K.S. Sarkaria, "The De Rham Cohomology of Foliated Manifolds", Thesis S.U.N.Y. Stony Brook (

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The DeRham Cohomology of Foliated Manifolds

Introduction (A) We first point out in broad outline the object of our study, and why it is worth studying.

Think of a p-covector at $x \in M$, where M is a smooth m dimensional manifold, as a skewsymmetric and multilinear map $T_x \times \cdots \times T_x$ (p times) $\xrightarrow{\omega_x} R$ T_x being the tangent space of x. We shall say that ω_x is of filtration $\geq i$ with respect to a subspace $D_x \subset T_x$ if it vanishes whenever $p-i+1$ of the arguments are in D_x. Let us now suppose that M is supplied with a tangent subbundle $D \subset T$. Then a form ω will be said to be of filtration $\geq i$ if ω_x is such for each $x \in M$. Thus we get a decreasing sequence of vector spaces $\Lambda = \Lambda_0 \supset \Lambda_1 \supset \cdots \supset \Lambda_c \supset \Lambda_{c+1} = 0$, Λ_i being all smooth forms of filtration $\geq i$. Here c = codimension of D in T.

The deRham cohomology of M is the homology of Λ under the exterior derivative $d: \Lambda \to \Lambda$; $H(\Lambda) = \ker d/\mathrm{Im}\, d$. Our starting point is the simple observation that D is involutive if and only if the exterior derivative preserves the filtration. So each Λ_i is now a subcomplex of Λ, and Λ is a filtered complex. On the other hand by Frobenius theorem--see, e.g. [34]--D is involutive if and only if it is tangent to a foliation; in other words, M can be covered by coordinate

neighborhoods $x_1, \ldots, x_l, x_{l+1}, \ldots, x_m$ such that locally $\partial/\partial x_1, \ldots, \partial/\partial x_l$ form a basis for D_x. In this way M is partitioned into l-dimensional manifolds called the Leaves of the foliation; each leaf being a maximal connected sub-manifold given locally by some constant values for $x_{l+1}, \ldots, x_m$.

Hence to each pair (M,foliation)--i.e. to a foliated manifold M--is attached in a natural way a filtered complex Λ. Our object is to study this filtered complex. We recall that by standard homological algebra, as in [7], one can attach to any filtered complex an object called its spectral sequence. It consists of a sequence E_r of graded groups, each of which is the homology of the preceding under a differential d_r, such that E_0 is the graded group of Λ (under above filtration) and the final term E_∞--which is attainable in a finite number of steps--is the graded group of $H(\Lambda)$ (under the obvious induced filtration).

A better idea of the importance, and scope, of such an investigation can be formed by considering the analagous case of complex manifolds. In this case we have a similar spectral sequence with $E_\infty = H(\Lambda_{\underline{C}})$--i.e., the complex deRham cohomology--and E_1 is the so called

Dolbeaut cohomology. A vast theory is centered around this cohomology (see, e.g. [15]). Deep results requiring both analytical and algebraic techniques have been attained. For instance one has the finiteness theorem (i.e., if M is compact E_1 is finite dimensional), the duality theorem (i.e., for M^m compact $E_1^{p,q} \cong E_1^{m/2-p, m/2-q}$); and, on the algebraical side, one can mention results involving characteristic classes (e.g., for M compact $\Sigma (-1)^q e_1^{0,q} = \text{ch} M \ \text{td} M$, $e_1 = \dim E_1$; this is called the Riemann Roch theorem). These three specimen results due-respectively-to Cartan-Serre, Serre and Hirzebruch have in turn led to very interesting generalizations due to Grothendieck-Grauert, Grothendieck and Grothendieck-Atiyah-Singer and others.

The results, concepts and constructions occurring in this work should all be viewed as part of an ongoing and extensive programme whose goal is to build a similar body of knowledge for foliated manifolds.

(B) The following is a summary of the contents.

Sections 1-5 give a rapid review of the apparatus of our spectral sequence, following Cartan-Eilenberg [7].

In sections 6-7 we show that $E_1^{p,q} \cong H^q(M, \underline{D}^p)$. Here $\underline{D}^p$ is the sheaf of germs of smooth forms of degree p which are transverse and invariant: a p form ω is called transverse if $\omega \in \Lambda_p$, it is called invariant if $L_X\omega = 0$ for any vector field $X \in C^\infty(D)$. $\underline{D}^0$--or just $\underline{D}$--is simply the sheaf of germs of smooth functions which are constant on leaves. It plays an important role in many places.

Section 8 studies complex line bundles. Let $\underline{C}_S^*$ (resp. $\underline{C}_D^*$) denote the sheaf of germs of smooth nonzero complex valued functions (resp. those that are constant on leaves). Now, the isomorphism classes of such bundles form a group under tensor product, viz., $H^1(M, \underline{C}_S^*)$; and the sheaf inclusion $\underline{C}_D^* \subset \underline{C}_S^*$ gives us a map $H^1(M, C_D^*) \to H^1(M, \underline{C}_S^*)$ whose image gives us the invariant line bundles. The first chern class of a line bundle $\xi \in H(M, \underline{C}_S^*)$ is an element $c_1(\xi) \in H^2(M, Z)$. We shall say that a chern class vanishes in $\underline{D}$ if it is killed by the induced map $H^2(M, Z) \to H^2(M, \underline{D})$. We see that the first chern class of a line bundle vanishes in $\underline{D}$ if and only if it is invariant. For an analagous result for analytic line bundles over a complex manifold, see [15].

In sections 9, 10 we study two notions of homotopy. In the category of foliated manifolds the morphisms are those which map leaves into leaves. Two such maps $f,g:M \to M'$ are called <u>k-homotopic ($k=1,2$)</u> if they can be extended to a morphism $F:M \times I \to M'$ where $M \times I$ carries the <u>k-foliation ($k=1,2$)</u>. Here the 1-foliation is given by multiplying each leaf of M by I, and the 2-foliation is the natural 1-dimensional foliation on $M \times I$. Then <u>we construct a k-chain homotopy (between the induced maps $f,g:\Lambda(M') \to \Lambda(M)$ of filtered complexes) by using a k-homotopy</u>; i.e., a chain homotopy that "disturbs" filtration by $k-1$ units. Thus we can think of the spectral sequence E_r, for $r \geq k$, as a functor attached to the k-homotopy category ($k=1,2$) of foliated manifolds, by usual homological algebra ([7]).

Sections 11, 12 involve some simple observations about the relationship of the exterior product to the filtration. Since our filtration is of length c, if a form of filtration i is multiplied by a form of filtration j and $i+j>c$ we get 0. Using this we see that <u>an odd dimensional foliation can have $\dim H^{2k}_{\{\frac{c}{2}\}}(\Lambda) \geq \frac{1}{2}\dim H^{2k}(\Lambda)$ only if sign $M=0$</u>. Here $H_j(\Lambda)$ is the part of $H(\Lambda)$ of filtration $\geq j$; $\{\frac{c}{2}\}$ is the first integer after $\frac{c}{2}$ and sign M is the signature of M. We shall say that a foliation obeys <u>Serre duality</u> if $E_\infty^{p,q} \cong E_\infty^{c-p,1-q}$. It is clear that such foliations satisfy the above inequality.

In section 13 we extend the definition of section 8 to define <u>invariant complex vector bundles</u>: Let G_S (resp G_D) denote the sheaf of germs of smooth function with values in $GL(n,\underline{C})$ (resp. those that are constant on leaves). Now the isomorphism classes form only a set $H^1(M,G_S)$, and we consider the image of $H^1(M,G_D) \to H^1(M,G_S)$. [Note that with a stricter definition of "isomorphism" $H^1(M,G_D)$ is the set of isomorphism classes of all invariant bundles: this notion shall play a role in section 19. Further it plays a role in a natural K theory of invariant bundles.] We show that <u>the real chern ring of an invariant complex vector bundle vanishes in dimensions $\geq 2c$</u>. Applying this result to the complexification of the bundle $D^\perp$ of transverse 1-forms one gets Bott's vanishing theorem [4]. Note that no connection theory is used in the proof.

Corresponding to any two projection maps $P_D,P_N:T \to T$ with images D and N s.t. $D+N=T$, one has a natural <u>bigradation</u> of Λ, and the differential d is the sum of three differentials d_{01}, d_{10}, d_{2-1} of bidegrees (0,1), (1,0), (2,-1) respectively. See [13]. In section 14 we show that $\underline{E}_1 \cong H_{d_{10}}(\Lambda)$ and $\underline{E}_2 \cong H_{d_{10}} H_{d_{01}}(\Lambda)$. [Following [7], this means that E_1, E_2 are the first 2 terms of the spectral sequence of the double complex $(\Lambda, d_{01}, d_{10})$.] The *complex of sheaves* $\underline{D} \xrightarrow{d} \underline{D}^1 \xrightarrow{d} \cdots \xrightarrow{d} \underline{D}^c \to 0$ (which is in fact exact) has 2 standard spectral sequences; see, e.g., [35]. Using the notations of [35], the E_2 term of the "second" spectral sequence is same as E_2 of our spectral sequence. This is shown in section 15. We see from these that *if the foliation arises from a fibration the $\underline{E}_2$ term is the same as that in Serre's spectral sequence of a fibering [31].*

Section 16 covers the functional analysis that is relevant to finiteness theorems. *We put on Λ the usual Frechet space topology.* So it induces on each E_r a topological vector space topology. We recall the usual definition of *a smoothing map (= integral operator) $s: \Lambda \to \Lambda$.* Any K-chain homotopy between 1 and s will be called a *k-parametrix* (of d). We show, under some additional conditions, *that on a compact foliated manifold, the existence of a k-parametrix implies that $\underline{E}_k$ is finite dimensional.* This result is the reason

why we shall be interested in k-parametrices. We also point out in this section that if $\dim E_r = e_r < \infty$, then $\chi(M) = \Sigma\, (-1)^{p+q} e_r^{p,q}$. Here $\chi(M)$ is the Euler characteristic of M.

In section 17 we construct, by modifying techniques given in [11], a 2-parametrix in the following special case: *assume that there exists a continuous uniformly transitive map $F: R^m \to C_1^\infty(M,M)$, with $F(0) = 1$, then there exists a 2 parametrix.* Here $C_1^\infty(M,M)$ consists of all smooth maps $M \to M$ which map leaves into leaves; it is given the usual C^∞ topology. By *uniformly transitive* we mean that there is a neighborhood U of $0 \in \underline{R}^m$ which, for all $x \in M$, gives us a diffeomorphism F_x of U onto a neighborhood of x by $F_x(\eta) = F(\eta)(x)$. We point out that *this hypothesis is fulfilled if a compact foliated manifold M has a global parallelism by vector fields which are infinitesimal transformations of the foliate structure.*

Section 18 records some tid-bits which may be of value. E.g., we point out that an obvious extension of a "patching argument" of [11] allows us to show that *if the foliation arises from a fibration then there is a 2-parametrix.* Again, now let's suppose that M is oriented. If the differentials $d_0, d_1, \ldots$ of the spectral sequence are assumed to be topological homomorphisms then the foliation obeys Serre's duality. This hypothesis is satisfied if dim $E_1 < \infty$. Hence odd dimensional foliations with dim $E_1 < \infty$ exist only if signature (M) = 0. We shall also recall in this article an example which shows that for almost all irrational flows on the torus dim $E_1 < \infty$. But there exist irrational flows for which this is no longer true.

Section 19 studies the inter-relationships between reduction of the structure sheaf of a bundle--see [12a] and sections 8,13 above--and connection theory. In section 19a we think of a connection on a real vector bundle W over M as a derivation of $\Lambda(W)$, $\partial : \Lambda(W) \to \Lambda(W)$, lying above d (p.76). Here $\Lambda(W)$ are forms on M with coefficients in W. Now the "structure sheaf" of W consists of the smooth germs with values in $GL(w)$, $w = \dim W$. Again, as in sections 8,13, W will be called an invariant vector bundle if this sheaf $GL(w)_S$ can be reduced to $GL(w)_D$, the subsheaf of germs constant on leaves. We put a bigradiation in the manner of section 14 now and thus ∂ splits up into three derivations $\partial_{01}, \partial_{10}, \partial_{2-1}$. Then W is invariant if and only if one has a connection for which $\partial_{01}^2 = 0$. Such a connection will be called a Bott connection. Note that one can now define $E_1(W)$, the homology of $\Lambda(W)$ under ∂_{01}. Also if W^2 is the natural w^2 dimensional bundle associated to W, (p.81), we can define the group $E_1(W^2)$. For each $\xi \in H^1(M, GL(w)_D)$ we shall also define the notions ξ-Bott connection (resp. ξ-invariant connection) by requiring that the local connection matrices of 1-forms (with respect to trivializations of W agreeing with ξ) consist of transverse (resp. transverse invariant) 1-forms. Any vector bundle W associated to ξ admits a ξ-Bott connection, but it need not admit an ξ-invariant connection. Note that our definition of Bott connection merely says that the curvature form (which is a 2 form with coeffs in W^2) is of filtration ≥ 1. If we satisfy "filtration ≥ 2" we shall say that it is an invariant connection. Finally, we call a bundle which admits an ξ-invariant connection as a stiff bundle. Starting with any Bott connection one can define an element $[\Omega_{1,1}]$ of $E_1^{1,1}(W^2)$ by using the part of the curvature which is of bidegree 1,1. In complete analogy with Atiyah [1] it will be seen that W is stiff if and only if $[\Omega_{1,1}] = 0$.

