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Tetrahedron ABCD of Width 1 with Minimum AB+BC+CD

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A woodborer, freshly hatched at a random point within a slab of timber of unit thickness, wants to tunnel its way out by making at most two changes in its randomly chosen initial direction, in such a way that the worst case distance is minimized. What strategy does it adopt? We present the solution of this problem, also partial results about the analogous higher-dimensional problems.

Tetrahedron *ABCD* of Width 1 with Minimum $AB + BC + CD$

Karanbir S. Sarkaria

1. INTRODUCTION. It all began with Gunjeet Kaur, a calculus student, coming to me with a problem—which she had seen on page 77 of [**1**], where it appears without attribution—that I soon began to think about as an evolutionary fable.

Turtle Island. Forever and ever, mother turtles have laid eggs at random all over a long strip of land of unit width, having water on both sides. For survival, the newly hatched (and quite blind) baby turtles must reach water within a certain time. Eons ago, they probably just kept on walking stubbornly in a fixed randomly chosen direction, till they either found water or death. Much, much later, a mutant turtle appeared that was not averse to making a single right turn in case it had not found water within a certain distance. Natural selection favoured this mutation, so much so that today only these one-turn turtles exist and they now have the uncanny habit of turning right through precisely 120°, after having travelled exactly $2 \cdot 3^{-1/2}$ units in vain! Prove that these choices are governed by the following:

Maxim. Minimize the worst-case distance you may have to travel.

Proof. As Figure 1 (which illustrates the three generic possibilities) shows, a 1-turn path $\overrightarrow{ABC}$ is congruent to one embedded within Turtle Island if and only if one of the altitudes $\{A^\perp, B^\perp, C^\perp\}$ of the triangle ABC is less than 1 (this fact also justifies the definition of width (ABC) that we will give presently). Indeed, this is true for the altitude from the particular vertex with the property that the line through it and normal to the parallel lines passes through the triangle. So any triangle ABC with

$$\text{width } (ABC) := \min\{A^\perp, B^\perp, C^\perp\} \geq 1$$

and A in the interior of the strip would have B or C in the strip's boundary or exterior, i.e., the path $\overrightarrow{ABC}$ would suffice to exit the island. Since a regular (that is, equilateral) triangle with altitude 1 has all exterior angles equal to 120° and all sides of length

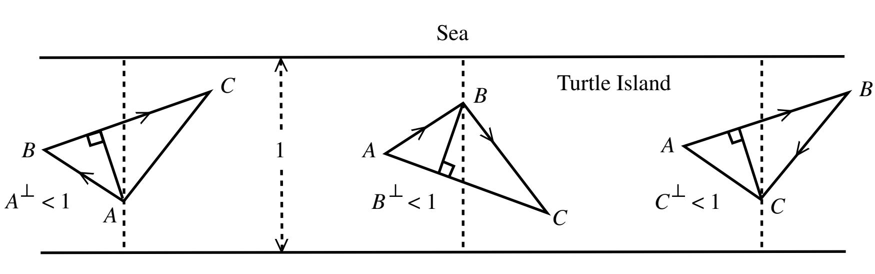

Figure 1.

$2 \cdot 3^{-1/2}$, we are reduced to demonstrating the proposition that *a triangle ABC of width 1 with $AB + BC$ minimum is regular.*

To prove this, note that if $\theta = \angle ABC$, then $AB = A^{\perp} \csc\theta$ and $BC = C^{\perp} \csc\theta$. As we run over all triangles ABC with width at least 1 and a fixed $\angle ABC$, $AB + BC$ takes its smallest value $2\csc\theta$ when both the altitudes $A^{\perp}$ and $C^{\perp}$ are equal to 1. Then $AB = BC = \csc\theta$ and

$$B^{\perp} = \csc\theta \cdot \cos(\theta/2) = 1/2\csc(\theta/2) \geq 1$$

gives $\theta \leq 60^\circ$, so the equilateral case $\theta = 60^\circ$ gives the minimum value $4 \cdot 3^{-1/2}$ of $AB + BC$. ∎

Many questions arose as I discussed this problem with my calculus class—especially with Dippy Aggarwal, Neha Behl, and Gunjeet. For example, in analogy with the foregoing proposition, we inquired whether a tetrahedron $ABCD$ of width 1 with minimum $AB + BC + CD$ is regular? Our objective in this article is to present *the rhombic tetrahedron* that we found, showing thereby that the answer to this question is negative (section 2); and in section 3 we give partial results for the similar, but harder, higher-dimensional problem.

At this point the class learned of (and enjoyed the information posted on) Finch's aptly named website "Lost in a Forest" [**4**]. Very briefly, Bellman proposed in a 1956 paper [**3**] the problem of finding paths that minimize worst case exit distance, as well as the harder problem of finding those that minimize expected exit distance (realistic evolutionary models also involve probabilistic considerations). For the strip, Bellman's first problem was solved by Zalgaller [**6**]. Since then, Zalgaller's arch-shaped curve of width 1 having least length has been independently rediscovered at least twice, in particular by Adhikari and Pitman [**2**]. However Bellman's second, harder problem still defies complete solution, even for the case of a strip, though much is known (see Zalgaller [**7**]). And, for most regions other than the strip, even Bellman's first problem remains open. For instance, it is still unsolved for an equilateral triangular forest!

At the end of [**2**], after raising the (still open) problem of finding the shortest width 1 curve in dimensions $d \geq 3$, Adhikari and Pitman aver that, for $d = 3$, *"presumably the solution is shaped something like three connected sides of a regular tetrahedron of altitude one inch, but we have no idea of what the exact shape must be"* [**2**, p. 326]. Modulo an obvious error—the width (see section 2 for the definition) of a regular tetrahedron equals the distance between its opposite edges (i.e., between any two edges that don't meet) which is *less* than its altitude—they clearly seem to be of the opinion that a tetrahedron $ABCD$ of width 1 with minimum $AB + BC + CD$ is regular. We show that this is false.

2. RHOMBIC TETRAHEDRA.

We recall that the *join* of two objects in Euclidean three-space is the union of all line segments having an endpoint in each of the objects. In particular, note that a *regular* tetrahedron of width 1 is the join of nonparallel diagonals of opposite faces of a cube of side 1 (see Figure 2, which shows, more generally, a tetrahedron $ABCD$ as the join of the diagonals AD and BC of opposite faces of any *box*, that is, any parallelepiped). So, *any* of its three pairs of opposite edges has the properties: (i) the vectors determined by the two edges are orthogonal; (ii) the segment joining their midpoints has length 1 and is perpendicular to both; and (iii) the squared reciprocals of their lengths have sum 1 (this because a square with side 1 has both diagonals equal to $2^{1/2}$).

