* $\overline{\mathrm{vert}K}$ (the *inverse vertices*) which are supported on the simplices of K. Since the lower central series of a free group as the free Lie ring as graded group the connection which we'll uncover below with the Poincaré-Birkhoff-Witt theorem is to be expected.

Then we'll end by giving the closely related simplicial group GK of **Kan** which yields the following pleasant definition of homotopy groups :

$$H_i^{\mathrm{Moore}}(GK) \cong \pi_i(|K|) .$$

REMARK. The lower central series of FK and GK play a big rôle in homotopy theory, e.g. (excluding some initial terms) the **Adam's spectral sequence** (which converges towards the homotopy groups and which is one of the principal tools for computing them) arises from this filtration.

Chapter 5.

REALIZATIONS

(5.1) Function algebra $\mathbb{R}[K]$. As in (2.11) we'll think of each vertex v as the function $\mathbb{R}^{\mathrm{vert}(K)}$ $(= \mathbb{R}^N) \longrightarrow \mathbb{R}$ defined by $v(f) = f(v)$, and we denote by $\mathbb{R}[\mathrm{vert}K]$ the graded algebra of all functions obtained from these, and the constant functions $\mathbb{R}$, by pointwise addition and multiplication. Note that an $f \in \mathbb{R}[\mathrm{vert}K]$ can be written uniquely as

$$f = \textstyle\sum_{\sigma \subseteq \mathrm{vert}K} f_\sigma . v^\sigma ,$$

where v^σ denotes the product of functions $\prod \{v \in \sigma\}$, and $f_\sigma \in \mathbb{R}$.

We'll denote by $I(K)$ the ideal of $\mathbb{R}[\mathrm{vert}K]$ consisting of all functions for which the coefficients f_σ are zero whenever σ is in K and by $\mathbb{R}[K]$ the quotient algebra obtained by dividing $\mathbb{R}[\mathrm{vert}K]$ out by $I(K)$. (Alternatively one can think of $\mathbb{R}[K]$ as the graded vector space $C_*(K_{\mathrm{comm}})$ equipped with a multiplication.)

Note that the zero set Lin(K) of I(K) in $\mathbb{R}^N$ contains $|K|$ and even Aff(K) : it is the cone of Aff(K) over the origin. On the other hand the zero set of I(K) in $\mathbb{RP}^{N-1}$ (lines of $\mathbb{R}^N$ through the origin) is the space Proj(K) considered in (2.11). It is these and such-like spaces (e.g. one can replace $\mathbb{R}$ by $\mathbb{C}$ or consider *the complement* of say Aff(K) in $\mathbb{R}^N$ etc.) which we want to consider further in this chapter.

Note that in each case the space in question occurs as the *union* (in case of complements, the *intersection*) of some **spaces which are functorially attached to the simplices** of K. For example $|K|$ arises from the functor $\sigma \longmapsto \mathrm{Conv}(\sigma)$ while Aff(K) is associated to the functor $\sigma \longmapsto \mathrm{Aff}(\sigma)$. We'll use this functorial nature of these spaces to calculate their singular homology (or singular local homology).

Since these spaces originated from a purely combinatorial object K, it makes sense to seek **purely combinatorial defining chain complexes for these homologies** (e.g. for $|K|$ the oriented chain complex, i.e. the normalized chain complex of K_{comm}, gives a neat solution of this problem.) One way of answering this question is to make use of the multiplication of $\mathbb{R}[K]$, i.e. by exploiting the finer **algebraic geometric structure** of these spaces. This gives (à priori) finer invariants of K since they depend on this structure; besides, these algebraic definitions usually work over any field $\mathbb{F}$ or its closure $\overline{\mathbb{F}}$, rather than just $\mathbb{R}$ or $\mathbb{C}$. Another way is to simply give functorial subdivisions of the spaces Sph(K), Proj(K), etc. We'll illustrate both these methods.

(5.2) **Visualizing categories and functors.** We have already used realizations of simplicial complexes, of posets (via their order complexes), and of semi-simplicial complexes (= contravariant functors $\mathcal{N} \longrightarrow \mathcal{S}ets$). A general way of visualizing *any* small category, or a functor defined on it, was given by GROTHENDIECK in 1959, but became popular only after SEGAL's beautiful paper (which makes modestly enough "no great claim to originality") in the *Pub. I.H.E.S* of 1968.

The basic idea is to think of any (always small) category $\mathcal{C}$ as the semi-simplicial complex whose n-simplices are all **trains of morphisms** $(\phi_1, \dots, \phi_n)$ of length n — i.e. ϕ_1 maps into domain of ϕ_2 which maps into domain of ϕ_3 etc. — with the n+1 **faces defined a la Hochschild** : i.e. the first one by dropping the first morphism, the next n-1 by replacing any two successive morphisms by their composition, and the (n+1)th by dropping the last morphism. The degeneracies are defined by insertions of identity morphisms. We define $|\mathcal{C}|$ to be the realization of this semi-simplicial complex. This notion becomes clear once one has a look at the following case.

Example. Consider the elements of a **poset** P as the objects of a category with $Mor(a_1, a_2)$ empty unless $a_1 \le a_2$ in which case it will consist of just one morphism. We defined K_{comm}, where K = sd(P), as the semi-simplicial complex whose n-simplices are increasing sequences $(a_0, a_1, \dots, a_n)$ of length n+1 in P, with the n+1 faces obtained by omissions. Consider now the corresponding train $(\phi_1, \phi_2, \dots, \phi_n)$ of morphisms $\phi_i : a_{i-1} \to a_i$. It is clear that the n+1 faces of our simplex correspond to the n+1 Hochschild faces of this train. Thus the semi-simplicial complex defined above is isomorphic to K_{comm}.

Exercise. Work out $|\mathcal{N}|$ where, as usual, $\mathcal{N}$ denotes the category of non-decreasing maps between the finite sets $[n]$, $n \geq 0$.

Exercise*. Let $\mathcal{C}yc$ be the category of *all* maps between the finite sets $[n]$, $n \geq 0$. What is $|\mathcal{C}yc|$? (This category will be important later.)

To visualize a covariant functor $D : \mathcal{C} \longrightarrow$ *Sets* we consider the semi-simplicial complex whose n-simplices are all pairs (tr,x), where $tr = (\phi_1, \ldots, \phi_n)$ is a train of $\mathcal{C}$, and $x \in D(\mathrm{Dom}\phi_1)$. The first face $\partial_0(tr,x)$ will be the pair $(\partial_0 tr, (D\phi_1)x)$, while for the other faces the second coordinate is just a passenger, i.e. $\partial_i(tr,x) = (\partial_i tr, x)$ (cf. definition of homology with local coefficients). Likewise the degeneracies are merely insertions of the identity morphisms in the first coordinate. We define $|D|$ to be the realization of this semi-simplicial complex. (The definition for D contravariant is similar, or else one can just replace $\mathcal{C}$ by its opposite category.)

Exercise. Check that for the case of a contravariant functor $D : \mathcal{N} \longrightarrow \mathcal{S}ets$ the above definition gives the usual Milnor realization $|D|$ of the semi-simplicial complex D.

For functors $D : \mathcal{C} \longrightarrow \mathcal{S}paces$ (e.g. the functors $\sigma \longmapsto \mathrm{Lin}(\sigma)$ etc. of 5.1) the above definition gives a contravariant functor $\mathcal{N} \longrightarrow \mathcal{S}paces$, i.e. a **semi-simplicial space**, and it will be understood, that while defining $|D|$, we use the topology of the second factor of each (standard n-simplex) $\times$ (*space* of n-simplices), before making the identifications stipulated by the face and degeneracy operators. (Covariant functors from small categories to $\mathcal{S}paces$ are frequently called **diagrams**, in fact that's why we used the letter D.)

Exercise. Let $f : X \longrightarrow Y$ be any continuous map. Consider a poset P, with just two elements $a \leq b$, as a category as in Ex. 1, and let $F : P \longrightarrow \mathcal{S}paces$ be the functor which takes the morphism $a \leq b$ to f. Then $|F|$ is the **mapping cylinder** (see 4.) of f.

Exercise. With f as above now let the poset P have three elements with relations $a \le b$ and $a \le c$, and let the functor F take $a \le b$ to f, and $a \le c$ to a constant map $X \longrightarrow \{pt\}$. Check that now $|F|$ is the so-called **mapping cone** of f.

(5.3) Local singular homology of Lin(K) **at the origin.** We now proceed to a GORESKY-MACPHERSON type formula (see the last chapter of their 1988 book on *Stratifed Morse Theory*) for the local (= relative) singular homology $H_*(X, X\backslash 0)$ at 0 of this contractible space X = Lin(K).

Theorem. *The ith local singular homology of* Lin(K) *at the origin is isomorphic to* $\bigoplus_{\sigma \in K} \bar{H}_{i-\dim\sigma}(Lk_K\sigma)$ (where $\sigma = \emptyset$ is allowed).

The following argument exemplifies a general methodology, due to ZIEGLER-ZIVALJEVIC, *Math. Ann.*, 1993, which can be used to study many other such spaces (e.g. all those of 5.1).

Proof. We denote by $Sph(\sigma)$ the unit sphere at the origin in $Lin(\sigma)$ and by Sph(K) the union of these spheres. Thus Sph(K) is the **link** of Lin(k) (no pun intended !) at the origin and $H_i(X, X\backslash 0) \cong \bar{H}_{i-1}(Sph(K))$.

Step 1. To compare the Grothendieck realization $|Sph|$, of the functor $Sph : K \longrightarrow \mathcal{S}paces$, with the space $Sph(K) = \bigcup_{\sigma\in K} Sph(\sigma)$, we note that the latter is the realization of the trivial functor $\{pt\} \longrightarrow \mathcal{S}paces$, $\{pt\} \longmapsto Sph(K)$. The constant map $K \longrightarrow \{pt\}$ and the inclusions $Sph(\sigma) \subseteq Sph(K)$ give a morphism π from Sph to this trivial functor, and so a surjection $\pi : |Sph| \longrightarrow Sph(K)$. It can be checked (*Exercise*) that this has contractible fibers and "so" is a homotopy equivalence.

(The "so" requires some work — cf. Segal, 4.1, and Z-Z, *op cit.* — and the assertion is not true in general, e.g. Alexandrov's example (2.19) is not homotopy equivalent to its x-projection even though the fibers are clearly contractible. However here we are dealing with CW complexes for which an upward induction on the skeletons will work.)

Step 2. We'll identify $\mathbb{R}^j$, for $j \le N$, with the subspace of $\mathbb{R}^N$ obtained by setting the last $N-j$ coordinates zero. It would be nice now if we could *deform* our diagram Sph continuously to the associated **flag diagram** $\mathrm{Sph}_{\mathrm{flag}} : K \longrightarrow \mathcal{S}paces$, i.e. the functor which maps *all* simplices of the same dimension i to the *same* unit sphere $S^i \subset \mathbb{R}^{i+1}$, and which maps each proper inclusion $\theta \subset \sigma$ to the *constant map* $\mathbf{1} : S^{\dim\theta} \longrightarrow S^{\dim\sigma}$, i.e. with constant value $\mathbf{1} = (1, 0, \ldots, 0)$.

