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Minimal Simplicial Self-maps of the 2-Sphere

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Abstract. For any d (resp. for almost all d) we compute the least number $\lambda(\mathrm{d})$ of vertices which a triangulation K of the 2-sphere (resp. any other orientable surface) must have in order that there exists a degree d simplicial map from K to the 4-vertex 2-sphere. We also prove an analogous result for uniquely 4-colourable Ks.

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There is an intimate connection between topology and combinatorics stemming from the fact that many spaces are, at least upto homotopy type, realizations of (finite) simplicial complexes. Though 'finding small triangulations' of such *spaces* is a topic of active current interest, analogous questions about *maps* between them have, somewhat surprisingly, remained neglected. To mention just one, for any homotopy class $\Im$ of maps $X \to Y = |L|$, what is the least number $\lambda_{\mathrm{L}}(\Im)$ of vertices of K, as K runs over all triangulations of X admitting a simplicial map $K \to L$ contained in $\Im$ (the simplicial approximation theorem ensures that such Ks do exist)?

The question just posed seems most alluring for maps into the sphere $Y = S^n$, from any polyhedron X of dimension n. This because now a *theorem of Hopf* – see, e.g., Hu [2], Chapter II, Section 8, for a very readable account – tells us that the homotopy classes $\Im$ are classified by the cohomology group $H^n(X; \mathbb{Z})$. This correspondence depends on the choice of an *orientation* of S^n, and is given by $\Im \leftrightarrow f^*(o^n)$, where o^n denotes the generator of $H^n(S^n; \mathbb{Z}) \cong \mathbb{Z}$ determined by the orientation, and $f : X \to S^n$ is any member of $\Im$. So, if X is an oriented n-manifold, then homotopy classes are classified by the integers $\mathbb{Z}$, by assigning to each its degree $d(\Im)$, defined by $f^*(o^n) = d(\Im) \cdot o^n_X$. Equivalently, $f_*(o^X_n) = d(\Im) \cdot o_n$, where o^X_n and o_n denote the dual generators of the nth homology groups, i.e. the fundamental cycles of X and S^n, respectively, determined by their orientations. So, if $f : K \to L$ is any simplicial map contained in $\Im$, then $d(\Im)$ must be – *only this simple definition of degree is used below* – the algebraic number of n-simplices of K which map onto any of the n-simplices of L.

There is clearly a degree d simplicial map from the double cone of a $3d$-gon to that of a 3-gon, however, as the following result shows, this uses too many vertices. Indeed we compute all the numbers $\lambda_L(\Im) = \lambda(\Im)$ – which we can, by above, write as $\lambda(d)$, where d denotes degree – for the case $X = Y = S^2$, with L being the minimal triangulation $S_4^2 = \{$proper faces of a tetrahedron ABCD$\}$ of the 2-sphere.

THEOREM 1. *If* $|d| \leqslant 1$, *then* $\lambda(d) = 4$; *if* $|d| = 2$, *then* $\lambda(d) = 7$; *and if* $|d| \geqslant 3$, *then* $\lambda(d) = 2 + 2 \cdot |d|$.

Proof. Case $|d| \leqslant 1$ follows because no triangulation of S^2 has less than 4 vertices, and a simplicial map $S_4^2 \to S_4^2$ has degree zero unless it permutes the vertices, and has degree $+1$ or -1, depending upon whether this permutation is even or odd. Since we can compose with such a degree -1 simplicial self-map of S_4^2, it also follows that from here on we can assume that $d \geqslant 2$.

