SIERKSMA'S DUTCH CHEESE PROBLEM

K. S. Sarkaria

Abstract. Consider partitions of a cardinality $(q-1)(d+1)+1$ generic subset of euclidean d-space, into q parts whose convex hulls have a nonempty intersection. We show that if these partitions are counted with appropriate signs ± 1 then the answer is always $((q-1)!)^d$. Also some other related results are given.

1. Introduction.

The object of this note is to prove the following, thus establishing a conjecture of Sierksma [10], 1979.

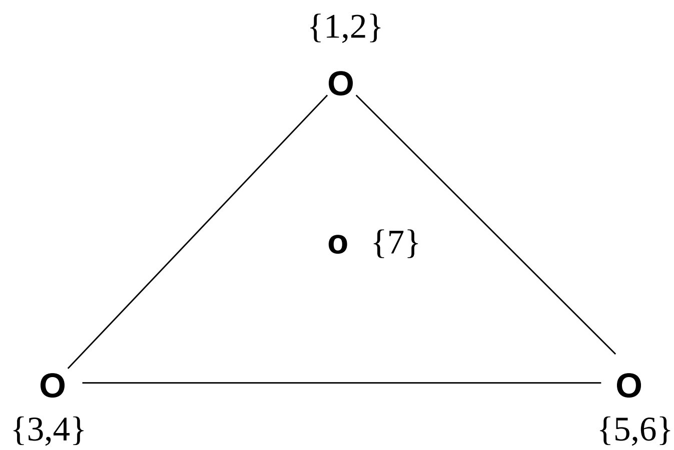
Theorem 1. *Let S be any cardinality $(q-1)(d+1)+1$ subset of a real affine d-dimensional space $\mathbb{A}^d$. Then there exist at least $((q-1)!)^d$ partitions of S into q disjoint subsets, $S = \sigma_1 \cup \cdots \cup \sigma_q$, with* $\operatorname{conv}(\sigma_1) \cap \cdots \cap \operatorname{conv}(\sigma_q)$ *nonempty.*

We remark that even the existence of *one* such partition is not obvious : it was conjectured by Birch [2] in 1958 and confirmed by Tverberg [13] in 1966 only by means of a fairly involved argument (but see also remark (e) of §5). Henceforth we will refer to partitions of the above kind as *Tverberg partitions* of S.

The proof of the above theorem is given in §4 and depends on Lemma 1 of §2 which verifies that the Euler number of a certain complex vector bundle $\mathfrak{L}^\perp$ on complex projective $(q-1)(d+1)$-space is $((q-1)!)^{d+1}$, and on Lemmas 2-4 of §3 which serve to relate the Tverberg partitions of S with the zeros of a section s of $\mathfrak{L}^\perp$. (For basic facts regarding characteristic classes see Steenrod [11], Hirzebruch [3], and Milnor-Stasheff [5].) We in fact arrive at a precise *index formula* which shows that if one counts the Tverberg partitions of a generic S with appropriate signs $\pm$ then the answer is always $((q-1)!)^d$. We note in §5 that our method also establishes the so-called *"continuous" Tverberg theorem for all q.*

The bound given by the above theorem is the best possible as can be seen by using Sierksma's configuration S_0 : take $q-1$ coincident points at each of $d+1$ affinely independent positions, and let the last point be at the barycenter (the case $d=2$, $q=3$ is shown below).

Part of this work was done at the Mathematical Sciences Research Institute, Berkeley. Research at MSRI is supported in part by NSF grant DMS-9022140.



To obtain a Tverberg partition of S_0, totally order each set of coincident points, and for each $1 \leq i \leq q-1$, let σ_i consist of the $d+1$ points which occur at the ith places in these orderings, and let $\sigma_q = \{\text{barycenter}\}$. The number of these partitions is $((q-1)!)^d$ and it is clear that there is no other Tverberg partition of S_0.

2. A Chern number

We denote by $\mathbb{Z}/q$ the group of the qth roots of unity

$$\{1, \omega = \exp(2\pi i/q), \omega^2, \ldots, \omega^{q-1}\},$$

and consider the *regular representation* $\mathbb{V}$ of this group, i.e. the vector space of all q-tuples of complex numbers equipped with the action of ω given by

$$(z_1, z_2, \ldots, z_q) \longmapsto (z_2, \ldots, z_q, z_1).$$

Now $\mathbb{V}$ is a direct sum $\oplus_{0 \leq k < q} \mathbb{V}_k$ of q inequivalent one dimensional representations, with the kth being given by

$$\mathbb{V}_k \;=\; \{(z_1, z_2, \ldots, z_q) : z_{i+1} = \omega^k . z_i \forall i\}.$$

We note that the diagonal $\mathbb{V}^0 = \mathbb{V}_0$ is given by $z_1 = \cdots = z_q$, while its complement $\mathbb{V}^\perp = \oplus_{1 \leq k < q} \mathbb{V}_k$ is given by $z_1 + z_2 + \cdots + z_q = 0$.

We note next that $\mathbb{Z}/q$ is a subgroup of the group S^1 of complex numbers of absolute value 1, and that its action on $\mathbb{V}_k$ extends in a natural way to an action of S^1 (or even of $\mathbb{C}^\times$) on $\mathbb{V}_k$:

$$g \bullet (z_1, z_2, \ldots, z_q) \;=\; (g^k z_1, g^k z_2, \ldots, g^k z_q) \;\; \forall g \in S^1.$$

Using this we'll consider each $\mathbb{V}_k$, and so also their direct sum $\mathbb{V}$, as a representation of S^1 (or even of $\mathbb{C}^\times$), and the $(d+1)$-fold direct sum $\mathbb{V} \oplus \cdots \oplus \mathbb{V}$ will be denoted $\mathbb{L}$. Likewise $\mathbb{L}^0 = \mathbb{V}^0 \oplus \cdots \oplus \mathbb{V}^0$ and $\mathbb{L}^\perp = \mathbb{V}^\perp \oplus \cdots \oplus \mathbb{V}^\perp$.

We now consider the *Hopf bundle* ξ, i.e. the principal S^1-bundle over $\mathbb{C}P^N$, $N = (q-1)(d+1)$, whose total space is the sphere

$$S^{2N+1} = \{(z_1, \ldots, z_{N+1}) \in \mathbb{C}^{N+1} : |z_1| + \cdots + |z_{N+1}| = 1\}$$

of $\mathbb{C}^{N+1}$, with the action of S^1 being given by multiplication. We denote by $\mathfrak{L}$ the quotient of $S^{2N+1} \times \mathbb{L}$ by the diagonal S^1-action, i.e. the total space of the complex vector bundle $\mathfrak{L} \to \mathbb{C}P^N$ associated to ξ by the S^1-representation $\mathbb{L}$.

Lemma 1. *If $c_N(\mathfrak{L})$ denotes the Nth Chern class of $\mathfrak{L}$, then*

$$< c_N(\mathfrak{L}), \mathbb{C}P^N > = \ ((q-1)!)^{d+1}.$$

Proof. We will use the fact that the first Chern class $c_1(\xi) \in H^2(\mathbb{C}P^N; \mathbb{Z})$ generates the cohomology ring $H^*(\mathbb{C}P^N; \mathbb{Z})$, i.e. that $H^*(\mathbb{C}P^N; \mathbb{Z})$ consists of all integral multiples of its powers $(c_1(\xi))^j$.

Let $\mathfrak{V}_k$ be the complex line bundle associated to ξ by the irreducible representation $\mathbb{V}_k, 0 \leq i < q$. We note that $\mathfrak{L}$ is a direct sum of $d+1$ copies of each of these q line bundles (so it has complex fibre dimension $N + (d+1)$). Thus $c_N(\mathfrak{L}) \in H^{2N}(\mathbb{C}P^N; \mathbb{Z})$ is the $(d+1)$th power of the cohomology class

$$c_1(\mathfrak{V}_1) \cdot c_1(\mathfrak{V}_2) \cdot \ldots \cdot c_1(\mathfrak{V}_{q-1}) \in H^{2(q-1)}(\mathbb{C}P^N; \mathbb{Z}).$$

Thus what we need to check is that this is $(q-1)!$ times the generator $(c_1(\xi))^{q-1}$ of $H^{2(q-1)}(\mathbb{C}P^N; \mathbb{Z}) \cong \mathbb{Z}$. This follows because $c_1(\xi) = c_1(\mathfrak{V}_1)$ and $c_1(\mathfrak{V}_k \cong \otimes^k \mathfrak{V}_1) = k.c_1(\xi)$. *q.e.d.*

We note that $c_N(\xi)$ is also the *Euler class* of the vector bundle $\mathfrak{L}^\perp \to \mathbb{C}P^N$ associated to ξ by the sub representation $\mathbb{L}^\perp$, and what we have calculated is the *Euler number* of this oriented vector bundle.