However since we merely want to deform the space $|\mathrm{Sph}|$ to $|\mathrm{Sph}_{\mathrm{flag}}|$, it turns out (cf. Segal, 2.1, and Z-Z) that it suffices to show that there is a natural transformation $H : \mathrm{Sph} \longrightarrow \mathrm{Sph}_{\mathrm{flag}}$ such that each $H(\sigma) : \mathrm{Sph}(\sigma) \longrightarrow S^{\dim\sigma}$, $\sigma \in K$, is a homotopy equivalence.

This H can be defined thus : Choose in the non-singular part of $\mathrm{Sph}(\sigma)$ a small disk. To define $H(\sigma)$ map the centre of this disk to the point $\mathbf{1}$, stretch the rest of the interior of the disk bijectively over $S^{\dim\sigma}\backslash\mathbf{1}$, and map its complement to 1. Clearly $H(\sigma)$ is a homotopy equivalence, and since its restriction to any smaller dimensional $\mathrm{Sph}(\theta)$ is the constant map $\mathbf{1}$, clearly H is a natural transformation.

Step 3. Finally we compute $|\mathrm{Sph}_{\mathrm{flag}}|$. For this note that the set of trains of the poset K starting from σ identifies with $\mathrm{Lk}_K\sigma$, and that our functor assigns to the domain of any such train the space $S^{\dim\sigma-1}$. Also remember that each inclusion $\theta \subset \sigma$ becomes a constant map under the functor. So (cf. Z-Z for more details) $|\mathrm{Sph}_{\mathrm{flag}}|$ has the homotopy type of the wedge $\bigvee_{\sigma\in K} (\mathrm{Lk}_K(\sigma).S^{\dim\sigma-1})$, which implies the result by repeatedly using $H_*(A) \cong H_{*+1}(A.S^0)$. *q.e.d.*

Corollary. *A simplicial complex* K *is Cohen-Macaulay over* $\mathbb{R}$ *iff the local singular homology of* Lin(K) *at* 0 *with coefficients* $\mathbb{R}$ *is zero in all dimensions* $\le \dim K$.

Here, by saying that K is **Cohen-Macaulay** over $\mathbb{F}$, we mean that the reduced homology over $\mathbb{F}$ of each link $\mathrm{Lk}_K\sigma$ lives only in the top most dimension $\dim K - \sigma - 1$.

REMARK. Note from its definition and from above corollary that Cohen-Macaulayness of K is a statement about the "smoothness" of (the affine variety Lin(K) attached to) K. Also note that *it is a purely topological property* of K, but that *the ground field* $\mathbb{F}$ *is important.* For example, a triangulation of $\mathbb{R}P^2$ is not Cohen-Macaulay over $\mathbb{F}_2$, but is Cohen-Macaulay over $\mathbb{R}$. Later on we'll look at equivalent algebraic and algebro-geometric reformulations of this notion. This will at once yield important information about the combinatorics of such K's.

Exercise. We know that $\mathrm{Aff}(K) \simeq |K|$. Verify this again using the Z-Z method, and also check that the analogously defined space $\mathrm{Aff}_{\mathbb{C}}(K) \subset \mathbb{C}^N$ is contractible. (*Hint.* $\bigcap_{\sigma \in K} \mathrm{Aff}_{\mathbb{C}}(\sigma)$ is *nonempty*, e.g. by HILBERT's **Nullstellensatz,** which applies because $\mathbb{C}$ is algebraically closed.)

Exercise. Show that the ith reduced cohomology of the complement of Aff(K) in $\mathbb{R}^N$ is $\oplus_{\sigma \in K} \overline{H}_{N-i-2-\dim\sigma}(\mathrm{Lk}_K \sigma)$ where now $\sigma \neq \emptyset$. (*Hint.* By Alexander duality the problem reduces to computing the (N-i-1)th homology of the *one point compactification* $\overline{\mathrm{Aff}(K)} \subset S^N$).

Theorem. *The ith homology of* $\mathrm{Proj}_{\mathbb{C}}(K) \subset \mathbb{C}P^{N-1}$ *is isomorphic to* $\oplus_{j \geq 0} H_{i-2j}(K^{(j)})$*, where* $K^{(j)}$ *denotes the jth* **co-skeleton** *of* K (i.e. the sub poset consisting of all simplices of dimensions $\geq$ j). *On the other hand* $H_i(\mathrm{Proj}_{\mathbb{R}}(K); \mathbb{Z}/2) \cong \oplus_{j \geq 0} H_{i-j}(K^{(j)}; \mathbb{Z}/2)$.

Proof (following Z-Z). We can again check that Proj(K) has the same homotopy type as the realization $|\mathrm{Proj}_{\mathrm{flag}}|$ of the flag diagram associated to the functor Proj : K $\longrightarrow$ *Spaces.* This flag diagram is defined just as before, only now, to each inclusion $\theta \subset \sigma$, we have to associate the homotopically non-trivial inclusion $\mathbb{P}^{\dim\theta} \subset \mathbb{P}^{\dim\sigma}$ of (complex or real) projective spaces.

The proof of $\mathrm{Proj}(K) \simeq |\mathrm{Proj}_{\mathrm{flag}}|$ is same as before, except that now H is a natural transformation only "upto homotopies"; however it turns out that this suffices for Step 2.

More generally the above holds even if K is replaced by any of the sub posets $K^{(t)}$, $t \geq 0$. Further a downward induction on t shows that the homology of the corresponding flag diagram is $H_i(K^{(t)}) \oplus H_{i-2}(K^{(t)}) \oplus \ldots \oplus H_{i-2t}(K^{(t)}) \oplus (\oplus_{j>t} H_{i-2j}(K^{(j)}))$. The required result corresponds to the case $t = 0$ of this. (Cf. Z-Z prop 2.15, and proof of prop. 2.12, which however both contain some misprints.) *q.e.d.*

Exercise*. Relate the above computations to the Serre spectral sequence of the quotient map $Sph_{\mathbb{C}}(K) \longrightarrow Proj_{\mathbb{C}}(K)$, a locally trivial fibration with S^1 as fiber which *generalizes the Hopf fibration.*

REMARK. The cited paper of Ziegler and Zivaljevic also gives some other such examples; however they don't give cohomology of the complement of $Proj_{\mathbb{C}}(K)$ in $\mathbb{C}P^{N-1}$, for this see Goresky-Macpherson. Note also that the above answers do, à *posteriori*, give combinatorial chain complexes which give the homology in question, but the second question we had posed in (5.1) was to get *functorial* definitions of such complexes. Lastly we remark that Z-Z and G-M work in more generality, and in fact *do not even mention the examples* Lin(K), Aff(K), Proj(K), Sph(K), etc.; however these pretty spaces are in fact the universal examples for the kinds of problems considered by them.

REMARK. Since the **irrelevant ideal** (= all polynomials with constant term zero) and the defining ideal I(K) respectively determine 0 and Lin(K) it is reasonable to hope that *an algebraic computation of the local homology at* 0 *of the affine variety* Lin(K) can be made by using these : we'll see in (5.5) that this is indeed the case. Likewise we'll give a way of *algebraically computing* Proj(K) using its defining homogenous ideal I(K). This algebraical method of calculating homology also brings to light *finer invariants of* K and it is to the definition of these that the next section is devoted.

(5.4) Sheaf cohomology. The "finer invariants" we just mentioned are some cohomologies with coefficients in a **sheaf** $\mathcal{O}$, i.e. a space which comes equipped with a *surjective local homeomorphism* $\mathcal{O} \longrightarrow X$. Moreover $\mathcal{O}$

is called a sheaf *of groups, rings, etc.*, iff each stalk (= fiber) $\mathcal{O}_x$ is a group, etc., with operations continuous in an obvious sense.

REMARK. The above simple definition of LAZARD made its appearance first in the *Seminaire Cartan* of 1950-51, also it is the definition which SERRE adopts in *Faisceaux Algèbriques Cohérents*, the landmark *Annals* 1955 paper which we are now following. The notion of a sheaf was invented (in a P.O.W. camp !) by LERAY, and its first use was the following clean definition of a **smooth manifold** X of dimension n : *it is a sheaf* $\mathcal{C}^\infty(X) \longrightarrow X$ *locally isomorphic to* $\mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}^n$, *the sheaf of* **germs** *of smooth functions on* $\mathbb{R}^n$. This made it evident that other "model sheaves" could be used analogously to define other structures on X.

Let $\mathbb{F}$ be any field. We equip $\mathbb{F}^N$ (resp. $(\mathbb{F}^{N-1} \setminus \{0\})/\mathbb{F}^* = \mathbb{F}P^{N-1}$) with the **Zariski topology**, i.e. a subset will be deemed closed iff it is an affine (resp. projective **variety**), i.e. the zero set of some polynomials (resp. homogenous) polynomials in N variables. The algebra (resp. graded algebra) of all (resp. homogenous) polynomials in N variables will be denoted R (resp. S).

For any subset U of $\mathbb{F}^N$ (resp. $\mathbb{F}P^{N-1}$) we will denote by $\mathcal{O}(U)$ (resp. $\mathcal{O}_m(U)$) — the sheaf of germs of all (resp. homogenous of degree m) **rational functions** on U. So R (resp. S) consists of all the sections of $\mathcal{O}(\mathbb{F}^N)$ (resp. $\oplus_m \mathcal{O}_m(\mathbb{F}P^{N-1})$). More generally we'll also consider coherent sheaves $\mathcal{F}$: these are modules (resp. graded modules) over these sheaves $\mathcal{O}$ (resp. $\hat{\mathcal{O}} = \oplus_m \mathcal{O}_m$) of algebras (resp. graded algebras), and **coherent** means that the sections of $\mathcal{F}$ form an R-module M (resp. graded S-module M $= \oplus_m M_m$) which is of finite type.

Lemma. *For any affine variety* X *over an algebraically closed field* $\mathbb{F}$ *and any coherent sheaf* $\mathcal{F}$ *over* X *one has* $H^q(X,\mathcal{F}) = 0$ *for all* $q \geq 1$.

We omit the proof of this (see F.A.C., p. 239) but the reader should compare this with an Exercise re $\mathrm{Aff}_{\mathbb{C}}(K)$ in (5.3) : the *Nullstellensatz* is used in a similar way here. Also note that here, and below, we use **Cech cohomology** (see 2.19).

REMARK. Before F.A.C. sheaf cohomology had been defined à la Leray for *analytic varieties*. These definitions can't work now for our Zariski spaces because they need *partitions of unity*. Serre observed that the right thing was to go back to Cech's nerves. Very soon Serre's definitions itself were superseeded to some extent by those of Grothendieck who interpreted sheaf cohomology as derived functors. The analytic analogue of the above lemma is called the CARTAN-OKA theorem.