Section 19B is devoted to examining the Weil homomorphisms, and is basically a dual of section 19A. Let G be a lie group. (For simplicity we think of G as a matrix group.) By a <u>G-algebra</u> we mean a graded anticommutative algebra (over some commutative ring) which is provided with (a) a differential d, (b) for each left invariant vector field x on G a skew derivation i_x of degree -1 and (c) a derivation L_X of degree zero such that $i_x^2 = 0$, $L_{[X,Y]} = [L_X, L_Y]$, $i_{[X,Y]} = [L_X, L_Y]$, $i_x d + d i_x = L_X$. For example the <u>Weil Algebra</u> W(G) of G is a G-algebra--see [6]. Another example arises from $\Lambda(P)$, the deRham complex of a principal G-bundle P over M. In this case each left invariant vector field X gives us in a natural way a vector field--also called X--along the fibres and i_X, L_X are defined to be the usual inner product and Lie differentiation respectively. We can <u>define a connection to be a G-algebra morphism of W(G) into some other algebra</u>. There always exist such morphisms $W(G) \to \Lambda(P)$: this definition is known to be same as for section 19A for $G = GL(w)$. P inherits from M a codimension c foliation: so $\Lambda(P)$ is a filtered complex. W(G) is also a filtered complex if we set $W_i(G)$ as all those terms containing polynomials of degree $\geq 2i$: this is called the <u>1-filtration</u> of W(G): when we are considering W(G) with this filtration we'll

write (1) W(G). Then we will see that <u>A principal G-bundle is invariant if and only if there is a connection (1) $W(G) \xrightarrow{f} \Lambda(P)$ preserving the filtrations</u>. Of course "invariant" means, as before, that we can reduce the structure sheaf G_S to G_D. [This result is simply the dual of the first in the above paragraph: in this new setting these are the Bott Connections.] For each $\xi \in H^1(M, G_D)$--i.e., each "invariance isomorphism class" ξ--one has a family of <u>ξ-Bott connections</u> $f(\xi): (1)W(G) \to \Lambda(P)$ which are associated to ξ [They arise in the proof of above proposition]. We show that any 2 such connections $f(\xi)$, $g(\xi)$ are 1-homotopic (in the sense of section 9). Thus to <u>each $\xi \in H^1(M, G_D)$ there is associated a 1-homotopy class $[f(\xi)]: (1)W(G) \to \Lambda(P)$ of Bott connections</u>. This implies that, for $r \geq 1$, the induced maps $(1)E_r(G) \to E_r(P)$ depend only on ξ: we can denote it by $\bar{\xi}$. Then dualising a result of section 19A we can state that, <u>an invariant bundle $\xi \in H^1(M, G_D)$ is stiff iff the map $\bar{\xi}: (1)E_1^{1,1}(G) \to E_1^{1,1}(P)$ is zero</u>. We now define the <u>2-filtration</u> of W(G) by putting $(2)W_{2i-1} = (2)W_{2i} = (1)W_i$. Then we can deduce that: <u>A principle G-bundle is stiff if and only if we have a connection $(2)W(G) \xrightarrow{f} \Lambda(P)$ preserving the filtration</u>. [These are what were in section 19A the <u>invariant</u>

connections. Note that a Bott connection is simply one in which the curvature is of filtration ≥ 1, while an invariant connection is one in which it is of filtration ≥ 2]. For each $\xi \in H^1(M,G_D)$ one has a family of ξ-invariant connections $f(\xi)$. We show that any 2 of them are 2-homotopic (section 9). Thus to each $\xi \in H^1(M,G_D)$ there is associated a 2-homotopy class $[f(\xi)]:(2)W(G) \to \Lambda(P)$ of invariant connections. In particular, for $r \geq 2$, the induced map $(2)E_r(G) \to E_r(P)$ depends only on ξ: we can denote it by $\bar{\bar{\xi}}$. Note that for a point foliation, $G_D = G_S$, and for each differentiable structure $\theta \in H^1(M,G_S)$, one has the well-known map $(2)E_2^{*,0}(G) \xrightarrow{\bar{\bar{\theta}}} E_2^{*,0}(P_f) = H^*(M,\underline{R})$ called the Chern-Weil homomorphism: here P_f means P considered with the foliation arising from the fibration $P \xrightarrow{f} M$.

In section 20 we study some aspects of linear connections that are relevant to our study--and find use in section 21--we follow standard terminology, as in [17]. A linear connection is any connection on the principal tangent bundle (and so on its associated vector bundles, T, T* etc.). Given any pair (M,D), i.e., a(manifold, plane field) one says that a linear connection is a Walker Connection--see [37], [38]--if it is torsionless and reducible to $D^\perp$ (i.e., keeps the plane field D parallel). Then a walker connection exists if and only if D is involutive. With section 19 in mind it is natural to say that a linear connection which reduces to $D^\perp$ (an invariant bundle) is a Bott connection if it restricts to such a connection on $D^\perp$. Every Walker connection is a Bott connection. This section points out that foliated manifolds may be studied in the context of torsionless G-structures: however I have not pursued this aspect in this work.

Section 21 is occupied with constructing 2-parametrices (see section 17 above) for foliated principal bundles. Special hypotheses are needed for these constructions. E.g., the hypothesis made in section 21a is that--in the terminology of section 20--the foliation arises from a torsionless $\mathfrak{g}(1,m)$-structure. Here $\mathfrak{g}(1,m)$ is the Lie algebra of all endomorphisms of $\underline{R}^m$ with image in $\underline{R}^1$. Let Q denote the principal bundle of compatible frames, provided with the natural codimension c foliation. Then we employ the canonical parallelism of Q by such a torsionless connection (assumed complete) to construct a 2 parametrix on Q a la section 17. In section 21b we assume that $D^\perp$ is stiff: then we can have a Walker connection whose restriction to $D^\perp$ is an invariant connection (i.e., filtration of curvature ≥ 2: see section 19): a linear connection with the latter property can be called an invariant connection. Let P denote the principal bundle of tangent frames compatible with the foliate structure. Then we employ the canonical parallelism of P by such an invariant torsionless connection (assumed complete) to construct a 2-parametrix for P. In this result P is not foliated in w dimension c. Instead we show that there is always a natural $l + lm$ dimensional foliation of P sitting above the foliation of M. This is the foliation that occurs in the above result. In section 21c we use averaging process (over a Haar measure) to get a 2-parametrix for the subcomplex Λ_G of forms which are right invariant over G.

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1. Let M be a smooth compact m dimensional manifold. We shall denote its tangent bundle by T. The dual cotangent bundle is T^*. By p-forms we shall understand smooth sections of $\lambda^p T^*$, the bundle of p-covectors; the space of all p forms is denoted by Λ^p, and the space of all forms by Λ. Let us now assume given, once and for all, a subbundle $D \subset T$ of fibre dimension l and codimension c; so $l + c = m$. We now define a filtration of Λ,

$$\Lambda \supset \Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_2 \supset 0 \qquad (1)$$

in the following way: we think of a p-form ω as a multi-linear skewsymmetric form on p tangent vector fields, $\omega(X_1,\ldots,X_p)$, which commutes with the action of $C^\infty(M)$, i.e., $\omega(X_1,\ldots,fX_i,\ldots,X_p) = f\omega(X_1,\ldots,X_p)$. The values of this form are in $C^\infty(M)$. We now say that Λ_i^p consists of all those p-forms which vanish at $x \in M$ whenever $p - i + 1$ of the vector fields $X_1, X_2, \ldots, X_p$ lie in D at x. If $i \geq c + 1$ it is seen that $\Lambda_i^p = 0$. We shall understand by Λ_i the space of forms lying in Λ_i^p for some p. If $i \leq 0$, $\Lambda_i = \Lambda$ and if $i \geq c + 1$, $\Lambda_i = 0$. Another way of looking at this filtration is this. We have an isomorphism $\lambda^p T^* \cong (\lambda^p T)^*$; at $x \in M$, an elt. in the latter bundle is a linear map $\lambda^p T_x \to \underline{R}$. By putting the requirement that this linear map vanish on $V_1 \wedge \cdots \wedge V_p$ whenever $p - i + 1$ of the vectors are in D we get a subbundle $\lambda_i^p T^*$. The space of all sections of this subbundle is Λ_i^p. We also have the bundle $\lambda_i T^*$ formed from the whitney sum $\bigoplus_p \lambda_i^p T^*$. The space of all sections of this is Λ_i.

2. Now we assume that the subbundle D is involutive, i.e., if two tangent vector fields X and Y take their values in D so does their Lie bracket $[X,Y]$. In the real vector space Λ we have the exterior derivative, $d:\Lambda \to \Lambda$ which is given by the following formula (thinking of forms as skewsymmetric multilinear maps $C^\infty(T) \times \cdots \times C^\infty(T) \to C^\infty(M)$ as above explained).

$$(d\omega)(X_0,X_1,\ldots,X_r) = \frac{1}{r+1}\Big\{(-1)^i X_i\big(\omega(X_0,\ldots,\hat{X}_1,\ldots,X_r)\big)$$

$$+ \sum_{0 \leq i < j \leq r} \omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_r)\Big] \qquad (2)$$

Though the notion of d goes back at least to E. Cartan, the intrinsic definition (2) was given first by Palais [22].

Proposition 1. D is involutive if and only if the filtration (1) commutes with the endomorphism d, i.e., $d(\Lambda_i) \subset \Lambda_i$ for all i.

Proof: If D is involutive we easily see that if $(r+1) - i + 1$ of the vectors $X_0, X_1, \ldots, X_r$ are in D then $r - i + 1$ of the vectors $X_0, \ldots, \hat{X}_i, \ldots, X_r$ and

$[X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_r$ are also in D. So we see that if $\omega \in \Lambda_1^r$ then $d\omega \in \Lambda_1^{r+1}$ by using formula (2). Conversely let's simply suppose that $d(\Lambda_1^1) \subset \Lambda_1^2$ and take any $\omega \in \Lambda_1^1$. It is a 1-form which vanishes on D. Using formula (2) we have

$$(d\omega)(X,Y) = \tfrac{1}{2}\Big\{X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega([X,Y])\Big\}$$

Pick X and Y to be two tangent vector fields with values in D. So this becomes simply

$$(d\omega)(X,Y) = -\tfrac{1}{2}\omega([X,Y])$$

But $d\omega \in \Lambda_1^2$. So it vanishes if both vectors are in D. Hence we get $\omega([X,Y]) = 0$; this being true for all 1-forms ω vanishing on D. It implies that $[X,Y]$ itself takes its values in D, i.e., D is involutive. QED

3. From now on we shall suppose that D is involutive, i.e., that the filtration (1) commutes with the endomorphism d. This is the setting in which a homological algebra can be used to obtain information about $H(\Lambda)$; an algebraical machinery called the spectral sequence is available which allows us to obtain information about $H(\Lambda)$ from the fact that the filtration commutes with d, i.e., from involutivity of D. The following definitions are adapted from Cartan and Eilenberg [7], henceforth abbreviated as CE.