3. A deleted join

We will denote by K the simplicial complex consisting of all subsets of the standard basis $\{e_1, \ldots, e_{N+1}\}$ of $\mathbb{C}^{N+1}$. The sphere S^{2N+1} is the join of the circles ge_α, $g \in S^1$, i.e. it consists of all points $P = \sum_\alpha t_\alpha(P) g_\alpha(P) e_\alpha$. Here (and in all similar sums) α runs over some subset $[P]$ of cardinality $|P|$ of $\{1, 2, \ldots, N+1\}$, $g_\alpha(P) \in S^1$, and the $t_\alpha(P)$'s are positive reals with sum 1. (Frequently we'll drop the (P)'s and simply write $\sum_\alpha t_\alpha g_\alpha e_\alpha$ etc.) Collecting together terms having equal g_α's we will also sometimes write $P = \sum_{i=1}^{i=r(P)} t_i(P) g_i(P) x_i(P)$ where $1 \leq r(P) \leq |P| \leq N+1$ denotes the number of *distinct* $g_\alpha(P)$'s, $t_i(P) = \sum_{g_\alpha = g_i} t_\alpha(P)$'s are positive numbers having sum 1, and $x_i(P) = \frac{\sum_{g_\alpha = g_i} t_\alpha(P) e_\alpha}{\sum_{g_\alpha = g_i} t_\alpha(P)}$'s are points belonging to *pairwise disjoint* faces of the geometrical simplex $|K| = \mathrm{conv}\{e_1, \ldots, e_{N+1}\}$.

Recall now that the simplices $(\sigma_1, \sigma_2, \ldots, \sigma_q)$ of the join $K \cdot K \cdot \ldots \cdot K$ of q disjoint copies of K are obtained by taking unions of q simplices one from each copy. We equip it with the $\mathbb{Z}/q$-action $(\sigma_1, \sigma_2, \ldots, \sigma_q) \mapsto (\sigma_2, \ldots, \sigma_q, \sigma_1)$. The q-fold *deleted join* $K * K * \cdots * K$ is the free $\mathbb{Z}/q$-subcomplex of $K \cdot K \cdot \ldots \cdot K$ consisting of all simplices $(\sigma_1, \sigma_2, \ldots, \sigma_q)$ for which the σ's are disjoint in K.

Taking the jth copy, $1 \leq j \leq q$, of K to be that having vertices $\omega^{j-1} e_\alpha$, we will identify the geometrical realization of $K * \cdots * K$ with the free $\mathbb{Z}/q$-subspace of S^{2N+1}

consisting of all points $P = \sum_{i=1}^{i=r(P)} t_i(P)g_i(P)x_i(P)$ with $g_i(P) \in \mathbb{Z}/q \subset S^1$ (so $r(P) \leq q$ here). We'll denote by $K\#K\#\cdots\#K$ the q-fold *deleted product* of K, i.e. the subspace of $|K*\cdots*K|$ consisting of all points P of the form $\sum_{i=1}^{q} \frac{1}{q} g_i(P)x_i(P)$, $g_i(P) \in \mathbb{Z}/q$.

We will consider $\mathbb{L} = \mathbb{V} \oplus \cdots \oplus \mathbb{V}$ as *the vector space* $\mathbb{C}^{d+1} \otimes \mathbb{V}$ *of all* $(d+1) \times q$ *matrices* over $\mathbb{C}$, and $\mathbb{A}^d$ (*resp.* $\mathbb{R}^{d+1}$, *resp.* $\mathbb{C}^{d+1}$) as the affine (*resp.* linear, *resp.* complex linear) span in $\mathbb{L}$ of the $d+1$ matrices whose sole nonzero entry is 1 and lies in the first column. A matrix will often be denoted by the sequence of its column vectors.

The given points $S = \{s_1, \ldots, s_{N+1}\}$ of $\mathbb{A}^d \subset \mathbb{L}$ determine the affine linear map $|K| \xrightarrow{s} \mathbb{A}^d$, $s(e_\alpha) = s_\alpha$, which we'll extend to the continuous S^1-map $s : S^{2N+1} \to \mathbb{L}$ defined by $s(P) = \sum_\alpha t_\alpha(P)(g_\alpha(P) \bullet s_\alpha)$. The decomposition $\mathbb{L} = \mathbb{L}^0 \oplus \mathbb{L}^\perp$ then determines an S^1-map $s : S^{2N+1} \to \mathbb{L}^\perp$ whose zeros are the principal items of interest for us.

Lemma 2. *Any zero P, $s(P) = 0$, of the S^1-map $s : S^{2N+1} \to \mathbb{L}^\perp$ must either lie on an S^1-orbit passing through the q-fold deleted product $K\#\cdots\#K$, or else be such that $r(P) > q$ with the projections of the points $\{g_\alpha(P) \bullet s_\alpha : \alpha \in [P]\}$* not *in general position on $\mathbb{L}^\perp$.* (Here as usual "general position" means that any $i \leq$ dimension of the vectors are linearly independent.)

Proof. Using the definitions of the matrix space $\mathbb{L}$ and the S^1-map $s : S^{2N+1} \to \mathbb{L}$ we have

$$s(\sum_\alpha t_\alpha g_\alpha e_\alpha) = \frac{1}{q} \sum_{l=0}^{q} \{\sum_\alpha t_\alpha {g_\alpha}^l s_\alpha (1, \omega^l, \ldots, \omega^{l(q-1)})\}.$$

The orbit of $P = \sum_\alpha t_\alpha g_\alpha e_\alpha \in S^{2N+1}$ is mapped by s to a single point iff the right side of the above equation is in $\mathbb{L}^0$, i.e. iff only the first of its summands is nonzero, i.e. iff

$$\sum_{\alpha=1}^{N+1} t_\alpha {g_\alpha}^l s_\alpha = 0, \quad 1 \leq l \leq q-1.$$

Alternatively, using $P = \sum_{i=1}^{r} t_i g_i x_i$ (recall that here the g_i's are distinct), these equations read

$$\sum_{i=1}^{r} t_i {g_i}^l y_i = 0, \quad 1 \leq l \leq q-1,$$

where $y_i = s(x_i)$.

CASE $r(P) < q$. This cannot happen because now the equations $\sum_{i=1}^{r} z_i {g_i}^l = 0$, $1 \leq l \leq q-1$, have only the trivial solution, whereas $t_i y_i \in \mathbb{R}^{d+1} \setminus \{0\}$.

CASE $r(P) = q$. We will denote the Vandermonde determinant $|{g_i}^l|$, $1 \leq l \leq q-1$, $1 \leq i \leq q-1$, by $\Delta(g_1, \ldots, g_{q-1})$ or just Δ. It is related to its complex conjugate by

$$\overline{\Delta(g_1, \ldots, g_{q-1})} = (-1)^{\binom{q-1}{2}} (g_1 \cdots g_{q-1})^{-q} \Delta(g_1, \ldots, g_{q-1}).$$

This follows from

$$\Delta(g_1, \ldots, g_{q-1}) = g_1 \cdots g_{q-1} \prod_{i>j} (g_i - g_j)$$

and the fact that the complex conjugate of any $g \in S^1$ is the same as its inverse.

By Cramer's rule any solution of the system of equations $\sum_{i=1}^{q} z_i {g_i}^l = 0$, $1 \leq l \leq q-1$, is of the form $z_m = -\frac{\Delta_{m,q}}{\Delta} z_q$, $1 \leq m \leq q$, with z_q arbitrary. Here $\Delta_{m,q}$ denotes the determinant obtained from Δ by replacing g_m by g_q. *These determinantal ratios are all real only if $\{g_1, \ldots, g_q\}$ is a coset of $\mathbb{Z}/q$ in S^1.* This is so because by above the complex conjugate of $\frac{\Delta_{m,q}}{\Delta}$ is $(\frac{g_m}{g_q})^q \frac{\Delta_{m,q}}{\Delta}$ and so for reality we must have $(\frac{g_m}{g_q})^q = 1$.