So cohomology (not *local* cohomology) of affine varieties is uninteresting over an algebraically closed $\mathbb{F}$. We now move on to projective varieties.

Koszul complexes. We denote by $(C_k^*(M),\delta)$ the cochain complex, where $C_k^q(M)$ = all alternating maps from length q+1 sequences of $\{1, 2, \dots, N\}$ to $S_{k(q+1)}$, and the coboundary δ is defined as follows

$$(\delta m)(i_0, \dots, i_{q+1}) = \sum_j (-1)^j (x_{i_j})^k\, m(i_0, \dots, \hat{i}_j, \dots, i_{q+1}) .$$

Theorem. *For any projective variety* X *over an algebraically closed field* $\mathbb{F}$ *and any coherent sheaf* $\mathcal{F}$ *on* X *the sheaf cohomology* $H^q(X,\mathcal{F})$ *is isomorphic to the direct limit, as* k *approaches infinity, of the* q*th cohomology of the Koszul complexes defined above.*

This is one of the principal results of F.A.C. and the following is merely intended to delineate the main lines of Serre's argument. (See especially no. 64 of F.A.C.)

Proof. There is no loss of generality in assuming $X = \mathbb{F}P^{N-1}$. This is so because we can always extend $\mathcal{F}$ to all of $\mathbb{F}P^{N-1}$ by defining it to be zero outside the closed set X, clearly this does'nt change M.

Now let $\mathcal{U} = \{U_i\}$ be the open covering by the N open sets $x_i \neq 0$. Consider a q-dimensional alternating Cech cochain of $\mathcal{U}$: it assigns to each $(i_0, \dots, i_{q+1})$ a section of the sheaf $\mathcal{F}_m$ over $U_{i_0} \cap \dots \cap U_{i_q}$. Multiplying it by a suitable, say kth, power of $x_{i_0} x_{i_1} \dots x_{i_q}$, one gets a section of $\mathcal{F}_{m+k(q+1)}$ which extends to all of X, i.e. one gets an element of $M_{m+k(q+1)}$. Thus we have a cochain of $C_k^q(M_m)$ for all k big. Conversely a cochain $C_k^q(M_m)$ gives such a Cech cochain. It can be seen that the Koszul coboundary becomes Cech coboundary under this. So we have *the isomorphism of cochain complexes :*

$$C^*(\mathcal{U},\mathcal{F}) \cong \varinjlim C_k^*(M) .$$

This gives $H^*(\mathcal{U},\mathcal{F}) \cong \varinjlim H_k^*(M)$. Since each U_i is a copy of the affine (N-1)-space, we know by the Lemma that each $H^*(U_i,\mathcal{F})$ is zero in positive dimensions. So using the standard acyclic model (or Léray nerve) type of argument we also have $H^*(X,\mathcal{F}) \cong H^*(\mathcal{U},\mathcal{F})$. This gives the required result. *q.e.d.*

Corollary. $H^*(X,\mathcal{F}) \cong \varinjlim \mathrm{Ext}_S^*(J_k,M)$ *where* J_k *denotes the graded ideal of* S *generated by the* k*th powers of the variables* $x_1, \ldots , x_N$.

Proof (cf. F.A.C. no. 69). To see this check that $C_k^*(M)$ equals $\mathrm{Hom}_S(\ ,M)$ applied to a resolution of J_k. *q.e.d.*

REMARK. The computation of the local sheaf cohomology at 0 of affine varieties is similar to the one above (see remark to no. 69 of F.A.C.). It seems likely that an interpretations of $\varinjlim\ \mathrm{Tor}_S^*(M,J_k)$ in terms of some sheaf homology is also known ?

(5.5) Two computations of Hochster. We now go back to the special varieties Proj(K) and Lin(K) and compute the above Ext's and Tor's. We will see that we get the same results as those in (5.3) for singular homology. This will give us the alternative algebraic computation of these homologies that we were seeking.

(to be written)

(5.6) Cohen-Macaulayness via deleted functors. For any simplicial complex K we'll denote by $K \bullet K$ the subcomplex, of the join of K and a disjoint copy of K, consisting of all simplices (σ,θ) where $\sigma \in K$ and $\theta \in K$ are disjoint with $\sigma \cup \theta \in K$. Thus if we omit the very last condition we get the bigger *deleted join* $K\#K \supseteq K\bullet K$ of 4.11.

Theorem. *$K\bullet K$ is a $\mathbb{Z}/2$-triangulation of $\mathrm{Sph}_{\mathbb{R}}(K)$.*

Here of course the $\mathbb{Z}/2$-action on $\mathrm{Sph}_{\mathbb{R}}(K)$ (i.e. the intersection of Lin(K) and the unit sphere) is the antipodal action and that of $K\bullet K$ is the switching action $(\sigma,\theta) \longleftrightarrow (\theta,\sigma)$.

Proof. Place each negative vertex (i.e. of the second copy of K) at unit distance from the origin on each negative axis. It is easily seen that Sph(K) is homeomorphic to the boundary of the convex hull of the 2N positive or negative vertices. Thus $K\bullet K = \bigcup_{\sigma\in K} \sigma_{\#}$, where $\sigma_{\#}$ is the **octahedral sphere** of dimension $\dim\sigma$ determined by the vertices of σ and their negatives (= the deleted join of the closed simplex σ) is a $\mathbb{Z}/2$-triangulation of $\bigcup_{\sigma\in K} \mathrm{Sph}(\sigma) = \mathrm{Sph}(K)$. *q.e.d.*

It follows that the local singular homology $H_*(\mathrm{Lin}K, 0^c)$ can be computed by using the chain complex of $K\bullet K$ and that the homology $H_*(\mathrm{Proj}_{\mathbb{R}}K)$ can be computed by using the *equivariant chain complex* (see 4.9) of this free $\mathbb{Z}/2$-complex $K\bullet K$.

Corollary. *For any simplicial complex K we have*

$$\bar{H}_*(K\bullet K) \cong \oplus_{\sigma\in K} \bar{H}_{*-\dim\sigma-1}(K) .$$

So K is Cohen-Macaulay iff the reduced homology of $K\bullet K$ is zero in all dimensions less than dimK.

Proof. Follows at once by using the computation of (5.3). *q.e.d.*

REMARK. We note that the above purely combinatorial formula can also be checked directly. For example one can use a spectral sequence arising from the filtration (indexed by K !) defined by stipulating that a chain is of *filtration* $\leq \sigma$ iff it is a linear combination of simplices $(\alpha,\beta) \in K\bullet K$ with $\alpha \subseteq \sigma$. One can check (*Exercise*) that this spectral sequence degenerates at E_1 = the right side of the formula. We note also the close affinity of the above formula with Bier's homology (2.18). Maybe this points to some relationship between Sph(K) = $|K\bullet K|$ and the geometrical realizations $|K_{\mathrm{comm},r}|$?

REMARK. As remarked in Chapter 4 one can also likewise define deleted functors over groups G other than $\mathbb{Z}/2$. For example we can define $K\bullet_G K$ to consist of all functions ϕ from G to K such that the simplices $\phi(g)$, $g \in G$, are disjoint, and their union is also a simplex of K. We note that this is simplicial complex has the same dimension as K though of course *for* G *infinite* it will be an infinite simplicial complex. In case G is *a topological group*, we'll equip the set of all q-simplices of $K\bullet_G K$ with the topology induced by that of G, thus turning it into a *simplicial space*.

Of particular interest in the context of the above is the simplicial space $K\bullet_{S^1} K$ whose realization can be checked to be homeomorphic to $\mathrm{Sph}_{\mathbb{C}}(K)$. Thus we can compute the homology of $\mathrm{Sph}_{\mathbb{C}}(K)$ from its chain complex, likewise the associated S^1-*equivariant chain complex* $K\bullet_{S^1} K$ computes the homology of $\mathrm{Proj}_{\mathbb{C}}(K)$. Note that this is only a *partial* combinatorialization of this computation, however in Chapter 6 we'll improve this (using Connes' cyclic model of S^1) to obtain a completely combinatorial functorial way of computing $H_*(\mathrm{Proj}_{\mathbb{C}} K)$.

Interest focusses most on the **smooth** (= all proper links Cohen-Macaulay) case because then the homology of $K\bullet K$ is essentially that of K and so we are roughly able to equip the latter with an S^1-structure which gives interesting information about the combinatorics of K.

Chapter 6. *

OPERATOR ALGEBRAS

(6.1) Quantization. The "classical" picture of $C_*(K_{comm})$ as an algebra of some *functions* on an explicit space is no longer available to us for the non-commutative but associative graded algebra $C_*(K_{assoc})$. So this time we'll assign to each vertex v a (linear first order partial differential) *generic operator* X_v (acting on all polynomial functions in N variables) and thus will think of $C_*(K_{assoc})$ as a quotient of the algebra generated by such operators.

The usefulness of this "quantization" stems from a famous theorem of POINCARE, 1899 — see *Oeuvres*, vol. 3, p.172 — now called the **Poincaré-Birkhoff-Witt** or PBW theorem, which gives the structure of this operator algebra : it consists of an anticommutative or LIE part, with the rest being obtained from this via a commutative extension. This fact ties up $C_*(K_{assoc})$ with the **de Rham complex** (of piecewise polynomial forms of K) of THOM, yields a new proof of **de Rham's theorem**, as well as a computation of the **cyclic homology** of (2.18), and finally pulls back the **rational homotopy type** of SULLIVAN to this associative complex, thus relating this construction to KALAI's **shifting**.

REMARK. The idea of **genericity** was much favoured by Poincaré : as mentioned in Ch.3 he had, in his *Analysis Situs* of 1895, stated de Rham's theorem in these terms, and had also defined the fundamental group in a similar way. (It might even be that *Poincaré suspected a connection between PBW and de Rham :* the date 1899, and the cyclic notation he used for d in 1895, do leave open this possibility !)

However before having a look back at this 1899 paper we'll quickly review in the next section (mainly from SERRE's, *Lie Algebras and Lie Groups*) a very small part of the amazingly vast present-day ramifications of the PBW theorem.

(6.2) Enveloping Lie Algebra. We recall that an **algebra** A is a module (over some commutative ring $\mathbb{F}$ which will usually be a field of characteristic zero) together with a bilinear map $A \times A \longrightarrow A$. In case

* This chapter especially has undergone drastic changes & additions & should be considered very provisional !

this bilnear map is anticommutative and obeys the **Jacobi identity**

$$(ab)c + (bc)a + (ca)b = 0,$$

then A is called a **Lie algebra**.