In the following H will denote the homologies induced by d. In some other differential δ enters the picture we will use the notation $H_\delta(\ )$. Also in the following we'll frequently use the triangle lemma (CE, p. 316) which says that if (in the figure shown), the bottom row is exact then $\frac{\text{Im }\varphi}{\text{Im }\varphi'} \cong \text{Im }\psi$.

$$\begin{tikzcd}
 & C \arrow[d, "\varphi"] \arrow[dr, "\psi"] & \\
A' \arrow[ur] \arrow[r, "\varphi'"'] & A \arrow[r, "\eta"'] & A''
\end{tikzcd}$$

Since our filtration is compatible with d, we have an induced filtration

$$H(\Lambda) \supset H_1(\Lambda) \supset \ldots \supset H_c(\Lambda) \supset 0 \qquad (3)$$

Of the homology of Λ, where $H_i(\Lambda) = \text{Im }\{H(\Lambda_i) \to H(\Lambda)\}$, the homorphism inside the bracket being induced by the inclusion $\Lambda_i \subset \Lambda$. The 2 filtrations (1) and (3) give rise to the associated quotients $E_0^i = \frac{\Lambda_i}{\Lambda_{i+1}}$ and $E_\infty^j = \frac{H_j(\Lambda)}{H_{j+1}(\Lambda)}$. And furthermore for each $r \geq 1$ we put

$$E_r^i = \frac{\ker\left\{H\left(\frac{\Lambda_i}{\Lambda_{i+1}}\right) \to H\left(\frac{\Lambda_i}{\Lambda_{i+r}}\right)\right\}}{\text{Im}\left\{H\left(\frac{\Lambda_{i-r+1}}{\Lambda_i}\right) \to H\left(\frac{\Lambda_i}{\Lambda_{i+1}}\right)\right\}} \qquad (4)$$

where the 2 morphisms are the connecting homomorphisms in the exact homology sequences arising from the exact sequences

$$0 \to \frac{\Lambda_{i+1}}{\Lambda_{i+r}} \to \frac{\Lambda_i}{\Lambda_{i+r}} \to \frac{\Lambda_i}{\Lambda_{i+1}} \to 0$$

and $$0 \to \frac{\Lambda_i}{\Lambda_{i+1}} \to \frac{\Lambda_{i-r+1}}{\Lambda_{i+1}} \to \frac{\Lambda_{i-r+1}}{\Lambda_i} \to 0 \qquad \text{respectively}$$

The numerator and denominator in (4) shall be denoted Z_r^i and B_r^i respectively. One notes that $B_{r_1}^i \subset Z_{r_2}^i$ for any r_1, r_2; that the spaces B_r are increasing with r while the spaces Z_r^i are decreasing with r. If r is bigger than or equal to either of the two numbers $i+1$ and $c-i+1$ (or briefly, if r is big) then these 2 spaces stabilize and (4) reads

$$\frac{\operatorname{Im}\left\{H(\Lambda_i) \to H\left(\frac{\Lambda_i}{\Lambda_{i+1}}\right)\right\}}{\operatorname{Im}\left\{H\left(\frac{\Lambda}{\Lambda_i}\right) \to H\left(\frac{\Lambda_i}{\Lambda_{i+1}}\right)\right\}} .$$

(To see this read numerator of (4) as $\operatorname{Im}\left\{H\left(\frac{\Lambda_i}{\Lambda_{i\ r}}\right) \to H\left(\frac{\Lambda_i}{\Lambda_{i+1}}\right)\right\}$; take r big now). Since $0 \to \frac{\Lambda}{\Lambda_i} \to \frac{\Lambda_i}{\Lambda_{i+1}} \to \frac{\Lambda}{\Lambda_{i+1}} \to 0$ is exact, this coincides with $\operatorname{Im}\left\{H(\Lambda_i) \to H\left(\frac{\Lambda}{\Lambda_{i+1}}\right)\right.$. This in turn coincides with $\frac{\operatorname{Im}\{H(\Lambda_i) \to H(\Lambda)\}}{\operatorname{Im}\{H(\Lambda_{i+1}) \to H(\Lambda)\}}$, i.e., E_∞^i. So we see that if r is big $E_r = E_\infty$. This property is called the <u>convergence</u> of the spectral sequence.

In addition our vector space Λ is graded by the degree of forms. And we have an induced filtration

$$H^p(\Lambda) \supset H_1^p(\Lambda) \supset \dots \supset H_c^p(\Lambda) \supset 0 \qquad (3')$$

for each p, where $H_i^p(\Lambda) = \operatorname{Im}\{H^p(\Lambda_i) \to H^p(\Lambda)\}$. Similarly for each p we have the filtration

$$\Lambda^p \supset \Lambda_1^p \supset \dots \supset \Lambda_c^p \supset 0 \qquad (1')$$

Associated to these 2 sets of fibrations are the quotients $E_0^{i,p-i} = \frac{\Lambda_i^p}{\Lambda_{i+1}^p}$ and $E^{i,p-i} = \frac{H_i^p(\Lambda)}{H_{i+1}^p(\Lambda)}$. And for each $r \geq 1$ we have

$$E_r^{i,p-i} = \frac{\operatorname{Im}\left\{H^p\left(\frac{\Lambda_i}{\Lambda_{i+r}}\right) \to H^p\left(\frac{\Lambda_i}{\Lambda_{i+1}}\right)\right\}}{\operatorname{Im}\left\{H^p\left(\frac{\Lambda_{i+r-1}}{\Lambda_r}\right) \to H^p\left(\frac{\Lambda_i}{\Lambda_{i+1}}\right)\right\}} \qquad (4')$$

Once more, we point out that the vector spaces $E_r^{p,q}$ owe their existence to the involutivity of D.

<u>4</u>. One knows from deRham's theorem that $H(\Lambda)$ is just $H^*(M,\underline{R})$, the real cohomology of the manifold, which is a topological invariant. We see thus that

$$H^p(M,\underline{R}) \cong \bigoplus_{i+j=p} E_\infty^{i,j} . \qquad (5)$$

The whole idea of the spectral sequence is to find the interplay between E_∞, and E_r for low r. For this purpose more algebra is introduced. As explained below, in each E_r a differential d_r of degree r is introduced (i.e., $d_r^2 = 0$ and $d_r(E_r^i) \subset E_r^{i+r}$) and it is shown that $H(E_r) = E_{r+1}$. Many times this statement itself--quite independent of the nature of the differentials d_r--suffices to calculate some of the groups entering into the spectral sequence. For example, let us consider Reeb's [25] foliation of S^3, when $E_r^{i,j}$ can be non-zero only if $0 \leq i \leq 1$ and $0 \leq j \leq 2$. From (5) we see that $E_\infty^{0,0} = E_\infty^{1,2} = \underline{R}$ and the other $E_\infty^{i,j} = 0$. Supposing we have seen that $E_1^{0,0} = \underline{R}$ (Section (6) below). Then $d_1 : E_1^{0,0} \to E_1^{1,0}$ must be the null map and we must have $E_1^{1,0} = 0$.

5. To wind up the algebraic machinery of the spectral sequence we recall the definition of d_r. For this one sees, using the exactness of

$$0 \to \frac{\Lambda_i}{\Lambda_{i+r+1}} \to \frac{\Lambda_i}{\Lambda_{i+1}} \to \frac{\Lambda_{i+1}}{\Lambda_{i+r+1}} \to 0,$$

that

$$\frac{Z_r^i}{Z_{r+1}^i} = \frac{\mathrm{Im}\left\{H\left(\frac{\Lambda_i}{\Lambda_{i+r}}\right) \to H\left(\frac{\Lambda_i}{\Lambda_{i+1}}\right)\right\}}{\mathrm{Im}\left\{H\left(\frac{\Lambda_i}{\Lambda_{i+r+1}}\right) \to H\left(\frac{\Lambda_i}{\Lambda_{i+1}}\right)\right\}}$$

coincides with

$\mathrm{Im}\left\{H\left(\frac{\Lambda_i}{\Lambda_{i+r}}\right) \to H\left(\frac{\Lambda_{i+1}}{\Lambda_{i+r+1}}\right)\right\}$. Using a similar argument this in turn coincides with $\frac{B_{r+1}^{i+r}}{B_r^{i+r}}$. Hence these two spaces are isomorphic (With r, Z_r^i decreases as fast as B_r^{i+r} increases). The differential d_r is defined as the following composition

$$E_r^i = \frac{Z_r^i}{B_r^i} \to \frac{Z_r^i}{Z_{r+1}^i} \cong \frac{B_{r+1}^{i+r}}{B_r^{i+r}} \to \frac{Z_r^{i+r}}{B_r^{i+r}} = E_r^{i+r} \qquad (6)$$

As we saw in 3) for r big $Z_r^i = Z_{r+1}^i$; so then this is a zero endomorphism. It is clear from (6) that $\ker\{d_r : E_r^i \to E_r^{i+r}\}$ is $\frac{Z_{r+1}^i}{B_r^i}$ and $\mathrm{im}\{E_r^{i-r} \xrightarrow{d_r} E_r^i\}$ is $\frac{B_{r+1}^i}{B_r^i}$. Hence we see that $\ker d_r \subset \mathrm{Im}\, d_r$, that is to say $d_r^2 = 0$. Dividing we see that $\frac{\ker d_r}{\mathrm{Im}\ d_r}$ equals $\frac{B_{r+1}^i}{Z_{r+1}^i}$, i.e., E_{r+1}^i. This shows that $H_{d_r}(E_r)$ equals E_{r+1}.

6. We shall now evaluate the E_1 term of the spectral sequence. For this purpose we introduce the notion of <u>invariant transverse</u> forms. Given any subbundle D of T, the tangent bundle, a form $\omega \in \Lambda$ os called transverse if $i_X\omega = 0$ for any $X \in D$. Here i_X is the interior product (see eqn. (13) below). A form is

called invariant if $L_X\omega = 0$ for any $X \in C^\infty(D)$. When D is involutive, we can (by Frobenius Theorem) find local coordinates $x_1, x_2, \ldots, x_l; x_{l+1}, \ldots, x_m$ such that the leaves are given by assigning some constant values to the last c of these. In terms of these local coordinates an invariant transverse p form will appear as $\sum_\alpha f_\alpha(x_{l+1}, \ldots, x_m) dx_{\alpha_1} \wedge dx_{\alpha_2} \wedge \ldots \wedge dx_{\alpha_p}$ where $l+1 \le \alpha_1 \le \ldots \le \alpha_p \le m$. One notes that the coeffs. f_α are constant on each leaf. When we change coordinates to another compatible system of same type the transformation matrix thus consists of functions which are constant on leaves.

Now we define D^r to be the vector space of all invariant transverse r-forms. The corresponding sheaf of germs of such focus will be denoted by $\underline{D}^r$. In particular $\underline{D}^0$, or just $\underline{D}$, is the sheaf of germs of functions which are smooth and constant on leaves. As usual if a sheaf S sits over our manifold M, $H^*(M,S)$ shall denote the Čech cohomology of M with coefficients in the sheaf S.

We remark in passing that the transverse invariant forms are those which remain parallel along leaves with respect to any Bott connection on the

bundle of transverse forms (see defn. of Bott connections in section 19). Also we will notice as we proceed further that functions which are constant on leaves play a part as important for foliated manifolds as that of analytic functions in complex manifolds.

The following proposition gives the first term of the spectral sequence.

<u>Proposition 2</u>. $E_1^{p,q}$ is isomorphic to $H^q(M, \underline{D}^p)$. The next section deals with the proof of this proposition. Note that the Reeb foliation of S^3 and more generally the foliations of S^{odd} given by Lawson [20], Tamura [36] are such that any global smooth function which is constant on leaves is simply a constant. In other words $E_1^{0,0} = H^0(M, \underline{D}) = \underline{R}$ for such foliations. Note that by 'thickening' the compact leaves one can destroy this property.

7. Proof of Proposition 2. We recall that $E_0^{p,q} = \dfrac{\Lambda_p^{p+q}}{\Lambda_{p+1}^{p+q}}$. We now construct a sheaf $\mathcal{E}_0^{p,q}$ in the following way. We take the presheaf which assigns to each open set U of M the vector space $E_0^{p,q}(U)$. If $U \subset V$ we have natural restriction maps $\Lambda_p^{p+q}(V) \to \Lambda_p^{p+q}(U)$ which yield homomorphisms $E_0^{p,q}(V) \to E_0^{p,q}(U)$. $\mathcal{E}_0^{p,q}$ is taken to be

the sheaf determined by this presheaf. If U has local coordinates, $x_1, x_2, \ldots, x_c, y_1, \ldots, y_c$ we may represent the stalk at x by expressions of the type $\omega_{\alpha,\beta}(x,y)\,dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_q} \wedge dy_{\beta_1} \wedge \ldots \wedge dy_{\beta_p}$. It is understood that for the multi-indices α and β we have $\alpha_1 < \ldots < \alpha_q$ and $\beta_1 < \ldots < \beta_p$. In this local representation the zeroth differential $d_0 : E_0^{p,q}(U) \to E_0^{p+1,q}(U)$ is given by

$$d_0\{\omega_{\alpha,\beta}(x,y)\,dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_q} \wedge dy_{\beta_1} \wedge \ldots \wedge dy_{\beta_p}\}$$

$$= \sum_{k=1}^{1} \frac{\partial\omega_{\alpha,\beta}(x,y)}{\partial x_k}\, dx_k \wedge dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_q} \wedge dx_{\beta_1}$$

$$\ldots \wedge dx_{\beta_p} \tag{7}$$

And so we also have a parallel sheaf homomorphism $d_0 : \mathcal{E}_0^{p,q} \to \mathcal{E}_0^{p+1,q}$. Using this we construct the following sequence

$$0 \to \underline{D}^p \subset \mathcal{E}_0^{p,0} \xrightarrow{d_0} \mathcal{E}_0^{p,1} \xrightarrow{d_0} \ldots \to \mathcal{E}_0^{p,m-p} \to 0 \tag{8}$$

Note that $\mathcal{E}_0^{p,0}$ is simply the sheaf of germs of smooth p-forms which vanish where one of the vectors is in D. This explains the first inclusion. We prove now that this sequence (8) is exact. An element of $\underline{D}^p$ is given locally by sum of terms of the type $\omega_\beta(y)\,dy_{\beta_1} \wedge \ldots \wedge dy_{\beta_p}$. It is clear that d_0 will kill it. Conversely an element of $\mathcal{E}_0^{p,0}$ is sum of terms of the type $\omega_\beta(x,y)\,dy_{\beta_1} \wedge \ldots \wedge dy_{\beta_p}$ and it is clear from (7) that d_0 will kill such a sum only if each ω_β is a function of y alone. This shows exactness at first place. Since d_0^2 is zero in U by (7), to show exactness at other places, we assume that

$$d_0\{\sum_{\alpha,\beta} \omega_{\alpha,\beta}(x,y)\,dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_q} \wedge dy_{\beta_1} \wedge \ldots \wedge dy_{\beta_p}\} = 0$$

which can happen only if

$$d_0\{\sum_\alpha \omega_{\alpha,\beta}(x,y)\,dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_q}\} = 0$$

for each multi-index β. Using Poincare's lemma for each y one can find a q-1 form $\sum_\gamma \theta_{\gamma,\beta}(x,y)\,dx_{\gamma_1} \wedge \ldots \wedge dx_{\gamma_{q-1}}$ such that

$$d_0\{\sum_\gamma \theta_{\gamma,\beta}(x,y)\,dx_{\gamma_1} \wedge \ldots \wedge dx_{\gamma_{q-1}}\}$$

$$= \sum_\alpha \omega_{\alpha,\beta}(x,y)\,dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_q}. \tag{9}$$

