SOME SIMPLICIAL (CO)HOMOLOGIES

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Septembre 1995

IHES/M/95/83

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by

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(Lectures given in the Instructional Workshop on cyclic (Co)homology at Chandigarh on April 20 and 24, 1995)

Introduction. A ubiquitous player in many arguments is the humble SIMPLICIAL COMPLEX, i.e. a (*usually finite*) set K of finite (*nonempty*) sets which is closed under inclusion. This homely object is used most often to define and compute some (CO)HOMOLOGY which measures the discrete ambiguity of the (continuous) phenomenon being studied.

My object in these talks is simply to bring out the combinatorial beauty of this methodology. This I'll do by giving *many off-beat, but nevertheless elegant and natural, definitions of simplicial (co)homologies, and results concerning their computation.* Some of this material - e.g. the cyclic cohomologies of §§ 1 and 7, the philosophy of §§ 3-4, the cyclotomic homology of § 6, and the zeta function computation of § 8 - is new. More details regarding these results will be given in the lecture notes of my seminar of 1994-95.

I remark that while giving these examples I'll mostly pass over their "standard variations" (dualizing, relativisation, localization, changing coefficients, etc.) in silence, and it will be understood, unless otherwise stated, that the coefficients are always taken from a *field* $\mathbb{F}$ *of characteristic zero.*

§ 1. **Partial skewsymmetry.** Let me begin by recalling how the **usual** (= oriented) cohomology $H^*(K)$ of a simplicial complex K is defined. We consider the set K_{assoc} of all **associative monomials** (= finite sequences) in the vertices which are supported on simplices of K, and define $C^*(K)$ to consist of all functions $f : K_{assoc} \longrightarrow \mathbb{F}$ which are *totally skewsymmetric*, i.e. such that

$$f(v_0 v_1 \,..\, v_q) \;=\; (-1)^{\pi}\, f(v_{\pi 0} v_{\pi 1} \,..\, v_{\pi q})$$

for *all* $\pi \in S_{q+1}$. We then define $\delta : C^*(K) \longrightarrow C^{*+1}(K)$ by

$$(\delta f)(v_0 v_1 \,..\, v_{q+1}) \;=\; \Sigma_i \;(-1)^i\, f(v_0 \,..\, \hat{v}_i \,..\, v_{q+1}),$$

check $\delta \circ \delta = 0$, and set $H^*(K) = \dfrac{\ker\delta}{\mathrm{im}\,\delta}$.

If we now drop the above skewsymmetry condition altogether, i.e.

consider the (much bigger !) graded vector space $C^*(K_{assoc})$ of *all* functions $f : K_{assoc} \longrightarrow \mathbb{F}$, and equip it with δ defined just as above, then this extended map $\delta : C^*(K_{assoc}) \longrightarrow C^{*+1}(K_{assoc})$ still clearly satisfies $\delta \circ \delta = 0$. We'll denote its cohomology by $H^*(K_{assoc})$.

What is $H^*(K_{assoc})$? More generally, we can ask when *partial skewsymmetry*, i.e. with respect to permutations π belonging to some subgroups $G_{q+1} \leq S_{q+1}$, yields a cochain subcomplex of $C^*(K_{assoc})$, and as to what these cohomologies are ? An answer is given by the following.

Theorem 1. *Partial skewsymmetry with respect to the sequence* $\{G_{q+1}\}$ *of permutation groups yields a sub cochain complex of* $C^*(K_{assoc})$ *for all* K *if and only if* $\{G_{q+1}\} = \{id\}$ *or* $\{\mathbb{Z}/2\}$ (= reversals) *or* $\{C_{q+1}\}$ (= rotations) *or* $\{D_{q+1}\}$ (= reversals and rotations) *or* $\{S_{q+1}\}$. *The cohomologies of these five cochain complexes* — $C^*(K_{assoc})$, $C^*_{rev}(K_{assoc})$, $C^*_{cyc}(K_{assoc})$, $C^*_{dih}(K_{assoc})$ *and* $C^*_{alt}(K_{assoc})$ — *are :*

$$H^*(K_{assoc}) \cong H^*_{rev}(K_{assoc}) \cong H^*_{alt}(K_{assoc}) \cong H^*(K),$$

$$H^*_{cyc}(K_{assoc}) \cong \oplus_{j\geq 0} H^{*-2j}(K),$$

$$H^*_{dih}(K_{assoc}) \cong \oplus_{j\geq 0} H^{*-4j}(K).$$

Regarding the proof — for more details re all results of this talk see [15] — note that $C^*_{alt}(K_{assoc}) = C^*(K)$ by definition, while $H^*(K_{assoc}) \cong H^*(K)$ is a result of Eilenberg-Steenrod [2]. Using this it follows easily that the spectral sequence of the (B,b) double complex degenerates at the first term, which gives the cyclic cohomology $H^*_{cyc}(K_{assoc})$. The computation of $H^*_{dih}(K_{assoc})$ is similar.