Since $t_i y_i \in \mathbb{R}^{d+1}$ it follows that $\{g_1, \ldots, g_q\} = g^{-1} \cdot \mathbb{Z}/q$ for some S^1. In this case any solution of our system of linear equations satisfies $z_1 = \cdots = z_q$, so the $t_i y_i$'s must be equal to each other, which implies that the points $y_i \in \mathbb{A}^d$ must coincide and the t_i's must be equal to each other. So our orbit contains the point $\frac{1}{q} g g_1 x_1 + \cdots + \frac{1}{q} g g_q x_q$ of $K\# \ldots \#K$.

CASE $r(P) > q$. If the projections of the $|P|$ vectors $\{g_\alpha(P) \bullet s_\alpha : \alpha \in [P]\}$ on the N-dimensional vector space $\mathbb{L}^\perp$ are in general position, $\sum_{\alpha \in [P]} t_\alpha(P)(g_\alpha(P) \bullet s_\alpha)^\perp = 0$ is possible only when $|P| = N+1$, which we'll assume from here on.

We will denote the kth coordinate of $s_\alpha \in S \subset \mathbb{A}^d \subset \mathbb{R}^{d+1}$ by $s_{k,\alpha}$, and by $D(P,S)$ or just D the $N \times N$ determinant $|{g_\alpha}^l s_{k,\alpha}|$, with $1 \leq \alpha \leq N$ indexing the columns, and the rows indexed, in lexicographic order, by the ordered pairs (k,l), $1 \leq k \leq d+1$, $1 \leq l \leq q-1$. Grouping together terms involving the first $q-1$ rows, then the next $q-1$ rows, etc., we see that it has Laplace expansion

$$D = \sum_{\pi} (-1)^\pi \{\Delta(g_{\pi_1}, \ldots, g_{\pi_{q-1}}) s_{1,\pi_1} \cdots s_{1,\pi_{q-1}}\} \cdot \{\cdots$$

$$\cdots\} \cdot \{\Delta(g_{\pi_{N-q+1}}, \ldots, g_{\pi_N}) s_{d+1,\pi_{N-q+1}} \cdots s_{d+1,\pi_N}\},$$

where π runs over all permutations of $\{1, \ldots, N\}$ such that $\pi_1 < \cdots < \pi_{q-1}$; $\pi_q < \cdots < \pi_{2(q-1)}$; etc. So using the conjugation rule of Vandermonde determinants, we see that

$$\overline{D} = (-1)^{(d+1)\binom{q-1}{2}} (g_1 \cdots g_N)^{-q} D.$$

More generally we'll denote by $D_{m,N+1}$, $1 \leq m \leq N+1$, the determinant obtained by replacing the mth column of D by $[{g_{N+1}}^l s_{k,N+1}]$ (so $D_{N+1,N+1} = D$) and use their Laplace expansions and conjugation rules too.

We'll equip $\mathbb{L}$ with the *Fourier basis* consisting of the matrices $M_{l,k}$: all rows zero except the kth which equals $(\frac{1}{q}, \frac{1}{q}\omega^l, \ldots, \frac{1}{q}\omega^{l(q-1)})$. Note that $(g_\alpha \bullet s_\alpha)^\perp = \sum_{\substack{1 \leq l < q \\ 1 \leq k \leq d+1}} {g_\alpha}^l s_{k,\alpha} M_{l,k}$; so our general position hypothesis is equivalent to saying that *all* the determinants $D_{m,N+1}$ are nonzero. So (this step works even if *any one* of the determinants is nonzero) the real solution $< t_\alpha >$ of the N equations

$\sum_{\alpha=1}^{N+1} t_\alpha {g_\alpha}^l s_{k,\alpha} = 0$ obeys $t_m = -\frac{D_{m,N+1}}{D} t_{N+1}$ for $1 \leq m \leq N$. But these determinantal ratios are real only if $(\frac{g_m}{g_{N+1}})^q = 1\ \forall m$ which is impossible because there are $r(P) > q$ distinct g_α's. *q.e.d.*

REMARK 1. *The above determinant $D(P,S)$ is zero for all $S \subset \mathbb{A}^d \subset \mathbb{R}^{d+1}$ if and only if $P \in S^{2N+1}$, $|P| = N+1$, is such that a $g_\alpha(P)$ repeats more than $d+1$ times for* $1 \leq \alpha \leq N$ (likewise for the other determinants $D_{m,N+1}$). 'If' is obvious, and for 'only if' note that if no g_α repeats more than $d+1$ times, then the $N = (d+1)(q-1)$ α's can be partitioned off into $d+1$ parts, of cardinality $q-1$ each, such that the g_α's corresponding to each part are distinct. Assume without loss of generality that $\{g_1, \ldots, g_{q-1}\}$ are all distinct, $\{g_q, \ldots g_{2(q-1)}\}$ are all distinct, and so on. Now, choosing $s_1 = \cdots = s_{q-1} = (1, 0, \ldots, 0)$, $s_q = \cdots = s_{2(q-1)} = (0, 1, 0, \ldots, 0)$ and so on, we get $D(P,S) = \Delta(g_1, \ldots, g_{q-1}) \cdots \Delta(g_{N-q+1}, \ldots, g_N) \neq 0$. *q.e.d.*

We'll denote by $\mathbb{L}_\mathbb{R}$ and $\mathbb{L}_\mathbb{R}^\perp$ the real subspaces of $\mathbb{L}$ and $\mathbb{L}^\perp$ consisting of all *real matrices*. Note that a matrix $\lambda \in \mathbb{L}$ is real iff its Fourier components $\lambda_{l,k}$, with respect to the basis $\{M_{l,k}\}$, satisfy the conditions $\lambda_{l,k} = \overline{\lambda_{q-l,k}}$. Indeed, as follows at once from $\lambda = \sum_{l,k} \lambda_{l,k} M_{l,k}$ and $M_{l,k} = \overline{M_{q-l,k}}$, the complex conjugate $\overline{\lambda}$ of any $\lambda \in \mathbb{L}$ has Fourier components $\overline{\lambda}_{l,k} = \overline{\lambda_{q-l,k}}$. Note that these real subspaces are preserved only by the $\mathbb{Z}/q$ action, and likewise that complex conjugation commutes with the $\mathbb{Z}/q$, but not with the S^1-action. Generalizing an argument used in the proof of the previous lemma we'll now check the following.

Lemma 3. *A point $P \in S^{2N+1}$ lies on an S^1-orbit passing through the q-fold deleted join if and only if there exist $v_\alpha \in \mathbb{L}_\mathbb{R}^\perp$, $\alpha \in [P]$, with* $\operatorname{rank}\{g_\alpha(P) \bullet v_\alpha : \alpha \in [P]\} = |P| - 1$ *and* 0 *lies in the open convex hull of $\{g_\alpha(P) \bullet v_\alpha : \alpha \in [P]\}$.*

Proof. Given any $P \in S^{2N+1}$ we can certainly find $|P|$ affinely independent vectors $w_\alpha \in \mathbb{L}_\mathbb{R}^\perp$ such that 0 lies in their open convex hull. (Moreover, these vectors can be so chosen that 0 has any prescribed positive barycentric coordinates t_α, say $t_\alpha = t_\alpha(P)$, with respect to them.) When $P \in |K * \cdots * K|$, i.e. when $g_\alpha(P) \in \mathbb{Z}/q$, we now get the required $v_\alpha \in \mathbb{L}_\mathbb{R}^\perp$ by solving $w_\alpha = g_\alpha(P) \bullet v_\alpha$. The same v_α's will work also for any other point gP of the orbit $\overline{P}$ through P.