We'll generally denote a Lie algebra by $\mathfrak{g}$ and its bilinear map will be denoted $[a,b]$. It is easy to see (*Exercise*) that $\mathfrak{g}$ can be enveloped (i.e. embedded) in an **associative algebra** $U\mathfrak{g}$ with identity such that $[a,b] = ab - ba$ in $U\mathfrak{g}$; furthermore, if we assume (as we'll always do) that $U\mathfrak{g}$ is universal (i.e. biggest) with respect to this property, then (*Exercise*) $U\mathfrak{g}$ is *unique* in the obvious sense.

Example 1. Recall that a tangent **vector field** X of a manifold M is a **derivation** of algebra $A = C^{\infty}(M)$ of its smooth functions. (This means $X\colon A \longrightarrow A$ is $\mathbb{R}$-linear and obeys the *product rule* $X(fg) = X(f)g + fX(g)$.) Clearly $[X,Y] = XY - YX$ is also a derivation. In case the manifold is a **Lie group** G (i.e. a differentiable manifold equipped with a group structure for which $(g,h) \longmapsto gh^{-1}$ is differentiable) then we'll denote by $\mathfrak{g}$ the (finite-dimensional) Lie subalgebra of all **left-invariant vector fields** of G. Now $U\mathfrak{g}$ consists of all linear **left-invariant differential operators** acting on the ring of smooth functions of G.

Example 2. A very different kind of Lie algebra (over $\mathbb{Z}$) is provided by the graded group $\mathfrak{g} = \oplus_p\ (G_p/G_{p+1})$ of the **lower central series** $\ldots \supseteq G_p \supseteq G_{p+1} \supseteq \ldots$ of any (discrete) group G, with the bracket $[\bar{a},\bar{b}]$ now defined as the coset of $a^{-1}b^{-1}ab$. This time $U\mathfrak{g}$ can be seen to be isomorphic to $\mathbb{Z}[G]$, the **group ring** of G.

Being a quotient of $T\mathfrak{g}$ there is an obvious increasing **filtration** of $U\mathfrak{g}$ and a general version of PBW is the following ($\mathbb{F}$ a field here).

Theorem. *The graded algebra determined by the above filtration of* $U\mathfrak{g}$ *is isomorphic to the polynomial algebra* $S\mathfrak{g}$ *over* $\mathfrak{g}$.

We note that *commutativity* of this graded algebra is clear because the "error" ab − ba resulting from any transposition is of lesser degree, equally it is easy to see that the obvious map $S\mathfrak{g} \longrightarrow gr(U\mathfrak{g})$ is surjective; the job is to show that it is injective.

Example 3 (POINCARE, 1899). Let L_V be the **free Lie algebra** generated by a cardinality N set V : i.e. mod out the **free non-associative algebra** A_V (= all linear combinations of *non-associative words* of V) by the ideal generated by all expressions ab + ba and (ab)c + (bc)a + (ca)b. Then its envelope UL_V = the **free associative algebra** Ass_V generated by V (the quotient of A_V by ideal generated by all expressions (ab)c − a(bc)) and its graded algebra is isomorphic to the symmetric or **polynomial algebra** SL_V over L_V .

REMARK. The free Lie algebra L_V is interesting *number theoretically* because its **dimension function** is given by

$$\dim L_X{}^n = \frac{1}{n} \sum_{m|n} \mu(m)\, N^{n/m} \text{ , where}$$

$$\frac{1}{\zeta(s)} = \sum \frac{\mu(n)}{n^s} \text{ ,}$$

$\zeta(s)$ being the famous **zeta function** of RIEMANN ! It is interesting *combinatorially* because enumeration of words is often facilitated (see e.g. GARSIA in *Analysis, Etcetera*) by using the known graded bases — **Hall basis, Lyndon basis**, etc. — of L_V . For *group theory* its importance rests on the fact (see Example 2) that it is the Lie algebra arising from the lower central series of the **free group** F_V on the set V.

REMARK. There is a group homomorphism of F_V into the multiplicative semigroup of **formal associative series over** V (i.e. *completion* $\widehat{Ass}_V$ of Ass_V) defined by $v \longmapsto 1 + v$, $\forall\ x \in V$. The nth **dimension subgroup** consists of elements whose series have non-constant terms of degree ≥ n. In this free case (but not in general !) this subgroup coincides with the nth term of the lower central series.

REMARK. Note that the usual **exp** and **log** series set up a one-one correspondence between all formal power series with no constant terms and ones starting with 1. In his 1899 paper Poincaré uses PBW to check that there is a formal power series $z \in \hat{L}_{\{x,y\}}$ such that $z = \log(\exp(x)\exp(y))$. (This result had been proved before by CAMPBELL, and the explicit expression for z, which starts

$$z(x,y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] + \text{(quite hard to guess !)},$$

is called the **Campbell-Hausdorff-Baker-Dynkin formula.**) Note that this series obeys the "group-like" rules :

$$z(x,0) = x,\ z(0,y) = y,\ z(z(w,x),y) = z(w,z(x,y)).$$

(We met these formulae once before while doing a proof in (4.2). This notion of BOCHNER, now called **formal group**, crops up in many places in topology !) Using now some differential equation theory Poincaré integrates z(x,y) obtaining, thanks to above rules, a **chunk** near identity of *a Lie group* G *having a pre-assigned Lie algebra* $\mathfrak{g}$, and so by analytic continuation, the entire Lie group G. This result, called **Lie's third theorem**, is the main object of Poincaré's paper, with PBW only taking up § 3 of this classic. (The modern proofs of Lie's theorem — see e.g. Serre — also run on these same lines.)

(6.3) Poincare's proof. There are now numerous proofs of PBW (see e.g. the books of SERRE, CARTAN-EILENBERG, JACOBSON, MAGNUS, VARADARAJAN, and the paper of GARSIA, and references given there); however Poincaré's exposition still remains one of the best.

By composing and adding his N generic *elementary* operators (say $X = \sum X_i \frac{\partial}{\partial x_i}$ with all X_i's algebraically independent) Poincaré forms non-commutative **symbolic polynomials**. These are called **regular** iff each monomial occurs together with all its re-arrangements, and two symbolic polynomials are called **equivalent** iff they are obtainable from each other by using the relations

$$XY - YX = [X,Y].$$

Any symbolic operator is equivalent to a regular one.

Proof. Make polynomial regular by adding all missing re-arrangements. What we have added is equivalent to a lower degree operator (use fact that in above relations the right side has lesser degree) and so we are through by induction. *q.e.d.*

Now Poincaré asks (a "homological" question !) : *is this reduction unique* or, put another way, should a **regular sum** of **trinomial products**, $P(XY - YX - [X,Y])Q$, be necessarily null ?

To answer that it must be, Poincaré first looks in it at the sum of the highest degree **binomial products**

$$P(XY - YX)Q$$

contributed by its trinomials. By degree considerations this part is surely null, which implies that all these terms can be rearranged as a sum of circuits, where a **circuit** means a cyclic sequence

$$\ldots , P(XY - YX)Q, Q(YX - XY)R, \ldots$$

Using this the question boils down to showing that *any circuit occurs thus as all the highest degree terms of some null regular sum of trinomial products* (again note "homological" flavour !) and once this, the main step in the proof, is done, he is through by an induction.

We omit how Poincaré does this "main step" and instead just mention an example which he gives at the very outset.

Example.

(6.4) De Rham theory of Lie groups. The *T.A.M.S.* 1948 paper of CHEVALLEY-EILENBERG (which begins modestly enough with a disclaimer to any "deep originality" !) is a very good entré into de Rham theory, the reason being that this becomes easy and very beautiful for Lie groups, and this case also illuminates the general case vividly.

For example, for a general closed M, it is not à *priori* obvious why the infinite dimensional (over $\mathbb{R}$) de Rham complex should even have a finite dimensional homology ! However for the Lie group case there is a ready explanation ... but before we get into that lets recall the basic

DEFINITIONS. By a degree r **form** of a manifold M is meant any multilinear skew-symmetric A-linear (here $A = C^{\infty}(M)$, as in Example 1 of 6.2) map $\omega : Der(A) \times \ldots \times Der(A) \longrightarrow A$. We'll denote by $\Omega^r(A)$ the A-module of all these r-forms, and the direct sum $\Omega = \oplus_{r\geq 0}\Omega^r(A)$ will be considered in the usual way as an **exterior algebra** of the A-module of 1-forms.

For any $f \in A$ let $df \in \Omega^1(A)$ be the 1-form given by $df(X) = Xf$ (so in local coordinates $df = \frac{\partial f}{\partial x_1}.dx_1 + \ldots + \frac{\partial f}{\partial x_m}.dx_m$). Note that $\Omega^1(A)$ is the A-module generated by the symbols df, $f \in A$, subject to the relations $d(fg) = df.g + f.dg$. (This point shows how to extend these definitions to *any* commutative algebra A.) More generally the **exterior derivative** $d : \Omega^r(A) \longrightarrow \Omega^{r+1}(A)$ is defined by

$$f_0.df_1 \wedge \ldots \wedge df_n \longmapsto df_0 \wedge df_1 \wedge \ldots \wedge df_n .$$

Thus $(\Omega^*(A),d)$ becomes a graded-commutative graded differential algebra (DGA). (Here superfixes are used because the construction is *contravariant in* M, note that it is *covariant in* A.)

The cohomology algebra of $(\Omega^*(A),d)$ is called **de Rham cohomology** and is denoted $H^*_{DR}(M)$ or $H^*_{DR}(A)$. It will be considered as a functor by associating to each smooth function $f : M \longrightarrow N$ the algebra morphism f^* : $C^{\infty}(N) \longrightarrow C^{\infty}(M)$, $g \longmapsto g\circ f$, which extends to a DGA morphism f^* : $\Omega(N) \longrightarrow \Omega(M)$.

With these definitions out of the way now lets look at the case M = G of a Lie group. We have now, within $\Omega(A)$, the DGA $(\Omega(\mathfrak{g}),d)$ of all **left-invariant forms** of G. Since this is clearly finite-dimensional (it can be identified with the exterior algebra of the tangent space at $1 \in G$) the finite-dimensionality of $H^*_{DR}(G)$ is explained by the following.

Theorem. *For any closed and connected Lie group* G *the inclusion*

$\Omega(\mathfrak{g}) \longrightarrow \Omega(G)$ *induces an algebra isomorphism in cohomology.*

Proof. The point is that G (assumed oriented) has a unique bi-invariant **Haar measure** (in fact a smooth degree r form) vol(G) with integral $\int_G \mathrm{vol}(G) = 1$. Also note (because of connectedness) that each left translation L_g is homotopic to identity. Using these it follows that within any de Rham cohomology class $[\omega]$ we have the left-invariant closed form $\int_G L_g^*(\omega).\mathrm{vol}(G)$. Finally note that if there is any form θ with $d\theta$ equal to a left-invariant form ω, then we'll also have $d\Theta = 0$, where once again Θ is the "averaged" (and so left-invariant) form $\int_G L_g^*(\theta).\mathrm{vol}(G)$. *q.e.d.*

We next sharpen this to the following beautiful

Theorem. $H^*_{DR}(G)$ *is isomorphic to the subalgebra of* Ω *consisting of all* **bi-invariant forms** (i.e. both left and right invariant).