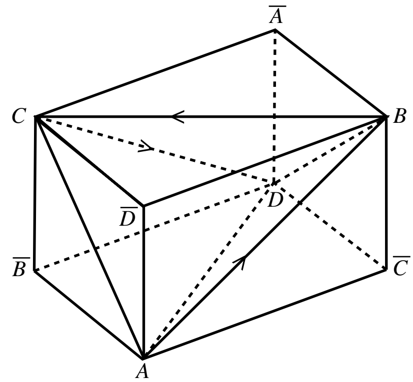

Figure 2.

A *rhombic tetrahedron ABCD* is one that enjoys properties (i)–(iii) for at least one pair of opposite edges, say AD and BC. Using property (iii) (i.e., $(AD)^{-2} + (BC)^{-2} = 1$), we see that $ABCD$ determines, and is determined by, a unique θ in $(0, \pi/2)$ such that $AD = \csc\theta$ and $BC = \sec\theta$, so we will sometimes denote this rhombic tetrahedron by T_θ. (Let $[AD]$ and $[BC]$ denote the faces of the box—see Figure 2 again—having the edges AD and BC as diagonals. These are now congruent rhombi, and 2θ is the angle of the rhombus $[BC]$ opposite its diagonal BC.) Note that $\theta = \pi/4$ gives the regular tetrahedron. This 1-parameter deformation T_θ, $0 < \theta < \pi/2$, of the regular tetrahedron has the following properties.

Theorem 1. *Any pair of opposite edges of T_θ are at distance 1 from each other; however, the width of T_θ is 1 if and only if θ belongs to $[\pi/6, \pi/3]$. There is a unique θ in $(\pi/6, \pi/4)$ such that $t = \cos^2\theta$ satisfies*

$$(2t-1)^2\big[(1-t)^{-3} + 1\big] = 2,$$

and the corresponding T_θ is, up to congruence, the unique tetrahedron ABCD of width 1 for which $AB + BC + CD$ is minimal.

It is this minimizing T_θ that was called *the* rhombic tetrahedron in section 1. We lay the groundwork for the proof of Theorem 1 in several lemmas.

We recall that the *width* of a tetrahedron is the minimum distance between pairs of parallel planes containing it, which is the same as the smallest of the distances between the seven pairs of parallel planes containing {edge, opposite edge} or {triangular face, opposite vertex}. Using only the first three of these we define the *edge width* of a tetrahedron to be the minimum distance between pairs of parallel planes containing opposite edges.

In Lemmas 1–3, ABCD will denote an ordered tetrahedron with edge width at least 1 for which the edge sum $AB + BC + CD$ is minimal. That such a tetrahedron exists follows by a compactness argument; our aim is to determine $ABCD$, and show that it has width 1. The analysis will make use of the concomitant box $[ABCD]$—see Figure 2—formed by the three pairs of parallel planes containing opposite edges. We begin by noting that AB and CD are nonparallel diagonals of parallel faces $[AB]$ and $[CD]$ of this box, and, in case the face $[BC]$ of the box is nonrectangular, BC must be its shorter diagonal; otherwise $BADC$ would have a smaller edge sum $BA + AD + DC$ than $ABCD$.

Lemma 1. *The top face [BC] of the box for which the minimum value $AB + BC + CD$ is attained is perpendicular to its adjacent faces and at distance* 1 *from the parallel bottom face [AD].*

Proof. Choose rectangular coordinates such that $[BC]$ is on $z = 0$, $[AD]$ lies below it on $z = -w$, and one of the vertices of $[BC]$ is the origin, with $B = (m, 0, 0)$ on the positive x-axis and $C = (n, p, 0)$. Then, if $A = (u, v, -w)$, the coordinates of the box are as shown in Figure 3.

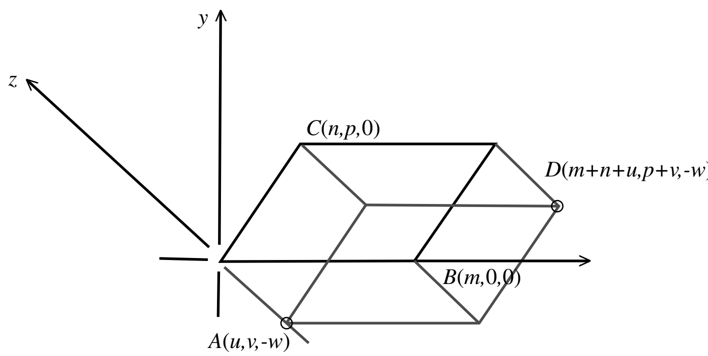

Figure 3.

We note that the length BC does not depend on the variables u, v, and w, while

$$\begin{aligned} AB + CD &= \left[(m-u)^2 + v^2 + w^2\right]^{1/2} + \left[(m+u)^2 + v^2 + w^2\right]^{1/2} \\ &\geq \left[(m-u)^2 + 1\right]^{1/2} + \left[(m+u)^2 + 1\right]^{1/2} \geq 2(m^2+1)^{1/2}. \qquad (1) \end{aligned}$$

The first inequality in (1) holds because $v^2 \geq 0$ and $w \geq 1$. To justify the second inequality, we note that for m fixed the middle term of (1) considered as a function of u attains its minimal value at the point $u = 0$. Indeed, this is the only point where its derivative is zero, and its second derivative is positive there.

Both inequalities in (1) become equalities only if $u = v = 0$ and $w = 1$. We assert that these must be the values for our minimizing box $[ABCD]$. Otherwise, compare the box with another one, namely, the straight (= rectangular) prism with the same top face $[BC]$ and height 1. It follows from the foregoing argument that the value of $AB + CD$ for the second box is strictly smaller than for the initial one. ∎

The second lemma describes the shape of the top face of the box associated with the extremal tetrahedron of edge width at least 1.

Lemma 2. *The top face [BC] is a rhombus with both altitudes* 1.