We'll first check that $\lambda(d) \geqslant \max\{7, 2 + 2d\}$ *for* $d \geqslant 2$. Let $g : G \to S_4^2$ be a simplicial map of degree d, and let λ, μ and ν be the number of vertices, edges and triangles of G : we have $2\mu = 3\nu$ and $\lambda - \mu + \nu = 2$ (Euler's formula). The required inequality $\lambda(d) \geqslant 2 + 2d$ is thus equivalent to $\nu \geqslant 4d$, which is obvious, for indeed *there are at least d triangles of* G *which are mapped positively by* g *onto each of the triangles of* S_4^2. This follows because, by definition of degree, the excess over d of these numbers equals the numbers of triangles which are mapped negatively (i.e. in an orientation reversing way) onto the four triangles of S_4^2. Further, we note that two triangles of G sharing an edge can not map positively to the same triangle of S_4^2, i.e. *two triangles mapping positively to the same triangle of* S_4^2 *can share at most one vertex*. This gives $\lambda \geqslant 7$, for, if a vertex of S_4^2 has at least two vertices of G in its pre-image, then each of the other three vertices of S_4^2 has at least two vertices of G in its pre-image.

Conversely, seven vertices suffice for degree $d = 2$. Let G_7 be any 7-vertex triangulation of S^2, having a vertex ∞ which is adjacent to all the others. The orientation of S^2 determines a cyclic ordering of the links of ∞ in G and of D in S_4^2; let the latter be (CBA). Now image ∞ to D, and image every third vertex of the cyclically ordered hexagon Link (∞) to C, B, and A respectively. This gives a simplicial map $g : G_7 \to S_4^2$ of degree $d = 2$. Likewise one can define a degree $d \geqslant 3$ simplicial map g from any analogous $(3d + 1)$-vertex triangulation of S^2, and at first sight it seems one can't do much better than this.

Actually, just 2d+2 vertices suffice for degree $d \geqslant 3$. To define our G we will think of S^2 as $\mathbb{R}^2 \cup \{\infty\}$ and begin with a regular d-gon surrounding the origin 0 of $\mathbb{R}^2$. On each edge of this d-gon we mount a triangle having a distinct new vertex. *When d is even* we will embed these triangles in $\mathbb{R}^2 \backslash \{0\}$ in such a way that the new vertices are alternately in the exterior and the interior of the d-gon $\{$Figure 1(b) shows case $d = 8\}$. *When d is odd*, we will first place three consecutive triangles

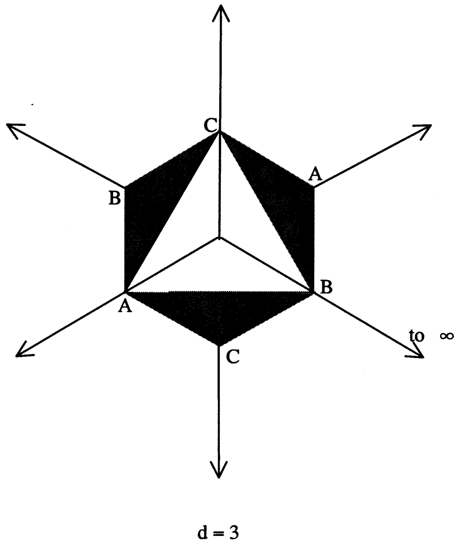

Figure 1a.

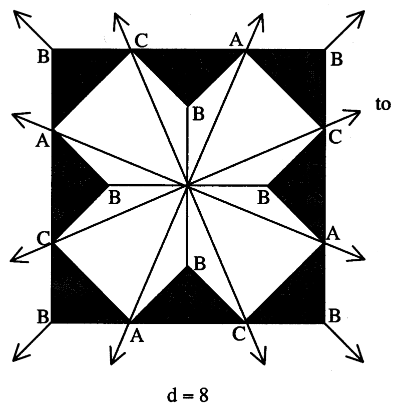

Figure 1b.

in the exterior of the d-gon, and then the remaining d-3 triangles alternately in the interior and exterior (so for case $d = 3$, Figure 1(a), all three triangles are in the exterior). Having embedded this complex consisting of d triangles and their faces in $\mathbb{R}^2\backslash\{0\}$, we now cone its inner boundary over the origin 0 to get a simplicial 2-cell with $2d + 1$ vertices. Next, coning the boundary of this 2-cell over $\{\infty\}$ we obtain a triangulation G_{2+2d} of S^2 having $2d + 2$ vertices.