Proof. For this consider the action of G×G on G given by $(g,h).k = gkh$. Forms invariant under it are precisely the bi-invariant forms of G. So we can argue just as above using averaging over this action. To check that *each bi-invariant form is closed* use the formula $[d,\iota_v] = L_v$ (see 6.4) and the fact (immediate from definition of L_v) that a left-invariant form is right invariant iff $L_v\omega = 0 \ \forall \ v \in \mathfrak{g}$. *q.e.d.*

REMARK. In case G is **semi-simple**, i.e. has a finite fundamental group, it admits a bi-invariant Riemannian metric $< \ , \ >$. The bi-invariant forms are precisely those which are **harmonic** with respect to it, i.e. those which satisfy the generalized **Laplace equation** $dd^* + d^*d = 0$, where the adjoint differential operator $d^* : \Omega^r \longrightarrow \Omega^{r-1}$ is defined by $\int_G <d\omega,\theta>.\mathrm{vol} = \int_G <\omega,\delta\theta>.\mathrm{vol}$. A theorem of HODGE says that for *any* Riemannian manifold M the vector space of all harmonic forms has the same dimension as $H^*_{DR}(M)$, but note that in general (unlike this case $M = G$) the exterior product of harmonic forms need not be harmonic.

As the proof above shows it is useful to consider actions of G on manifolds M other than G also. Another point is to also consider (cf. 4.9) forms $\Omega(M;V)$ with coefficients in any G-vector space V. Defining d

: $\Omega(M;V) \longrightarrow \Omega(M;V)$ in the expected way one now has a cohomology $H^*_{DR}(M;V)$. This time one has the distinguished subcomplex of **equivariant forms**, i.e. those which transform under the left action L_g of G on M as per the given action P_g of G on V. However, for non-trivial representations V, the **equivariant de Rham cohomology** $H^*_{DR}(M;V,G)$ thus defined can be *much* smaller than $H^*_{DR}(M;V)$.

Theorem. *If V is an irreducible non-trivial representation of G then* $H^*_{DR}(M;V,G)$ *is zero.*

Proof. Let $[\omega]$ be any equivariant de Rham class and z any smooth singular cycle. By **de Rham's theorem** (see Chapter 3 and 6.5 below) it suffices to check $\int_z \omega = 0$ which follows from the given hypotheses on V because

$$P_g(\int_z \omega) = \int_z P_g\omega = \int_z (L_g)^*\omega = \int_{L_g z} \omega = \int_z \omega ,$$

where the last step again uses $L_g \simeq \mathrm{id}$. *q.e.d.*

Corollary. *For any representation* V, *the equivariant de Rham cohomology* $H^*_{DR}(M;V,G)$ *consists of as many copies of the homology of the Lie group, as the number of copies of the trivial representation* $\mathbb{R}$ *in* V.

Thus, thanks to de Rham's theorem (which was conjectured by E.CARTAN for just this purpose !) the calculation of the real cohomology of a Lie group G (even with non-trivial coefficients V) has now been reduced to a problem of linear algebra, viz. of finding all the bi-invariant forms of G, a problem which we'll re-formulate still more explicitly in the next section.

REMARK. However the homology of the classical Lie groups was first calculated by the young (and already blind) PONTRJAGIN by using an explicit and elegant cell subdivision (shortly after he had attended, with his mother as interpreter for him, a talk which E.Cartan had given in Moscow). Likewise, HOPF had proved that *the real homology of a Lie group is that of a product of some odd dimensional spheres*, without using any de Rham theory.

(6.5) De Rham theory of Lie algebras. Let $\Omega(\mathfrak{g};V)$ be the forms on G equivariant in the sense of (6.3) with respect to the left action L_g of G on G, and the action $P_g \in \mathrm{Aut}(V)$ of G on V. We'll denote by $p_v \in \mathrm{End}(V)$ the action of $\mathfrak{g}$ on V obtained by differentiating the action P_g.

Theorem. *The exterior derivative* d : $\Omega^r(\mathfrak{g};V) \longrightarrow \Omega^{r+1}(\mathfrak{g};V)$ *is given by*

$$(d\omega)(v_1, \ldots, v_{r+1}) = \sum_i (-1)^{i+1} p_{v_i} \omega(v_1, \ldots, \hat{v}_i, \ldots, v_{r+1}) +$$

$$\sum_{i<j} (-1)^{i+j+1} . \omega([v_i,v_j], v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{r+1}).$$

Proof. Despite its forbidding appearance this formula is not very hard. For example, for trivial coefficients $V = \mathbb{R}$, only the second term on the right remains, and is easily seen to be (like d) a derivation of Ω of degree +1. So it will suffice to do the case r = 1, called the **Maurer-Cartan formula**, which follows (*Exercise*) by a short local computation. *q.e.d.*

Note that each $v \in \mathfrak{g}$ determines a tangent vector at $1 \in G$ which in turn determines a *right* invariant vector field of G. We'll denote the derivative with respect to this flow by L_v ; clearly this **Lie derivative** is a derivation of Ω of degree 0. Also one has for each $v \in \mathfrak{g}$ the derivation of degree −1, called **interior product** with v, which takes any 1-form ω to the number $\omega(v)$.

Exercise. Work out the following closed formulae for these derivations :

$$(L_v\omega)(v_1, \ldots, v_r) = \sum_i \omega(v_1, \ldots, [v,v_i], \ldots, v_r),$$

$$(i_v\omega)(v_1, \ldots, v_{r-1}) = \sum_i (-1)^i . \omega(v_1, \ldots v_{i-1}, v, v_{i+1}, \ldots, v_{r-1}).$$

Exercise. Show that, amongst the derivations ι_v , L_v and d of graded-commutative Ω, one has the following (signed) bracketing rules :

$$[\iota_v , \iota_w] = 0, \qquad [L_v , L_w] = L_{[v,w]}, \qquad [d,d] = 0 ,$$

$$[\iota_v , L_w] = \iota_{[v,w]}, \qquad [L_v , d] = 0, \qquad [\iota_v, d] = L_v .$$

(*Hint.* Check equality on generators of Ω.)

REMARK. These derivations ι_v , L_v (and of course d) all make sense for the algebra $\Omega(M)$ of *any* manifold M and for *any* vector field v of M. All the above bracket formulae are valid, and one has similar closed formulae for the three derivations, with vector fields being now arbitrary. For instance, the closed formula for d is exactly as given in above theorem, with $V = C^\infty(M)$ now, and action $p_v f = v(f)$. Thus (*think this one out !*) an arbitrary smooth form on M can be heuristically considered as an equivariant form on Diff(M) (a "Lie group" whose "Lie algebra" consists of all vector fields on M !) with coefficients in $C^\infty(M)$. (It is this heuristic analogy — thinking of *any* M as a homogenous space of the "Lie group" Diff(M) — which makes looking at the case of Lie groups so instructive for understanding de Rham theory of *any* M.)

To save time we don't want to enter into the details of Elie Cartan's programme (e.g. the explicit computation of the bi-invariant forms of a semi-simple $\mathfrak{g}$), but we note that, thanks to the above formula for d, it now makes sense to talk of the **cohomology** $H(\mathfrak{g},V)$ **of any Lie algebra** $\mathfrak{g}$ with coefficients in any representation V of $\mathfrak{g}$.

Theorem. *For any* $\mathfrak{g}$ *we have* $H^q(\mathfrak{g},U\mathfrak{g}) = 0$ *for all* $q \geq 1$ *provided* $U\mathfrak{g}$ *is equipped with the* $\mathfrak{g}$*-action* $a \longmapsto v\circ a \ \forall\, v \in \mathfrak{g}$.

Proof. Identifying skew-symmetric maps $\mathfrak{g} \times \dots \times \mathfrak{g} \longrightarrow U\mathfrak{g}$ with $U\mathfrak{g}\otimes\Lambda\mathfrak{g}$ we see that the assertion is equivalent to the the acyclicity of $(U\mathfrak{g}\otimes\Lambda\mathfrak{g},\ d_*)$ where d_* is defined by

$$d_*(a;\ v_1\wedge \dots \wedge v_r) = \textstyle\sum_i (-1)^{i+1}(v_i.a;\ v_1 \wedge .. \wedge \hat{v}_i \wedge ..\ v_r)$$

$$+ \sum_{i<j} (-1)^{i+j} (a;\ [v_i, v_j],\ v_1,\ \dots,\ \hat{v}_i,\ \dots,\ \hat{v}_j,\ \dots,\ v_r).$$

To check this we filter $(U\mathfrak{g}\otimes\Lambda\mathfrak{g},\ d_*)$ by total degree. By the PBW theorem, the (E_0, d_0) of the resulting spectral sequence is $S\mathfrak{g}\otimes\Lambda\mathfrak{g}$ equipped with the Koszul differential, which we know (see Chapter 5) is acyclic. So E_1 and thus E_∞ is trivial. *q.e.d.*

Exercise*. Is the above vanishing theorem *equivalent* to the Poincaré-Birkhoff-Witt theorem ? (If so, one can *deduce the PBW theorem as a consequence of De Rham's theorem,* for the case when $\mathfrak{g}$ is the Lie algebra of a compact Lie group, by noting that the above representation $U\mathfrak{g}$ does not contain the trivial representation.)

Given any representation V of the Lie algebra $\mathfrak{g}$ we'll consider it as a $U\mathfrak{g}$-module under the unique associative algebra homomorphism $U\mathfrak{g} \longrightarrow \mathrm{End}(V)$ which extends the given Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathrm{End}(V)$. Now, since the above proof tells us that, in the category of $U\mathfrak{g}$-modules, $(U\mathfrak{g}\otimes\Lambda\mathfrak{g},\ d_*)$ is a resolution of $\mathbb{F}$, we obtain the following.

Corollary. $H^*(\mathfrak{g}, V) \cong \mathrm{Ext}_{U\mathfrak{g}}(\mathbb{F}, V)$.

Thanks to this result of KOSZUL and CARTAN-EILENBERG (see Ch. XIII of their book — this is now *Henri* Cartan, Elie's son), we can apply to Lie algebra cohomology all the standard tricks of **homological algebra** (i.e. the subject dealing with *derived functors*, i.e. the subject created by this 1956 C-E book and GROTHENDIECK's 1957 *Tohuku* paper).

Now let us turn to $U\mathfrak{g}^{ad}$, the representation of $\mathfrak{g}$ on $U\mathfrak{g}$ defined by $a \longmapsto v\circ a - a\circ v$ (as against $a \longmapsto v\circ a$ above) for all $v \in \mathfrak{g}$.

Theorem. *The Lie algebra cohomology* $H^q(\mathfrak{g}, U\mathfrak{g}^{ad})$ *is isomorphic to the Hochschild cohomology* $HH^q(U\mathfrak{g})$ *of the associative algebra* $U\mathfrak{g}$.