Proof. The distance between any two opposite faces of $[BC]$ is at least 1 because they lie on parallel planes at distance at least 1 from each other. If $[BC]$ were not a rhombus with altitudes 1, we could shrink our box to make it one, say with a corner at B and the other two sides tangent to the circle of radius 1 with B as centre, and this would produce a tetrahedron with edge width 1 having a smaller $AB + BC + CD$. To check this, note that if one side of $[BC]$ is longer, then shrinking it first to the length of the other side—see Figure 4—gives $BC' < BC$ because the angle $BC'C$ is obtuse: in fact if 2θ is the acute angle of the parallelogram $[BC]$, then $\angle BC'C = \pi/2 + \theta$. ∎

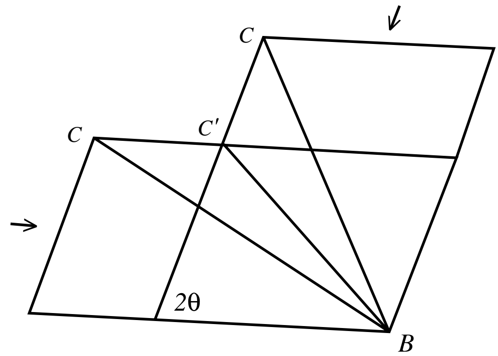

Figure 4.

Finally, we pin down the precise geometry of the top face in a third lemma.

Lemma 3. *The acute angle 2θ of the rhombus (the top face of the box [ABCD] for which $AB + BC + CD$ is minimal) is uniquely determined by the condition $(2t - 1)^2[(1 - t)^{-3} + 1] = 2$, where $t = \cos^2\theta$. This angle lies between $\pi/3$ and $\pi/2$.*

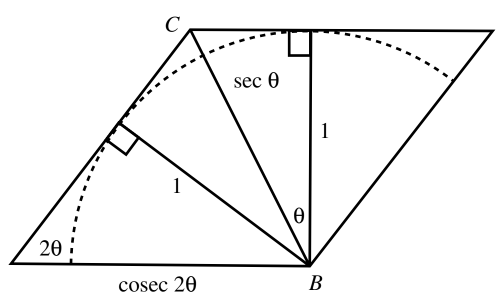

Figure 5.

Proof. We compute

$$AB + BC + CD = \sec\theta + 2(\csc^2 2\theta + 1)^{1/2},$$

for the rhombus (Figure 5) has side $\csc 2\theta$ and shorter diagonal $\sec\theta$. Minimality gives

$$d/d\theta\left[\sec\theta + 2(\csc^2 2\theta + 1)^{1/2}\right] = \sec\theta\cdot\tan\theta - 4(\csc^2 2\theta + 1)^{-1/2}\cdot\csc^2 2\theta\cdot\cot 2\theta$$
$$= 0,$$

i.e.,

$$1/4(\sec\theta\cdot\tan\theta\cdot\sin^2 2\theta)\cdot(1 + \sin^2 2\theta)^{1/2} = \cos 2\theta,$$

which translates to

$$\sin^3\theta\cdot(1 + 4\sin^2\theta\cdot\cos^2\theta)^{1/2} = 2\cos^2\theta - 1.$$

Squaring and putting $\cos^2\theta = t$, $\sin^2\theta = 1-t$, we get $(2t-1)^2[(1-t)^{-3}+1] = 2$. Since θ is in $(0, \pi/4]$, t lies in $[1/2, 1)$. On this interval the expression $(2t-1)^2 \cdot [(1-t)^{-3}+1]$ is strictly increasing—because its derivative is positive on $(1/2, 1)$—from 0 to $+\infty$, so there is a unique t in $(1/2, 1)$ for which it equals 2. The concluding assertion follows by noting that when $\theta = \pi/6$, $t = \cos^2\theta = 3/4$, whence $(2t-1)^2[(1-t)^{-3}+1] = 65/4 > 2$. ■

Having done the necessary preparations we can now wrap up Theorem 1.

Proof of Theorem 1. For any θ in $(0, \pi/2)$, the box of the tetrahedron $T_\theta = ABCD$ has height 1, is straight, and has rhombi with altitudes 1 as its top and bottom faces, with 2θ the angle of the top face opposite the (now not necessarily shorter) diagonal $BC = \sec\theta$. For the other diagonal, we have $AD = \csc\theta$. The reflections in the vertical planes through AD and BC preserve this box, so $A \leftrightarrow D$ and $B \leftrightarrow C$ are symmetries of T_θ (that is, permutations of the ordered tetrahedron induced by isometries of three-space). Also, there is the (non-order-preserving) congruence $A \leftrightarrow B'$, $B \leftrightarrow A'$, $C \leftrightarrow D'$, $D \leftrightarrow C'$ between $T_\theta = ABCD$ and $T_{\pi/2-\theta} = A'B'C'D'$, so in our computation of the four altitudes—coordinates being as in Figure 6, view from above—we may assume as before that θ belongs to $(0, \pi/4]$.

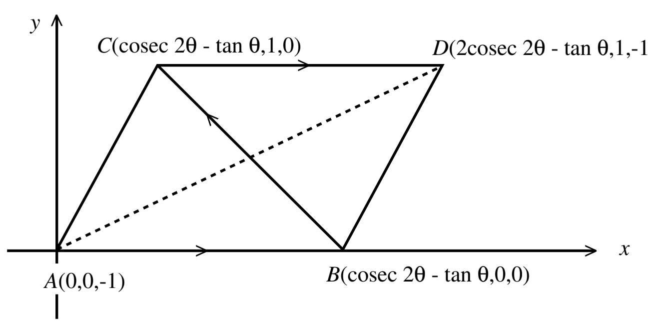

Figure 6.

We denote by $A^\perp$ the distance from the vertex A to the plane containing the opposite face BCD; $B^\perp$, $C^\perp$, and $D^\perp$ are defined likewise. We have

$$A^\perp = D^\perp = \frac{\overrightarrow{AD} \cdot (\overrightarrow{AC} \times \overrightarrow{AB})}{|\overrightarrow{AC} \times \overrightarrow{AB}|},$$

where

$$\overrightarrow{AD} = (2\csc 2\theta - \tan\theta, 1, 0)$$

and

$$\overrightarrow{AC} \times \overrightarrow{AB} = (\csc 2\theta - \tan\theta, 1, 1) \times (\csc 2\theta, 0, 1) = (1, \tan\theta, -\csc 2\theta).$$

This gives

$$A^\perp = D^\perp = 2\csc 2\theta \cdot (\sec^2\theta + \csc^2 2\theta)^{-1/2} = 2(1 + 4\sin^2\theta)^{-1/2}.$$

These two altitudes thus remain no smaller than 1 for all θ in $(0, \pi/4]$, with minimum value $2(3)^{-1/2}$ at $\theta = \pi/4$.