We'll assume that $S^2 = \mathbb{R}^2 \cup \{\infty\}$ has been given the clockwise orientation. To define the map $g : G_{2+2d} \to S^2_4$ we image both 0 and ∞ to D, and the remaining vertices will all go to (i.e. will be labeled or coloured with) A, B or C. For $d = 3$, this will be done as in Figure 1(a), and for any odd $d > 3$, the 7 vertices of '3 exterior triangles' on three consecutive edges of the d-gon will be labelled similarly, with label A given to both the end points on the d-gon. For odd $d > 3$ the remaining d-9 vertices, and for $d \geqslant 4$ all the vertices other than 0 and ∞, will be coloured as in Figure 1(b), i.e. the vertices of the d-gon are alternately A and C, and all the other vertices go to B. This g has degree d because exactly d triangles (the shaded ones of Figures 1(a) and 1(b)) are imaging to ABC, each in an orientation preserving way. □

The *four colour theorem*, of Kempe, . . . , Appel and Haken [1], says that to the vertices of any triangulation of S^2 one can assign four colours $\{A, B, C, D\}$ in such a way that adjacent vertices are assigned different colours. We note that *the maps* $g : G_{2+2d} \to S^2_4$ *are induced by such four colourings*. Moreover, the case $d = 3$ is particularly nice, for then g is, upto permutations of the colours, the unique four

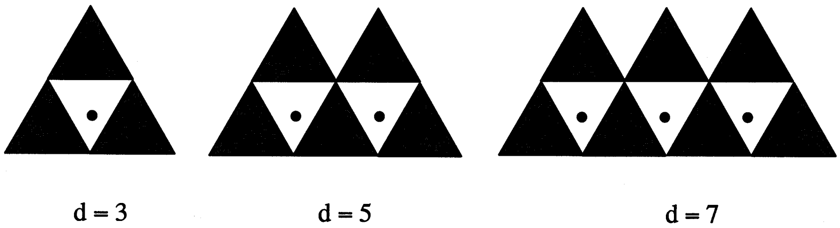

Figure 2.

colouring of G_8. The next result is the analogue of Theorem 1 if one is allowed only to use these very special simplicial maps.

THEOREM 2. *For each odd* $d \geqslant 1$, *one has a uniquely four colourable* $2{+}2d$ *vertex triangulation* H_{2+2d} *of the 2-sphere whose four colourings, considered as simplicial maps* $H_{2+2d} \to S_4^2$, *are of degree* $\pm d$. *Also, for each even* $d \geqslant 2$, *one has a uniquely four colourable 3+2d vertex triangulation* J_{3+2d} *of* S^2 *with four colourings of degree* $\pm d$, *but one has no such triangulation with* 2+2d *vertices.*

Proof. For $d = 1$ take $H_4 = S_4^2$. For odd $d = 3, 5, 7, \ldots$ we start with a 2-cell, as in Figure 2 above, made up of d black, and $\frac{d-1}{2}$ white, equilateral triangular tiles.

We now derive (i.e. stellarly subdivide) the white tiles at their barycenters (as indicated by dots in Figure 2) and then cone the boundary of our 2-cell over ∞. This gives the required $(2 + 2d)$-vertex triangulation H_{2+2d} of S^2 (see $H_8 = G_8$).

It is easily seen that H_{2+2d} has a unique four colouring, under which the barycenters get the same colour as ∞, while the d black tiles all get the remaining three colours, and are oriented concordantly by a cyclic ordering of these three colours (the uniqueness being of course upto a permutation of the colours). We remark that topologically our 4-colouring $H_{2+2d} \to S_4^2$ is a d-fold covering, *branched* over the 4-vertices of S_4^2, each of which has $\frac{d+1}{2}$ pre-images.

For any odd $d \geqslant 3$, we define $J_{3+2(d-1)}$ like H_{2+2d} except that *we do not derive one of the white tiles* (note that $J_7 = G_7$). Again, it is easily seen that it has a unique four colouring under which ∞ and the barycentres get the same colour. Now, besides the d black tiles, the underived white tile also gets the remaining 3 colours, but gets oriented oppositely to the d black tiles, so degree is $\pm(d - 1)$.

