Straight to Mecca

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May 30, 2013. It was 1953, and many in Washington felt that a mosque being built there was all wrong, for it was pointing to the *north* of east, while Mecca is to the south of east. Luckily nothing was wrong, this mosque too—like hundreds all over the world for hundreds of years!—was pointing straight to Mecca, i.e., precisely in the direction of the great circle path which minimizes the distance to Mecca. A doubting diplomat was enabled to see this for himself by *running a thread tightly on a globe between two tacks placed at* W *and* M. Besides, central projection onto the tangent plane at the north pole gives, ***a map of the northern hemisphere*** *in which great circle paths are straight line segments*, and we see that going straight from W to M entails going through northerly latitudes :-

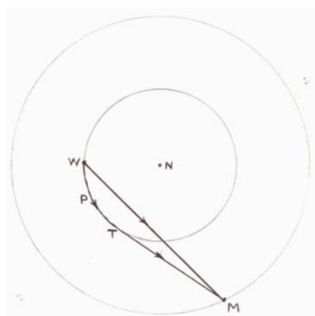

Figure 1

This story suggests a problem : *find an optimal path from* W *to* M ***with latitude non-increasing***. Now W and M are *any* two points in above map with latitude(W) > latitude(M). To *minimize distance*, as another thread experiment suggests, we should – Figure 1 – remain on the latitude of W till a point P = T such that PM does not meet this latitude in another point, and then go straight to M. However, unless W = T, this path is far from straight, it has uncountably many bends P. So one may prefer to interpret an 'optimal path' as one having the *fewest bends*, and in this context we'll now show that, *there are paths from* W *to* M *with latitude non-increasing and having finitely many bends.*

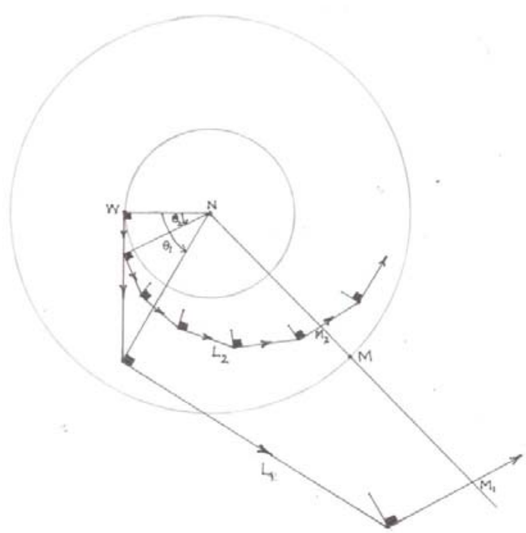

Figure 2

For any acute angle θ define an anticlockwise broken-line **spiral** L(θ) from W – Figure 2 – thus : remain on the tangent line to the latitude of the initial point for the first θ degrees of longitude; then turn to, and remain on, the tangent line to the latitude of this point for the next θ degrees of longitude; etc. If $\theta = \theta_1$ is big the first hit M_1 of this spiral L_1 on the longitude of M will be to its south. On the other hand, if $\theta = \theta_2$ is small enough, the spiral L_2 will initially stay arbitrarily close to the latitude of W, and this first hit M_2 will be to the north of M. So, *an angle* θ *intermediate between* θ_1 *and* θ_2 *gives us a spiral from* W *through* M *which is a path of the required kind.* Indeed, at each bend the next segment makes a right angle with the longitude, while for latitude non-increasing we only need that this angle be not acute. However this improvement is minor : *given a latitude non-increasing path, we can always push the last bend, at which this angle is obtuse, southwards on the same longitude to make it right, etc.* The number of bends in our spiral path is the largest k such that kθ is less than the longitude of M, and it seems no latitude non-increasing path has fewer bends if M is situated to the right of W in the lower half of the last picture.