Similarly

$$C^{\perp} = B^{\perp} = \frac{\overrightarrow{AB} \cdot (\overrightarrow{AD} \times \overrightarrow{AC})}{|\overrightarrow{AD} \times \overrightarrow{AC}|},$$

where $\overrightarrow{AB} = (\csc 2\theta, 0, 1)$. In this case

$$\begin{aligned}\overrightarrow{AD} \times \overrightarrow{AC} &= (2\csc 2\theta - \tan\theta, 1, 0) \times (\csc 2\theta - \tan\theta, 0, 1)\\ &= (1, \tan\theta - 2\csc 2\theta, -\csc 2\theta),\end{aligned}$$

which gives

$$C^{\perp} = B^{\perp} = 2\csc 2\theta \cdot (-\sec^2\theta + 5\csc^2 2\theta)^{-1/2} = 2(1 + 4\cos^2\theta)^{-1/2}.$$

As θ grows from 0 to $\pi/4$, the last expression increases strictly from $2(5)^{-1/2}$ (< 1) to $2(3)^{-1/2}$ (> 1), taking the value 1 at $\theta = \pi/6$. It follows that the width of T_θ is 1 if and only if θ is in $[\pi/6, \pi/3]$, which completes the proof because this interval contains the minimizing value of θ given by Lemma 3. ∎

Numerical values. A little work with a pocket calculator shows that, for the minimizing rhombic tetrahedron, $t = \cos^2\theta$ lies between 0.64583 and 0.64584, and thus $\theta \simeq 36.521°$. So $BC = \sec\theta \simeq 1.24434$, $AD = \csc\theta \simeq 1.68034$, and the remaining four edges of the tetrahedon (AB, DB, AC, and DC) have the same length $(\csc^2 2\theta + 1)^{1/2} \simeq 1.44671$. The minimum attained by $AB + BC + CD$ is thus about 4.13776, which can be compared with the value $3\sqrt{2} \simeq 4.24264$ for the regular tetrahedron of width 1. Regarding the angles, the three needed later are $\angle ABC = \angle BCD \simeq 64.5287°$ and $\angle ACD \simeq 71.0079°$; the remaining angles are easily computed by taking advantage of the tetrahedron's symmetries. We note that 4.1377... is the minimum length of a 3-link polygonal path of width 1, but can be easily beaten if the width 1 path is required only to be continuous. The best result so far in this direction is the "L_3" of Zalgaller [**8**] (see also [**9**]), which has length 3.92154..., but most probably the sought-for shortest width 1 space curve is shorter still.

The three-dimensional fable. This variation on our theme has a woodborer, freshly hatched at a random point within a slab of timber of unit thickness and tunneling its way out by making at most two changes in its initially chosen direction, in such a way that the worst case distance is minimized. What strategy does it adopt?

The rhombic tetrahedron ABCD gives the solution. The borer continues in an initially chosen random direction up to a distance AB (unless of course it has exited before this). Then it turns, with new direction making interior angle $\angle ABC \simeq 64.5289°$ with the initial direction; note that there are infinitely many such choices making a half cone around the initial direction as axis. It then persists in this direction up to (its exit or) a distance of at most BC. At this point, it makes its second and last turn, into either one of the half-spaces into which 3-space is separated by the plane of the path (congruent to) $\overrightarrow{ABC}$ already traversed, the exact direction being determined uniquely in each case by the requirement that the internal angles it makes with BC and AC should be $\angle BCD \simeq 64.5287°$ and $\angle ACD \simeq 71.0079°$, respectively. Continuing along this final direction, the borer exits within a distance at most CD. Theorem 1 ensures that all other two-turn strategies (i.e., those corresponding to tetrahedra of width 1 not congruent to $ABCD$) give a worst case path longer than $AB + BC + CD$.

The woodborer thus has an $\mathbb{O}(2)$ *worth of best strategies.* Here $\mathbb{O}(n)$ denotes the space of orthogonal $n \times n$ matrices, so $\mathbb{O}(2)$ is the union of two disjoint circles (the matrices in $\mathbb{O}(2)$ with determinant 1 correspond to rotations of $\mathbb{R}^2$, hence form a circle; the other circle consists of determinant -1 matrices). In this context note that, even for the two-dimensional fable, there is a twofold ambiguity (also, $\mathbb{O}(1)$ has only two members): the turtles could equally well have made a single *left* turn. The well-documented effect of the earth's magnetic field on turtles—see, for example, Melton [**5**]—leads us to suspect that, down under in the Southern Hemisphere, there is another equally fabulous island, where turtles do exhibit such leftist tendencies!

Remark. Some other minimization problems can be solved by the same method. For example, Theorem 1 generalizes as follows: *for each* $p \geq 1$*, there is up to congruence a unique tetrahedron* $ABCD$ *of width* 1 *for which* $AB^p + BC^p + CD^p$ *is minimal, namely, the rhombic tetrahedron* T_θ *for which*

$$(1 + \sin^2\theta)^{1-p/2} \cdot 2^{p-1} \cdot \sin^{2+p}\theta = \cos 2\theta.$$

Furthermore, the minimizing value of θ, which lies in $(\pi/6, \pi/4)$, is strictly increasing as a function of p and approaches $\pi/4$ (regularity) as p becomes very large. Another example: *the regular tetrahedron* $ABCD$ *of width* 1 *is the unique one for which* $AB + BC + CD + DA$ *is minimal.* This can be proved by an argument quite similar to, but even simpler than, that given earlier.

3. IN HIGHER DIMENSIONS. The results that follow were inspired by the reported curious behaviour of some (turtle-like!) *traflamadorians*, which, we recall, are ethereal beings inhabiting the space $\mathbb{R}^n$ of all n-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of real numbers. Here $n \geq 4$, so we human beings can hope to "see" these goings-on only as word-pictures, and, for this, it is necessary to recall some terminology.

Preliminaries. The geometry (distance, perpendicularity, angles) of $\mathbb{R}^n$ is determined in the usual way by its *dot product*

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n.$$

Also, we will employ the standard terminology regarding linear and affine dependence in $\mathbb{R}^n$. A subset K of $\mathbb{R}^n$ is *convex* if all line segments joining pairs of points of K are contained in K. The convex hull of (i.e., the smallest convex set containing) $m + 1$ affinely independent points ($m \leq n$) of $\mathbb{R}^n$ is called an m-simplex of this space. The $m + 1$ points are its *vertices*, a segment joining any two of these is an *edge* of the simplex, and more generally a t-simplex ($t \leq m$) determined by any cardinality $t + 1$ subset of vertices is a *face* (or *facet*, in case $t = m - 1$) of the simplex. A sequence of directed edges, with initial vertex of each the same as the final vertex of the previous, is an *edge path* of the simplex, and such a path is called *Hamiltonian* if it visits each vertex once and only once. Note that the simplex can be recovered from such a path as its convex hull.