For the last part we have no elementary argument, however it follows easily from an old conjecture of Fiorini and Wilson, recently proved by Fowler – see [5], pp. 857–858 – via an improvement on the computer assisted proof of the four colour theorem: *a triangulation* U *of* S^2 *is uniquely four colourable iff it is obtainable from* S_4^2 *by repeatedly deriving triangles*. Since each such derivation changes the degree of the associated four colouring by ± 1, we see that if the number of vertices of U is even, this degree has to be odd. □

We turn now to the more general problem of computing $\lambda(\Im) = \lambda_L(\Im)$ when X is any (closed, connected) surface M^2, with Y and L being again S^2 and S^2_4. First, recall that the least number of vertices required to triangulate M^2, i.e. $\lambda(\Im)$ for the trivial homotopy class $M^2 \to S^2$, is given by *the map colour theorem* of Heawood, . . ., Youngs and Ringel [4]. Indeed, *for a non-orientable M^2, the map colour theorem theorem settles the problem of computing $\lambda(\Im)$ for all homotopy classes $M^2 \to S^2$*. This follows because $H^2(M^2; \mathbb{Z}) \cong \mathbb{Z}/2$, so there are just two homotopy classes, and, by labelling all the vertices D, except those of one triangle, which are labelled A, B, C, one can define a cohomologically non-trivial simplicial map from any triangulation of M^2 to S^2_4.

However, for an *orientable surface M^2*, the problem of computng $\lambda(\Im)$ for all homotopy classes $M^2 \to S^2$ is much deeper than the map colour theorem, but the asymptotic part of theorem 1 can be generalized quite independently of the map colour theorem.

THEOREM 3. *For any orientable surface M^2, and for almost all homotopy classes $\Im$ of maps $M^2 \to S^2$, one has $\lambda(\Im) = 2|d(\Im)| + 2 - 2\gamma(M^2)$ where $\gamma(M^2)$ denotes the genus of the surface M^2.*

Proof. As before, we can assume $d > 0$, and $\lambda \geqslant 2d + 2 - 2\gamma$ follows because, by Euler's formula, it is equivalent to the obvious inequality $v \geqslant 4d$. We want to show that there exist a minimum C_γ such that for all degrees $d \geqslant C_\gamma$ one has $\lambda = 2d + 2 - 2\gamma$. The case $\gamma = 0$ has already been done and we know from Theorem 1 that $C_0 = 3$. We'll now assume $\gamma \geqslant 1$ and show below that $C_\gamma \leqslant 9(2\gamma - 1)$.

The equality $\lambda = 2d + 2 - 2\gamma$ holds for $d = 9(2\gamma - 1)$. To see this we recall that the genus γ surface can be obtained by identifying the opposite sides of a 4γ-gon. As this 4γ-gon we take a rectangle of size $1 \times (2\gamma - 1)$ with perimeter subdivided into 4γ edges of length 1, two of these being sides of this rectangle. We first triangulate M^2 by subdividing the rectangle into 9 $(2\gamma - 1)$ squares of size $1/3 \times 1/3$ by lines parallel to its sides, and then further into twice as many triangles by parallel 45° lines (Figure 3(a) shows the case $\gamma = 2$).

Next we assign three labels $\{A, B, C\}$ to the vertices of this triangulated rectangle in such a way that adjacent vertices have different labels (Figure 3(b) shows the case $\gamma = 2$). We note that this three colouring is unique upto permutations, and that it assigns the same label to vertices of the perimeter which get identified when we pass from the rectangle to M^2. The cyclic order (ABC) assigns the counter clockwise orientation to half the triangles (shown shaded in Figure 3(b)). We derive the remaining triangles (the new vertices are indicated by dots in Figure 3(b)) and assign the label D to these new vertices, to obtain the desired labelled triangulation $\mathbb{K}$ of M^2 having $1 + 17(2\gamma - 1)$ vertices, whose labelling determines a simplicial map $\mathbb{K} \to S^2_4$ of degree $9(2\gamma - 1)$. This follows because precisely $9(2\gamma - 1)$ triangles (the shaded ones) map onto ABC, all positively.