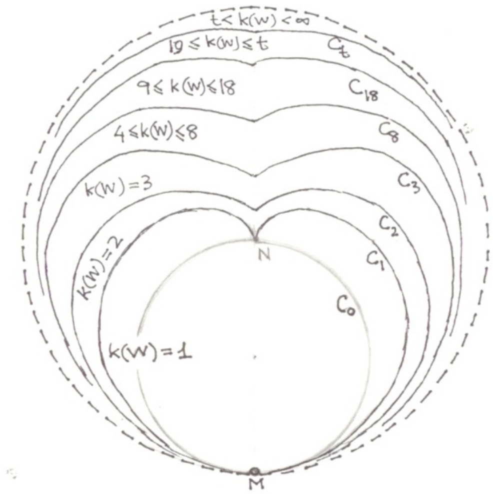

Figure 3

Anyway, *the* ***bend-distance to*** M, the least k(W) > 0 such that there is a latitude non-increasing path from W to M with at most k bends, *stratifies its northerly points as shown above.* We can go straight to M iff the angle subtended on NM is not acute, i.e., iff W is in the disk D_0 with diameter NM. So {W: k(W) ≤ 1} is the union of disks with diameters NP, P ϵ D_0, and more generally, {W: k(W) ≤ k+1} *is the union of disks with diameters* NP, P ϵ {W: k(W) ≤ k}. The same union is obtained if P runs only over the boundaries C_0 of D_0 and C_k of P ϵ {W: k(W) ≤ k}, or even if 'union of disks' is replaced by 'union of circles'. This shows that, *the stratification is unchanged if we insist that each segment of our path be tangent to the latitude at its initial point*, for example, if we are in the interior of the disk D_0 we can take this tangent in either direction till we arrive at a point on the circle C_0 from where we go straight to M. These *envelopes* C_{k-1}, k ≥ 2, *of families of circles have polar equations* $r = NM \cos^k(\theta/k)$, where θ is the longitude from M, and though terminology varies a lot, these "sinusoidal spirals" are sometimes called the "pedal curves" of the "cardioid" C_1.

This heart-shaped region plays a key role in that masterly 1957 paper of Gleason's when he proves the crucial and ***very startling fact*** : *if a frame function on the unit sphere*—that is, a function $f : S^2 \to R$ such that the sum of its values on all orthonormal frames is the same—*is non-negative, then it must be continuous.* The job is to find, *for each* ϵ > 0, *a non-empty open subset* U *of* S^2 *in which the oscillation of* f *is less than* ϵ. For it is easy—cf. Note 5—that a point orthogonal to a point of U has then a neighbourhood in which the oscillation of this frame function is less than 2ϵ; and of course, given any two directions, cross product gives us a third orthogonal to both.

We assume that the infimum of f is 0, and denote by ω its *weight*, that is, the sum of its values on any orthonormal frame. Take any N ϵ S^2 as *north pole,* and let σ be a polar rotation by π/2. Then g(x) = f(x) + f(σx) *is a pointwise bigger* *frame function of weight* 2ω *which has constant value* ω – f(N) *on the equator.* On any other W, say in the northern hemisphere, its value is at most 2f(N) more : choose an orthogonal $W^\perp$ on the equator, so g(W) + g($W^\perp$) ≤ 2ω, that is, g(W) ≤ ω + f(N). Indeed g(W) ≤ g(P) + 2f(N) *for any point* P *on the great circle through* W *and* $W^\perp$: for g(W) + ω – f(N) = g(P) + g(Q), where Q is a point on this great circle orthogonal to P, and the right hand side is at most g(P) + ω + f(N). We note that, *the northern half of this circle is the tangent line at* W *to its latitude in our map of this hemisphere.* Which shows that, *for any two points in this hemisphere with* latitude(W) > latitude(M) *we have* g(W) ≤ g(M) + 2(k+1)f(N)*, where* k *denotes the bend-distance to* M *of* W, because there is a path from W ≠ N to M having at most k+1 segments, each tangent to the latitude of its initial point. *For instance* g(W) ≤ g(M) + 4f(N) *if* W *is in the heart-shaped open set bounded by the cardioid* C_1. So, if the value of g on M is within δ of its infimum on more northerly points W ≠ N, the oscillation of g is less than 4f(N) + δ in this nonempty open set. By the remarks made in the previous paragraph, it follows that any point, in particular the north pole N, has a neighbourhood U in which the oscillation of g is less than four times 4f(N) + δ. But g(N) = 2f(N), so g(x) – and therefore also f(x) – lies between 0 and 18f(N) + 4δ for x ϵ U. Which does the job because we can choose N, M and δ such that this number is less than the given ϵ.