Thus the **generalized De Rham theorem** of [16], § 5, is equivalent to saying that above combinatorially defined cohomologies coincide with the corresponding "De Rham" cohomologies defined there.

There are variants of the above game — cf. § 7 — which apply when there is some additional structure. For example if the vertices of K are *colored*, then demanding rotational skewsymmetry only with respect to vertices of the same color will give a sub cochain complex.

§ 2. **Partial normalization.** Many homologies are defined by using a total order on the set of vertices of K. For example using this the set K_{comm} of **commutative monomials** supported on simplices of K can be identified with the subset of K_{assoc} consisting of all increasing vertex sequences. Now $C_*(K_{assoc})$, the vector space spanned by K_{assoc}, comes equipped with the boundary ∂ dual of δ. This is given by

$$\partial(v_0 v_1 \,..\, v_q) \;=\; \Sigma_{i\geq 0}\,(-1)^i.\, v_0 \,..\, \hat{v}_i \,..\, v_q \;,$$

and so clearly ∂ maps the subspace $C_*(K_{comm})$ into itself.

As you know *the homology* $H_*(K_{comm})$ *of* $(C_*(K_{comm}),\partial)$ *is* *isomorphic to the usual homology* $H_*(K)$. However the game is just begun! There are numerous other natural sub chain complexes of $(C_*(K_{assoc}),\partial)$ to consider. For instance, for each $r \geq 1$, we have the sub chain complex $(C_*(K_{comm,r}),\partial)$ spanned by all *commutative monomials in which each vertex occurs* $\leq r$ *times*. What are these homologies $H_*(K_{comm,r})$? The answer is given by the following striking result of T.Bier [1].

Theorem 2. *For r odd* $H_*(K_{comm,r}) \cong H_*(K)$ *but for r even* $H_*(K_{comm,r}) \cong \oplus_{\sigma\in K} \tilde{H}_{*-r|\sigma|}(Lk_K\sigma)$.

Here the tildes signal "reduced" homologies, i.e. that *the empty set* $\emptyset$ *is being considered as a simplex*. So for r even one of the above summands is $\tilde{H}_*(K)$ because $Lk_K\emptyset = K$. Regarding these links $Lk_K\sigma$ we recall that they are defined as follows. We identify K with the corresponding element of $C_*(K_{comm,1})$ and divide out completely by σ to get $K = \sigma Q + R$ (so σ does not divide R). Then $Q = Lk_K\sigma$.

Curiously it is only the r odd case of Th. 2 which requires some work. Then the r even case follows easily — again see [15] for details — by using the fact that, while calculating ∂, we can pull out even powers of vertices (like we pull out constants while differentiating).

Remark. The total order of the vertices made K_{comm} into a complete semi-simplicial complex with *normalization* (= non-degenarate simplices) $K_{comm,1}$, so one can call the above $K_{comm,r}$'s *partial normalizations of* K_{comm}. Note that these are semi-simplicial complexes, but in general the non-degenerate simplices of a c.s.s.c. (e.g. those of K_{assoc}) need not constitute a semi-simplicial complex.

§ 3. **Deleted joins.** Many homologies occur in the following obvious way : one performs a natural combinatorial operation on K and *then* calculates homology. This calculation can range from "easy" to "hard" to "very hard".

For instance the join K.L of two disjoint simplicial complexes (now $\emptyset$ is a simplex) consists of the unions (σ,θ) of all ordered pairs $\sigma \in K$, $\theta \in L$. It is easy enough to compute its homology. One gets for K.K, the join of two disjoint copies of K, the formula

$$\tilde{H}_*(K.K) \cong \oplus_{p+q+1=*} \tilde{H}_p(K) \otimes \tilde{H}_q(K).$$

On the other hand there is no nice formula known for the homology of the (maximal) deleted join K*K, i.e. the subcomplex of K.K consisting of all (σ,θ) with $\sigma \cap \theta = \emptyset$. In fact as I showed in [13] even the case dimK = 1 is non-trivial and very interesting.

Remark. K*K is the maximal subcomplex of K.K on which the *$\mathbb{Z}/2$-action* $(\sigma,\theta) \longleftrightarrow (\theta,\sigma)$ is *free*. This is what makes K*K very useful — see e.g. [11] – [13] — for numerous questions re embeddings and colorings of K. We note also that K*K has the $\mathbb{Z}/2$-homotopy type of the space $|K.K|$ *minus* its diagonal, and that a nice formula (due to

Richardson-Smith and involving homology operations) is known for the $\mathbb{Z}/2$-homology with mod 2 coefficients of $|K.K|$ *mod* its diagonal.

We turn now to a smaller deleted join, viz. the subcomplex $K \bullet K$ of $K * K$ consisting of all (σ,θ) with $\sigma \cup \theta \in K$, for which I got the following formula (it was also noted independently by A.Björner).