The equality $\lambda = 2d + 2 - 2\gamma$ holds also for all d bigger than $9(2\gamma - 1)$. We construct, for each $k \geqslant 1$, a labelled triangulation $\mathbb{K}_k$ of M^2 by modifying $\mathbb{K}$ only within

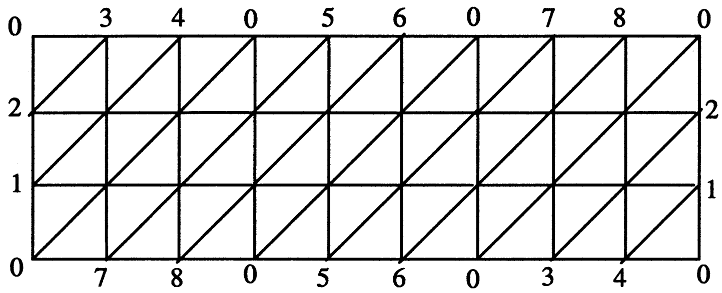

Figure 3a.

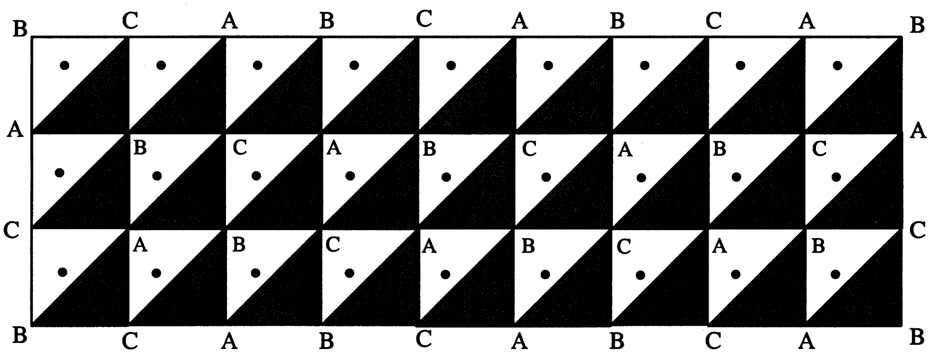

Figure 3b.

'two squares': for $k = 1$ this 'local modification' is shown in Figure 4a; for *even* $k \geqslant 2$ we stretch a vertex into an edge on which we mount k shaded triangles facing alternately up and down as in Figure 4(b); while for *odd* $k \geqslant 3$ the first three face the same way and then the remaining ones alternate as in Figure 4(c) (all this is just as in the proof of Theorem 1). As before, it is understood that the boundary of the unshaded cell is to be coned over the 'D' vertex inside it, while the other vertices are labelled A, B, C, such that (ABC) assigns the anti-clockwise orientation to each shaded triangle.

The required result now follows because $\mathbb{K}_k$ has $2k$ vertices more than $\mathbb{K}$, and its labelling determines a simplicial map $\mathbb{K}_k \to S_4^2$ of degree k more than that of the map $\mathbb{K} \to S_4^2$. □

The bound $C_\gamma \leqslant 9\,(2\gamma - 1)\ \forall\ \gamma \geqslant 1$ proved above is not the best possible, e.g. Figure 5(a) shows a 12 vertex degree 6 labelled triangulation of the torus. Using its $(2\gamma - 1)$-fold concatenation – Figure 5(b) depicts this for the orientable surface with $\gamma = 2$ – instead of $\mathbb{K}$, the above proof gives the *improved bound* $C_\gamma \leqslant 6$