The construction of these functions $\theta_{\gamma,\beta}(x,y)$--see, e.g., Sternberg [34]--only involves integrating the smooth functions $\omega_{\alpha,\beta}(x,y)$ and their derivatives over x. So these functions can be chosen to be smooth in both x and y. Since we have

$$d_0\left\{\sum_{\gamma,\beta} \theta_{\gamma,\beta}(x,y)dx_{\gamma_1}\wedge\ldots\wedge dx_{\gamma_{q-1}}\wedge dx_{\beta_1}\wedge\ldots\wedge dx_{\beta_p}\right\}$$

$$= \sum_{\alpha,\beta}\omega_{\alpha,\beta}(x,y)dx_{\alpha_1}\wedge\ldots\wedge dx_{\alpha_q}\wedge dx_{\beta_1}\wedge\ldots\wedge dx_{\beta_p}$$

by using (9) and (7) it follows that the sheaf sequence (8) is exact. Note that $\mathcal{E}_0^{p,q}$ arises also as the sheaf of germs of smooth cross-sections of a vector bundle. So it is fine, i.e., any local cross-section of $\mathcal{E}_0^{p,q}$ can be extended globally. Again for the same reason the space of smooth sections of $\mathcal{E}_0^{p,q}$ is precisely $E_0^{p,q}$. These two facts, the fine sheaf resolution (8), and standard sheaf theory--see e.g., Hirzebruch--now imply that the cech cohomology $H^q(M,\underline{D}^p)$ coincides with the homology of the complex

$E_0^{p,0} \xrightarrow{d_0} E_0^{p,1} \xrightarrow{d_0} E_0^{p,2} \to \ldots \to E_0^{p,m-p}$, which is precisely E_1. QED

8. We shall now show how the chern classes of certain line bundles vanish in $H^2(M,\underline{D})$. In a subsequent section this leads to a generalized Bott vanishing theorem. We have to introduce some notations. We denote by $\underline{C}_S$ the sheaf over M of smooth complex valued function germs; and by $\underline{C}_D$ the subsheaf made up of those germs which are constant on leaves. Similarly $\underline{C}_S^*$ and $\underline{C}_D^*$ denote the (multiplicative) sheaves of non-zero complex smooth germs and those which are constant on leaves. We have now the two exact sheaf sequences

$$0 \to \underline{Z} \to \underline{C}_S \xrightarrow{e^{2\pi i(\ )}} \underline{C}_S^* \to 0$$

$$\text{and}\quad 0 \to \underline{Z} \to \underline{C}_D \xrightarrow{e^{2\pi i(\ )}} \underline{C}_D^* \to 0 \qquad (10)$$

where the first maps denote inclusions of the constant sheaf $\underline{Z}$. Note also that only the sheaf $\underline{C}_S$ is fine in (10). Elements of $H^1(M,\underline{C}_S^*)$ are called (equivalence classes of) smooth complex line bundles. See Hirzebruch [15] for the motivation for this terminology. By invariant line bundles we shall understand those lying in $\mathrm{Im}\,\{H^1(M,\underline{C}_D^*) \to H^1(M,\underline{C}_S^*)\}$ where the morphism is induced by the inclusion $\underline{C}_D^* \subset \underline{C}_S^*$. The first chern class $c_1(\xi)$ is defined for each line bundle $\xi \in H^1(M,\underline{C}_S^*)$ as an element of $H^2(M,\underline{Z})$ in the following way. The first of the sequences (10) gives us a long exact cohomology sequence, some of whose terms are

$$\to H^1(M,\underline{C}_S) \to H^1(M,\underline{C}_S^*) \to H^2(M,\underline{Z}) \to H^2(M,\underline{C}_S) \to$$

The two groups at the ends are zero, as the sheaf $\underline{C}_S$ is fine. The connecting isomorphism in the center is called $c_1(\cdot)$. If S is any sheaf containing $\underline{Z}$, by the first chern class in S, we will understand the composition of the above morphism with the map $H^2(M,\underline{Z}) \to H^2(M,S)$ induced by the inclusion $\underline{Z} \to S$. For example to get real chern class one takes $S = \underline{R}$.

Proposition 3. The first chern class of an invariant complex line bundle vanishes in $\underline{D}$.

Proof: We look at the following diagram:

$$\begin{tikzcd} H^1(M,\underline{C}^*_S) \arrow[r, "c_1(\cdot)", "\cong"'] & H^2(M,\underline{Z}) \arrow[r] \arrow[d, equal] & H^2(M,\underline{D}) \arrow[d] \\ H^1(M,\underline{C}^*_D) \arrow[u] \arrow[r, "\delta"] & H^2(M,\underline{Z}) \arrow[r] & H^2(M,\underline{C}_D) \end{tikzcd} \tag{11}$$

where δ is the connecting homomorphism in the exact sequence induced by the second sequence in (10). The unnamed maps arise from inclusion. This diagram obviously commutes. The bottom row is zero due to exactness. So the proposition is proved if we can see that $H^2(M,\underline{D}) \to H^2(M,\underline{C}_D)$ is a monomorphism. Take a sufficiently small open cover U of M. Suppose that we have a 1-cochain g, $U_i \cap U_j \xrightarrow{g_{ij}} \underline{C}$, on it whose coboundary is h, $U_i \cap U_j \cap U_k \xrightarrow{h_{ijk}} \underline{R}$, where g_{ij}, h_{ijk} are smooth and constant on leaves. So we have $h_{ijk} = g_{ij} + g_{jk} + g_{ki}$. Since h_{ijk} is real we also have $h_{ijk} = \mathrm{Re}g_{ij} + \mathrm{Re}g_{jk} + \mathrm{Re}g_{ki}$, showing that the cochain Reg, $U_i \cap U_j \xrightarrow{\mathrm{Re}g_{ij}} \underline{R}$ also has h as coboundary.

QED

We can complement the above proposition by including the converse statement, which follows immediately from the exactness of bottom row in (11).

Proposition 4. The first chern class of a smooth complex line bundle vanishes in $\underline{D}$ if and only if it is invariant.

In this form this proposition should be compared with a theorem of Lefschetz, Hodge, Kodaira, Spencer and Dolbeault--Thm. 15.9.1. in Hirzebruch--which characterizes complex analytic line bundles over a complex manifold.

We have, as in (5), that $H^2(M,\underline{R}) = E_\infty^{0,2} + E_\infty^{1,1} + E_\infty^{2,0}$. In this decomposition, proposition 3 implies that the first real chern class of an invariant line bundle does not lie in $E_\infty^{0,2}$. Equivalently if one looks at the diagram

$$\begin{tikzcd} H^1(M,\underline{C}^*_D) \arrow[d] & & H^2(\Lambda) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] \arrow[dr, phantom, "\#"] & \dfrac{H^2(\Lambda)}{H^2_1(\Lambda)} \arrow[d, phantom, "i\cap"] \\ H^1(M,\underline{C}^*_C) \arrow[r, "c_1(\ )", "\cong"'] & H^2(M,\underline{Z}) \arrow[r] & H^2(M,\underline{R}) \arrow[r] & H^2(M,\underline{D}) \cong E_1^{0,2} \end{tikzcd} \tag{12}$$

where the inclusion i results since $E_\infty^{0,2} = \dfrac{H^2(\Lambda)}{H^2_1(\Lambda)}$ is gotten from $E_1^{0,2}$ by successively taking the kernel under the differentials $d_1, d_2, d_3, \ldots$ of section 5; then proposition 3 shows that the bottom row evaluates to zero. The commutavity of the rectangle thus shows that if ξ is invariant $c_1(\xi)$ projects to zero under the natural map $H^2(\Lambda) \to \dfrac{H^2(\Lambda)}{H_1^2(\Lambda)}$. In other words $c_1(\xi) \in H_1^2(\Lambda)$ We record this as a Corollary 5. If ξ is invariant $c_1(\xi) \in H_1^2(\Lambda)$. So it can be represented by a closed form $\in \Lambda_1$. The converse statement is also true.

We remark that the rectangle in (12) commutes for the following reason: the map $H^2(\Lambda) \to E_1^{0,2}$ in this rectangle arises from the projection map $(\Lambda, d) \to \left(\frac{\Lambda}{\Lambda_1}, d_0\right)$. These 2 complexes provide resolutions of the two sheaves $\underline{R}$ and $\underline{D}$ which commute with the inclusion $\underline{R} \to \underline{D}$

$$\begin{tikzcd}
\underline{R} \arrow[r, hook] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\#"] & \mathcal{A}^0 \arrow[r, "d"] \arrow[d] \arrow[dr, phantom, "\#"] & \mathcal{A}^1 \arrow[r, "d"] \arrow[d] \arrow[dr, phantom, "\#"] & \mathcal{A}^2 \arrow[r] \arrow[d] & \cdots \\
\underline{D} \arrow[r, hook] & \mathcal{E}_0^{0,0} \arrow[r, "d_0"] & \mathcal{E}_0^{0,1} \arrow[r, "d_0"] & \mathcal{E}_0^{0,2} \arrow[r] & \cdots
\end{tikzcd}$$

9. In this section the question of homotopy invariance of the spectral sequence shall be posed and solved. Suppose that $M_a \xrightarrow{f} M_b$ is a smooth map between two foliated manifolds which takes leaves into leaves; or, to be precise is such that the induced map $T_a \xrightarrow{f} T_b$ satisfies $f(D_a) \subset D_b$. One now has another induced map $\Lambda_b \xrightarrow{f} \Lambda_a$ which is a vector space homomorphism presenting the filtrations. Thus we have induced homomorphism $E_{r,b} \xrightarrow{f} E_{r,a}$, commuting with the differentials d_r (see Cartan and Eilenberg). Suppose we have two homomorphisms $f, g: \Lambda_b \to \Lambda_a$ and we can find a chain homotopy between them. $s: \Lambda_b \to \Lambda_a$ such that $d_a s + s d_b = g - g$, then the two induced maps $f, g: H(\Lambda_b) \to H(\Lambda_a)$ are identical. And the induced maps $f, g: E_{r,b} \to E_{r,a}$ are the same if r is big enough (section 3). Let us now put on the chain homotopy the additional requirement that $s(\Lambda_{i,b}) \subset \Lambda_{i-k,a}$ for all i, i.e., that the homotopy disturbs the filtration by at most k units, then one can see from CE, p. 321, that for $r > k$, the 2 induced maps $f, g: E_{r,b} \to E_{r,a}$ are identical.

Now the product $M_a \times I$ can be foliated in two natural ways. We'll say that it is 1-foliated if its leaves are gotten by multiplying the leaves of M_a by I. And, we'll say that it is 2-foliated if its leaves are just: (leaf of M_a, {t}). So in the first case the codimension is unchanged, while in the second case the dimension is unchanged.

We say that 2 maps $f, g: M_a \to M_b$ which map leaves into leaves are k-homotopic ($k = 1,2$) if we can find a smooth map $S: M_a \times I \to M_b$ which takes the k-leaves into leaves, with $s_0 = f$ and $s_1 = g$.

<u>Proposition 6</u>. If $f,g: M_a \to M_b$ are k-homotopic (in the above sense) one can find a chain homotopy for which $s(\Lambda_{i,a}) \subset \Lambda_{i-k+1,b}$. (Such an s is called a <u>k-chain homotopy</u>) $(k = 1,2)$.

As pointed out above this will have the following consequence.

<u>Corollary 7</u>. The spectral sequence is a k-homotopy invariant from the E_k term onwards $(k = 1,2)$.

One recalls that the spectral sequence of a fibering is stable from the E_2 term on. The notion of fibre homotopy coincides with the notion of 2-homotopy which has been introduced above. The next section deals with the proof of proposition 6. It involves a construction which will be helpful subsequently in building up a parametrix for d.