Theorem 3. $\tilde{H}_*(K \bullet K) \cong \oplus_{\sigma\in K} \tilde{H}_{*-|\sigma|}(Lk_K\sigma)$.

This is of course remarkably like Bier's theorem except that there r was even and here $r = 1$ (however the combinatorial proof we have is not nice enough to explain this similarity). In § 5 below we'll see that above also follows easily by using Goresky-Macpherson [4].

Recall next that K is called **Cohen-Macaulay** iff the *local homologies* $\tilde{H}_*(Lk_K\sigma)$ vanish in dimensions less than $\dim K - |\sigma|$. This concept, which is of great importance in combinatorics, has thus the following interesting reformulation in terms of the above homology.

Corollary 4. *A simplicial complex is Cohen-Macaulay if and only if* $\tilde{H}_*(K \bullet K)$ *vanishes in dimensions less than* $\dim K$.

Remark. In the above we only considered the case $G = \mathbb{Z}/2$ but one can analogously define various "deleted G-joins", G being now any discrete (or even continuous) group, and these too are useful for embedding/coloring questions. We note also that the infinite join EG =

BGG... of Milnor [9] is a deleted G-join of the infinite simplex. Thus there is an obvious tie-up between deleted joins, "classifying spaces" $BG = EG/G$, and the "configuration spaces" of Segal about which Prof. Burghelea has talked in this Workshop.

§ **4. Assorted visualizations.** To understand $K \bullet K$ better (and to define some more "simplicial" homologies !) we now move briefly to the other side of the double-arrow DISCRETE $\longleftrightarrow$ CONTINUOUS.

We begin by recalling how the **usual realization** $|K|$ is defined. One thinks of the N vertices as the standard bases vectors of $\mathbb{R}^N$, each subset σ of these is replaced by its *convex hull* $\mathrm{Conv}(\sigma)$, and one sets

$$|K| = \mathrm{Conv}(K) = \bigcup_{\sigma \in K} \mathrm{Conv}(K).$$

The point which I want to make now is that *there is really nothing so sacred about convex hulls* !! It is in my opinion at least equally natural to also consider the following **affine, linear, spherical, and projective realizations** of K.

$$\mathrm{Aff}(K) = \bigcup_{\sigma \in K} \mathrm{Aff}(\sigma), \qquad \mathrm{Lin}(K) = \bigcup_{\sigma \in K} \mathrm{Lin}(\sigma),$$

$$\mathrm{Sph}(K) = \bigcup_{\sigma \in K} \mathrm{Sph}(\sigma), \qquad \mathrm{Proj}(K) = \bigcup_{\sigma \in K} \mathrm{Proj}(\sigma).$$

Here of course $\mathrm{Aff}(\sigma)$ denotes the *affine hull* of σ in $\mathbb{R}^N$, likewise $\mathrm{Lin}(\sigma)$ is its *linear hull* (= coordinate subspace), $\mathrm{Sph}(\sigma)$ its *spherical hull* (= unit sphere of $\mathrm{Lin}\sigma$), and $\mathrm{Proj}(\sigma)$ its *projective hull* (= $\mathrm{Lin}\sigma$ divided out by the scalar multiplication of $\mathbb{R}^{\times}$).

We note that $\mathrm{Aff}(K)$ has the same homotopy type as $\mathrm{Conv}(K)$ and that $\mathrm{Lin}(K)$ is contractible. However the latter has an interesting singularity at the origin with link homeomorphic to $\mathrm{Sph}(K)$. Note also that $\mathrm{Sph}(K)$ contains a dense subspace homeomorphic to $\mathrm{Aff}(K)$, likewise the projective realization $\mathrm{Proj}(K)$ is also a compactification of $\mathrm{Aff}(K)$.

Proposition 5. *$K\bullet K$ is a $\mathbb{Z}/2$-triangulation of* $\mathrm{Sph}(K)$.

So the homology of $K\bullet K$ coincides with the singular homology of $\mathrm{Sph}(K)$, and the equivariant homology of $K\bullet K$ gives that of $\mathrm{Proj}(K)$. Further we see that K *is Cohen-Macaulay iff the singularity of* $\mathrm{Lin}(K)$ *at the origin is homologically trivial in dimensions less than* $\dim K$.

The above proposition follows easily by noting that $K\bullet K$ is a union of the octahedral spheres $\bar{\sigma}\bullet\bar{\sigma}$, $\sigma \in K$, and we can use these to triangulate the spheres $\mathrm{Sph}(\sigma)$ of $\mathrm{Sph}(K)$.

We note next that there are *analogous visualizations* $\mathrm{Aff}_{\mathbb{F}}(K)$, $\mathrm{Lin}_{\mathbb{F}}(K)$, *etc. over other fields* $\mathbb{F}$ (for $\mathrm{Conv}_{\mathbb{F}}K$ and $\mathrm{Sph}_{\mathbb{F}}K$ only we need to assume $\mathbb{F}$ ordered and normed respectively).