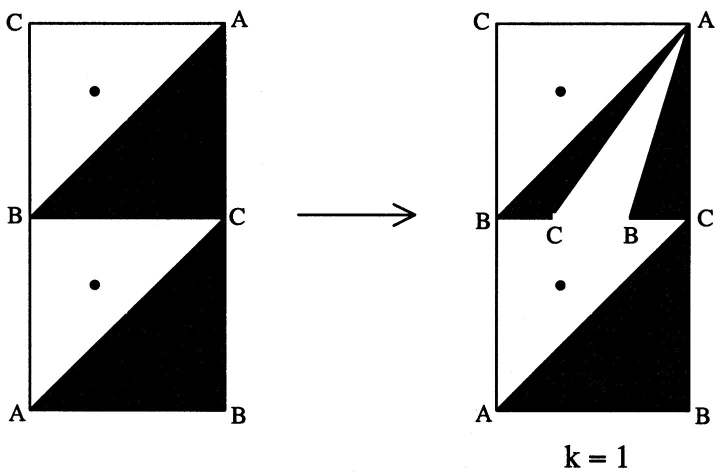

Figure 4a.

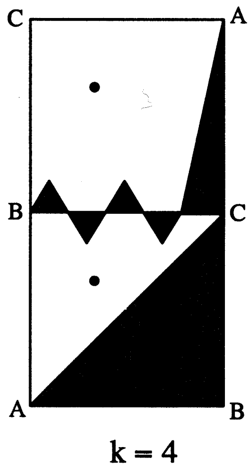

Figure 4b.

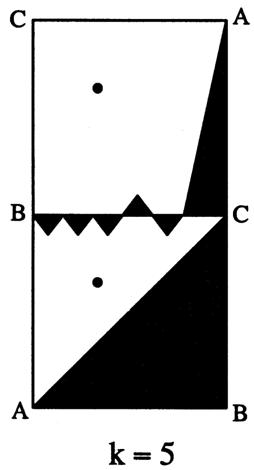

Figure 4c.

$(2\gamma - 1)\ \forall\ \gamma \geqslant 1$. For $\gamma = 1$ it is easily checked that this is the best possible, i.e. that $C_1 = 6$; however for $\gamma \geqslant 2$, there is room for further improvement.

It seems to us that the task of understanding all triangulations of a surface for which $\lambda = 2d + 2 - 2\gamma$ holds, and so of determining C_γ for all γ, is probably much easier than that of computing $\lambda(\Im)$ for all homotopy classes $M^2 \to S^2$. This because such triangulations are very special, e.g. they are precisely those which admit a (necessarily unique upto permutations) *'nice' four colouring*, i.e. one in which all the four vertices in the closed star of any edge have different colours. Also, the

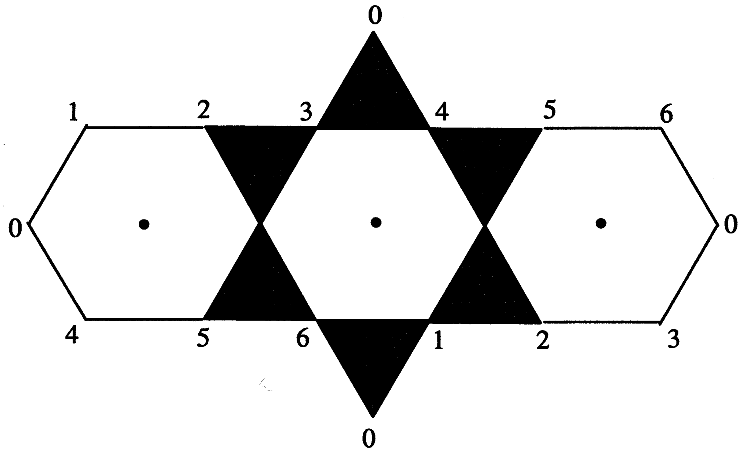

Figure 5a.