Proof. Any $U\mathfrak{g}$-linear map $U_+\mathfrak{g} \times \dots \times U_+\mathfrak{g} \longrightarrow U\mathfrak{g}^{ad}$ gives by restriction a multi-linear map $\mathfrak{g} \times \dots \times \mathfrak{g} \longrightarrow U\mathfrak{g}$, and conversely any such map, extends uniquely to a $U\mathfrak{g}$-multi linear map $U_+\mathfrak{g} \times \dots \times U_+\mathfrak{g} \longrightarrow U\mathfrak{g}$ (here $U_+\mathfrak{g}$ denotes operators having no constant terms).

This allows us to identify the defining cochain complex $C^*(\mathfrak{g}, U\mathfrak{g}^{ad})$ for $H^*(\mathfrak{g}, U\mathfrak{g}^{ad})$ as the **skew-symmetric subcomplex of the defining normalized Hochschild complex** $C^*(U_+\mathfrak{g}, U\mathfrak{g})$ **for** $HH^q(U\mathfrak{g})$. (This uses the fact that $U\mathfrak{g}$ carry the adjoint representation ad.)

Exercise. Check that ι_v and L_v also extend in a natural way to all $U\mathfrak{g}$-multi linear maps $U_+\mathfrak{g} \times \ldots \times U_+\mathfrak{g} \longrightarrow U\mathfrak{g}$ (i.e. all **left-invariant covariant tensors** of degree ≥ 1 of G) and that now one has $\iota_v.\delta + \delta.\iota_v = L_v$ where δ = Hochschild coboundary (cf. C-E, p. 278.)

Further, *this inclusion* $C^*(\mathfrak{g}, U\mathfrak{g}^{ad}) \longrightarrow C^*(U_+\mathfrak{g}, U\mathfrak{g})$ *induces an isomorphism in cohomology.* This is checked (see LODAY, pp. 97-98) by a spectral sequence argument analogous to that of the last theorem. One filters by total degree, and notes that now, for the left side, one has $(E_0, d_0) = (S\mathfrak{g}\otimes\Lambda\mathfrak{g}, \text{zero})$, while for the right side one has (E_0, d_0) = normalized Hochschlid complex for the polynomial algebra $S\mathfrak{g}$. Now one uses the fact that *the Hochschild homology of a polynomial algebra is isomorphic to the algebra of its differential forms* to obtain an isomorphism of the E_1, and thus the E_∞ terms. *q.e.d.*

Likewise (cf. LODAY) we can see that *the Hochschild cohomology of* $U\mathfrak{g}$ *with coefficients in any bi-module* M *coincides with the Lie algebra cohomology of* $\mathfrak{g}$ *with coefficients in the representation* M^{ad} *of* $\mathfrak{g}$ *given by restricting to* $\mathfrak{g}$ *the difference of the left and right actions of* $U\mathfrak{g}$.

Since in particular the trivial $U\mathfrak{g}$-bimodule $\mathbb{R}$ gives the trivial representation $\mathbb{R}$ of $\mathfrak{g}$, we obtain the following striking result.

Corollary. *The real cohomology of a compact Lie group* G *coincides with the Hochschild cohomology with coefficients* $\mathbb{R}$ *of the algebra of all left-invariant differential operators of* G.

REMARK. More generally CONNES saw that the real cohomology of *any* closed manifold M can be calculated directly (via *the cyclic quotient of its Hochschild complex*) from a suitable operator algebra, e.g. (a closure of) the **algebra of smoothing operators** of M. The exciting thing is that this algebra generalizes (one has to take a little care vis-à-vis holonomy) from this "one-leaf case" to any **foliation** (but now need *not* be stably equivalent to any commutative algebra) and it turns out that this **cyclic cohomology** still gives topologically meaningful results whereas classical theory does not.

(6.6) De Rham theorem. The 1933 proof of DE RHAM had used a C^∞ triangulation of M (an assumption which was shown to be justified only later). So WEIL had reformulated the argument in terms of open coverings instead of triangulations. This in turn was later reformulated in the language of **sheaves** as follows (cf. HIRZEBRUCH, *Topological methods in Algebraic Geometry*, 1956, pp. 36-37).

We note that the vector space $\Omega^r(M)$ of all smooth r-forms of smooth manifold M consists of all the **sections** of the sheaf $\Omega^r_{sh}(M)$ of germs of all smooth r-forms on M. Thus the De Rham complex $(\Omega^*(M),d)$ is obtained by applying the functor "section" to the complex $(\Omega^*_{sh}(M),d)$ of sheaves.

Theorem (Poincaré's Lemma). *The complex* $(\Omega^*_{sh}(M),d)$ *is exact.*

Proof. Clearly this amounts to saying that $H^*_{DR}(\mathbb{R}^m) = 0$ in all positive dimensions. For m = 1 this is just a homological reformulation of the **fundamental theorem of calculus** viz. that the derivative of an indefinite integral equals its integrand. An induction which we'll omit (*Exercise,* also cf. Poincaré 1882) starting from this case m = 1 now establishes it for all m ≥ 2. *q.e.d.*

Passing now from Euclidean spaces to an arbitrary M we know already that the real singular cohomology coincides with its Cech cohomology (with trivial coefficients $\mathbb{R}$). Using the above lemma we will now check the same for De Rham cohomology.

Theorem. *De Rham cohomology coincides with Cech cohomology.*

Proof. We first note (use partition of identity) that each of the sheaves $\Omega^r_{sh}(M)$ is **fine** : this means that given any locally finite open cover $\mathcal{U}$ of M, we can find endomorphisms of the sheaf supported on the members of $\mathcal{U}$, which add up to the identity morphism of the sheaf.

Composing local sections with these endomorphisms and adding (cf. Hirzebruch, p.34) one can define a chain contraction of the Cech cochain complex $C^*(M;\Omega^r_{sh}(M))$. So it follows that *the Cech cohomology of each of the sheaves* $\Omega^r_{sh}(M)$ *is trivial.*

The remaining argument uses only this and is purely homological (cf. 2.20 or Hirzebruch, p. 36). *q.e.d.*

REMARK. The above proof is also very close to the main ideas of GROTHENDIECK's 1957 *Tohuku* paper. He showed that in the category of all sheaves on any space M, one can find, for each object S, an exact sequence $S \rightarrow S_0 \longrightarrow S_1 \longrightarrow \dots$ where each sheaf S_i is an *injective object* of this Abelian category. Also he showed that, *for a paracompact space M, Cech cohomology of* $H^*(M,S)$ *coincides with the derived functors of the section functor* : i.e. it is the cohomology of the cochain complex obtained by applying the section functor to the above exact sequence of injective sheaves $S_0 \longrightarrow S_1 \longrightarrow \dots$ (so for arbitrary spaces the right way of defining sheaf cohomology is as these derived functors). To see this one notes that Grothendieck's cohomology is unaffected if we merely assume that these S_i's are such that these derived functors are trivial for them, and then one uses the paracompactness of M to define such S_i's for which one recovers the Cech cochain complex.

REMARK. Thus Poincaré's Lemma serves to show that De Rham cohomology is a derived functor, just as the PBW theorem served to show that (the intimately related) Lie algebra cohomology is a derived functor. Also we have indicated in (6.4) how in one case PBW probably follows from De Rham's theorem, and as mentioned in (6.1) will also show that conversely PBW implies a De Rham theorem. To get to that stage we have *to make the definition of De Rham cohomology more combinatorial* and so we now take the first step in this direction.

(6.7) Differential forms on simplicial complexes. In his 1957 lectures at the University of Chicago, THOM showed how to compute the real cohomology of any simplicial complex K by means of forms. (Much of this was also anticipated in WHITNEY's 1956 book.)

REMARK. An account of Thom's work was published only much later by SWAN (see *Topology* of 1975) who pointed out that Thom's proof works over $\mathbb{Q}$ also. In the meanwhile SULLIVAN rediscovered Thom's complex, and used this explicit solution of the "commutative cochain problem" (see below : this had been solved in a more abstract way by QUILLEN who too was unaware of Thom's work) to build up his De Rham Homotopy Theory.

The key point which Thom noted was that one should not use just smooth forms (i.e. restrictions to $|K|$ of smooth forms of an ambient Euclidean space) but piecewise smooth forms. Here a **piecewise smooth r-form** ω on K is one which associates to each simplex σ of K a smooth r-form ω_σ of $\mathrm{Aff}(\sigma)$, in such a way that whenever $\theta \subseteq \sigma$, then under the map of forms induced by the inclusion $\mathrm{Aff}(\theta) \subseteq \mathrm{Aff}(\sigma)$, the form ω_σ should image to ω_θ. Obviously if $\omega = \{\omega_\sigma\}$ is piecewise smooth, then so is $d\omega := \{d\omega_\sigma\}$.

Note that the complex of piecewise smooth forms is bigger than that of smooth forms : e.g. if K is n-dimensional a piecewise smooth form is obtained by assigning to each $\sigma \in K$ *any* n-form ω_σ on $\mathrm{Aff}(\sigma)$.

Note also that it is meaningful to ask that the coefficients of each ω_σ be in the rational polynomial ring $\mathbb{Q}[\mathrm{vert}K]$; if this holds then we'll say that we have a **piecewise polynomial form** ω over $\mathbb{Q}$.

Theorem. *For any simplicial complex* K *the cohomology algebra of the De Rham complex* $C^*_{Th}(K)$ *of all piecewise polynomial forms of* K *over* $\mathbb{Q}$ *is functorially isomorphic to the rational cohomology algebra* $H^*(K;\mathbb{Q})$.

Since furthermore this DGA $C^*_{Th}(K)$ over $\mathbb{Q}$ is graded commutative and such that any inclusion $L \subseteq K$ induces a surjection $C^*_{Th}(K) \longrightarrow C^*_{Th}(L)$ follows that C^*_{Th} is a solution of the **"commutative cochain problem"** (i.e. the problem of finding a functor from simplicial complexes to DGA's over $\mathbb{Q}$ having all the stated properties).

Proof. We'll follow Thom and use the Eilenberg-Steenrod axioms.

q.e.d.

REMARK: One can regard the above as a *combinatorialization of the De Rham complex* analogous in some sense to the combinatorialization K_{pl} of the singular complex of K. Later we'll consider another more striking solution of the commutative cochain problem, due to CONNES, which is akin to the combinatorialization K_{kan} of the singular complex considered in Chapter 4.

Exercise. Show that the commutative cochain problem *over* $\mathbb{Z}$ has no solution. (*Hint.* Take any K with non trivial Steenrod squares.)

(6.8) Cyclic vector spaces. In his search for an analogue of De Rham cohomology for a certain non-commutative algebra (namely the convolution algebra of smoothing operators along the leaves of a foliation) CONNES discovered cyclic cohomology. To understand the homological algebraic aspects (= Connes' *C.R. note* of 1983, also see LODAY's book) of this construction we will begin by working out as an example a (co)homology that we had defined in (2.18).