For example, $\mathrm{Sph}_{\mathbb{C}}(K) \subset \mathbb{C}^N$ is a union of $(2\dim\sigma+1)$-dimensional spheres $Sph_{\mathbb{C}}(\sigma)$, with S^1 acting freely on each of these odd spheres via complex multiplication, so the quotient $Sph_{\mathbb{C}}(K) \longrightarrow Proj_{\mathbb{C}}(K)$ is sort of like a "generalized Hopf fibration".

A by-product of all of the above is that *we have now a host of new "simplicial" homologies to consider, viz. these visualizations' singular* (or even l-adic !) *homologies.* Many of these, e.g. $H_*(Sph_{\mathbb{R}}K)$, can be computed most conveniently by using the topological methods of either Goresky-Macpherson [4] or Ziegler-Zivjaljevic [19] (rather than an explicit triangulation like that of $Sph_{\mathbb{R}}K$ given above).

The most interesting of these computations — see §§ 7 and 8 — is undoubtedly that of $H_*(Proj_{\mathbb{C}}K)$. We'll give in § 7 a small "cyclic" set which triangulates the free S^1-space $Sph_{\mathbb{C}}K$, thus re-interpreting $H_*(Proj_{\mathbb{C}}K)$ as a purely combinatorially defined "cyclic" homology.

Also in § 10 we'll relate the above to the local cohomology of the algebraic variety Lin(K) at the origin. This cohomology in fact pertains to the commutative algebra $\mathscr{A}(K)$ of all polynomial functions on Lin(K). Note that $\mathscr{A}(K) = \{C_*(K_{comm}),$ equipped with the obvious multiplication of commutative monomials$\}$.

It would be interesting to compute the Hochschild and cyclic homologies of $\mathscr{A}(K)$, as well as that of the associative algebra $\mathscr{A}_{assoc}(K)$, obtained by equipping $C_*(K_{assoc})$ with the multiplication of associative monomials. Since this remains to be done, I will now give instead a known and very striking non-commutative example.

§ 5. **Non-abelian chains.** In this section the coefficients are from $\mathbb{Z}$. So $CK = C_*(K_{comm})$ is the *free Abelian group* generated by K_{comm}. We now consider a definition due to Moore which makes sense even for *the free (non-Abelian) group* $FK = F_*(K_{comm})$ generated by K.

For this we note that the boundary of $C_*(K_{comm})$ equals

$$\partial \quad = \quad \partial_0 - \partial_1 + \partial_2 - \dots ,$$

where the *face homomorphisms* ∂_i: $C_*(K_{comm}) \longrightarrow C_{*-1}(K_{comm})$ are given by $v_0 v_1 \dots v_q \longmapsto v_0 \dots \hat{v}_i \dots v_q$. So ∂ maps the subgroup

$$C_*^{Moore}(K_{comm}) \quad = \quad (\ker\partial_1) \cap (\ker\partial_2) \cap \dots$$

into itself, and on it coincides with ∂_0. The key observation of Moore was that *the inclusion of chain complexes* $(C_*^{Moore}(K_{comm}),\partial_0) \longrightarrow (C_*(K_{comm}),\partial)$ *induces an isomorphism in homology* $H_*^{Moore}(CK) \cong H_*(K)$.

This is very interesting because (unlike the bigger chain complex) the non-Abelian analogue of this smaller chain complex, i.e. $(F_*^{Moore}(K_{comm}) = (\ker\partial_1) \cap \ker(\partial_2) \cap \dots, \partial_0)$, *is still a chain complex*, whose homology will be denoted $H_*^{Moore}(FK)$.

That ∂_0 is a differential of $F_*^{\text{Moore}}(K_{\text{comm}})$ is clear from $\partial_0\partial_0 = \partial_0\partial_1$, and to check that its image is a normal subgroup one needs a small verification which uses the usual relations between the ∂_i's and the *degeneracy homomorphisms* s_j: $F_*(K_{\text{comm}}) \to F_{*+1}(K_{\text{comm}})$, $v_0 v_1 \ldots v_q \longmapsto v_0 \ldots v_j v_j \ldots v_q$.

Thus we can in fact define **Moore homology** $H_*^{\text{Moore}}(G_\bullet)$ for any *simplicial group* $G_\bullet$, i.e. a sequence G_n, $n \geq 0$, of groups equipped with face and degeneracy homomorphisms satisfying the usual relations.

With the definition made, there arose the usual question : *what is* $H_*^{\text{Moore}}(FK)$? The striking answer was soon discovered by Milnor [10].

Theorem 6. $H_*^{\text{Moore}}(FK) \cong \pi_{*+1}(S^0.K)$, *i.e. the Moore homology gives the homotopy groups of the suspension of* K.

We note that the above — see Kan [6] for more — gives a *purely combinatorial definition of these homotopy groups* which is much more satisfying than the obvious one (e.g. by using a huge "Kan extension" of K_{assoc}) suggested by the simplicial approximation theorem. Also it tends to show that, despite appearances, the homotopy groups are indeed the natural "non-Abelian analogues" of the homology groups !