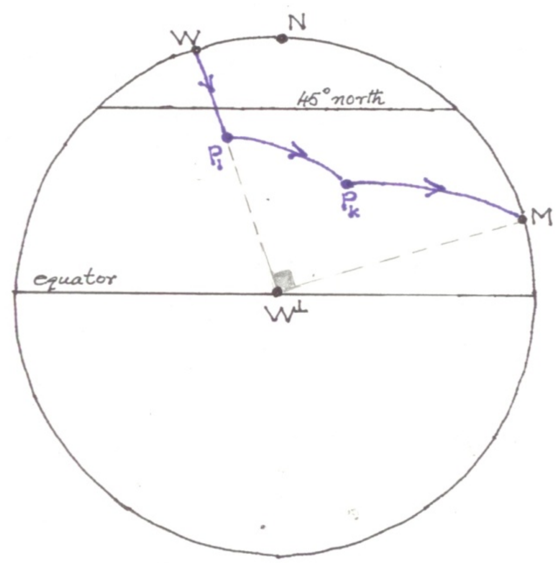

Figure 4

If a frame function g *is* 0 *on the north pole and has a constant value* ω ≥ 0 *on the equator, then for any two points in the northern hemisphere with* latitude(W) > latitude(M) *we have* g(W) ≤ g(M). This is easier – only the first few lines of this page are needed – but suffices for S^2 **is startling** : *there is no function* g : S^2 → {0,1} *which is* 0 *exactly once on each orthonormal frame.* This because we cannot have g(W) = 1 within 45 degrees of an N with g(N) = 0 : for g($W^\perp$) = 1 would imply g(M) = 0 where M is orthogonal to both W and $W^\perp$ – see Figure 4, note M is at longitude 180 degrees from W – but M is more southerly than W. Again there is an associated construction of **finite startling subsets**. Let k be the bend-distance to M of W. We note, using Figure 3, that k increases from 2 to infinity as latitude(W) decreases to 45 degrees. Choose a path – in blue in Figure 4 which shows case k = 2 – from W to M having k+1 great circle arcs, each tangent to the latitude of its initial point. Let F be the finite subset of S^2 consisting of N, W; the points $W^\perp$, P_1, Q_1 of the first great circle, the points $P_1^\perp$, P_2, Q_2 of the second great circle, etc.; their poles M, $P_1 \times P_1^\perp$, ...; etc. Then the same argument shows there is no g: F → {0,1} of the above kind, *with* g(N) = 0 *and* g(W) = 1. This extra condition on the truth functions can be removed, just as in *Startling Logic*, by using a bigger but still finite subset.

Notes

1. That mosque had made quite a stir in the U.S. capital then—see for example, *You can't build that mosque with a compass,* Washington Daily News, April 15, 1953—so Mackey and Gleason had probably heard of it, but I don't know if it actually inspired the latter to his solution of the problem posed by the former.

2. This problem appeared in print only after it had been solved, see page 51 of Mackey, *Quantum mechanics and Hilbert space,* Amer. Math. Monthly 64 (1957) pp. 45-57. The footnote on this page suggests that Gleason first classified only those frame functions f which have fourier expansions in spherical harmonics. So the idea of using great circle arcs tangent to latitude at their initial points, on which his delicate proof of the automatic continuity of f hinges—and on which alone we focus—probably came to him later. But even after he had these "E-W great circles" in hand, it seems from lines 3-4 on the key page 889 of Gleason's paper, that he first assumed that f attains its minimum, so proved that S^2 is startling, before turning his attention to the technical finesse that obviated this assumption.