Conversely, writing the given convex dependency $\sum_{\alpha \in [P]} t_\alpha (g_\alpha(P) \bullet v_\alpha) = 0$ in components, we get $\sum_{\alpha \in [P]} t_\alpha {g_\alpha}^l v_{\alpha,l,k} = 0$, $1 \leq k \leq d+1$, $1 \leq l < q$, i.e. N equations in $|P|$ positive variables $< t_\alpha >$. Since $v_\alpha \in \mathbb{L}_\mathbb{R}^\perp$, multiplying the αth column of the coefficient matrix $[{g_\alpha}^l v_{\alpha,l,k}]$ by ${g_\alpha}^{-q}$ replaces the element ${g_\alpha}^l v_{\alpha,l,k}$ of its (k,l)th row by ${g_\alpha}^{l-q} v_{\alpha,l,k}$ which is the complex conjugate of the element ${g_\alpha}^{q-l} v_{\alpha,q-l,k}$ of its $(k, q-l)$th row. This gives the conjugation rule

$$(\prod_{\alpha \neq \beta} (g_\alpha)^{-q}).D_{R,\beta} = \pm \overline{D_{JR,\beta}}\ ,$$

where the sign is independent of β, $D_{R,\beta}$ denotes determinant obtained by omitting the βth column and using a cardinality $|P| - 1$ subset R of rows (k,l), and JR denotes the corresponding subset of rows $(k, q-l)$. By hypothesis we can choose R so that at least one of these determinants $D_{R,\beta}$ which we call D_R is nonzero. Then the corresponding D_{JR} from the $D_{JR,\beta}$'s is also nonzero. We now apply Cramer's rule, to the rows R, and also to the rows JR, to solve for $|P|-1$ of the t_α's in terms of one of them. This shows that we must have $\frac{D_{R,\beta}}{D_R} = \frac{D_{JR,\beta}}{D_{JR}}$ and that these ratios are real. Applying the above conjugation rule it now follows that all ratios of $g_\alpha(P)$'s must be in $\mathbb{Z}/q$, i.e. that P must be on an S^1-orbit passing through $|K * \cdots * K|$. *q.e.d.*

Our map $s : S^{2N+1} \to \mathbb{L}^\perp$ was defined starting from $\{s_\alpha^\perp\}$ which was the projection of a subset constrained to be in $\mathbb{A}^d$. Taking our cue from the above lemma it is better to consider *the class of all subsets* $\{v_\alpha : 1 \leq \alpha \leq N+1\}$ *of* $\mathbb{L}_\mathbb{R}^\perp$. This gives us more "elbow room" e.g. by perturbing the set slightly we can now assume (as we do in the following) that, for all $g_\alpha \in \mathbb{Z}/q$, all minors of the $N \times (N+1)$ matrices $[g_\alpha \bullet v_\alpha]$ are nonzero.

Lemma 4. *For any subset $\{v_\alpha\} \subset \mathbb{L}_\mathbb{R}^\perp$ which is generic in the above sense, there exists an S^1-map $f : S^{2N+1} \to \mathbb{L}^\perp$, with zeros only on orbits passing through the deleted join, and such that*

$$f(P) = \sum_{\alpha \in [P]} t_\alpha(P)(g_\alpha(P) \bullet v_\alpha)$$

for all $P \in K * \cdots * K$.

Proof[1]. We'll only use the weaker hypothesis that $\mathrm{rank}\{g_\alpha \bullet v_\alpha : 1 \leq \alpha \leq N+1\} = N$ for all $g \in \mathbb{Z}/q$, however some steps become slightly simpler under the stronger assumption that all minors of these matrices are nonzero.

The basic idea in constructing f is to use a *subset* $\{f_\alpha(P)\} \subset \mathbb{L}_\mathbb{R}^\perp$ *"moving" with* $P \in S^{2N+1}$ which coincides with the given "constant" $\{v_\alpha\}$ on the deleted join. Here the $f_\alpha(P)$'s are continuous and invariant under the S^1-action : $f_\alpha(gP) = f_\alpha(P)\ \forall g \in S^1$. This ensures that the same recipe,

$$f(P) = \sum_{\alpha \in [P]} t_\alpha(P)(g_\alpha(P) \bullet f_\alpha(P)),$$

continues to define a continuous S^1-map $S^{2N+1} \to \mathbb{L}^\perp$.

We'll equip the base space of our Hopf fibration $\pi : S^{2N+1} \to \mathbb{C}P^N$, $P \mapsto \overline{P}$, with the *toric subdivision* $\mathbb{C}P^N = \cup_{\sigma \in K} \mathbb{C}T^\sigma$, $\mathbb{C}T^\sigma = \{\overline{P} : [P] = \sigma\}$. Note that each $\mathbb{C}T^\sigma$ is homeomorphic to a product of $|\sigma| - 1$ copies of $\mathbb{C}^\times$, and that the fibration is trivial over this "complex torus". In fact for each $\gamma \in \sigma$ the condition $g_\gamma(P) \equiv 1$ fixes a continuous right inverse $\gamma : \mathbb{C}T^\sigma \to S^{2N+1}$ of π. Let $\mathbb{C}_t^N = \cup_{|\sigma| \leq t} \mathbb{C}T^\sigma$. The extension f will be constructed inductively over these skeletons using the fact that an S^1-map $S^{2N+1} \to \mathbb{L}^\perp$ is same thing as a section of the bundle $\mathfrak{L}^\perp \to \mathbb{C}P^N$.

The main thing is to ensure at each step that the f_α's satisfy the rank condition of Lemma 3 for then the reality of the f_α's would imply that the S^1-map f has no new zeros. This process is easier to indicate for the *case q even* which we'll assume now.

[1]Sketchy ... needs clarifying and simplifying ... comments most welcome ...

Note that the (k,l)th component of $g_\alpha(P)^{-q/2}g_\alpha(P)\bullet f_\alpha(P)$ is $g_\alpha^{-q/2}g_\alpha^l f_{\alpha,l,k}$ which is the complex conjugate of its $(k,q-l)$th component $g_\alpha^{-q/2}g_\alpha^{q-l}f_{\alpha,q-l,k}$. So like $f_\alpha(P)$ this vector is also in $\mathbb{L}_{\mathbb{R}}^{\perp}$. Moreover clearly any minor of $[g_\alpha(P)^{-q/2}g_\alpha(P)\bullet f_\alpha(P)]$ is nonzero iff the corresponding minor of $[g_\alpha(P)\bullet f_\alpha(P)]$ is nonzero.

So we can work with the space of all $N\times(N+1)$ "real" (but written in Fourier component) matrices $[w_\alpha : w_\alpha \in \mathbb{L}_{\mathbb{R}}^{\perp}]$ of rank N. Now this space has the homotopy type of $GL_+(N+1,\mathbb{R})\simeq SO(N+1,\mathbb{R})$ which is fairly complicated. But we don't really have to deal with obstruction-theoretic problems arising from this because to start with we only have *finitely many* such matrices $[g_\alpha\bullet v_\alpha : g_\alpha\in\mathbb{Z}/q, 1\leq \alpha\leq N+1]$ and so we can work within an open *contractible* subspace Ω of rank N matrices of size $N\times(N+1)$ containing these finitely many initial ones. (For Theorem 3 a slightly different Lemma 4 would be needed because the set of initial matrices is not finite but they are still defined over a space – the union of the open top-most simplices of the deleted join – having the homotopy type of finitely many points.) Restricting to columns indexed by $\alpha\in\sigma$ we get a space of size $N\times|\sigma|$ matrices which we'll denote by Ω_σ. Note that these have rank at least $|\sigma|-1$. We'll denote by Ω_σ^0 the subspace of those having rank $|\sigma|$.

We'll assume that our inductive construction has taken care that over any $\mathbb{C}T^\sigma$ with $|\sigma|\leq t$ the matrices $[g_\alpha(\gamma\overline{P})^{-q/2}g_\alpha(\gamma\overline{P})\bullet f_\alpha(\overline{P})]$ are in Ω_σ (or in Ω_σ^0 if we start with the all-minors-nonzero hypothesis). Over the boundary of a $\mathbb{C}T^\theta$ with $|\theta|=t+1$ we perturb the $f_\alpha(P)$'s to raise, if possible, the rank of the similar matrix from $|\theta|-2$ to $|\theta|-1$ (or from $|\theta|-1$ to $|\theta|$ if doing construction with the stronger hypothesis on the v_α's). This is possible as long as there is a row available outside a biggest sized nonzero minor because then by perturbing the $f_{\alpha,l,k}$'s of this row we can raise the rank.