Let σ be the set of vertices of an m-simplex in $\mathbb{R}^n$. A $\{p, q\}$*-partition* of σ is a pair of nonempty, disjoint subsets α and β of σ, of cardinality p and q, respectively, such that $p + q = m + 1$, i.e., such that $\sigma = \alpha \cup \beta$. The geometrical simplex, the convex hull of σ, is also typically denoted σ; likewise α and β will do double duty, signifying both sets of vertices and the faces (of dimensions $p - 1$ and $q - 1$, respectively) determined by them. Thus σ is the join $\alpha * \beta$ of these *complementary faces* α and β. We next recall something that is geometrically very charming: within the affine hull of (i.e., the affine subspace generated by) the simplex, *each partition determines a unique pair of parallel codimension one hyperplanes through the two complementary faces*, viz., those generated by the union of each face with an intersecting translate of the other face. We refer to the direction (unique up to sign) normal to these hyperplanes as *the direction determined by the partition.* The shortest distance between the parallel hyperplanes is denoted $\langle \alpha, \beta \rangle$, or $\langle A_1 A_2 \dots A_p, B_1 B_2 \dots B_q \rangle$ in case the two sets in the $\{p, q\}$-partition are specified by listing their vertices. For the case $m = n$ this can be computed as follows:

$$\langle A_1 A_2 \dots A_p, B_1 B_2 \dots B_q \rangle = \frac{|\mathbf{v} \cdot \overrightarrow{A_i B_j}|}{|\mathbf{v}|}$$

(the result is independent of i and j), where $\mathbf{v}$ is a nonzero vector orthogonal to each of the $n - 1$ vectors $\overrightarrow{A_i A_{i+1}}$ and $\overrightarrow{B_j A_{j+1}}$. We call $\langle \alpha, \beta \rangle$ a $\{p, q\}$-*altitude* of the simplex. The minimal $\langle \alpha, \beta \rangle$ as α and β range over all partitions of σ is called the *width* of σ; it is the same as the minimum distance between any pair of parallel codimension one affine hyperplanes (of the affine hull of σ) that enclose σ.

General problem. *Determine the shortest Hamiltonian edge path in the collection of n-simplexes of width* 1. This problem remains open for $n \geq 4$, but we will give some partial results. In particular, we will show that, just as for $n = 3$, the answer is not the *regular* simplex of width 1, the n-simplex of width 1 having all edges equal to each other. We begin work by looking first at such simplexes.

The standard simplex Σ^n. We denote by Σ^n the convex hull of the $n + 1$ unit vectors $e_1, e_2, \dots, e_{n+1}$ in $\mathbb{R}^{n+1}$, where e_i has a 1 in its ith coordinate and 0s elsewhere. Note that Σ^n is regular with edge $\sqrt{2}$. If (α, β) is a $\{p, q\}$-partition of these $n + 1$ vertices, the barycenter $\hat{\alpha}$ of α (recall that a subset $\{A_1, \dots, A_t\}$ of $\mathbb{R}^n$ has *barycenter* $(1/t) \cdot \sum_{i=1}^{t} A_i$) has p nonzero coordinates, all equal to $1/p$; further, these coordinates are zero for $\hat{\beta}$, while the remaining coordinates of $\hat{\beta}$ are all $1/q$. Since $\hat{\alpha}$ and $\hat{\beta}$ are the nearest pair of points on the respective faces, one has

$$\langle \alpha, \beta \rangle = (p/p^2 + q/q^2)^{1/2} = (1/p + 1/q)^{1/2},$$

i.e., the $\{p, q\}$-*altitudes of* Σ^n *are* $(1/p + 1/q)^{1/2}$. This achieves its minimum when $|p - q|$ is smallest, so *the standard n-simplex has width* $(2/p)^{1/2}$ *if* $n = 2p - 1$ *and width* $((2p + 1)/p(p + 1))^{1/2}$ *if* $n = 2p$. Scaling by the reciprocals of these numbers, we obtain the *regular simplex of width 1*: $(p/2)^{1/2} \cdot \Sigma^{2p-1}$ or $(p(p + 1)/(2p + 1))^{1/2} \cdot \Sigma^{2p}$, depending on the parity of n. Observe that it has edge length $p^{1/2}$ if $n = 2p - 1$, and $(2p(p + 1)/(2p + 1))^{1/2}$ if $n = 2p$.

We note that these *midaltitudes* (meaning $\{p, p\}$-altitudes if $n = 2p - 1$ and $\{p + 1, p\}$-altitudes if $n = 2p$) are strictly smaller than the other altitudes of Σ^n. So, when we deform Σ^n slightly, these other altitudes will stay bigger than the midaltitudes, i.e., the width will still be determined by the midaltitudes, and if we could somehow maintain these at their standard value, it would remain constant. We show in Theorem 2 that there does exist a beautiful deformation of this kind. The definition that follows was suggested by the pretty coordinates $B((\sec\theta)/2, 0, 0)$, $C(-(\sec\theta)/2, 0, 0)$, $A(0, (\csc\theta)/2, 1)$, $D(0, -(\csc\theta)/2, 1)$ for the rhombic tetrahedron $T_\theta = ABCD$ of section 2.

Rhombic simplexes. Let $p \geq q \geq 2$. Given a $\{p, q\}$-partition (α, β) of the vertices of the regular n-simplex of width 1 and θ in $(0, \pi/2)$, move each vertex A of α (respectively, B of β) on the line joining it to the barycenter $\hat{\alpha}$ (respectively, $\hat{\beta}$) to a position whose distance is $2^{-1/2} \sec\theta$ (respectively, $2^{-1/2} \csc\theta$) times its original $(\theta = \pi/4)$ distance from this barycenter. These new positions of the vertices define an n-simplex that we denote by $T_\theta^{p,q}$. Note that it is the simplex obtained by uniformly deforming the squares $A_1B_1A_2B_2$ of the regular simplex into rhombi of the same altitude; in particular, one has $T_\theta^{2,2} = T_\theta$, the rhombic tetrahedron. Observe as well that the pair of rhombic simplices $T_{\pi/4\pm\phi}^{p,p}$ are congruent to each other. We show next that a partial analogue of Theorem 1 is valid in dimension 5.

Theorem 2. *For each θ in $(0, \pi/2)$ the ten $\{3, 3\}$-altitudes of the rhombic 5-simplex $T_\theta^{3,3}$ are all 1, and $T_\theta^{3,3}$ has width 1 if and only if $\cos^{-1}(5/8)^{1/2} \leq \theta \leq \sin^{-1}(5/8)^{1/2}$. Furthermore, there is up to congruence a unique rhombic 5-simplex $T_\theta^{3,3}$ having a Hamiltonian path of shortest length, namely, the simplex that corresponds to the θ in the interval $(\cos^{-1}(5/8)^{1/2}, \pi/4)$ satisfying*

$$2^{3/2}(\sec^2\theta + \csc^2\theta + 2)^{-1/2}(\csc^2\theta \tan\theta - \sec^2\theta \cot\theta) = (3/2)^{1/2} \sec\theta \tan\theta$$

We do not know if, even amongst *all* 5-simplexes of width 1, $T_\theta^{3,3}$ is the one having the shortest Hamiltonian path. We remark that, in order to cut down on messy details, we have treated only dimension 5. Similar facts hold in higher odd dimensions.