<u>10</u>. Proof of proposition 6: Given a vector field X on the manifold, one has a skew-derivation $i_X: \Lambda \to \Lambda$ of degree -1 called the interior product with respect to X. We recall--Kobayashi and Nomizu [17], p. 35--that if ω is an r-form the definition of i_X is

$$(i_X\omega)(Y_1,\ldots,Y_{r-1}) = r\cdot\omega(X,Y_1,\ldots,Y_{r-1}) \tag{13}$$

The property of the interior product needed to construct a chain homotopy is

$$L_X = di_X + i_Xd \tag{14}$$

Now turning to the two given maps $f,g: M_a \to M_b$ and their homotopy $S: M_a \times I \to M_b$ we have the induced morphisms $f,g: \Lambda_b \to \Lambda_a$ and $S: \Lambda_b \to \Lambda(M_a \times I)$ each preserving the filtration. Let us now take the vector field to be the lift of the standard vector field $\frac{\partial}{\partial t}$ on I: X is a vector field in $M_a \times I$. Let us define the morphism $\Lambda(M_a \times I) \xrightarrow{i_X} \Lambda(M_a \times I)$ by the formula (13). From this formula, and the fact that $M_a \times I$ carries the k-foliation it follows that this map disturbs filtration by k-1 units only. Now we define one more homomorphism $\Lambda(M_a \times I) \to \Lambda(M_a)$ in the following way. Let $i_t: M_a \to M_a \times I$ by the map $x \to (x,t)$. Then $\int_0^1 \omega = \int_0^1 (i_t^*\omega)\,dt$. Finally we define a chain homotopy $s: \Lambda \to \Lambda$ to be the following composition

$$\Lambda_b \xrightarrow{S} \Lambda(M_a \times I) \xrightarrow{i_X} \Lambda(M_a \times I) \xrightarrow{\int_0^1} \Lambda(M_a) \tag{15}$$

It is clear that this map disturbs the filtration by k-1 units, and is of degree -1. The formula $ds + sd = g - f$ follows by integrating (14):

$$ds + sd = d\int_0^1 i_XS + \int_0^1 i_XSd = \int_0^1 di_XS + \int_0^1 i_XdS$$

as d commutes with the induced map S and with $\int_0^1$ which was defined by the induced maps i_t^*. So

$$ds + sd = \int_0^1 L_XS = S_1 - S_0 = g - f$$

as X was the lift of the vector $\frac{\partial}{\partial t}$ on I. QED

11. We now study the behaviour of the exterior product with respect to our filtration. Let us suppose that $\omega \in \Lambda_i^p$ and $\sigma \in \Lambda_j^q$. Then the p + q form $\omega \wedge \sigma$ is defined by

$$(\omega \wedge \sigma)(X_1, X_2, X_3, \dots, X_{p+q})$$

$$= \frac{1}{(p+q)!} \sum_{\pi} \varepsilon(\pi)\omega(X_{\pi(1)}, \dots, X_{\pi(p)})$$

$$\sigma(X_{\pi(p+1)}, \dots, X_{\pi(p+q)}) \qquad (16)$$

where π is a permutation of the set $\{1, 2, \dots, p+q\}$ and $\varepsilon(\pi)$ denotes the parity of the same. Now suppose that $p+q-i-j+1$ of the vectors $X_1, X_2, \dots, X_{p+q}$ lie in D. This implies that whenever $\leq p-i$ of the vectors $X_{\pi(1)}, \dots, X_{\pi(p)}$ are in D, $\geq q-j+1$ of the vectors $X_{\pi(p+1)}, \dots, X_{\pi(p+q)}$ are in D. Hence one of the two factors in each term of (16) is always zero. Thus $\omega \wedge \sigma \in \Lambda_{i+j}^{p+q}$. The product above defined is related to the exterior derivative in the following well known way

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^p \omega \wedge d\pi \qquad (17)$$

This fact is expressed by saying that d is a skew derivation. From this fact it follows that a product,

the 'cup' product, is induced in $H(\Lambda)$; this product obeys the well known anticommutavity rule

If $a \in H^p(\Lambda)$, $b \in H^q(\Lambda)$, then $ab \in H^{p+q}(\Lambda)$ and $ab = (-1)^{pq} ba$ (18)

Proposition 8. If $a \in H_i(\Lambda)$ and $b \in H_j(\Lambda)$, then $ab \in H_{i+j}(\Lambda)$. In particular, if $i+j > c$, the codimension of the foliation, $ab = 0$.

Proof: Immediately follows from definitions. $a \in H_i(\Lambda)$ means $a \in \mathrm{Im}\{H(\Lambda_i) \to H(\Lambda)\}$ and so a can be represented by a closed form $\omega_a \in \Lambda_i$. Similarly we have another closed form $\omega_b \in \Lambda_j$ which represents b. So ab is represented by the closed form $\omega_a \wedge \omega_b$, which by above discussion lies in Λ_{i+j}. It represents also a homology class in $H(\Lambda_{i+j})$ which is mapped to ab under the inclusion $\Lambda_{i+j} \to \Lambda$ i.e., $ab \in H_{i+j}(\Lambda)$ QED

This proposition puts strong conditions on the ring structure of $H^*(M, \underline{R})$; these will give "vanishing theorems" (see below). Note that we have thus a multiplication induced in the E_∞ term of the spectral sequence: If $\alpha \in E_\infty = \frac{H_i(\Lambda)}{H_{i+j}(\Lambda)}$ has a representative $a \in H_i(\Lambda)$ and similarly $b \in H_j(\Lambda)$ represents $\beta \in E_\infty^j = \frac{H_i(\Lambda)}{H_{j+1}(\Lambda)}$, then $\alpha\beta \in E_\infty^{i+j}$ is represented by $ab \in H_{i+j}(\Lambda)$. It is clear that the coice of representatives is irrelevant. We shall denote this induced product in E_∞ by $\alpha \wedge \beta$ also. Note that one may very well have non-zero elts. a & b in $H_i(\Lambda)$ and $H_j(\Lambda)$ with $ab \neq 0$ and still have $\alpha \wedge \beta = 0$; for, ab could lie in $H_{i+j+1}(\Lambda)$ and we have $E_\infty^{i+j} = \frac{H_{i+j}(\Lambda)}{H_{i+j+1}(\Lambda)}$.

But if $i+j>c$, one can conclude that $ab = 0$ if $\alpha \wedge \beta = 0$ in E_∞. In an entirely analogous fashion the exterior product in Λ induces a product in E_0 which pairs E_0^i and E_0^j to E_0^{i+j}. The equation (7) leads to the equation $d_0(\omega \wedge \sigma) = d_0\omega \wedge \sigma + (-1)^p\omega \wedge d_0\pi$ for $\omega \in E_0^{i,p-1}$ and $\sigma \in E_0^{j,q+j}$. We have $\omega \wedge \sigma$ lying in $E_0^{i+j,p+q-i-j}$. Hence this product in turn will induce a product in the homology of E_0, viz., E_1 and the differential d_1 will be a skew derivation with respect to this induced product. The above remark holds at every stage: if $i+j>c$ then vanishing in E_{r+1} implies vanishing in E_r.

12. As a first example of exploiting the above proposition, we shall study the signature of a foliated 4k dimensional oriented manifold M. We recall the definition of the signature. The cup product provides us with a bilinear symmetric (by (18)) quadratic form on the vector space $H^{2k}(M,\underline{R})$ obtained by evaluating the product of any 2 classes in $H^{2k}(M,\underline{R})$ on the

orienting 4k-cycle of the manifold. So if we denote this form by F, $F(x,y) = xy[M]$. The signature of F is called sign M and is a topological invariant.

Proposition 9. If c is odd, and $\dim H^{2k}_{\{\frac{c}{2}\}}(\Lambda) \geq \frac{1}{2} \dim H^{2k}(\Lambda)$, then sign M = 0.

Proof: Here $\{\frac{c}{2}\}$ denotes the first integer after $\frac{c}{2}$. We define the cone of F as the subset of $H^{2k}(M,\underline{R})$ given by the condition $F(x,x) = 0$. It shall be denoted by $\gamma(F)$. Let us denote by p and q respectively the number of positive and negative values when F is reduced--by changing bases--to a diagonal form. It is well-known that that p and q are independent of the reduction process. And $p - q$ = sign M. By Poincare duality we have $p + q = \dim H^{2k}(\Lambda)$, as F is a non-singular quadratic form. Now we notice, by using Proposition 5, that $H^{2k}_{\{\frac{c}{2}\}}(\Lambda) \subset \gamma(F)$. And therefore, since it is well known, that either of the 2 nos. p, q is at least equal to the maximal dimension of subspace in $\gamma(F)$, we get the two inequalities $p \geq \dim H^{2k}_{\{\frac{c}{2}\}}$ and $q \geq \dim H^{2k}_{\{\frac{c}{2}\}}$. Using the given hypothesis it follows that $p \geq \frac{1}{2} \dim H^{2k}$ and $q \geq \frac{1}{2} \dim H^{2k}$. So it follows that $p = q = \frac{1}{2} \dim H^{2k}$ and sign M = 0 QED

Some foliations obey a type of duality which resembles that observed by Serre [30] in complex manifolds. We'll call this Serre duality. It states that $E_\infty^{a,b} \cong E_\infty^{c-a,1-b}$. Using this and (5) we see that the hypothesis of proposition 6 are satisfied.

Corollary 10. A 4k dimensional oriented manifold admits a foliation with odd codimension which satisfies Serre duality, only if the signature vanishes.

It is known that fibered manifolds in which the fundamental group of the base space acts trivially on the cohomology groups of the fibre obey Serre duality. (See [8].) Serre duality undoubtedly holds in other instances; but the work in this regard is as yet unfinished.

13. In this section we state and prove a generalized Bott vanishing theorem. We recall--[15]--that $H^1(M,S)$ is defined, as a set with distinguished element, even if S is a sheaf of nonabelian groups. The important cases in this section are when S is either the sheaf G_S of smooth germs from M into $GL(n,\underline{C})$ or else the sheaf G_D of such smooth germs which are constant on leaves from M into $GL(n,\underline{C})$. The elements of $H^1(M,G_S)$ are called smooth $GL(n,\underline{C})$-bundles over M. Note that given a space Y and an action of $GL(n,\underline{C})$ on Y, one can

construct for each $\xi \in H^1(M,G_S)$, a fibre bundle with group G and fibre Y: simply take an open cover U in which a cocyle $g_{ij}: U_i \cap U_j \to G$ representing ξ can be found and use the g_{ij}'s as coordinate transformations. Of course G can be chosen as another Lie group, and the definitions hold. If $H \subset G$ then we have $H_S \subset G_S$ and $H_D \subset G_D$ and these sheaf inclusions induce morphisms $H^1(M,H_S) \to H^1(M,G_S)$ and $H^1(M,H_D) \to H^1(M,G_D)$ etc. We shall denote bundles lying in $\operatorname{Im}\{H^1(M,G_D) \to H^1(M,G_S)\}$ by the name invariant G-bundles. And we shall say that the group of such a bundle can be invariantly reduced if the bundle lies in $\operatorname{Im}\{H^1(M,H_D) \to H^1(M,G_S)\}$. We note in passing that the group of a $GL(n,\underline{C})$ bundle on M can be always reduced to $U(n,\underline{C})$, but the group of an invariant $GL(n,\underline{C})$ bundle on M need not be invariantly reducible to $U(n,\underline{C})$. However for the bundle of transverse 1-forms, the group can be invariantly reduced from $GL(c,\underline{R})$ to $O(c,\underline{R})$ if the manifold admits a bundle-like metric. (The concept of a bundle-like metric for a foliation is due to Reinhart [27]. It is a riemannian metric for which there exist coordinate systems such that g_{ij}, for i, j > 1, is constant on leaves.) We shall however not introduce any metrics; and the only invariant reduction which we encounter shall happen in a natural way soon. We remark again that the bundle of transverse 1-forms is invariant as we have local trivialisations by invariant and transverse 1-forms, and the coordinate transformations of these are functions $U_i \cap U_j \xrightarrow{g_{ij}} GL(c,\underline{R})$ which are constant on leaves. We recall--from Hirzebruch's book [15]--how characteristic classes can be defined using a theorem of Borel's. For a given $\xi \in H^1(M,G_D)$ one constructs an associated principal bundle, i.e., take the fibre as G and the action as left translation. We denote this bundle by $P \to M$. Hence we have an induced bundle $p^*\xi \in H^1(P/\Delta,G_D)$. Here we denote by $\Delta(n,\underline{C})$--or just Δ--the subgroup of $GL(n,\underline{C})$ consisting of triangular matrices (i.e., elements below the diagonal vanish).

<u>Proposition 11</u>. The group of $p^*\xi$ can be invariantly reduced to Δ.

Proof: If the word invariant is dropped this is a standard theorem from Steenrod [33]. The same proof works even now. We have a canonically given bundle with group and fiber Δ sitting over P/Δ viz., $P \to P/\Delta$. Obviously since the coordinate transformations of P over M can be chosen constant on leaves, so can those of this bundle (Note that P/Δ is a

compact smooth manifold. The fibration $P/\Delta \xrightarrow{p} M$ picks the foliation of M up to a foliation of P/Δ. The fiber of this fibration $P/\Delta \xrightarrow{p} M$ is precisely G/Δ, the manifold of flags: each element of G/Δ is a sequence of subspaces $0 \subset E_1 \subset E_2 \subset \dots \subset \underline{C}^n$ of $\underline{C}^n$). Let $\eta \in H^1(P/\Delta,\Delta_D)$ be associated to this bundle, then we assert that the induced map arising from $\Delta_D \subset G_{pD}$ sends η to $p^*\xi$. The proof of this fact can be found in Hirzebruch [15].