Note. The reader should keep in mind that the argument given below is absolutely general in the sense that it applies to any "cyclic vector space" (a notion that we'll define below) excepting the very last step which is particular for this example.

Theorem. *Over rational coefficients one has*

$$H_*^{cycl}(K_{assoc}) \cong H_*(K) \oplus H_{*-2}(K) \oplus H_{*-4}(K) \oplus \dots \quad .$$

Proof. The set of rational vector spaces $C_q = C_q(K_{assoc})$ constitute a (complete semi-) **simplicial vector space** with face and degeneracy operators $\partial_i : C_q \to C_{q-1}$ and $s_j : C_q \longrightarrow C_{q+1}$ being the linear maps defined on vertex sequences by omissions and repetitions. Moreover for each q we define linear map $t_q : C_q \longrightarrow C_q$ by

$$t_q\,(v_0 v_1 \ldots v_q) = (-1)^q.(v_q v_0 \ldots v_{q-1}).$$

It is easily checked (*Exercise*) that this turns this simplicial vector space into a **cyclic vector space** i.e. that the following relations hold (cf. Loday, p.75, a categorical reformulation of this definition will be given later) :

$$t_n^{n+1} = \mathrm{id} \quad , \quad \partial_0 t_n = (-1)^n.\partial_n \quad , \quad s_0 t_n = (-1)^n.t_{n+1}^2.s_n \ , \ \text{and}$$

$$\partial_i t_n = -\, t_{n-1}\partial_{i-1} \quad , \quad s_i t_n = -\, t_{n+1}.s_{i-1} \quad \text{for } 1 \le i \le n.$$

It follows in a straightforward way from these (*Exercise* or see Loday, pp.76, 52-53) that the following diagram is a **bicomplex**, i.e. the horizantals ∂_{hor} (= id - T or N) and verticals ∂_{ver} (= ∂ or $-\,\partial'$) obey the relations :

$$\partial_{hor}\partial_{hor} = 0 \ , \ \partial_{hor}\partial_{ver} + \partial_{hor}\partial_{ver} = 0 \ , \ \partial_{ver}\partial_{ver} = 0.$$

$$\begin{tikzcd}
{} \arrow[d, "\partial"] & {} \arrow[d, "-\partial'"] & {} \arrow[d, "\partial"] & {} \arrow[d, "-\partial'"] & {} \arrow[d, "\partial"] & \\
C_2 \arrow[d, "\partial"] & C_2 \arrow[l, "\mathrm{id}-t"'] \arrow[d, "-\partial'"] & C_2 \arrow[l, "N"'] \arrow[d, "\partial"] & C_2 \arrow[l, "\mathrm{id}-t"'] \arrow[d, "-\partial'"] & C_2 \arrow[l, "N"'] \arrow[d, "\partial"] & {} \arrow[l, "\mathrm{id}-t"'] \\
C_1 \arrow[d, "\partial"] & C_1 \arrow[l, "\mathrm{id}-t"'] \arrow[d, "-\partial'"] & C_1 \arrow[l, "N"'] \arrow[d, "\partial"] & C_1 \arrow[l, "\mathrm{id}-t"'] \arrow[d, "-\partial'"] & C_1 \arrow[l, "N"'] \arrow[d, "\partial"] & {} \arrow[l, "\mathrm{id}-t"'] \\
C_0 & C_0 \arrow[l, "\mathrm{id}-t"'] & C_0 \arrow[l, "N"'] & C_0 \arrow[l, "\mathrm{id}-t"'] & C_0 \arrow[l, "N"'] & {} \arrow[l, "\mathrm{id}-t"']
\end{tikzcd}$$

Here ∂ denotes the usual alternating sum $\sum_i\ (-1)^i\partial_i$ of face operators, while ∂' denotes a similar alternating sum *of all except the last* of the face operators ; on the other hand t denotes t_q's and N denotes the sum $id + t_q + t_q^2 + \dots + t_q^q$.

The "rows-first" spectral sequence. Since we are working over the rationals the rows (note that the qth row is the *periodic resolutions* of the $\mathbb{Z}/q+1$ - vector space C_q) are acyclic. So the first term of this spectral sequence is zero except for the first column which becomes

$$(C_*^{cycl}(K_{assoc}),\partial) \; := (C_*(K_{assoc}) /\mathrm{im}(id-t) \, , \partial).$$

So this spectral sequence's second term is final and equals the left-hand side $H_*^{cycl}(K_{assoc})$ of the desired formula. Of course this final term is also equal to the **total homology** of the bicomplex, i.e. the homology of the corresponding singly-graded complex (with grading constant on lines x+y = const. of above picture) under the the differential $\partial_{tot} = \partial_{hor} + \partial_{ver}$. Now we turn to the other spectral sequence of this bicomplex, which also of course converges to this same total homology.

The "columns-first" spectral sequence. The *odd* columns of the first term of this sequence now consist, by definition, of the homology groups $H_*(C) = H_*(K_{assoc})$. On the other hand *the even columns become zero* : this follows because the **last degenaracies** s supply us (*Exercise* or Loday, pp. 77, 47) with a chain contraction : $s\partial' + \partial's = id$.

The second term of this spectral sequence is thus as follows :

$$
\begin{tikzcd}[column sep=small]
H_2(C) & 0 & H_2(C) & 0 & H_2(C) & 0 & H_2(C) \\
H_1(C) & 0 & H_1(C) \arrow[ull, "d_2"'] & 0 & H_1(C) \arrow[ull, "d_2"'] & 0 & H_1(C) \arrow[ull, "d_2"'] \\
H_0(C) & 0 & H_0(C) \arrow[ull, "d_2"'] & 0 & H_0(C) \arrow[ull, "d_2"'] & 0 & H_0(C) \arrow[ull, "d_2"']
\end{tikzcd}
$$

To proceed further we will use the fact (: see Loday, pp. 77-78) that d_2 *is induced by the* **Connes' boundary map** $B : C_q \longrightarrow C_{q+1}$, which is defined by $B = (-1)^{q+1}(id-t_{q+1})sN$. (*We emphasize that upto here the argument works for* any *cyclic vector space.*)

Now we use the fact that **skew-symmetrization** $C_*(K_{assoc}) \longrightarrow C_*^{alt}(K_{assoc}) = C_*(K)$ (= oriented chain complex of K) induces an isomorphism in homology. (This we had proved in 2.18 by using the Eilenberg-Steenrod uniqueness theorem.) So it follows firstly that the odd columns of the above picture comprise of the usual simplicial homology $H_*(K)$, and secondly, since the action of B on any vertex sequence results in vertex sequences with repeating vertices (which die under skew-symmetrization) that the d_2's are all zero. So the above second term is indeed the final term. Summing along the lines x+y = constant we obtain the right side of the desired formula. *q.e.d.*

Exercise*. Show that $H_*^{dih}(K_{assoc}) \cong \bigoplus_{k\geq 0} H_{*-4k}(K)$.

REMARK. We note that $K_{assoc} = K_{\mathcal{F}}$ was an *$\mathcal{F}$-object* i.e. a contravariant functor from the category $\mathcal{F}$ of all set maps $[p] \longrightarrow [q]$. This category happened to contain five "nice" subcategories containing $\mathcal{N}$ (that of increasing maps) and this had given us the five cochain complexes :

$$C^*(K_{assoc}) \supseteq C^*_{rev}(K_{assoc}),\ C^*_{cyc}(K_{assoc}) \supseteq C^*_{dih}(K_{assoc}) =$$

$$C^*_{rev}(K_{assoc}) \cap C^*_{cyc}(K_{assoc}) \supseteq C^*_{alt}(K_{assoc}) .$$

More generally *for any group* G we have the category $\mathcal{F}_*$ of all maps $[p]_* \longrightarrow [q]_*$ between **deleted joins** (see 4.12) of the standard simplices [p] (for G = 1 one gets $\mathcal{F}_* = \mathcal{F}$) and one can likewise associate to any simplicial complex K an $\mathcal{F}_*$ – object $K_{\mathcal{F}_*}$. One can work out completely (see FIEDOROWICZ-LODAY, *op. cit.*) all the "nice" subcategories of $\mathcal{F}_*$ (it turns out that only the case G = $\mathbb{Z}/2$, viz. of **octahedral spheres** $[p]_*$, needs to be worked out by hand). For each of these one gets a (co)homology of the simplicial complex K : thus one gets a whole slew of (co)homologies of K which includes the equivariant cohomologies and many more whose computation seems a natural problem.

To see what all this has to do with the De Rham cohomology of K (see 6.7) we need to look at **a different cyclic vector space** :

For this we'll use the commutative algebra A = AK of all polynomial functions with rational coefficients on $|K|$ (or Aff(K)). We now put $C_q(AK)$ = (q+1)-fold tensor product of A (over the base field $\mathbb{Q}$). The face operators are define à *la Hochschild* i.e. one composes with the element cyclically in front. The degeneracies are defined by insertions of 1, and rotations with signs $(-1)^n$ give the t_n's.

The ordinary homology of this cyclic vector space (i.e. of the odd columns of the bicomplex) is called the **Hochschild homology** of K (or of the algebra AK) and will be denoted $HH_*(K)$ (or $HH_*(AK)$) while the cyclic homology (the final first column of the "rows-first" spectral sequence of the bicomplex) will be called the **Connes' homology** and denoted $HC_*(K)$ (or $HC_*(AK)$). It connects with 6.7 as follows.

Theorem. *If all the proper links of a simplicial complex* K *are Cohen-Macaulay over* $\mathbb{Q}$ *then its cyclic homology over* $\mathbb{Q}$ *is given by*

$$HC_r(K) \;\cong\; \frac{\Omega^r K}{d\Omega^{r-1}K} \oplus H^{r-2}_{DR}(K) \oplus H^{r-4}_{DR}(K) \oplus \dots .$$

So, in conjunction with Thom's de Rham theorem (6.7) we obtain, for all such simplicial complexes K (e.g. for simplicial manifolds)

$$HC_r(K) \;\cong\; H^r(K^r) \oplus H^{r-2}(K) \oplus H^{r-4}(K) \oplus \dots ,$$

which coincides, except for the first summand, with $H^{cyc}_r(K_{assoc})$.

Proof. The main point is that under the given hypothesis on K the commutative algebra A = AK is a **smooth algebra** in the sense of Loday, p. 101. Thus § 3.4 of Loday applies and computes the "columns-first" spectral sequence of the bicomplex as follows.