§ 6. **Higher order boundaries.** Let $p \geq 2$ and let us assume that the characteristic zero coefficients $\mathbb{F}$ contain the pth roots of unity.

We now equip $C_*(K_{assoc})$ with the *cyclotomic boundary* defined by

$$\partial(v_0 v_1 \ldots) = \Sigma_r (\omega)^r (v_0 v_1 \ldots \hat{v}_r \ldots),$$

where $\omega = \exp(2\pi i/p)$, $p \geq 2$. This boundary also obviously preserves the subspace $C_*(K_{comm})$ spanned by monomials which are increasing with respect to a given total order of the vertices. Moreover an easy calculation, using $(1 + \omega + \omega^2 + \ldots + \omega^{p-1}) = 0$ shows $\partial^p = 0$.

So for each ordered pair (r,s) of positive integers with sum $r + s = p$ we can define the cyclotomic homology groups

$$H_{*;r,s}(K_{comm}) = \frac{\ker(\partial^r)}{\operatorname{im}(\partial^s)}.$$

Theorem 7. *The non-zero cyclotomic homology is as follows :*

$$H_{kp+r-1;r,s}(K_{comm}) \cong H_{kp+s-1;s,r}(K_{comm}) \cong H_{2k}(K),$$

$$H_{kp-1;r,s}(K_{comm}) \cong H_{kp-1;s,r}(K_{comm}) \cong H_{2k-1}(K).$$

For the proof see [15]. Over cyclotomic *integers* this cohomology is more involved and not yet fully worked out.

I found the above definition — actually a more general one using *any non-trivial character* ω of any finite group — while going over the first *Complément* of Poincaré's *Analysis Situs* during my seminar [14] of 1993-94. Thus this definition was inspired by the very first combinatorial definition of homology !

Remark. If we use mod p *coefficients*, and replace -1 in the usual definition of ∂ by (the trivial character) 1, then too we have $\partial^p = 0$. This *Mayer homology* [8] was first computed by Spanier [17]. The results are similar to those given above. It seems that the game of § 1 can also be generalized to group characters other than -1.

§ 7. **Cyclic deleted joins.** We want to combinatorialize the free S^1-space $\mathrm{Sph}_{\mathbb{C}}K$. For $K = pt$, $\mathrm{Sph}_{\mathbb{C}}(K) \cong S^1$, and we'll use the discrete circle of Connes, i.e. the completion $\mathbb{S}$ of the semi-simplicial complex having one vertex 0 and one edge (00), together with a rotational group structure $\cong \mathbb{Z}/r+1$ on the $r+1$ r-simplices of $\mathbb{S}$ for each $r \geq 0$. The definition of these "*rotations*" t should be clear from the following — here $0(00)0 = s_2 s_0(00)$ etc. — which shows the case $r = 3$.

$$\sigma = (00)00,\ t(\sigma) = 0(00)0,\ t^2(\sigma) = 00(00),\ t^3(\sigma) = 0000.$$

For a general K we use a union of some joins of disjoint copies (each augmented by $\emptyset$) of $\mathbb{S}$ to define

$$K \bullet_{\mathbb{C}} K = \bigcup_{\sigma \in K} \left[\prod_{v \in \sigma} \mathbb{S}_v \right].$$

Now let $\delta : C^*(K \bullet_{\mathbb{C}} K) \longrightarrow C^{*+1}(K \bullet_{\mathbb{C}} K)$ be the sum of the maps induced by the coboundaries $\delta_v : C^*(\mathbb{S}_v) \longrightarrow C^{*+1}(\mathbb{S}_v)$. One has $\delta^2 = 0$ and δ preserves the subspace $C^*_{pcycl}(K \bullet_{\mathbb{C}} K)$ of cochains f *rotationally* *skewsymmetric over each* $\mathbb{S}_v$, i.e. such that $f(\sigma)$ only gets multiplied by $(-1)^{\dim_v}$ when the *part* σ_v of σ is "rotated" to $t(\sigma_v)$. We'll denote the cohomology of $(C^*_{pcycl}(K\bullet_{\mathbb{C}}K),\ \delta)$ by $H^*_{pcycl}(K\bullet_{\mathbb{C}}K)$.

Theorem 8. *The partially cyclic cohomology* $H^*_{pcycl}(K\bullet_{\mathbb{C}}K)$ *gives the singular cohomology of the projective realization as follows :*

$$H^*_{pcycl}(K\bullet_{\mathbb{C}}K) \cong \oplus_{j\geq 0} H^{*-2j}(\mathrm{Proj}_{\mathbb{C}}K).$$

One also has $H^*(K\bullet_{\mathbb{C}}K) \cong H^*(\mathrm{Sph}_{\mathbb{C}}K)$, in fact *the Milnor realization* $|K\bullet_{\mathbb{C}}K|$ *of* $K\bullet_{\mathbb{C}}K$ *is a cell subdivision of* $\mathrm{Sph}_{\mathbb{C}}K$ *which "complexifies" the triangulation* $K\bullet K$ *of* SphK as follows.