Now we resume our definition of characteristic classes. For each $k = 1,2,\dots,n$ we have a map $\Delta \xrightarrow{\varphi_k} \underline{C}^*$ which picks out the kth diagonal element of the triangular matrix. It thus induces a map $H^1(P/\Delta,\Delta_D) \xrightarrow{\varphi_k} H^1(M,\underline{C}^*_D)$ and corresponding to $p^*\xi \in H^1(P/\Delta,\Delta_D)$ we get k complex line bundles $\varphi_k(\xi)$, all invariant. Then one can check, as in [15], that $p^*\xi$ is continuously isomorphic to the whitney sum $\varphi_1(\xi) \oplus \dots \oplus \varphi_k(\xi)$. We remark that this is not an isomorphism as invariant bundles: however this is all we need. Now one defines the chern class $c(p^*\xi) \in H^{even}(P/\Delta,\underline{Z})$ by the formula $c(p^*\xi) = c(\varphi_1\xi)c(\varphi_2\xi)\dots c(\varphi_k\xi)$ employing the cup product; the chern class $1 + c_1$ of a line bundle having been defined already in section 8. Finally we appeal to the theorem of Borel which says that the projection $p: P/\Delta \to M$ induces a monomorphism $p^*: H^*(M,\underline{Z}) \to H^*(P/\Delta,\underline{Z})$ to pull this class to M. Thus $c(\xi) = p^{*-1}c(p^*\xi)$--the theorem of Borel also ensures that $c(p^*\xi)$ lies in Imp^*.

Proposition 12. The real chern ring of an invariant $GL(n,\underline{C})$ bundle over M vanishes in dimensions $> 2c$.

Proof: The fact that $p^*: H^*(M,\underline{R}) \to H^*(P/\Delta,\underline{R})$ is a monomorphism allows us to assume that the given invariant bundle $\xi \in H^1(M,G_D)$ can be invariantly reduced to Δ. So it is the continuous Whitney sum of invariant line bundles $\xi = \xi_1 + \dots + \xi_k$, $\xi_i \in H^1(M,\underline{C}_D^*)$. Using corollary 5 of section 8 $c_1(\xi_i) \in H_1^2(\Lambda)$. Then we use prop. 5 to conclude that an r-fold product of these classes will vanish if $r > c$. This proves the above theorem. QED

Note that unlike Prop. 4 we used the fact that Λ has a filtration of length c in an essential way. If one filters (for a continuous foliation) the complex of singular cochains we are not sure that the spectral groups $E_\infty^{i,j}$ vanish for $i > c$. So the above result is valid only for smooth foliation, whereas the vanishing theorem of prop. 4 is valid for topological foliations.

Bott [4] proved this theorem, only for $D^\perp \subset T^*$, and by a completely different way. This other method, which uses the Chern-Weil map, fits in naturally with our spectral sequence and shall be developed further in section 19.

Of course, as Bott and Heitsch [5] have pointed out, Theorem 2 breaks down for the integral chern ring. The reason for the (real) Bott vanishing theorem can be traced back to the second of the exact sequences in (10), which gives us the exactness in the bottom row of (11).

We remark that one can build up a K-theory for invariant bundles; just as one has the K-ring of complex analytic bundles.

14. In this section, we suppose that we have chosen a complementary subbundle N, i.e., $D + N = T$, and 2 projection maps $P_D: T \to T$ and $P_N: T \to T$ with images D and N respectively. Then--following a paper of Gugenheim and Spencer [13]--we define for each pair r,s such that $r + s = p$, a bundle map $\Pi_{r,s}: \lambda^p T \to \lambda^p T$ as follows: if $v_1 \wedge v_2 \wedge \dots \wedge v_p$ is a p-covector write it down as $(P_D v_1 + P_N v_1) \wedge \dots \wedge (P_D v_p + P_N v_p)$ and pick only

those terms in which $P_N v_i$ occurs r times and $P_D v_i$ occurs s times. We now think of Λ^p as $(\lambda^p T)^*$ and so we have induced maps $\Pi^*_{r,j}:\Lambda^p \to \Lambda^p$. The fixed points of this endomorphism form a subspace which we denote by $\Lambda^{r,s}$. From the definition of E_0, viz., $E_0^{p,q} = \dfrac{\Lambda_p^{p+q}}{\Lambda_{p+1}^{p+q}}$ it follows that we have an isomorphism $E_0^{r,s} \cong \Lambda^{r,s}$. We have some more simple relations: $\Lambda^p \cong \bigoplus_{r+s} \Lambda^{r,s}$, $\Lambda \cong \bigoplus_{\substack{r,s\\=p}} \Lambda^{r,s}$, $\Lambda_i = \bigoplus_{j\geq i} \Lambda^{i,s}$ etc. Now our filtration is preserved by the exterior-derivative, $d(\Lambda_i) \subset \Lambda_i$. It follows that $d(\Lambda^{0,0}) \subset \Lambda^{0,1} + \Lambda^{1,0}$ and $d(\Lambda^{1,0} + \Lambda^{0,1}) \subset \Lambda^{1,1} + \Lambda^{2,0} + \Lambda^{0,2}$. Again the exterior product obviously pairs $\Lambda^{r,s}$ and $\Lambda^{a,b}$ with $\Lambda^{r+a,s+b}$. Finally the endomorphism d is a skew derivation with respect to the exterior product. It follows from these remarks that in the bigraded module $\Lambda = \bigoplus_{r,s} \Lambda^{r,s}$, d can be thought of as the sum of three endomorphisms $d_{0,1}, d_{1,0}$ and $d_{2,-1}$ of degrees (0,1), (1,0) and (2,-1) respectively. Now the equation $d = 0$ can be written $(d_{0,1} + d_{1,0} + d_{2,-1})^2 = 0$. Equating terms of the same bidegree to zero we get $d_{01}^2 = d_{10}^2 = d_{2,-1}^2 = d_{0,1}d_{1,0} + d_{1,0}d_{0,1} = d_{1,0}d_{2,-1} + d_{2,-1}d_{10} = d_{2,-1}d_{0,1} + d_{01}d_{2,-1} = 0$. In other words d becomes

the sum of three endomorphisms, each of order 2, and any two of these commute up to sign.

<u>Proposition 13</u>. $d_{2,-1} = 0$ if and only if N is involutive.

Proof: Follows immediately from Proposition 1 applied to the filtration gotten from N in place of D. QED

Using this bigrading we can algebraically characterize E_1 and E_2. The endomorphism $d_{0,1}$ of order 2 acting on the above bigraded space $\bigoplus \Lambda^{r,s}$ gives us its homology which we denote by $H_{0,1}(\Lambda)$. Again since $d_{1,0}$ anticommutes with $d_{0,1}$ it induces a differential on this new bigraded space $\bigoplus H_{0,1}^{r,s}(\Lambda)$ of degree (1,0). If we take homology with respect to this differential we get a new bigraded space $H_{1,0}H_{0,1}(\Lambda)$. Similarly we have yet another bigraded space $H_{2,-1}H_{1,0}H_{0,1}(\Lambda)$.

<u>Proposition 14</u>. We have $E_1 \cong H_{0,1}(\Lambda)$ and $E_2 \cong H_{1,0}H_{0,1}(\Lambda)$.

Proof: The proof follows that in Cartan and Eilenberg p. 330. The zeroth differential in our spectral sequence, $d_0:E_0^{i,p-i} \to E_0^{i,p-i+1}$, i.e., $\dfrac{\Lambda_i^p}{\Lambda_{i+1}^p} \to \dfrac{\Lambda_i^{p+1}}{\Lambda_{i+1}^{p+1}}$ is the map induced from d, and so was the $d_{0,1}$ above. So they coincide and we get $E_1 \cong H_{0,1}(\Lambda)$. Now--see CE p. 319--the first differential in our spectral sequence, $d_1 : E_1^{i,p-i} \to E_1^{i+1,p-i}$ can be seen to be the same as the connecting homomorphism induced by the following exact sequence: $0 \to \frac{\Lambda_{i+1}}{\Lambda_{i+2}} \to \frac{\Lambda_i}{\Lambda_{i+2}} \to \frac{\Lambda_i}{\Lambda_{i+1}} \to 0$. $\{$Note, from (4'), that $E_1^{i,p-i}$ is same as $H^p\left(\frac{\Lambda_i}{\Lambda_{i+1}}\right)\}$. Now the connecting homomorphism $H^p\left(\frac{\Lambda_i}{\Lambda_{i+1}}\right)$

$\to H^{p+1}\left(\frac{\Lambda_{i+1}}{\Lambda_{i+2}}\right)$--given as we pass to homology under d, is obviously the same as the map induced by $d_{1,0}$ which is the part of d having degree 1,0 (as $d_{2,-1}$ would take us to Λ_{i+2} and thus play no role in above connecting morphism). So it enables us to identify d_1 and d_{10} and see that E_2 is same as $H_{1,0}H_{0,1}(\Lambda)$. QED

It is however not true that $E_3 = H_{2,-1}H_{1,0}H_{0,1}(\Lambda)$: the differential d_3 of the spectral sequence is quite different from the knight move.

15. The rather algebraical interpretation of the E_2 term, given by Prop. 14 is not altogether satisfactory. A more geometric result is the following.

Proposition 15. The sheaf sequence

$$0 \to \underline{R} \hookrightarrow D^0 \xrightarrow{d} \underline{D}^1 \to \dots \to \underline{D}^c \to 0 \tag{19}$$

is exact. For each $q \geq 0$, we have the induced chain complex

$$H^q(M,\underline{D}^0) \xrightarrow{d^t} H^q(M,\underline{D}^1) \xrightarrow{d^t} \dots \to H^q(M,\underline{D}^c) . \tag{20}$$

Under the isomorphism of prop. 2 this complex is same as

$$E_1^{0,q} \xrightarrow{d_1} E_1^{1,q} \to \dots \xrightarrow{d_1} E_1^{c,q} .$$

Thus the E_2 term can be thought of as the homology of the seqn. (20).

Proof: First we demonstrate the exactness. To do this we employ the classical Poincare's lemma in the following way: a local section of $\underline{D}^p$, i.e., a local invariant transverse p-form ω looks like $\sum_\alpha \omega_\alpha(y)dy_{\alpha_1} \wedge \dots \wedge dy_{\alpha_p}$ in coordinates $X_1, X_2, \dots, X_1, Y_1, \dots, Y_c$ compatible with the foliation, and $d(\omega)$ is just

$$\sum_\alpha \sum_k \frac{\partial\omega_\alpha(y)}{\partial y_k} dy_k \wedge dy \quad \wedge \dots \wedge dy_{\alpha_p} .$$

Using the fact that in $\underline{R}^p$ any closed form is exact, we are through.

For the second part we recall that the isomorphism of prop. 2 resulted from the fine sheaf resolutions occurring in the rows of the following sign-commutative diagram of sheaves

$$\begin{array}{ccccccccc}
0 \to \underline{D}^i & \hookrightarrow & \mathcal{E}_0^{i,0} & \xrightarrow{d_{0,1}} & \mathcal{E}_0^{i,1} & \to \dots \to & \mathcal{E}_0^{i,1} & \to 0 \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d_{1,0}} & & \downarrow & & \downarrow{\scriptstyle d_{1,0}} & \\
0 \to \underline{D}^{i+1} & \hookrightarrow & \mathcal{E}_0^{i+1,0} & \xrightarrow{d_{0,1}} & \mathcal{E}_0^{i+1,1} & \to \dots \to & \mathcal{E}_0^{i+1,1} & \to 0
\end{array}$$

Here I think of the sheaf $\mathcal{E}_0^{p,q}$ as the sheaf of germs of forms in $\Lambda^{p,q}$ (see Section 14 above). Only the commutavity of the first square could be non-obvious. It follows by noting that on invariant forms, $d_{0,1} = 0$. And since $Kt : \Lambda^{p,q} \to \Lambda^{p+2,q-1}$ is obviously zero for $q = 0$, we see that $d : \underline{D}^i \to \underline{D}^{i+1}$ is same as $d_{1,0} : \underline{D}^i \to \underline{D}^{i+1}$ and so the first square commutes.

Due to naturality, the second assertion follows from this commutative diagram. QED

In a well known special case, Serre [31] was able to give a better description of the spaces $E_2^{p,q}$. We shall now obtain his results. So we suppose that our foliation arises from a smooth fibration $M \xrightarrow{p} B$ with fiber F; here F and B are smooth manifolds, etc. Before taking up the general case we note that the case $q = 0$ is very easy.

Corollary 16. In this fibration case $E_2^{p,0} \cong H^p(B,\underline{R})$.

Proof: For $q = 0$, (20) is just the chain complex of sections arising from the differential sheaf $\underline{D}^i$. Now the sheaf $\underline{D}^i$ is simply the pull-back of the sheaf ${}^i(B)$ of i-forms on the base space. Hence (20) coincides with the deRham complex on B and the result follows. QED

For the general case we define a sheaf $H_B^q(F)$

on B from the following pre-sheaf: to each open set U of B we associate the vector space $H^q(p^{-1}U,\underline{R})$, and to each inclusion map $W \subset U$ the induced homorphism $H^q(p^{-1}W,\underline{R}) \leftarrow H^q(p^{-1}U,\underline{R})$ of $p^{-1}W \subset p^{-1}U$. Now we have the cohomology of B with coefficients in this sheaf, viz., $H^p\big(B, H_B^q(F)\big)$. It is usual to call this as the cohomology of B with local coefficients in $H^q(F)$. We then have the following:

Proposition 17. In the fibration case $E_2^{p,q} \cong H^p\big(B, H_B^q(F)\big)$.