Using the contractibility of Ω_θ we now extend this matrix $[g_\alpha(\gamma\overline{P})^{-q/2}g_\alpha(\gamma\overline{P})\bullet f_\alpha(\overline{P})]$ to a matrix $A_\alpha(\overline{P})\in\Omega_\theta$ defined continuously over all of $\mathbb{C}T^\theta$. Solving $A_\alpha(\overline{P})=g_\alpha(\gamma\overline{P})^{-q/2}g_\alpha(\gamma\overline{P})\bullet f_\alpha(\overline{P})$ we then get the required extensions of the functions f_α over $\mathbb{C}T^\theta$. Using these extend the S^1-map f. Applying Lemma 3 (with the stronger hypothesis one needs to use this lemma only when $|\theta|=N+1$) it follows that no new zeros have been introduced in this process. *q.e.d.*

REMARK 2. We note that for q even the action $[w_\alpha]\mapsto[g^{-q/2}g\bullet w_\alpha]$ of $g\in S^1$ on matrices multiplies each minor by a nonzero constant and preserves the value of all $N\times N$ minors because $(g^{-q/2})^N(g^1g^2\ldots g^{q-1})^{d+1}=1$. We note also that for the above argument it was necessary to use "moving" points because, for $q\geq 3$, *an f defined by a "constant" $\{v_\alpha\}\subset\mathbb{L}_{\mathbb{R}}^{\perp}$ can not satisfy the required rank condition at all P.* To see this, take any $1\leq k\leq d+1$, and inductively choose g_α as follows : in case $v_{\alpha,1,k}=\overline{v_{\alpha,q-1,k}}=0$ or if there is no $\beta<\alpha$ with $v_{\beta,1,k}=\overline{v_{\beta,q-1,k}}$ nonzero take any g_α, otherwise pick such a $\beta<\alpha$ and solve $\begin{vmatrix} g_\beta v_{\beta,1,k} & g_\alpha v_{\alpha,1,k}\\ {g_\beta}^{q-1}v_{\beta,q-1,k} & {g_\alpha}^{q-1}v_{\alpha,q-1,k}\end{vmatrix}=0$ to find $g_\alpha\in S^1$. This implies that all the 2×2 determinants formed from the $(k,1)$th and the $(k,q-1)$th rows of the $N\times N$ determinants $|{g_\alpha}^l v_{\alpha,l,k}|_{\alpha\neq m}$ are zero. So, at any P having such g_α's, these $N+1$ determinants are all zero. *q.e.d.*

4. Proof of Theorem 1

If $S \subset \mathbb{A}^d$ has t Tverberg partitions then any sufficiently close $p(S) \subset \mathbb{A}^d$ has $\leq t$ Tverberg partitions. This follows because if the disjoint partition $\sigma_1 \cup \cdots \cup \sigma_q$ of S is such that $\text{conv}(\sigma_1) \cap \cdots \cap \text{conv}(\sigma_q) = \emptyset$ then the same is true for the corresponding partition $p(\sigma_1) \cup \cdots \cup p(\sigma_q)$ of $p(S)$. So without loss of generality we can assume that $S \subset \mathbb{A}^d \subset \mathbb{R}^{d+1}$ is *generic*, i.e. that the coordinates $s_{k,\alpha}$, $1 \leq k \leq d+1$, of its points s_α are algebraically independent, but for the obvious relations $\sum_{k=1}^{d+1} s_{k,\alpha} = 1$, over the field of rationals.

Now consider the restriction $s^q : K \# \cdots \# K \to \mathbb{A}^d \times \cdots \mathbb{A}^d$ of the q-fold cartesian product of the linear map $s : |K| \to \mathbb{A}^d$ determined by S. We note that $K \# \cdots \# K$ is $(q-1)d$-dimensional, and can be equipped with the cell subdivision provided by all cells of the type $|\sigma_1| \times \cdots \times |\sigma_q|$, where the σ_i's are any pairwise disjoint nonempty subsets of $\{e_1, \ldots, e_{N+1}\}$. Since S is in general position on $\mathbb{R}^{d+1}$, the s^q-images of the lower dimensional cells miss the diagonal of $\mathbb{A}^d \times \cdots \times \mathbb{A}^d$, while the images of the top dimensional cells either miss the diagonal, or else cut it cleanly in just one interior point.

This last happens if and only if $(\sigma_1, \ldots, \sigma_q)$ is one of the $q!$ permutations of a Tverberg partition $\{\sigma_1, \ldots, \sigma_q\}$ of S. Thus the number of points of $K \# \cdots \# K$ which are mapped to the diagonal by s^q is $q!$ times the number of Tverberg partitions of S, and so the number of $\mathbb{Z}/q$ orbits of the $\mathbb{Z}/q$-space $K \# \cdots \# K$ which are imaged to a single point is $(q-1)!$ times the number of Tverberg partitions of S.

We extend the $\mathbb{Z}/q$-map $s^q : K \# \cdots \# K \to \mathbb{A}^d \times \cdots \times \mathbb{A}^d$ to the S^1-map s of §3 and consider its direct summand $s : S^{2N+1} \to \mathbb{L}^\perp$. Identifying this S^1-map with the section $\mathbb{C}P^N \xrightarrow{s} \mathfrak{L}^\perp$, $\overline{P} \mapsto [P, s(P)] \in (S^{2N+1} \times \mathbb{L}^\perp)/S^1 = \mathfrak{L}^\perp$, of the vector bundle $\mathfrak{L}^\perp \to \mathbb{C}P^N$, we see by using Lemma 2 that on $Q = \{\overline{P} : r(P) \leq q\}$ this section has only *Tverberg zeros* x, i.e. $x = \overline{P}$ for some $P \in K \# \cdots \# K$ which is mapped to the diagonal by s^q. The following lemma shows that these zeros are isolated, and so that the section s has no other zeros in a sufficiently small neighbourhood of Q.

Lemma 5. *For $S \subset \mathbb{A}^d$ generic, the section s of $\mathfrak{L}^\perp$ intersects its zero section z transversely at all Tverberg zeros x.* (Here as usual "transversely" means that the tangent space of $\mathfrak{L}^\perp$ is the sum of the subspaces tangent to the two sections at the intersection.)

Proof. Let $x = \overline{P}$ where $P \in K \# \cdots \# K$ is mapped to the diagonal by s^q. We must have $|P| = N+1$, for otherwise, a proper subset of S would have a Tverberg partition. For the same reason (use Carathéodory's theorem) all Tverberg parts σ_i must have cardinality $\leq d+1$. Hence, by Remark 1, the $N \times N$ determinants $D_{m,N+1}(P,S) = |{g_\alpha}^l s_{k,\alpha}|_{\alpha \neq m}$ must be nonzero polynomials in the $s_{k,\alpha}$'s.

The definitions $\overline{P} \mapsto [P, s(P)]$ and $\overline{P} \mapsto [P, 0]$ of s and z show that our assertion is equivalent to checking that the map $\overline{P} \mapsto s(P)$, $g_{N+1}(P) = 1$, has a nonsingular jacobian at x. To verify this we'll use, as local coordinates near x the $2N$ reals $< t_\alpha, \theta_\alpha >$, $1 \leq \alpha \leq N$, where $g_\alpha = \cos\theta_\alpha + i\sin\theta_\alpha$. In $\mathbb{L}^\perp$ we'll use as local coordinates the $2N$ reals given by the real and imaginary parts of the (l,k)th components, $1 \leq l < q$, $1 \leq k \leq d+1$, with respect to the Fourier basis. In these local coordinates our map reads

$$< t_\alpha, \theta_\alpha > \mapsto < s_{k,N+1} + \sum_\alpha t_\alpha \{\cos(l\theta_\alpha) s_{k,\alpha} - s_{k,N+1}\}, \sum_\alpha t_\alpha \sin(l\theta_\alpha) s_{k,\alpha} > .$$

Computing partial derivatives with respect to the t_α's and θ_α's we obtain the jacobian $2N \times 2N$ determinant

$$\begin{vmatrix} \cos(l\theta_\alpha)s_{k,\alpha} - s_{k,N+1} & -lt_\alpha \sin(l\theta_\alpha)s_{k,\alpha} \\ \sin(l\theta_\alpha)s_{k,\alpha} & lt_\alpha \cos(l\theta_\alpha)s_{k,\alpha} \end{vmatrix}.$$