We put the vertex 0_v of $\mathbb{S}_v$ on the identity $1_v \in \mathbb{C}_v$ of the vth coordinate axis of $\mathbb{C}^N$, and lay the edge $(00)_v$ of $\mathbb{S}_v$ over the unit circle S^1_v of $\mathbb{C}_v$. Then clearly the non-degenerate simplices of $K\bullet_{\mathbb{C}}K$, i.e. those with all parts equal to 0_v or $(00)_v$, furnish us with a cell subdivision *of* $\mathrm{Sph}_{\mathbb{C}}K$. The intersection of this cell subdivision with $\mathbb{R}^N$ gives a triangulation $\cong K\bullet K$ of SphK.

The proof of Theorem 8 is analogous to how cyclic cohomology is interpreted as an equivariant S^1 cohomology (see e.g. Loday [7]). We use a bigger (but homotopically equivalent) realization $\|K\bullet_{\mathbb{C}}K\|$. This turns out to be $\mathrm{Sph}_{\mathbb{C}}K \times ES^1$ and on it the combinatorial rotations translate into the diagonal S^1-action. So the required cohomology interprets as the singular homology of the quotient $\mathrm{Proj}_{\mathbb{C}}K \times BS^1$.

The combinatorial chain complex of Theorem 8 can probably be used to give a new proof of the following known result of Ziegler-Zivaljevic (and possibly Goresky-MacPherson and others).

Theorem 9. $H_*(\mathrm{Proj}_{\mathbb{C}}K) \cong \oplus_{j\geq 0} H_{*-2j}(K^{(j)})$.

Here $K^{(j)}$ is the jth co-skeleton of K, i.e. the poset (under $\subseteq$) of all simplices of K having dimensions $\geq j$, and $H_*(K^{(j)})$ denotes the homology of the order complex of this poset.

In fact the order complex is a key example of a **Grothendieck realization** $\|..\|$. It is the realization $\|K^{(j)}\|$ *of the category* obtained from $K^{(j)}$ by thinking of each $\subseteq$ as a morphism. (There are also $\|..\|$*'s of functors*, etc. According to Professor Burghelea this concept is essentially due to Ehresmann. Many people have contributed to the theory of these realizations, notably Segal and Bousfield-Kan.)

Since the Ziegler-Zivaljevic [19] proof of Theorem 9 also makes a heavy use of these realizations, a couple of quick words re them seem to be in order. Grothendieck's main idea was simply to use as simplices some trains of morphisms, with faces defined in the obvious way by "erasing" one object at a time. So e.g. the train

$$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet$$

has as faces the four trains shown below.

$$\bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet \qquad \bullet \xrightarrow{ba} \bullet \xrightarrow{c} \bullet$$

$$\bullet \xrightarrow{a} \bullet \xrightarrow{cb} \bullet \qquad \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$$

Thus "*Hochschild faces*" arise naturally if we visualize categories (or functors, etc.) à la Grothendieck et al.

§ 8. **Zeta functions.** In this section I'll use the finite field of q elements $\mathbb{F}_q$ (and its finite extensions $\mathbb{F}_{q^n}$). I'll give a result which has a well known "explanation" involving the (definition of the) *l-adic cohomology* of $\mathrm{Proj}_{\mathbb{F}_q} K$. Since this cohomology is isomorphic to $H^*(\mathrm{Proj}_{\mathbb{C}} K)$, it is natural to wonder whether this "explanation" can also be combinatorialized à la § 7 ?

Riemann's zeta function $\zeta(s) = \prod_{\mathcal{M}} \dfrac{1}{1 - p^{-s}}$ generalizes from $\mathbb{Z}$ to any ring R with finite quotient fields $R/\mathcal{M}$ by replacing the p's by the cardinalities $|R/\mathcal{M}|$. For example R can be the coordinate ring $\mathcal{A}$ of any *affine variety* (e.g. $\mathrm{Lin}_{\mathbb{F}_q} K$) over $\mathbb{F}_q$: now each $|R/\mathcal{M}| = q^n$ for some n, and $\mathcal{M}$ determines and is determined by a *point* of the variety over the finite extension $\mathbb{F}_{q^n}$. So the definition generalizes still further to all *projective varieties* (e.g. $\mathrm{Proj}_{\mathbb{F}_q} K$) over $\mathbb{F}_q$: use in place of $\mathcal{M}$'s points over the finite extensions $\mathbb{F}_{q^n}$ and in place of p's the cardinalities of the residue fields at these points.