Proof: We extend the construction above defined to the entire sequence (19), i.e., we construct sheaves $H_B^q(\underline{D}^p)$ on B from the presheaves which attach to each open set U of B the space $H^q(p^{-1}U,\underline{D}^p)$ and to each inclusion $W \subset U$ the induced map $H^q(p^{-1}W,D^p) \leftarrow H^q(p^{-1}U,\underline{D}^p)$. Now the morphisms in the sequence (19) induce sheaf homomorphisms $H_B^q(\underline{D}^p) \to H_B^q(\underline{D}^{p+1})$. The resulting sheaf sequence

$$0 \to H_B^q(F) \to H_B^q(\underline{D}^0) \to \dots \to H_B^q(\underline{D}^c) \to 0 \qquad (21)$$

is exact. Moreover each of the sheaves $H_B^q(\underline{D}^i)$ is fine as follows from noting that $H^q(M,\underline{D}^i)$ is a module over $C^\infty(B)$, the ring of functions constant on leaves. Finally the chain complex of sections arising from (21) coincides with (20). This proves the assertion. QED

We shall not pursue this special case further as it is well known and understood.

It should be pointed out however that the exact sheaf sequence (19) is the basic reason why one uses spectral sequences to study foliated manifolds. In the terminology of Swan [35] the "second" spectral sequence of (19)--with resolution by forms--is precisely the spectral sequence being studied. One can generalize this procedure to studying any structure on M, which gives us from the exact sequence

$$0 \to \underline{R} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \cdots \to \mathcal{A}^n \to 0$$

of sheaves another (semi-exact sequence) of subsheafs.

$$0 \to \underline{R} \to \mathcal{A}^0_G \xrightarrow{d} \mathcal{A}^1_G \xrightarrow{d} \cdots \to \mathcal{A}^n_G \to 0$$

In our case G is the foliate structure and the subsheaf $^{i}_{G}$ is $\underline{D}^i$, the sheaf of transverse and invariant i-forms, and this sequence is the exact sequence (19). (This general point of view can be seen in Spencer's work--see, e.g., [32].)

16. Let M be any smooth manifold, not necessarily compact, and suppose that we have a smooth vector bundle $V \xrightarrow{\pi} M$; the space of smooth sections is denoted by $C^\infty(V)$. By a local trivialisation of this bundle we mean a diffeomorphism χ of $\pi^{-1}(U)$ with the trivial bundle $\Omega \times \underline{R}^k$. (Here $\Omega \subset \underline{R}^m$ is an open set $\cong U$ and

k is the fibre dimension). If we look at a section $g \in C^\infty(V)$ under this trivialisation we get a map $g_\chi : \Omega \to \underline{R}^k$, whose ith coordinate shall be denoted by g_{χ_i}, $1 \le i \le k$. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a multi-index we shall denote by $g^{(\alpha)}_{\chi_i}$ the function $\Omega \to \underline{R}$ given by $\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} g_{\chi_i}$. (Here $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$).

Now choose a compact set L in Ω and set $|g|_{\chi,\alpha,L}$ equal to $\sup_{x \in L, 1 \le i \le k} |g^{(\alpha)}_{\chi_i}(x)|$. Then this is a semi-norm on the vector space $C^\infty(V)$. (A function $p : V \to [0,\infty)$ such that $p(x + y) \le p(x) + p(y)$, $p(\lambda x) = \lambda\, p(x)$ is called a semi-norm on the vector space V). Now we put on $C^\infty(V)$ the weakest topology which makes all the semi-norms $|\cdot|_{\chi,\alpha,L}$ continuous. One can in fact get this same topology by using only a countable number of these semi-norms. (Take a countable number of compact sets covering M with their interiors, each compact set lying in a U over which a trivialization is given). Also, all the semi-norms vanish only on the zero section. Hence this vector space is metrizable. [See p. 24 of L. Schwartz [29]. We'll refer to this book for all results on functional analyses.] In the preceding sections we have come across a number of vector spaces of this type. For example, Λ, Λ^p, E_0, $E_0^{p,q}$ are all spaces of sections of suitable vector bundles on M. These vector spaces, or for that matter any other such space of sections, shall be assumed to be topologized in the above manner.

Proposition 18. $C^\infty(V)$ is a Frechet space. [A Frechet space is a complete metrizable space. By complete we mean that every Cauchy sequence converges. A sequence v_i is called cauchy if $p(v_i - v_j) \to 0$ as $i,j \to \infty$ for each semi-norm p.]

Proof: We have already seen that its topology can be defined by a countable number of semi-norms. The completeness follows by noting that uniform limit of continuous functions is continuous. QED

Corollary. Λ, Λ^p; E_0, $E_0^{p,q}$ are Frechet spaces.

We remark that Λ is a hausdorff locally convex topological vector space--briefly, a HLCTVS--and, as such the theory of compact operators applies to it:

We recall that if E and F are 2 HLCTVSes then a linear map $E \to F$ is called compact if it is continuous and maps some neighborhood of zero onto a relatively compact set (i.e., a set with compact closure). The following basic theorem is the culmination

of the efforts of Fredholm, Hilbert, Riesz and Schwartz amongst others.

Proposition 19. If E is a Hausdorff locally convex topological vector space and $s: E \to E$ is compact then the map $1 - s: E \to E$ has a finite dimensional kernel, a closed image, and a finite dimensional cokernel.

We refer the reader to Schwartz [29], Theorems A-1, A-2 for even more general results. We shall need only the finiteness of codimension.

The volume bundle $\Omega \to M$ is the line bundle associated to the tangent bundle by the representation $GL(m) \to \underline{R}^+$ given by $A \to |\det A|$. It is clear that sections of Ω are smooth measures on M. A smooth kernel K--see Atiyah and Bott, Lefschetz Fixed pt. formula [2]--assigns smoothly to each point (x,y) of $M \times M$ a linear transformation $\lambda T_y^* \to \lambda T_x^* \otimes \Omega_y$. Thus for a given x, $K(x,y)(\omega(y))$ is a form at x times a measure. Integrating over y we shall get a form which we shall denote by $S_K\omega$. This integration is possible if K has a compact support in $M \times M$; or, even if, for x fixed, the set $\{y | K(x,y) \neq 0\} \subset M$ has a compact closure. A linear map $S_K: \Lambda \to \Lambda$ which arises from a smooth kernel K in the fashion described is called a <u>smoothing map</u>. We will write

$$(S_K\omega)(x) = \int_y K(x,y)\big(\omega(y)\big) \qquad (22)$$

We note that the definition of smoothing map holds even if we are working with an arbitrary vector bundle V in place of λT^*. Thus one can talk of smoothing maps $C^\infty(V) \xrightarrow{S_K} C^\infty(V)$. The right side of (22) continues to make sense even if ω is only continuous in y. Differentiation under the integral sign shows that if K is still a smooth kernel, $S_K\omega$ shall be smooth in x. In other words we have a natural extension $C^0(V) \xrightarrow{\bar{S}_K} C^\infty(V)$ of S_K and $S_K = \bar{S}_K \circ i$ where i denotes the inclusion map $C^\infty(V) \to C^0(V)$.

We shall topologise the vector space $C^K(V)$ of k-times continuously differentiable sections of a vector bundle $V \to M$ by the semi-norms $|\cdot|_{\chi,\alpha,L}$ with $|\alpha| \le K$. It is apparent that if M is compact then the topology is given by a finite number of such semi-norms: we take a finite number of compact sets L_i whose interiors cover M and a finite number of trivializations χ_i defined (resp.) on neighborhoods of L_i. Thus the topology is also given by a norm $\|\cdot\| = \sum_{i,\ \alpha \le k} |\cdot|_{\chi_i,\alpha,L_i}$. The space is clearly complete. Hence $C^k(V)$ has been topologised, for M

compact, as a Banach space.

<u>Proposition 20</u>. If M is compact $C^1(V) \hookrightarrow C^0(V)$ is compact.

Proof: Take any bounded neighborhood of zero in $C^1(V)$. As a subset of $C^0(V)$ it is equicontinuous and fibre-wise bounded. So by Ascoli's theorem--see, e.g., Schwartz, Th. 4-6--it has compact closure in $C^0(V)$ QED

This result shows that the inclusion i: $C^\infty(V) \hookrightarrow C^0(V)$ is also compact, as it factors through the above map. Hence we see that for M compact, a smoothing map $S_K : \Lambda \to \Lambda$ is always compact.

By a <u>parametrix ford</u>--or simply, a parametrix--we shall understand, as in [2], a linear map $\Lambda \xrightarrow{p} \Lambda$ such that $dp + pd = 1 - S$ where $\Lambda \xrightarrow{S} \Lambda$ is a smoothing map. When M carries a foliation we will also introduce a further refinement: a parametrix will be called a <u>k-parametrix</u> if s commutes with the filtration and $p(\Lambda_r) \subset \Lambda_{r-k+1}$. [One can always construct a parametrix for d(see e.g., sections 17 and 18 below) but it is not known whether a 3-parametrix is possible. An example due to Schwartz [28] implies that one need not have a 2-parametrix.] The important equation

$$1 - s = dp + pd \qquad (23)$$

shows that $1 - s$ maps Z ($Z = \{\omega \mid \omega \in \Lambda,\ d\omega = 0\}$) into B

($B = \{\omega \mid \omega \in \Lambda$ s.t. $\exists\, \theta$ with $d\theta = \omega\}$). But Z being a closed subspace of Λ is also Hausdorff, so a HLCTVS. Hence if M is compact, by Prop. 19, $Z \xrightarrow{1-S} Z$ has a finite codimension. Hence Z/B is finite dimensional, which of course is well known. Similarly we have the following.

Proposition 21. Suppose that a k-parametrix exists.

(a) Then if the space Z_k is Hausdorft in the induced topology and the induced map $s: Z_k \to Z_k$ is compact then E_k is finite dimensional.

(b) If M is compact and all the spectral sequence morphisms $d_0, d_1, \ldots$ are topological homomorphisms then the above hypothesis is satisfied.

Proof: By [7], prop. 3·1, page 321, the induced map $1-s: Z_k \to Z_k$ has image lying in B_k. Thus Z_k/B_k is finite dimensional by using hypotheses of (a) and proposition 19. This proves (a).

Now if M is compact the smoothing map $s: \Lambda \to \Lambda$ is compact. As $d_0, d_1, \ldots$ are topological homomorphisms the vector spaces $E_0, E_1, E_2, \ldots$ will all be Hausdorff locally convex in the induced topology. One sees that the induced map $s: Z_k \to Z_k$ is also compact. QED

The author feels that the hypotheses in (a) is not required for this finiteness result. However one would have to use more refined functional analysis to settle this point.

This proposition tells us that it is a good idea to construct k-parametrices for d. (This will be done in subsequent sections.) For example, a 2-parametrix would make E_2 finite dimensional under above hypotheses. Note however that $Z_2^{*,0}$ are always Hausdorff and the existence of a 2-parametrix would at once imply that $E_2^{*,0}$ are finite dimensional. Another simple observation is that E_r is finite dimensional--for M compact--for $r \geq c + 1$. This follows from the fact that E_∞ is finite dimensional and that $d_r = 0$ for $r \geq c + 1$, as the filtration is of length c. One can ask as to whether there exists an r, independent of c, for which $E_r(M)$ is finite dimensional; M being any compact foliated manifold. The recent counter-example of G. Schwartz [28] shows that if such an r exists it must be ≥ 3. This follows as the groups in [28] are precisely $E_2^{*,0}(M)$ and Schwarz gives foliations for which they fail to be finite dimensional. Note that [28] thus implies that one may not be able to construct a 2-parametrix.

We shall put

$$e_r^{p,q} = \dim E_r^{p,q} \qquad (24)$$

if the right side is a finite number

Proposition 22. If E_r is finite dimensional

$$\chi(M) = \sum_{p,q} (-1)^{p+q} e_r^{p,q} \qquad (25)$$

where $\chi(M)$ is the Euler-Poincare characteristic of M.

Proof: Since the ith Betti number of M is

$b_i = \sum_{p+q=i} e_\infty^{p,q}$, it follows that

$$\chi = \sum_{p,q} (-1)^{p+q} e_\infty^{p,q}$$

and so $= \sum_{p,q} (-1)^{p+q} e_N^{p,q}$ for N large enough.

But we know that the Euler characteristic of a finite complex is same as that of its graded homology. Hence the last expression equals

$$\sum_{p,q} (-1)^{p+q} e_r^{p,q} \qquad \text{QED}$$

One can pose more general index problems; e.g., calculate $\sum_q (-1)^q e_2^{p,q}$ (if E_2 is finite dimensional) in terms of characteristic classes. It is possible in some cases to guess at the probable expressions. But the general development in this direction is at the moment held up due to analytical difficulties: construction of parametrices etc., which we will

encounter in the following sections.

If we have a Hausdorff TVS V, equipped with a continuous differential $d: V \to V$ then it is clear that $\overline{d(V)} \subset d^{-1}(0)$ and so we can define the continuous homology of (V,d) as $\frac{d^{-1}(0)}{\overline{d(V)}}$; we denote it by $\overline{H}(V)$. Note that $\overline{H}(V)$ will be Hausdorff also. On the other hand the homology group H(V) need not be Hausdorff.