Proof. If $s = 2^{-1/2} \sec\theta$, $c = 2^{-1/2} \csc\theta$, then the six points

$$A_1 = (s, 0, 0, 0, 0), \quad A_2 = (-s/2, \sqrt{3}s/2, 0, 0, 0), \quad A_3 = (-s/2, -\sqrt{3}s/2, 0, 0, 0)$$
$$B_1 = (0, 0, c, 0, 1), \quad B_2 = (0, 0, -c/2, \sqrt{3}c/2, 1), \quad B_3 = (0, 0, -c/2, -\sqrt{3}c/2, 1)$$

of $\mathbb{R}^5$ span a $T_\theta^{3,3}$ (i.e., the six vertices of our simplex consist of the two copies of the three roots of unity in the complex subplanes $\mathbb{C} \times \{0\} \times \{0\}$ and $\{0\} \times \mathbb{C} \times \{1\}$ of $\mathbb{C} \times \mathbb{C} \times \mathbb{R} = \mathbb{R}^5$). Each AA-edge (joining two As) has length $\sqrt{3}s$, BB-edges have length $\sqrt{3}c$, and AB-edges have length $(s^2 + c^2 + 1)^{1/2}$. At $\theta = \pi/4$ these are all $\sqrt{3}$, the edge length of $\sqrt{3/2} \cdot \Sigma^5$, the regular 5-simplex of width 1.

Obviously $\langle A_1A_2A_3,\ B_1B_2B_3\rangle = 1$. To check the same for $\langle A_1A_2B_1,\ A_3B_2B_3\rangle$ note that the four vectors

$$\overrightarrow{A_1A_2} = (-3s/2, \sqrt{3}s/2, 0, 0, 0),\ \overrightarrow{A_2B_1} = (s/2, -\sqrt{3}s/2, c, 0, 1),$$
$$\overrightarrow{A_3B_2} = (s/2, \sqrt{3}s/2, -c/2, \sqrt{3}c/2, 1),\ \overrightarrow{B_2B_3} = (0, 0, 0, -\sqrt{3}c, 0),$$

are all normal to $\mathbf{v} = (1, \sqrt{3}, 2s/c, 0, -s)$. Dividing $\mathbf{v}$ by its length and taking the dot product of the resulting unit vector with $\overrightarrow{A_1A_3} = (-3s/2, -\sqrt{3}s/2, 0, 0, 0)$ show that

$$\begin{aligned}\langle A_1A_2B_1,\ A_3B_2B_3\rangle &= 3s/(4 + 4s^2/c^2 + s^2)^{1/2} = 3/(4/s^2 + 4/c^2 + 1)^{1/2} \\ &= 3/(4 \cdot 2 + 1)^{1/2} = 1.\end{aligned}$$

Note next that the symmetries $A_i \leftrightarrow A_j$ and $B_i \leftrightarrow B_j$ imply that the remaining eight $\{2, 2\}$-altitudes are of the same length $\langle A_1A_2B_1,\ A_3B_2B_3\rangle$, so these are also all equal to 1.

For θ in $(0, \pi/4)$ one has $s < 1 < c$, which gives

$$\sqrt{3}s < (s^2 + c^2 + 1)^{1/2} < \sqrt{3}c.$$

Accordingly, the shortest edge path must have at least one AA-edge. Being Hamiltonian, if it has two AA-edges it must also have a BB-edge. Now

$$2(s^2 + c^2 + 1)^{1/2} < \sqrt{3}c + \sqrt{3}s,$$

which is equivalent to

$$2s^2c^2 + 4 < 6sc,$$

holds if and only if $1 < sc = \csc 2\theta < 2$. So, for θ in $(\pi/12, \pi/4)$, a pair of AB-edges is shorter than an AA/BB pair, and our path must use one AA-edge and four AB-edges; e.g., $B_1A_1A_2B_2A_3B_3$. This has total length $4(s^2 + c^2 + 1)^{1/2} + \sqrt{3}s$. The derivative of this expression with respect to θ is

$$2^{3/2}(\sec^2\theta + \csc^2\theta + 2)^{-1/2}(\sec^2\theta\tan\theta - \csc^2\theta\cot\theta) + (3/2)^{1/2}\sec\theta\tan\theta,$$

while the second derivative is clearly always positive. At $\theta = \pi/4$ the derivative has the positive value $\sqrt{3}$, whence total path length is an increasing function of θ near $\theta = \pi/4$. This shows already that, as θ falls slightly below $\pi/4$, the width remains 1, whereas the shortest Hamiltonian path becomes smaller. A computation with a calculator indicates that the derivative is negative at $\theta = \pi/12$.

When $0 < \theta < \pi/12$ the shortest path has two AA-edges, two AB-edges, and one BB-edges (e.g., $B_1A_1A_2A_3B_2B_3$), so has length $3(s^2 + c^2 + 1)^{1/2} + 2\sqrt{3}s$. This expression behaves similarly to the earlier one: its derivative is negative at $\pi/12$, positive at $\pi/4$, while its second derivative is always positive. Accordingly we know that the shortest path is of the first kind and occurs for a unique value of θ that lies between $\pi/12$ and $\pi/4$.