Thanks to the fundamental theorem of arithmetic, Riemann's zeta is also given by $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$. Likewise, thanks to a unique factorization theorem for ideals, the aforementioned **zeta function** $\zeta_K(s)$ of $\mathrm{Proj}_{\mathbb{F}_q}(K)$ is also given by

$$\zeta_K(s) = \exp\left(\sum_{n\geq 1} a_n(K) \frac{(q^{-s})^n}{n} \right) ,$$

where $a_n = |\mathrm{Proj}_{\mathbb{F}_{q^n}} K|$.

Theorem 10 (with Anders Björner). *Let* $f_i(K)$ *denote the number of i-simplices of* K. *Then*

$$\zeta_K(s) = \prod_{j\geq 0} \frac{1}{(1 - q^j q^{-s})^{e_j(K)}} ,$$

where $e_j(K) = \sum_{i\geq j} (-1)^{i-j} \binom{i}{j} f_i(K)$ = *Euler characteristic of the order complex of the jth co-skeleton of* K.

For this we substitute $a_n(K) = \sum_{i\geq 0} f_i(K) \ (q^n - 1)^i$ in the formula for $\zeta_K(s)$ and simplify. The identification of the numbers $e_j(K)$ with the Euler characteristics $e(K^{(j)})$ requires some more work.

Alternatively, using $Z_K(t)$, the function of t obtained from $\zeta_K(s)$ by replacing q^{-s} by t, we can write the above formula as

$$Z_K(t) \;=\; \frac{P_1(t)\ P_3(t)\ P_3(t)\ \ldots}{P_0(t)\ P_2(t)\ P_4(t)\ \ldots}\,,$$

where $P_i(t)$ is the following integral polynomial of degree equal to the ith Betti number of $\mathrm{Proj}_{\mathbb{C}}K$:

$$P_i(t) = (1-t)^{b_i(K^{(0)})}\,(1-qt)^{b_{i-2}(K^{(1)})}\,(1-q^2t)^{b_{i-4}(K^{(2)})}\ \ldots\ .$$

Grothendieck's "explanation" of such factorizations runs thus : The cohomology of the complexified variety can be computed l-adically. If l does not divide q then $x \longmapsto x^q$ and its iterates act on this l-adic complex. Lefschetz's fixed point formula implies factorization.

We note that $\mathrm{Proj}_{\mathbb{C}}K$ is singular — however its singularities have codimension ≥ 2, i.e. it is a *pseudomanifold* — and that the functional equation and Riemann hypothesis do *not* hold for $\zeta_K(s)$. However these probably hold for a suitably modified zeta function based on the homologies considered in the next section.

§ 9. **Other skeleta.** Let π denote a *total order* (= permutation) of $[n] = \{0, 1, \ \ldots\ , n\}$. We define the ith **$\pi$-skeleton** of an n-dimensional simplicial complex K by

$$K_\pi^i = \{\sigma : \pi^{-1}(\dim\sigma) \le i\}.$$

For example, K_{id}^i is the (usual) ith *skeleton* K^i, while for the *opposite total order* $o(id) = (n, n-1, \ldots, 1, 0)$ one has $K_{o(id)}^i = K^{(n-i)}$, the (n-i)th *co-skeleton* of K.

For each i the inclusion $K_\pi^i \longrightarrow K_\pi^{i+1}$ induces a simplicial map in the order complexes of these posets. Using this we define the ith **π-homology** of K by

$$H_i^\pi(K) = \mathrm{im}\ \{H_i(K_\pi^i) \longrightarrow H_i(K_\pi^{i+1})\}.$$

Note that $H_i^{id}(K)$ identifies with the *usual homology* $H_i(K)$, which of course depends only on i and the underlying space of K. For $\pi \ne id$ only partial answers are known re the topological invariance of these homologies $H_i^\pi(K)$.

Theorem 11. *Let K be an n-dimensional pseudomanifold and let π or its opposite $o(\pi)$ be such that it ends with $(\ldots, n-1, n)$ and has just one local minimum. Then $H_i^\pi(K)$ depends only on π, i, and the underlying space of K, and one has $H_i^\pi(K) \cong H_{n-i}^{o(\pi)}(K)$.*

The above is due to Goresky-MacPherson [3] who call $H_i^\pi(K)$'s of the above kind the **intersection homologies** of K. (To correlate our definition with their's we note that each π of the above kind determines a "perversity" p under which our K_π^i becomes their "basic set" Q_i^p.)

If K is an *n-manifold*, i.e. if for each $\sigma \in K$ the trains of $\|K\|$ having all elements $\geq \sigma$ determine a **dual cell** of dimension n-dim(σ), then $H_{n-i}^{o(id)}(K)$ is isomorphic to the *usual cohomology* $H^{n-i}(K)$. Thus the above result contains *Poincaré's duality theorem* $H_i(K) \cong H^{n-i}(K)$. In fact intersection homology was found in the course of trying to "better understand" the cell/dual-cell proof of this theorem which Poincaré gave in the first *Complément* of his *Analysis Situs* !