We have the **Hochschild-Kostant-Rosenberg theorem** which says $HH_*(A) \cong \Omega^*(A)$ for such smooth algebras. So now the second term of this spectral sequence reads :

$$\begin{array}{ccccccc}
 & \nwarrow & & \nwarrow & & \nwarrow & & \nwarrow \\
\Omega^2(A) & 0 & \Omega^2(A) & 0 & \Omega^2(A) & 0 & \Omega^2(A) & \\
 & \nwarrow d & & \nwarrow d & & \nwarrow d & & \nwarrow \\
\Omega^1(A) & 0 & \Omega^1(A) & 0 & \Omega^1(A) & 0 & \Omega^1(A) & \\
 & \nwarrow d & & \nwarrow d & & \nwarrow d & & \nwarrow \\
A & 0 & A & 0 & A & 0 & A &
\end{array}$$

In the above we have put d for d_2 because it turns out that the latter, i.e. *the map induced by the Connes' boundary operator* B *now coincides with the De Rham derivative* d !!

Thus the third term of our spectral sequence coincides with the stated right side of our formula. The result follows because an easy argument shows (cf. Loday) that this must be the final term. *q.e.d.*

REMARK. Note that though De Rham cohomology is a contravariant functor from simplicial complexes K it is a covariant functor from their polynomial algebras AK. This explains the at first sight curious admixture of homology and cohomology in the above proof.

The most striking point in the above was : *the Connes' boundary map* B *is a lifting to* $C_*(AK)$ *of the De Rham derivative* d *of* $\Omega(AK)$. As Loday points out in his book this fact had actually been observed by Rinehart in a 1963 paper. We would like to point out that this was apparently known even to Poincaré as long ago as 1895 !! For this we will, in the next section, look carefully at a formula given in § 7 of the *Analysis Situs*, and then compare it with the B defined above.

(6.9) Non-commutative differential forms. In this section we'll define these, over any simplicial complex K, and using the P-B-W theorem, give a non-commutative version of De Rham's theorem, the usual one being its skew-symmetrization. By skew symmetrizing partially, i.e. only over rotations, we'll also define a cyclic De Rham cohomology which is related to the ordinary one just like $H_*^{cyc}(K_{assoc})$ was related to $H_*(K)$. We'll also look at the relation between it and the cyclic homology of the algebra of polynomial functions on K.

(*To be written.*)

(6.10) Shifting. We now give a construction of KALAI which associates, to any simplicial complex K, a combinatorially simpler simplicial complex ΔK, having, in each dimension, the same number of simplices, and the same Betti numbers (over a chosen field $\mathbb{F}$) as K.

Here, by *"combinatorially much simpler"*, we mean that the new complex Δ is **shifted**, i.e. that (with respect to some total order of its vertices) if it contains σ, and the equi-dimensional set θ of vertices has respective vertices $\leq$ those of σ, then Δ will also contain θ.

For the construction we'll think of $C^*(K)$, i.e. the vector space of alternating functions from vertK to $\mathbb{F}$ which vanish on sequences of vertices not supported on K, as the obvious quotient algebra of the free **exterior algebra** Λ generated by the vertices over $\mathbb{F}$. We'll later use the following formula for $\delta : C^*(K) \to C^{*+1}(K)$ which obviously holds :

$$\delta\omega \;=\; (\text{sum of all vertices}) \wedge \omega.$$

We will also use a graded algebra automorphism $X : \Lambda \longrightarrow \Lambda$. Note that it is determined by its restriction to the vertices $\{v_1, v_2, \dots, v_N\}$ — this order will be important in the following) — i.e. by the N×N matrix $[c_{ij}]$ over $\mathbb{F}$ defined by $X(v_i) = \sum_j c_{ij} v_j$. Given any subset σ of $\{1, 2, \dots, N\}$ we'll denote by v_σ the wedge, taken in order of increasing indices, of the vertices v_i, $i \in \sigma$. We'll denote $X(v_\sigma)$ also by x_σ : so it is the analogous wedge of the x_i's with $i \in \sigma$.

Under the vector space projection $\Lambda \longrightarrow C^*(K)$ this basis $\{x_\sigma\}$ projects to a graded spanning set of the graded vector space $C^*(K)$. From it *we seive out all elements which depend linearly on the lexicographically preceding elements.* Thus we obtain a basis of $C^*(K)$ which will be our ΔK. We now look at its salient properties.

▷ *In each dimension* ΔK *has as many simplices as* K. Obvious.

▷ *Like* K *this set of simplices* ΔK *is also a simplicial complex.* This follows because wedging a lexicographic dependency with a fixed exterior monomial x_σ gives another lexicographic dependency.

▷ *If* X *is generic then* ΔK *is shifted.* Here by **generic** we mean that the Galois group (of all field automorphisms of $\mathbb{F}$, we'll extend each of these to an $\mathbb{F}_0$– algebra automorphism of Λ in the obvious way) contains the group of all permutations of $\{x_i\}$. *From now on* X *will always be assumed generic.*

The above assertion follows because if we apply, to a q-dimensional lexicographic dependency, a (q, N-q) shuffle, then we still get a lexicographic dependency.

▷ *The qth Betti number of* ΔK *equals the number of its q-simplices* σ *not containing* x_1 *for which* $x_1.\sigma \notin \Delta K$. To see this note that the sphere $\partial(x_1.\sigma) \subseteq \Delta$, and this sphere does not bound, for otherwise we'll have $x_1.\sigma \in \Delta$. Also these cycles $x_1.\sigma$ are easily seen to be independent. In fact ΔK *has the homotopy type of this bouquet of spheres* $V_\sigma\ (x_1.\sigma)$. (The shifted nature of ΔK likewise immensely facilitates the verification of many assertions, so we'll generally be leaving these as simple Exercises from now on.)

Thus the homology of $C^*(\Delta K)$ (considered again as an exterior algebra) equals $\ker(x_1\wedge)/\mathrm{im}(x_1\wedge)$. We now check that same is true for K.

▷ *The homology of* $C^*(K)$ *equals* $\ker(x_1\wedge)/\mathrm{im}(x_1\wedge)$. To see this consider the algebra automorphism of Λ which multiplies each vertex v_i with the ith element of the first row of the matrix $[c_{ij}]$. Clearly this **diagonal automorphism** preserves the defining ideal of the quotient exterior algebra $C^*(K)$. The result follows now from the coboundary formula given above because under this automorphism of $C^*(K)$ the sum of the N vertices becomes x_1.

REMARK. This key property is quite similar to that used by WITTEN in his paper on Morse theory, we'll look at this point more later.

▷ ΔK *has the same Betti numbers as* K. Since it has the same face numbers clearly all that is needed is to check (because of the last two properties) that in each of them $\mathrm{im}(x_1\wedge)$ has the same dimensions. This follows easily because this equals the number of simplices of ΔK which contain the first vertex x_1.

REMARK. It follows that there exists a cochain isomorphism $C^*(K) \longrightarrow C^*(\Delta K)$. Sometimes it is useful (see below) to have an explicit cochain isomorphism : one such is given in my paper in the *Res. Bull. P.U.*, 1993.

As a typical combinatorial application of shifting one has the following result of BJORNER-KALAI.

Theorem. *For any* K *there exists a simplicial complex* E *such that the number of* i*-simplics of* E *is* $\kappa_i + \beta_i$ *of which precisely* β_i *are maximal* i*-simplices.*

Here β_i is the ith reduced Betti number of K and κ_i is the alternating sum $(f_i - \beta_i) - (f_{i+1} - \beta_{i+1}) + \ldots$ (so κ_0 is the usual Euler characteristic). The enumerative significance of the statement "there exists an s.c" of course stems from the Kruskal-Katona theorem which we mentioned previously in (4.6).

Proof. Take $E = \Delta K \setminus \mathrm{St}x_1$. An easy computation shows that this has the required properties. *q.e.d.*

REMARK. Björner-Kalai also showed conversely that the conditions given by the K-K theorem, the above result, and the Euler-Poincaré formula, characterize all pairs of sequences (f,β) which arise from a simplicial complex as (face sequence, Betti sequence). Starting from this a combinatorial analysis then enables them to work out the maximum value of β for a fixed f. This last result — the maximum occurs at the **compressed complexes**, i.e. those closed with respect to anti-lexicographic order — had been found previously by me in the *Jour. Combin. Theory* of 1988.

Problems :

o **22T** series etc, geometric interpns, enumeration ?

o $\mathbb{Z}/2$-minimal complex, $\mathbb{Z}/2$-homology, finding Kuratowski obs there ?

o A metric-dependent definition of homology of a Banach space which for case C(X) will give homology of X ? (cf. see Eilenberg, *Annals* 1942.)

o Connecting Z-Z computations with Hochster, FAC computations, singular Dolbeaut degenerates ?

o Representation of K_{assoc} by generic differential operators, definition of minimal model out of this ? connection with shifting ?

o Proof of de Rham via PBW ?

o Computation of Dyson homology ? Connection with any other, e.g. Poincaré's homology of a lattice ?

o PBW formulable as a homology vanishing theorem ?

o Can $H_*(K) \cong H_*(\Delta K)$ be "explained" by first identifying it with a derived functor ?

MEMO. Lectures 31/8, 2/9 (Po); 7/9, 9/9, 14/9, 16/9 (E-S); 21/9, 23/9 (Po) (first handout upto here); 27/9, 28/9, 3/10, 5/10, 10/10, 12/10 (E-Z/M/Z/Hu); 26/10, 28/10, 2/11, 9/11 (E; E-Z; E-M; P), 11/11 ($\mathbb{Z}/2$) (second handout approx. till here), 16/11 (Po+Kan/Moore), 18/11 (hol), 23/11 (May- 18/11 was hol), 29/11 (on Tue none on 25/11 D.S: May contd), 2/12 (realiz, Segal), 9/12 (7/12 was hol: Olym, Ramsey, realiz contd), 14/12 (vizualizing functors, Zieg-Ziv pf of Gor-Mac, CM, Hoch), 16/12 Ext's depth etc, FAC), 21/12 (K_{assoc}, PBW, Serre, new pf of de Rham ?, Dyson, UBC from old), 23/12 (Ch-Ei, C-E thm using PBW, de Rham sheaf, Thom's) (Total # 28).

Eilenberg's other works: a p Borsuk-Ulam theorem, L-S category of groups with Ganea, a fixed point theorem for multivalued functions with Montogomery, and a fundamental theorem of algebra for quaternions with Niven, etc.

Posnikov pts : p.10 "wild blossoming", n-types, "strongly connected", local coeff vs. "over a 1-cocycle" (multiplicative cohomology sets), why second derived needed.

Brown's 4CT ref., Wan (Thurst. BAMS) for finite field combin/ Weil, Abe's paper re non-Abelian π_q, Whitehead's exposition of duality (star or locally finite – closure finite)

Other BU applications : Zivjalj on Sierksma, Matousek, Tverberg. To complete :

(4.16) $\mathbb{Z}/2$ – minimal complexes
(4.17) Kuratowski obstructions
(4.18) A combinatorial classification theorem
(4.19) Van Kampen – Wu – Shapiro theorem
(4.12) Embedding 2-complex in 4-space