We now compute the $\{2, 4\}$-altitudes: these will determine the subinterval on which the width is 1. For $\langle B_1B_2, A_1A_2A_3B_3\rangle$ observe that

$$\overrightarrow{B_1B_2} = (0, 0, -3c/2, \sqrt{3}c/2, 0), \overrightarrow{A_1A_2} = (-3s/2, \sqrt{3}s/2, 0, 0, 0),$$
$$\overrightarrow{A_2A_3} = (0, -\sqrt{3}s, 0, 0, 0), \overrightarrow{A_3B_3} = (s/2, \sqrt{3}s/2, -c/2, -\sqrt{3}c/2, 1),$$

are all normal to $\mathbf{w} = (0, 0, 1, \sqrt{3}, 2c)$; taking the dot product of $\mathbf{w}/|\mathbf{w}|$ with $\overrightarrow{B_1B_3} = (0, 0, -3c/2, -\sqrt{3}c/2, 0)$ shows that

$$\langle B_1B_2, A_1A_2A_3B_3\rangle = 3c/2(1 + c^2)^{1/2}.$$

Next, for $\langle A_1A_2, B_1B_2B_3A_3\rangle$, note that $A_i \leftrightarrow B_i$ simply switches c and s, so this altitude equals $3s/2(1 + s^2)^{1/2}$. Similar reasoning suggests that $\langle A_1B_2, A_2A_3B_1B_3\rangle$ should be a symmetric function of s and c; and sure enough, a computation gives for it the constant value $3/2^{3/2} > 1$. The symmetries $A_i \leftrightarrow A_j$ and $B_i \leftrightarrow B_j$ reveal that the remaining $\{2, 4\}$-altitudes are also equal to one of these three values. We have both $3c/2(1 + c^2)^{1/2} \geq 1$ and $3s/2(1 + s^2)^{1/2} \geq 1$ if and only if $\cos^{-1}(5/8)^{1/2} \leq \theta \leq \sin^{-1}(5/8)^{1/2}$.

The $\{1, 5\}$-altitudes don't affect the width. The vectors

$$\overrightarrow{A_2A_3} = (0, -\sqrt{3}s, 0, 0, 0), \overrightarrow{A_3B_1} = (s/2, \sqrt{3}s, c, 0, 1),$$
$$\overrightarrow{B_1B_2} = (0, 0, -3c/2, \sqrt{3}c/2, 0), \overrightarrow{B_2B_3} = (0, 0, 0, -\sqrt{3}c, 0),$$

are normal to the unit vector $(1, 0, 0, 0, -s/2)/(1 + s^2/4)^{1/2}$, the absolute value of whose dot product with $\overrightarrow{A_1A_2} = (-3s/2, -\sqrt{3}s/2, 0, 0, 0)$ yields

$$\langle A_1, A_2A_3B_1B_2B_3\rangle = 3s/(4 + s^2)^{1/2} \geq 1$$

because $s^2 \geq 1/2$ for all θ. By symmetry, any $\{1, 5\}$-altitude has either this same value or the analogous value obtained by replacing s with c. Since $c^2 \geq 1/2$ is also true for all θ, we see that all these altitudes are always at least 1.

To conclude, we note that $\cos^{-1}(5/8)^{1/2} \simeq 37.76124°$ lies in $(\pi/12, \pi/4)$, and that the unique minimum of $4(s^2 + c^2 + 1)^{1/2} + \sqrt{3}s$ must be in the still smaller interval $(\cos^{-1}(5/8)^{1/2}, \pi/4)$, because its derivative (computed earlier) is negative even when $\cos^2\theta = 5/8$. ■

Space of simplexes. We remark that we have been dealing mostly with *ordered simplexes* (i.e., simplexes whose vertices are totally ordered in a specified way). Moreover, we have considered them only up to *congruence*: two ordered simplexes are deemed to be the same if, by some Euclidean motion (a composition of rotations, translations, or reflections), we can map the vertices of the first in order onto those of the second (we used *symmetry* to denote congruence of two ordered simplexes arising from the same unordered simplex). *The congruence classes of the n-simplexes in $\mathbb{R}^n$ can be organized into a space homeomorphic to an open ball of dimension $\binom{n+1}{2}$.* To be more precise, these congruence classes can be put into one-to-one correspondence with the set of all $n \times n$ upper triangular real matrices with positive diagonal entries. To see this, note that, by using appropriate Euclidean motions, we can put the first vertex of any given simplex at the origin, the second on the positive half of the first axis, the third on the plane determined by the first two axes in such a way that its second coordinate is positive, etc. (This space is thus the same as the coset space $\mathbb{GL}(n)/\mathbb{O}(n)$.) Our problem entails minimizing some quite simple functions on a (not quite so simple!) subspace of this space of upper triangular matrices, viz., that of all simplexes of width 1. For $n = 3$, the idea of a box gave us useful insight into the geometry of this topological open 3-ball. For $n \geq 4$, boxes are more intriguing.

Box of a simplex. We denote by $\text{box}(\sigma)$ the convex region enclosed by all hyperplanes arising from midpartitions (α, β) of an n-simplex σ. (One can also consider regions determined by other sets of partitions; e.g., if all partitions are used, then one gets the simplex itself.) We warn the reader that, for n different from 1 or 3, $\text{box}(\sigma)$ is *not* an n-parallelepiped (which has a pair of facets normal to each of n given directions). For example, for $n = 2p - 1$ with $p \geq 3$, $\text{box}(\sigma)$ has $\binom{2p}{p}$ facets, many more than the $2n$ of an n-parallelepiped. Even so, the *combinatorics of the box*—which, because of the affine nature of the definition, do not depend on the geometry of the simplex—are quite pleasant, and easy enough to work out. For any $n \geq 3$, the vertices of $\text{box}(\sigma)$ are the vertices v of σ and their *antipodes* $\overline{v}$, where $\overline{v}$ is the point on $v\hat{\sigma}$ produced for which

$$(n + 1)\hat{\sigma}\overline{v} = \big[(n + 1)/2\big]v\overline{v}.$$

Here $[x]$ denotes the greatest integer contained in x, so it is only for n *odd* that v and $\overline{v}$ are exactly equidistant from $\hat{\sigma}$. The facet containing the mid-dimensional simplex α is the join $\alpha * \overline{\partial\beta}$ with the antipode of the complementary face boundary, while all the lower dimensional faces are simplexes. Thus, for $n = 5$ each facet boundary gets triangulated as the join $\partial\alpha * \overline{\partial\beta}$ of two triangles (compare with the 3-dimensional Figure 2). The hope of course is that an analysis of the *geometry of the box* (a much harder task!) will help in generalizing the results of section 2 fully to all n. However, we will now wrap up the discussion by giving another, somewhat artificial, deformation that demonstrates that the solution to our problem is never regular.

Theorem 3. *For $n \geq 4$ and $\epsilon > 0$ sufficiently small, there exists a smooth 1-parameter family of n-simplexes $F_\theta^n = A_0A_1\cdots A_n$ in $\mathbb{R}^n$, with θ in $(\pi/4-\epsilon, \pi/4]$, such that $F_{\pi/4}^n = \Sigma^n$, the width of F_θ^n is independent of θ, and $A_0A_1 + A_2A_3 + \cdots + A_{n-1}A_n$ is a strictly increasing function of θ.*

The idea is simply to use the rhombic tetrahedral deformation $A_0A_1A_2A_3 = T_\theta$ on the first four vertices of Σ^n, simultaneously taking care to push the remaining vertices uniformly away, just far enough in the direction determined by the partition $\{A_0A_1A_2A_3, A_4A_5\ldots A_n\}$ so as to ensure constancy of width (a modicum of finesse is necessary because of the fact that the $\{1,3\}$-altitudes $B^\perp = C^\perp$ of T_θ decrease when θ falls below $\pi/4$. However the calculations are a little messy, so we present the details for the 4-dimensional case only.