§ **10. Algebras.** The next example is that of "*a cohomology with local coefficients*" (these go back to Čech and Steenrod). Let $\mathscr{A} = \mathscr{A}(K)$ be the *algebra of all polynomial functions defined on* Lin(K). For each simplex σ the coefficient ring $\mathscr{A}_\sigma$ will be zero unless $\sigma \in K$ when it will consist of *all rational functions defined on* Lin(σ).

Let $C^q(\mathscr{A})$ consist of all skew-symmetric functions X from vertex sequences with values $X(v_0 v_1 \ldots v_q)$ in $\mathscr{A}_\sigma$ where $\sigma = \{v_0, v_1, \ldots, v_q\}$. Let $\delta : C^*(\mathscr{A}) \longrightarrow C^{*+1}(\mathscr{A})$ be defined by

$$(\delta X)(v_0 v_1 \ldots v_q) = \Sigma_i \ (-1)^i \ \phi_i(X(v_0 \ldots \hat{v}_i \ldots v_q)),$$

where ϕ_i denotes the obvious map from the coefficient ring of the face of σ opposite v_i, to the coefficient ring of σ. One has $\delta \circ \delta = 0$ and $\frac{\ker\delta}{\mathrm{im}\delta}$ will be called, following Grothendieck, the **local cohomology** $H^*(\mathscr{A})$

of Lin(K) at the origin (or of $\mathscr{A}$ at its irrelevant ideal).

All nonzero elements of the type $[v_1^{\lambda_1} \dots v_N^{\lambda_N}] \in \mathscr{A}_\sigma$, $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N$, constitute a vector space basis of $\mathscr{A}_\sigma$. Using this we get a *finer* $\mathbb{Z}^N$*-grading* $C^q(\mathscr{A}) = \oplus_\lambda C^q_\lambda(\mathscr{A})$ and $H^q(\mathscr{A}) = \oplus_\lambda H^q_\lambda(\mathscr{A})$, where summation is over all $\lambda \in \mathbb{Z}^N$ with $|i : \lambda_i \neq 0| = q+1$. The following result is due to Hochster (see Stanley [18]).

Theorem 12. *$H^q_\lambda(\mathscr{A}K)$ can be nonzero only if all $\lambda_i \leq 0$ and $\sigma = \{v_i : \lambda_i \leq -1\} \in K$ in which case $H^q_\lambda(\mathscr{A}K) \cong \tilde{H}^{q-1-|\sigma|}(Lk_K\sigma)$.*

Using §§ 3–4 it follows that this algebraic local cohomology, though infinite dimensional, is determined by the singular local cohomology $H^q(\mathrm{Lin}K, \mathrm{Lin}K \setminus \{0\})$ of Lin(K) at $\{0\}$.

Note that *Krull dimension* $\dim\mathscr{A} = \dim K + 1$, and that $H^q(\mathscr{A}) = 0$ for all $q > \dim\mathscr{A}$. If $H^q(\mathscr{A}) = 0$ also for all $q < \dim\mathscr{A}$ then one says that $\mathscr{A}$ is a *Cohen-Macaulay ring* (there are various other equivalent reformulations of this concept). So the above theorem implies the following very useful result of Reisner.

Corollary 13. *The ring $\mathscr{A}(K)$ is Cohen-Macaulay if and only if the simplicial complex K is Cohen-Macaulay in the sense of § 3.*

We note that the defining cochain complex $C^*(\mathscr{A})$ of local cohomology is isomorphic to the limit of the cochain complexes $C^*_k(\mathscr{A})$, k

≥ 1, where $C_k^q(\mathscr{A})$ consists of all skew-symmetric functions X from length q+1 vertex sequences to $\mathscr{A}_\sigma$, and $\delta : C_k^*(\mathscr{A}) \longrightarrow C_k^{*+1}(\mathscr{A})$ is defined (somewhat like the "De Rham" δ of [16], § 5) by

$$(\delta X)(v_0 v_1 \dots v_q) = \Sigma_i (-1)^i (v_i)^k . X(v_0 .. \hat{v}_i .. v_q).$$

The resulting Koszul cohomologies $H_k^*(\mathscr{A})$, which are finite-dimensional approximations of the local cohomology $H^*(\mathscr{A})$, are also quite interesting combinatorially, e.g. Hochster showed that

$$H_1^*(\mathscr{A}K) \cong \oplus_\sigma \tilde{H}^{*-1-|\sigma|}(K\backslash\sigma),$$

where now σ runs over all subsets of $\{v_1, \dots, v_N\}$, and $K\backslash\sigma$ denotes the subcomplex of K consisting of all θ disjoint from σ.

There are also associated to $\mathscr{A}$ — and likewise to the algebras of the varieties Aff(K) and Proj(K) — many other (co)homologies, e.g. the algebraic De Rham cohomology, Hochschild homology, cyclic homology, etc., and there are some (or *many*, depending on one's viewpoint) results scattered in the literature regarding their computations.

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