Pulling the nonzero t_α's out from the columns, and doing some row transformations we see that it is $(\frac{1}{2})^N \prod_{\alpha=1}^N t_\alpha$ times

$$\begin{vmatrix} {g_\alpha}^l s_{k,\alpha} - s_{k,N+1} & l{g_\alpha}^l s_{k,\alpha} \\ -\overline{g_\alpha}^l s_{k,\alpha} + s_{k,N+1} & l\overline{g_\alpha}^l s_{k,\alpha} \end{vmatrix}.$$

Since, at x, $(g_\alpha)^q = 1 \ \forall \alpha$, on adding the (k,l)th bottom row of this determinant to its $(k, q-l)$th top row, we can change this to

$$\begin{vmatrix} {g_\alpha}^l s_{k,\alpha} - s_{k,N+1} & l{g_\alpha}^l s_{k,\alpha} \\ 0 & q\overline{g_\alpha}^l s_{k,\alpha} \end{vmatrix},$$

which equals $(D - D_{1,N+1} - \cdots - D_{N,N+1}) \cdot q^N \cdot \overline{D} \neq 0$, because the left side is a nonzero polynomial in the generic $s_{k,\alpha}$'s with coefficients in the algebraic number field $\mathbb{Q}[\omega]$. *q.e.d.*

Continuing the proof of Theorem 1 we now equip $\mathbb{C}P^N$, as well as the complex N-dimensional fibres of $\mathfrak{L}^\perp$, with the orientations determined by their complex structures. With respect to these orientations one can speak of the *local degree* d_x of the section s of $\mathfrak{L}^\perp \to \mathbb{C}P^N$ at each of its isolated zeros x, i.e. the degree of the obvious map from the link of x to the unit sphere of the fibre $\mathfrak{L}^\perp_x$ which is determined by s. From the above lemma it follows that this map is a diffeomorphism, so this degree is $+1$ or -1, depending on whether or not the diffeomorphism preserves or reverses orientation.

We now perturb $\{s^\perp_\alpha\} \subset \mathbb{L}^\perp_\mathbb{R}$ to a neighbouring $\{v_\alpha\} \subset \mathbb{L}^\perp_\mathbb{R}$ for which all minors of the matrices $[g_\alpha \bullet v_\alpha : g_\alpha \in \mathbb{Z}/q]$ are nonzero, and replace s by the section f supplied by Lemma 4. This only perturbs the existing zeros slightly (and they'll still be orbits passing through the deleted join but maybe not the the deleted product) and introduces no new ones. At each of the perturbed zeros one has the same local degree ± 1 as at the corresponding original one.

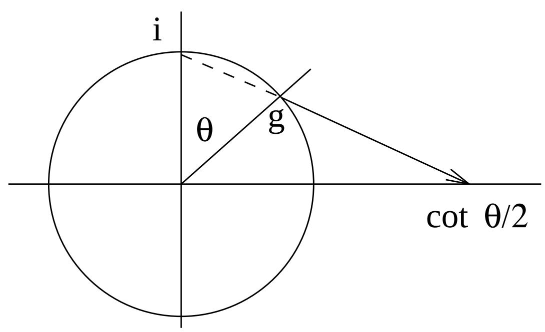
We recall now (this follows from the obstruction theoretic definition of the Euler class) the well-known *Poincaré-Hopf theorem* : the sum $\sum_x d_x$ of the local degrees of f coincides with the Euler number $< e(\mathfrak{L}^\perp), \mathbb{C}P^N >$. Here $e(\mathfrak{L}^\perp) = c_N(\mathfrak{L}^\perp) = c_N(\mathfrak{L})$, so this number is $((q-1)!)^{d+1}$ by Lemma 1. Since each $d_x = \pm 1$ it follows that the number of Tverberg zeros is at least $((q-1)!)^{d+1}$, and thus the number of Tverberg partitions of S is at least $((q-1)!)^d$. **q.e.d.**

REMARK 3. An interesting "stability" deserves to be mentioned : *in all of the above we could have assumed* $d >> q$. To see this, think of $\mathbb{A}^d$ as an affine hyperplane of $\mathbb{A}^{d+1}$, perturb one of the points v of the given general position $S \subset \mathbb{A}^d$ into $\mathbb{A}^{d+1}$, and take $q-1$ new points on the same side of $\mathbb{A}^{d+1}$. For a Tverberg partition of this subset of $\mathbb{A}^{d+1}$, the part which contains v cannot contain a new point; for then, one of the other parts will contain no new point, implying that the common point of the convex hulls is on $\mathbb{A}^d$, which is impossible for the proper subset $S\backslash\{v\}$ has (due to general position) no Tverberg partition. So each Tverberg partition of $S \subset \mathbb{A}^d$ can be associated with at most $(q-1)!$ Tverberg partitions of the bigger set of $\mathbb{A}^{d+1}$, viz. those obtained by adding one each of the new points to the $q-1$ parts not containing v. Furthermore, if the new $q-1$ points are coincident (or almost so) each of these is indeed a Tverberg partition of this subset of $\mathbb{A}^{d+1}$. *q.e.d.*

It is also worth mentioning that the ratios of the coefficients occuring in the expansion (cf. proof of Lemma 2) of $D(P,S)$, i.e. the *generalized cross ratios*,

$$\frac{\Delta(g_{\pi_1},\ldots,g_{\pi_{q-1}})\cdots\Delta(g_{\pi_{N-q+1}},\ldots,g_{\pi_N})}{\Delta(g_1,\ldots,g_{q-1})\cdots\Delta(g_{N-q+1},\ldots,g_N)},$$

are always real. In fact they remain the same under the stereographic projection $S^1\backslash\{i\} \xrightarrow{\cong} \mathbb{R}$, $g \mapsto \cot\frac{\theta}{2}$, shown below.



To see this, note that $\overline{g} = g^{-1}$ implies that these ratios are indeed real, and then use $|g_\alpha - g_\beta| = 2|\sin\frac{\theta_\alpha}{2}\cos\frac{\theta_\beta}{2} - \cos\frac{\theta_\alpha}{2}\sin\frac{\theta_\beta}{2}|$. The stereographic projection also shows that, when the $g_\alpha(P)$'s are in $\mathbb{Z}/q$, then the $D_{m,N+1}(P,S)$'s are sort of like the cotangent or "Dedekind sums" of the field $\mathbb{Q}[\omega]$, excepting that our $s_{k,\alpha}$'s are not rational.

Any cardinality $N+1$ set $S = \{s_\alpha\} \subset \mathbb{A}^d$ assigns one such set of $N+1$ determinants to each top dimensional simplex $(\sigma_1,\ldots,\sigma_q)$ of $K * \cdots * K$: simply use any point P of this simplex. When $(d+1)\binom{q-1}{2}$ is even, these determinants are all real, otherwise all are purely imaginary. We will also use the bigger determinant $\begin{vmatrix} 1\ldots 1 \\ {g_\alpha}^l s_{k,\alpha} \end{vmatrix}$, where $1 \leq \alpha \leq N+1$ indexes the columns, and the rows, from the second onwards, are indexed in lexicographic order by the pairs (k,l), $1 \leq k \leq d+1$, $1 \leq l < q$. Using the set $S \subset \mathbb{A}^d$ we now define an N-dimensional *characteristic cocycle* $\mathcal{X}_S(\sigma_1, \ldots, \sigma_q) \in \{-1, 0, +1\}$ of $K * \cdots * K$: we assign to each top simplex the sign of the real number $(-1)^N \begin{vmatrix} 1 \ldots 1 \\ {g_\alpha}^l s_{k,\alpha} \end{vmatrix} \cdot \overline{|{g_\alpha}^l s_{k,\alpha}|}_{\alpha \le N}$.

Theorem 2. *For any generic cardinality $N+1$ subset $S \subset \mathbb{A}^d$ one has*

$$\sum \mathcal{X}_S(\sigma_1, \ldots, \sigma_q) \;=\; ((q-1)!)^d,$$

where the summation is over all Tverberg partitions $\{s(\sigma_1), \ldots, s(\sigma_q)\}$ of S.