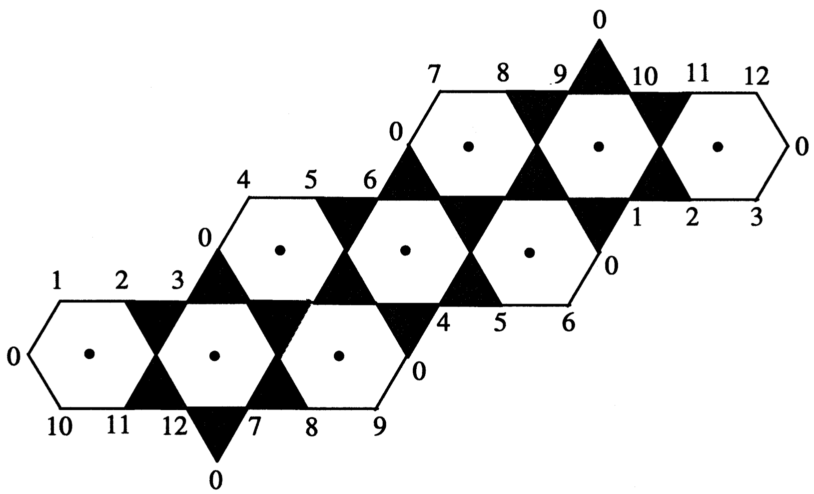

Figure 5b.

valence of any vertex must be a multiple of 3, and the simplicial map to S_4^2 given by 'nice' four colouring exhibits the surface as a d-fold covering of S^2 branched only at some verticies.

Such a branched covering f of $S^2 = \hat{\mathbb{C}}$ becomes complex analytic if we equip M^2 with the pulled-back complex structure, i.e. if we use, as a local complex coordinate, in the star of any vertex v of valence $3k$, the kth roots of a local complex coordinate in $St(f(v)) \subset \hat{\mathbb{C}}$. Thus the above simplicial maps are *minimal triangulations* (up to homeomorphisms) *of meromorphic functions*, e.g. the 8-vertex 2-sphere G_8 arises like this from the degree 3 rational function $f(z) = z^2/2 - 1/z$ (these remarks respond to questions which were posed to us by the referee). To see this note that $f(z)$ is regular everywhere in $\hat{\mathbb{C}}$, except for $-1, -\omega, -\omega^2$ and ∞, which are double points. At these, and at $2, 2\omega, 2\omega^2$ and 0 respectively, it takes the values $3/2, 3\omega^2/2, 3\omega/2$ and ∞. The 4 vertices of the triangulation S_4^2 of $\hat{\mathbb{C}}$ must be at these 4 singular values; as its 6 edges we take the 3 edges of the triangle $\{3/2, 3\omega^2/2, 3\omega/2\}$ and the 3 rays from the vertices of this triangle to ∞. The subdivision of $\hat{\mathbb{C}}$ is given by the inverse images of the simplices of S_4^2 is preserved by rotation through 120° – because $f(\omega z) = \omega^2 f(z)$ – and is easily verified to be a G_8. For example the triangle $\{3/2, 3\omega^2/2, 3\omega/2\}$ of S_4^2 pulls back to three curved triangles $\{2, -\omega, -\omega^2\}$, $\{-1, 2\omega, -\omega^2\}$ and $\{-1, -\omega, 2\omega^2\}$ of G_8, each of which contains one of the three simple zeros $2^{1/3}, 2^{1/3}\omega$ and $2^{1/3}\omega^2$ of $f(z)$, while the ray $\{3/2, \infty\}$ pulls back to the three straight edges $\{\infty, -1\}$, $\{-1, 0\}$ and $\{2, \infty\}$ of G_8.

Regarding homotopy classes $\Im$ of maps $S^k \to S^k, k \neq 2$, one obviously has $\lambda(\Im) = \max\{3, 3|d(\{\Im)|\})$ for $k = 1$, but for $k \geqslant 3$ we only have some estimates for these numbers $\lambda(\Im)$. Likewise, we have only partial results for homotopy classes of maps $S^3 \to S^2$, e.g. in [3] we showed that the Hopf map can be triangulated by 12 vertices, and used this to give a new construction of Kühnel's $\mathbb{C}P_9^2$.

Acknowledgement

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