3. Piron's beautiful reworking of these ideas brought the spirals L(θ) to the fore, but his coup de grâce – see page 79 of his *Foundations of Quantum Physics* (1976) – is the continuity almost everywhere of monotonic functions. Which reminds me, the insight which had led Gleason to his previous work on Hilbert's fifth problem was the differentiability almost everywhere of monotonic function. Also it brings back those days of long ago when I'd first learnt of such things about monotonic functions from that fantastic first chapter in F. Riesz and Sz-Nagy's *Functional Analysis.*

4. Indeed I also got to see my first love again, i.e., the classic from which I had taught myself mathematics during my teens, Goursat's *A Course in Mathematical Analysis*. Pages 432-433 of its first volume (Dover 1959 edition) will tell you a lot about envelopes of circles. See also Maschke, *A geometrical problem connected with the continuation of a power-series,* Annals of Math. 7 (1906) pp. 61-64, for another but related application of sinusoidal spirals.

5. *Preliminaries about frame functions*: they take same value on antipodal points, are closed under addition and composition with an isometry of S^2, and the restriction of a frame function to any great circle is a frame function. To see the assertion about oscillation, let q,v,p,u be four points in order on a great circle with q and v orthogonal to p and u which are both in U. Then there is a neighbourhood V of v such that, if we continue on the great circle from q through any point v' of V, the orthogonal points p' and u' are in U. Since $f(q) + f(p') = f(v') + f(u')$ and $f(q) + f(p'') = f(v'') + f(u'')$ imply $f(v') - f(v'') = f(p') - f(p'') + f(u'') - f(u')$ we see that the oscillation of f in V is at most 2ϵ.

6. The fact that Gleason's proof has in it a construction of finite startling sets was made explicit by Gill and Keane, *A geometric proof of the Kochen-Specker no-go theorem,* J. Phys. A : Math. Gen. 29 (1996) L289-291. Indeed, for each whole number $k \geq 2$ it gives a configuration, concatenating which one can obtain finite startling sets. More precisely, *if the angle between* N *and* W *in* Figure 4 *is* $\alpha < \pi/4$, *then the bend-distance from* W *to* M *is the smallest* k *such that* $\tan^2(\alpha) \leq \cos^{k+1}(\pi/(k+1))$. For, when we view them in our map of the northern hemisphere, the points W and M of Figure 4 are on opposite sides of N on the same straight line with NW = tan(α) and NM = cot(α), and the result follows from the polar equations of the spirals of Figure 3. So the smallest Gleason configuration, the one with k = 2, requires that the angle between N and W be at most that which satisfies $\tan^2(\alpha) = \cos^3(\pi/3) = 1/8$ i.e. $\tan(\alpha) = 1/\sqrt{8}$ i.e. $\alpha = \sin^{-1}(1/3)$, that is, *precisely the bound which Kochen and Specker need to make their configuration*! Not only that, the second smallest Gleason configuration, the one with k = 3, requires that the angle be at most that which satisfies $\tan^2(\alpha) = \cos^4(\pi/4) = 1/4$, i.e. $\alpha = \tan^{-1}(1/2)$, that is, *precisely the bigger bound permissible for Bell's slightly bigger configuration*! And besides these two, there is an infinity of steadily bigger Gleason configurations for k = 4, 5, ... Further, $\sin^{-1}(1/3)$ and $\tan^{-1}(1/2)$ were tied to the platonic subdivisions of the 2-sphere, and the angles α such that $\tan^2(\alpha) = \cos^{k+1}(\pi/(k+1))$ for k = 4, 5, ... may be tied to regular subdivisions of closed surfaces of higher genus. (*contd.*)