Proof. Our proof of Theorem 1 had given us the formula $\sum_x d_x = ((q-1)!)^{d+1}$ where x runs over all Tverberg zeros of s. From the definition of d_x given there it is clear that it is $+1$ or -1 depending on whether or not the jacobian of that locally defined map $\overline{P} \mapsto s(P)$, with respect to the coordinates mentioned, is positive or not. We recall also that this jacobian had turned out to be

$$(\frac{q}{2})^N \prod_\alpha t_\alpha (D - D_{1,N+1} - \cdots - D_{N,N+1})\overline{D} = (-\frac{q}{2})^N \prod_\alpha t_\alpha \begin{vmatrix} 1 \ldots 1 \\ {g_\alpha}^l s_{k,\alpha} \end{vmatrix} \overline{D}.$$

So it follows that $d_x = \mathcal{X}_S(\sigma_1, \ldots, \sigma_q)$ where $(\sigma_1, \ldots, \sigma_q)$ denotes the top simplex of the deleted join which contains P. We don't need to take g_{N+1} here because if each g_α is multiplied by ω the (k, l)th row of the bigger determinant is multiplied by ω^l and that of the smaller by ω^{-l}. Also zeros x interrelated by a reordering of S are related by composing s with an orientation preserving self-diffeomorphism of $\mathbb{C}P^N$ and so have same d_x. Dividing both sides of the formula by $(q-1)!$ we obtain the stated result. *q.e.d.*

REMARK 4. From the viewpoint of number theory it is the opposite case of *rational* $S \subset \mathbb{L}_{\mathbb{R}}^{\perp}$ which is most interesting. As the proof shows an index formula still holds provided the section s cuts the zero section z transversely and this happens not only for a general position S but in some other cases also. For example $\mathcal{X}_{S_o} = 1$ *on all Tverberg partitions of Sierksma's configuration* S_0, i.e. when $s_1 = \cdots = s_{q-1} = (1, 0, \ldots, 0)$; $s_q = \cdots = s_{2(q-1)} = (0, 1, \ldots, 0)$; and so on; with the very last point being $(\frac{1}{d+1}, \ldots, \frac{1}{d+1})$. For this note that at any Tverberg partition, all but the first of the entries of the last column of $\begin{vmatrix} 1 \ldots 1 \\ {g_\alpha}^l s_{k,\alpha} \end{vmatrix}$ can be made zero by multiplying this column by $d+1$ and adding all the other columns to it. So this bigger determinant is $(-1)^N \frac{N+d+1}{d+1}$ times the smaller and nonzero determinant $|{g_\alpha}^l s_{k,\alpha}|_{\alpha \le N}$. *q.e.d.*

The index formula probably gives non-trivial identities between the Dedekind sums of $\mathbb{Q}[\omega]$ for suitably chosen rational sets $S \subset \mathbb{L}_{\mathbb{R}}^{\perp}$ having more than $((q-1)!)^{d+1}$ zero orbits.

5. Concluding remarks

We'll give some more applications of the above ideas, followed by a few comments on their evolution.

(a) CONTINUOUS MAPS. Let $s : K \to \mathbb{A}^d$ be any continuous map, and let us say that a q-tuple $(x_1, \ldots, x_q)$ of points of $|K|$ is a *separated* q-tuple point of s if $s(x_1) = \cdots = s(x_q)$ and one can find pairwise disjoint subsets $\sigma_1, \ldots, \sigma_q$ of $\{e_1, \ldots, e_{N+1}\}$ such that $x_1 \in |\sigma_1|, \ldots, x_q \in |\sigma_q|$. Then one has the following "continuous" generalization of Tverberg's theorem. For *primes* q this was established by Bárány-Shlosman-Szücs [1], 1981, with a simpler proof (of which the following is an "S^1-version") given later in Sarkaria [6].

Theorem 3. *A continuous map from a $(q-1)(d+1)$-dimensional simplex to $\mathbb{A}^d$ has a separated q-tuple point.*

Proof. Consider the q-fold join

$$s^{(q)} : K * \cdots * K \to \mathbb{R}^{d+1} \times \cdots \times \mathbb{R}^{d+1} = \mathbb{L}_{\mathbb{R}},$$

of the given continuous $K \xrightarrow{s} \mathbb{A}^d$, i.e. the continuous $\mathbb{Z}/q$-map which images $P = \sum_{i=1}^{r(P)} t_i(P)(g_i(P) \bullet x_i) \in K * \cdots * K$ to

$$s^{(q)}(P) = \sum_{i=1}^{r(P)} t_i(P)(g_i(P) \bullet s(x_i)) = \sum_{\alpha \in [P]} t_\alpha(P)(g_\alpha(P) \bullet s_\alpha(P)),$$

where $s_\alpha(P) \in \mathbb{A}^d$ is defined, whenever $\alpha \in [P]$, by

$$s_\alpha(P) = s\left(\frac{\sum_{g_\beta(P)=g_\alpha(P)} t_\beta(P) e_\beta}{\sum_{g_\beta(P)=g_\alpha(P)} t_\beta(P)}\right).$$

Replacing each $s_\alpha(P)$ by its component $(s_\alpha(P))^\perp$ in the above formulae (use $\mathbb{L}_{\mathbb{R}} = \mathbb{L}^0_{\mathbb{R}} \oplus \mathbb{L}^\perp_{\mathbb{R}})$) gives the direct summand

$$s^{(q)} : K * \cdots * K \to \mathbb{L}^\perp_{\mathbb{R}}.$$

Assume, if possible, that s has no separated q-tuple points, i.e. that the last map has no zeros. The same will be true if we perturb $\{s_\alpha(P)^\perp\}$ to a neighbouring $\{v_\alpha(P)\} \subset \mathbb{L}^\perp_{\mathbb{R}}$ and use this to define our map $K * \cdots * K \to \mathbb{L}^\perp_{\mathbb{R}}$.

Choosing these $v_\alpha(P)$'s so that with $g_\alpha \in \mathbb{Z}/q$ the matrices $[g_\alpha \bullet v_\alpha(P)]$ have minors nonzero we can now extend this $\mathbb{Z}/q$-map, again using a Lemma 3 dependent construction analogous to that of Lemma 4, to an S^1-map $S^{2N+1} \to \mathbb{L}^\perp$, having no zeros anywhere on S^{2N+1}. This contradicts the fact that the Euler class of $\mathfrak{L}^\perp$ being nonzero, it admits no everywhere nonzero continuous section. *q.e.d.*

The Borsuk-Ulam theorem says that there is no continuous $\mathbb{Z}/2$-map from a free $\mathbb{Z}/2$-sphere to a lesser dimensional free $\mathbb{Z}/2$-sphere, and only essential ones to an equi-dimensional sphere. Indeed it is known that the same is true for all finite groups $G \neq 1$. The N-dimensional free $\mathbb{Z}/q$-complex $K * \cdots * K$ being $(N-1)$-connected, this immediately gave [6] the required contradiction, *for the case q prime,* because then the $\mathbb{Z}/q$-action on $\mathbb{L}^\perp_{\mathbb{R}} \backslash \{0\} \simeq S^{N-1}$ is free . A similar argument also gave a sharp generalization of the Van Kampen-Flores theorem to all primes $q \geq 2$. This approach also gives (for q prime) an estimate – see Vučić-Živaljević [15] – for the least number of Tverberg partitions which however falls short of the Sierksma number.

(b) PIECEWISE LINEAR MAPS. The main theorem generalizes to any subclass of continuous functions $s : K \to \mathbb{A}^d$ defined by a "local smoothness or finiteness" condition which is strong enough to guarantee transversality, and thus $d_x = \pm 1$, at the generic zeros, e.g. one has the following.

Theorem 4. *Any piecewise linear continuous map from a $(q-1)(d+1)$ dimensional simplex to $\mathbb{A}^d$ has at least $((q-1)!)^d$ separated q-tuple points.* (Here as usual "piecewise linear" means that the map is linear with respect to some finite subdivision.)

Proof. Let $s : K \to \mathbb{A}^d$ be linear on the subdivision K' of K. Clearly the number of separated q-tuple points can not increase if s is replaced by any neighbouring map also linear on K'. So we can assume that the set of coordinates of the values which s takes on the vertices of K' is generic in the sense of §4.