Proof for $n = 4$. The first four vertices of $F^4 = ABCDE$ will be

$$A = (c, 0, 0, 0), \quad B = (0, s, 1, 0), \quad C = (0, -s, 1, 0), \quad D = (-c, 0, 0, 0),$$

where $c = (\csc\theta)/2$ and $s = (\sec\theta)/2$—so clearly $ABCD = T_\theta$—and the fifth vertex will be “above” the barycenter of $ABCD$ at a distance $x = x(\theta)$ from it. Thus $E = (0, 0, 1/2, x)$ which, as we now show, is uniquely determined by the stated requirements.

We must have $x(\pi/4) = 5^{1/2}/2$, for at $\theta = \pi/4$ the edge-lengths $2s$, $2c$, $(s^2+c^2+1)^{1/2}$, $(s^2+x^2+1/4)^{1/2}$, and $(c^2+x^2+1/4)^{1/2}$ of $ABCDE$ must all be equal to $2^{1/2}$ (because we need $F_{\pi/4}^4 = \Sigma^4$). Now, when θ decreases below $\pi/4$, we want $x(\theta)$ to increase just enough to ensure that the width stays put at the standard value $(5/6)^{1/2}$.

Straightforward computations reveal that the length of any $\{2,3\}$-altitude is one of the following:

$$\left(1+\frac{1}{4}\cdot x^{-2}\right)^{-1/2}, 2\left(1+c^{-2}+\frac{9}{4}\cdot x^{-2}\right)^{-1/2}, 2\left(1+s^{-2}+\frac{9}{4}\cdot x^{-2}\right)^{-1/2}. \quad (2)$$

For instance, to compute $\langle BE, ACD\rangle$ note that $\overrightarrow{BE} = (0, -s, -1/2, x)$, $\overrightarrow{AC} = (-c, -s, 1, 0)$, and $\overrightarrow{CD} = (-c, s, -1, 0)$ are all orthogonal to $\mathbf{v} = (0, 1, s, 3s/2x)$. Taking the dot product of $\mathbf{v}/|\mathbf{v}|$ with $\overrightarrow{BC} = (0, -2s, 0, 0)$ leads to

$$\langle BE, ACD\rangle = 2\left(1+s^{-2}+\frac{9}{4}\cdot x^{-2}\right)^{-1/2}.$$

We note that the altitude $\langle CE, ABD\rangle$ has the same value because of the symmetry $B \leftrightarrow C$ (for $n > 4$, vertices from the fifth onwards are interchangeable, which speeds up calculations significantly).

Just $x(\theta) \geq x(\pi/4)$ suffices to make certain that neither of the first two expressions in (2) will decrease below its value $(5/6)^{1/2}$ at $\theta = \pi/4$. So the width of F_θ^4 will remain constant if and only if

$$2\left(1 + s^{-2} + \frac{9}{4} \cdot x^{-2}\right)^{-1/2} = \left(\frac{5}{6}\right)^{1/2},$$

which gives

$$x(\theta) = \frac{1}{2} \cdot 3\left(4\sin^2\theta - \frac{1}{5}\right)^{-1/2}.$$

For the last part it suffices to check that the derivative with respect to θ of the function

$$AB + BC + CD + DE = 2(c^2 + s^2 + 1)^{1/2} + 2s + \left(c^2 + \frac{1}{4} + x^2\right)^{1/2}$$

is positive at $\theta = \pi/4$. (For $n > 4$ the expression is not much more complicated because links from the fifth onwards have constant length $2^{1/2}$.) The derivative of $AB + CD$ is easily seen to be 0 at $\theta = \pi/4$. The derivative of $BC + DE$ works out to be

$$(\csc^2\theta + 1 + 4x^2)^{-1/2}\left[-\frac{1}{2}\csc^2\theta\cot\theta - 450\cos\theta\sin\theta(19 - 20\cos^2\theta)^{-2}\right]$$
$$+ \sec\theta\tan\theta,$$

which at $\theta = \pi/4$ has the value $2^{1/2} - (17/9) \cdot 2^{-1/2} > 0$. ■

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A Curiosity in Measure Theory

I ran across the following curiosity when teaching a graduate course in real analysis several years ago. I submitted it as a problem to the American Mathematical Monthly; the Problems Editor rejected it on the advice of both referees—one of whom thought the problem unfairly difficult, the other thought it trivial! What do you think?

As usual in this subject, we assume the axiom of choice, and say that a subset of $\mathbb{R}$ is *negligible* if it is Lebesgue measurable with measure zero. A family of subsets of some set X is said to be *totally ordered* if for every A, B in the family either $A \subseteq B$ or $B \subseteq A$.

Q: Suppose that $\mathcal{F}$ is a totally ordered family of negligible subsets of the unit interval [0, 1]. Must $\cup_{A\in\mathcal{F}} A$ be negligible?

A: No.

Proof. Assume that in fact $\cup_{A\in\mathcal{F}} A$ is negligible for any such $\mathcal{F}$. Then the family of *all* negligible subsets of [0, 1] would satisfy the hypotheses of Zorn's Lemma! There would then be a maximal negligible subset. But this is absurd. Indeed, suppose that S is a maximal negligible subset of [0, 1]. If there were x in [0, 1] not contained in S, then $S \cup \{x\}$ would be a negligible subset of [0, 1] strictly containing S. Hence there is no such x, and $S = [0, 1]$. But it is known that [0, 1] is not negligible. ∎

Remark. This proof is curiously nonconstructive even by the standards of the subject. It is true that this could be remedied by working through the proof of Zorn's Lemma to "construct" an example of a totally ordered family $\mathcal{F}$ of negligible subsets of [0, 1] whose union, call it U, is not negligible. But this still leaves the question of how "big" this U can be. For instance, can U be all of [0, 1]? If we also assume the Continuum Hypothesis, the answer is yes: we can then use a bijection of [0, 1] with the smallest uncountable ordinal to impose a total order $\prec$ on [0, 1] such that $I_x := \{y \in [0, 1] : y \preceq x\}$ is countable—and thus negligible—for all x in [0, 1], and set $\mathcal{F} = \{I_x : x \in [0, 1]\}$. Without the CH assumption, I do not know the answer.

Submitted by Noam D. Elkies, Harvard University