Using such an s now define S^1-map $f : S^{2N+1} \to \mathbb{L}^\perp$ as in the proof of Theorem 3. Considered as a section of $\mathfrak{L}^\perp$, the number of zeros of f on the projection (in $\mathbb{C}P^N$) of the qth deleted join will be $(q-1)!$ times the number of separated q-tuple points. One can check, by a calculation like that in the proof of Lemma 5, that at these points the section f cuts the zero section of $\mathfrak{L}^\perp$ transversely. Then the same argument as in §4 applies and gives the required result. *q.e.d.*

(c) A GLOBALIZATION. We now let K denote any N-dimensional simplicial complex on the set of vertices $\{e_1, \ldots, e_M\}$, where $M \geq N+1$. Let $\mathrm{Sph}(K) \subset S^{2M-1} \subset \mathbb{C}^M$ be the S^1-subspace consisting of all points $\sum_{\alpha \in [P]} t_\alpha(P) g_\alpha(P) e_\alpha$ with $[P] \subset \{e_1, \ldots, e_M\}$ a simplex of K. The space obtained by dividing this out by S^1 will be denoted $\mathrm{Proj}(K)$, and an oriented vector bundle $\mathfrak{L}^\perp \to \mathrm{Proj}(K)$ defined on it exactly as before.

Theorem 5. *For any linear map s from a $(q-1)(d+1)$ dimensional simplicial complex $|K|$ to $\mathbb{A}^d$ there exist at least $< e(\mathfrak{L}^\perp)$, $\mathrm{Proj}(K) > \div (q-1)!$ partitions $\sigma_1 \cup \cdots \cup \sigma_q$ of simplices of K into pairwise disjoint faces such that $\mathrm{conv}(s\sigma_1) \cap \cdots \cap \mathrm{conv}(s\sigma_q)$ is nonempty.*

Here we note that $\mathrm{Proj}(K)$ is a union of complex projective spaces and as such has a well defined fundamental class, which too has been denoted $\mathrm{Proj}(K)$ in the above statement. We omit the proof which is just as before; also it has a piecewise linear generalization like Theorem 4, and can be sharpened to an index formula like that of Theorem 2, with characteristic cocycle of a cardinality M subset of $\mathbb{A}^d$ defined just as before. Note that for $q = 2$ this index formula says that the algebraic number of circuits of the "oriented matroid" determined by this affine set coincides with the Euler number of a certain vector bundle.

One should think – for more on these lines see [9] and the lecture notes (under preparation) of my Panjab University Topology Seminar of 1994-95 – of $\mathrm{Sph}(K)$ as a "visualisation", just like the much better known $|K|$, of the simplicial complex K. Indeed there are many others, e.g. in a subsequent paper we'll give some interesting applications of S^1-versions of the "bigger" deleted joins used in Van Kampen's embedding theory. We note also that the incidence rule used in [9] to define a "cyclotomic homology" of simplicial complexes is like the definition of $\mathcal{X}_S$ in §4.

(d) A LOOK BACK. Shortly after writing [6] I had made an attempt [7] to interpret the Sierksma number as a certain invariant sum of local degrees at the intersections of $s^q(K\#\cdots\#K)$ with the diagonal of $\mathbb{A}^d\times\cdots\times\mathbb{A}^d$. This was interesting – e.g. a cycle over the cyclotomic field $\mathbb{Q}[\omega]$ was used – but unsuccessful: a computation (some initially overlooked sign changes were later pointed out to me by Kalai) using Sierksma's configuration S_0 shows that one only gets zero !

This had happened because a characteristic class of a *finite* group action was being evaluated on the aforesaid cycle; so it was clear from that point on that an infinite group was needed, with S^1 being the obvious choice. (Intuitively this "complexification" has the effect of changing the errant vertex transpositions into vertex-pair transpositions, and thus a total cancellation of the local degrees does not happen.) The mechanics of doing this became clear to me much later when I spoke about the aforementioned "visualisations" in my 1994-95 topology seminar, and the essentials of the above proof, including the key Vandermonde conjugation argument of Lemma 2, were in hand by the end of summer 1995. A more careful analysis – comments of Ofer Gabber were of great help in this regard – however showed that we were still short of a complete knowledge of all the zeros of the canonically defined section s. We have now side-stepped this difficulty by using moving subsets $\{f_\alpha(P)\}\subset\mathbb{L}^\perp_\mathbb{R}$.

We note that Lemma 1 also follows, by virtue of the Poincaré-Hopf theorem, from Remark 4 which showed that all the $((q-1)!)^{d+1}$ Tverberg zeros x of Sierksma's configuration S_0 have the same local degree $d_x=1$. Thus our proof is in the same spirit as the original inspiration of [7], viz. an argument given by Van Kampen in his amazing paper [14] of 1932. He showed that for $n\geq 2$ the algebraical number of separated general position self-intersections, of an n-complex immersed in $\mathbb{A}^{2n}$, is invariant under deformations. Then a computation, using a particular immersion of the n-skeleton of a $(2n+2)$-simplex, shows that this complex can not embed in $\mathbb{A}^{2n}$.

(e) WHAT LIES AHEAD ? We'll give in a sequel more regarding the combinatorics of the signs d_x, some generalizations to skeletons of simplices, and some interesting applications of our index formula. Indeed, since other *characteristic classes*, of say the pseudomanifolds $\mathrm{Proj}(K)$, can be defined inductively in terms of suitable split bundles, we hope that, like $e(\mathfrak{L}^\perp)$, these too contain analogous combinatorial cocycles, and that there are similar combinatorial *index formulas* for other numerical topological invariants.

We gave in [8] a very simple proof of Tverberg's theorem – see esp. Onn's remark (3), also see Kalai [4] – which again uses, like [6] and the proof of Theorem 2 above, the matrix space $\mathbb{L}^\perp_\mathbb{R}$, but avoids topology by exploiting instead the linearity of s via an elementary convexity argument of Bárány. The above proof of Theorem 1 can also probably be simplified to one using only the *representation theory* of S^1. Also we're trying to make another proof in which one replaces S^1 by $\mathbb{C}^\times$ and uses *field theory* : in this context we remark that apparently – cf. Sullivan [12] – the (discontinuous!) "Galois symmetries", of the $\mathbb{Z}/q^n$-deleted joins contained in their covers, have much to say about the homotopy and homeomorphism classification of the complex varieties $\mathrm{Proj}(K)$. Finally, we feel that *noncommutative* versions of these arguments, e.g. using $SU(2)$ instead of S^1, will be even more insightful.

References

1. I.Bárány, S.B.Shlosman and A.Szücs, *On a topological generalization of a theorem of Tverberg*, Jour. Lond. Math. Soc. **23** (1981), 158-164.
2. B.Birch, *On* $3N$ *points in a plane*, Proc. Camb. Phil. Soc. **55** (1959), 289-293.
3. F.Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer, 1966.
4. G.Kalai, *Combinatorics and Convexity*, Proc. I.C.M. Zürich, Birkhaüser, 1995, pp. 1363-1374.
5. J.W.Milnor and J.D.Stasheff, *Characteristic Classes*, Princeton, 1974.
6. K.S.Sarkaria, *A generalized van Kampen-Flores Theorem*, Proc. Amer. Math. Soc. **111** (1991), 559-565.
7. ——, *Sierksma's Dutch Cheese Problem*, unpublished ms. (1989).
8. ——, *Tverberg's theorem via number fields*, Isr. Jour. of Math. **79** (1992), 317-320.
9. ——, *Some simplicial (co)homologies*, I.H.E.S. preprint M/95/83 (1995).
10. G.Sierksma, *Convexity without linearity; the Dutch cheese problem*, Mimeographed Notes, University of Grönigen (1979).
11. N.E.Steenrod, *The Topology of Fibre Bundles*, Princeton, 1951.
12. D.P.Sullivan, *Galois symmetry in manifold theory at the primes*, Actes Congrès Inter. Math. Nice, 1970, pp. 169-175.
13. H.Tverberg, *A generalization of Radon's theorem*, Jour. Lond. Math. Soc. **41** (1966), 123-128.
14. E.R.Van Kampen, *Komplexe in euklidischen Räumen*, Abhand. math. Sem. Hamburg **9** (1932), 72-78 and 152-153.
15. A.Vučić and R.T.Živaljević, *Note on a conjecture of Sierksma*, Disc. Comput. Geom. **9** (1993), 339-349.

K. S. Sarkaria, Department of Mathematics, Panjab University, Chandigarh 160014, INDIA.