Proposition 23. The continuous homology of the deRham complex is same as the deRham cohomology.

Proof: In fact $B = d(A)$ can be characterized--by deRham's theorem--as those forms which have zero value on all cycles. Thus it will follows that $B = \overline{B}$ and the result is clear.

QED

Note that if M is compact, prop. 23 is trivial: B is of finite codimension in the Frechet space Z, so it must be closed.

This proposition and prop. 21 show that it might be desirable to replace the spectral sequence E_r by a continuous spectral sequence $\overline{E}$ in which at each step we take the continuous homology with respect to a continuous differential. The advantage of such a change would be that the Hausdorffness requirement can be dropped from prop. 21: If there is a k-parametrix, $\overline{E}_k$ is finite dimensional! By prop. 23 such a spectral sequence would also converge to $H^*(M,\underline{R})$. However we shall not go into this here because (1) a lot of preliminary continuous homological algebra is required; and (2) the Hausdorffness condition ought to be studied, (since it has connections with Serre duality), and not evaded. (See prop. 27 below.)

<u>17</u>. Let us denote by $C^\infty(M,M)$ the set of all smooth maps $M \to M$, and by $C_1^\infty(M,M)$ the set of all smooth maps $M \to M$ which map any leaf into another. Consider a function

$$F:\underline{R}^m \to C^\infty(M,M), \text{ with } F(0) = \mathrm{id}. \tag{26}$$

Let $f(\eta)$ be any function on $\underline{R}^m$ which is smooth, has compact support, and for which

$$\int_{\underline{R}^m} f(\eta)\,d\eta = 1 \tag{27}$$

Let us now write the formal expressions, with $\omega \in \Lambda$:

$$(s\omega)(x) = \int_{\underline{R}^m} \left(F(\eta)^*\omega\right)(x)\cdot f(\eta)\,d\eta \tag{28}$$

and

$$(p\omega)(x) = \int_{\underline{R}^m}\int_1^0 \left(\theta_t^* i_{\frac{\partial}{\partial t}} F(\eta t)^*\omega\right)(x) \cdot f(\eta)\,d\eta dt \tag{29}$$

Here the map $F(\eta t): M \times I \to M$ is defined by $(x,t) \mapsto F(\eta t)x$ and for each $t \in [0,1]$ we define the

injection $\theta_t: M \to M \times I$ by $x \mapsto (x,t)$. $\frac{\partial}{\partial t}$ is the standard vector on $M \times I$ along the t-direction. One should compare these formulae to those in Section (10).

We can topologize the set of all maps, $C^\infty(M,M)$, in a natural way with the C^∞ topology: so 2 maps are 'near' each other if (in local coordinates) all their derivatives are 'near' each other. The subset $C_1^\infty(M,M)$ shall be given the subspace topology.

Now if we require that the function F be continuous in (26) it follows immediately that the two integrands in (28) and (29) are continuous. Since they have compact support also both these expressions make absolutely good sense.

<u>Proposition 24</u>. p is a chain homotopy between 1 and s; i.e., $1 - s = dp + dp$. If the image of (26) lies in $C_1^\infty(M,M)$ then s preserves filtration and p is a 2-chain homotopy.

Proof: Since d commutes with induced maps we see that

$$dp\omega + pd\omega = \int_{\underline{R}^m}\int_1^0 \left(d\theta_t^* i_{\frac{\partial}{\partial t}} F(\eta t)^*\omega + \theta_t^* i_{\frac{\partial}{\partial t}} F(\eta t)^* d\omega\right) \cdot f(\eta)\,d\eta dt$$

equals

$$\int_{\underline{R}^m}\int_1^0 \left(\theta_t^*(d i_{\frac{\partial}{\partial t}} + i_{\frac{\partial}{\partial t}} d) F(\eta t)^*\omega\right)\cdot f(\eta)\,d\eta dt$$

which, by (14), is the same as

$$\int_{\underline{R}^m} \int_1^0 \left(\theta_t^* L_{\frac{\partial}{\partial t}} F(\eta t)^* \omega\right) \cdot f(\eta)\, d\eta\, dt \,.$$

Integrating with respect to t we get, as $F(0) = \mathrm{id}$,

$$\int_{\underline{R}^m} \omega \cdot f(\eta)\, d\eta - \int_{\underline{R}^m} \left(F(\eta)^* \omega\right) \cdot f(\eta)\, d\eta$$

ans so, by (27) and (28) this is the same as

$$\omega - s\omega \,.$$

Now, if $F(\eta): M \to M$ maps leaves into leaves then $F(\eta)^*: \Lambda \to \Lambda$ preserves the filtration. So, by (28) $s: \Lambda \to \Lambda$ preserves the filtration. Considering $M \times I$ to be carrying the 2-foliation (section 9) we see that $F(\eta t)^*: \Lambda(M) \to \Lambda(M \times I)$ also preserves the filtration. Since $\frac{\partial}{\partial t}$ is transverse to the 2-foliation of $M \times I$ it follows that the filtration of $i_{\frac{\partial}{\partial t}} F(\eta t)^* \omega$ is one unit less at most. Since $\theta_t^*: \Lambda(M \times I) \to \Lambda(M)$ preserves the filtration we thus see from (29) that $p: \Lambda \to \Lambda$ obeys the condition $p(\Lambda_i) \subset \Lambda_{i-1}$. QED

To construct a parametrix we need to choose F so as to make the map $s: \Lambda \to \Lambda$ a smoothing map. For the construction of a 2-parametrix we will have to ensure that the image of F lies in $C_1^\infty(M,M)$.

Consider the following situation: M is

parallelizable, and admits a global parallelism by m complete vector fields. This means that we have m globally defined tangent vector fields on M, which are linearly independent at each point, and which define 1-parameter groups of diffeomorphisms of M. Thus we have a continuous function

$$\overline{F}: \underline{R}^m \to C^\infty(T),\ \overline{F}(0) = 0, \tag{30}$$

with the right hand side given the usual Frechet topology. Let us denote the 1-parameter group of diffeomorphisms of $\overline{F}(\eta)$ by $F_t(\eta)$. Then by taking $F(\eta) = F_1(\eta)$ we get a continuous function

$$F: \underline{R}^m \to \mathrm{Diff}\,(M),\ F(0) = \mathrm{id}\,. \tag{31}$$

Note that if M is compact the completeness assumption can be dropped from the above. The space of all diffeomorphisms $M \to M$ which figures in the right side of (31) is topologized as a subspace of $C^\infty(M,M)$. We'll have occasion also to employ the space $\mathrm{Diff}_1\,(M)$ of diffeomorphisms of M which map leaves onto leaves.

Now choose an $x \in M$ and consider the map F_x of $\underline{R}^m$ into M given by $\eta \mapsto F(\eta)x$. This map takes 0 to x. Also it maps $T_0\underline{R}^m$, the tangent space to $\underline{R}^m$ at 0 isomorphically onto T_xM. In fact, using the canonical identification of $T_0\underline{R}^m$ with $\underline{R}^m$, this map is simply the map $\overline{F}_x: \underline{R}^m \to T_x$ given by $\eta \mapsto \overline{F}(\eta)(x)$; and, by hypothesis, it is an isomorphism. It follows therefore that $F_x : \underline{R}^m \to M$ maps some neighborhood of $0 \in \underline{R}^m$ diffeomorphically onto a neighborhood of $x \in M$. (In general we will say that the continuous map of (26) is <u>locally transitive</u> at $x \in M$ if $F_x : \underline{R}^m \to M$ has the above property.)

Let us consider also the product space $M \times M$ and let Δ denote the diagonal in this space. The map F leads to a natural map $F_\Delta : M \times \underline{R}^m \to M \times M$ given by $(x,\eta) \mapsto \big(x, F(\eta)x\big)$ which maps $(x,0)$ to $(x,x) \in \Delta$. Again it follows from the given hypothesis that the tangent space at the first point--$(x,0)$--is mapped isomorphically to that at (x,x). Hence some neighborhood of the first point is mapped diffeomorphically onto a neighborhood of the second. From this we conclude that given an $x \in M$ one can find a neighborhood V of x such that a neighborhood of V of $0 \in \underline{R}^m$ is mapped diffeomorphically onto some neighborhood of $y \in M$ by F_y for each $y \in V$. (This sentence makes sense for any continuous map $F : \underline{R}^m \to C^\infty(M,M)$. We will say then that F is <u>locally transitive</u> near x.)

Now if M is compact then the last sentence can be strengthened to read: There exists a neighborhood V of $0 \in \underline{R}^m$ which is mapped diffeomorphically to some neighborhood of $y \in M$ by any F_y, $y \in M$. (We will

say that (26) is <u>uniformly transitive</u> if it possesses the property expressed by this sentence.)

We will assume now that the function $f(\eta)$ used above in constructing s and p has its support inside the aforementioned neighborhood V of $0 \in \underline{R}^m$.

<u>Proposition 25</u>. If the continuous map $F : \underline{R}^m \to C^\infty(M,M)$ is uniformly transitive and if $f(\eta)$ has sufficiently small support, then s is a smoothing map: so p is a parametrix in Prop. 24. If further the image of F lies in $C_1^\infty(M,M)$ then it is a 2-parametrix.

Proof: By sufficiently small support we simply mean that the assumption just made above holds. Now $\bigcup_x \big(x, F_x(U)\big) = N$ gives us a neighborhood of the diagonal $\Delta \subset M \times M$ such that for each $(x,y) \in N$ we have a smooth linear transformation $K(x,y)$ of λT_y^* to $\lambda T_x^* \otimes \Omega_y$ given by $\omega(y) \mapsto \big(F(\eta)^*\omega\big)(x) \otimes \sigma$: here $\eta = F_x^{-1}(y)$ and σ is the measure in $F_x(U)$ corresponding to the measure $f(\eta)d\eta$ in U, under the diffeomorphism F_x. Defining $K(x,y) = 0$ for $(x,y) \notin N$ we thus get a smooth kernel k on $M \times M$. It is clear that the s given by eqn. (28) is the same as s_k, a smoothing map, given by (22). QED

The remark made above indicates that this result applies whenever M is a compact parallelizable manifold. If then the image of $\overline{F}$ lies in $C_1^\infty(T)$--the vector fields of the foliation--then we will get a 2-parametrix. If M is not complact, but still has a parallelism by complete vector fields (30), then F is only locally transitive near each x. This situation will arise later on when we take M to be a principal tangent bundle. However, in that case $\overline{F}$ is equivariant with respect to group action on the fibres. Due to this F will be uniformly transitive if the base space is compact. (This construction will be postponed to sec. 21 since it uses connection theory.)

18. In this section we will make a few comments which are not directly needed in the ensuing developments.

a. For non-parallizable manifolds we will rapidly sketch the modifications to be made on the above argument:

A. First we notice that if we have a finite number of linear maps s_i, p_i of Λ such that $1 - s_i = dp_i + p_i d$, then by putting $s = s_1 s_2 s_3 \ldots$ and $p = p_1 + s_1 p_2 + s_1 s_2 p_3 + \ldots$ we get $1 - s = dp + pd$. Hence it is enough to show that $s_1 s_2 s_3 \ldots$ is smoothing.

B. Now, given a manifold M and any $x \in M$ we can always find a continuous map $\overline{F}: \underline{R}^m \to C^\infty(T)$, $\overline{F}(0) = 0$ such that for all $y \in V$, a nhbd. of x, $\overline{F}_y: \underline{R}^m \to T_y$ is an isomorphism. From there we can find

a map $F: \underline{R}^m \to \mathrm{Diff}\,(M)$ which is locally transitive near x. We will take $\overline{F}(\eta)(y)$ to be zero in $M - V$; so $F(\eta)(y) = y$ outside of V. Now using this map F the proof of prop. 25 shall show that s is a smoothing map in some neighborhood of x, i.e., any C^0 form with support inside this neighborhood is mapped to a C^∞ form by s.

C. We now can find (as M is compact) a finite number of maps s_i, p_i of Λ such that we have (1) $1 - s_i = dp_i + p_i d$; (2) s_i smoothes forms inside the open set W_i^1 (where $\bigcup_i W_i^1 = M$); (3) and any form with support in $M - W_i^1$ is mapped into a form with support in $M - W_i^2$ (here $\overline{W}_i^2 \subset W_i^1$ and $\bigcup_i W_i^2 = M$). The last can be achieved by reducing the support of the $f_i(\eta)$ which are used to construct s_i. But from (1), (2), (3) it follows at once that $s = s_1 s_2 s_3 \ldots$ will make any C^0 form into a smooth form. Thus $1 - s = dp + pd$ is the required parametrix.

D. One notes that this patching argument will run into trouble if (with M foliated) we impose the requirement that the image of F lies in $\mathrm{Diff}_1(M)$. However for fibered manifolds this problem can be avoided. We take a product neighborhood V and a continuous map $\overline{F}: \underline{R}^m \to C_1^\infty(T)$ such that $\overline{F}_y: \underline{R}^m \to T_y$ is an isomorphism for $y \in V$: $\bar{F}(0) = 0$, $\bar{F}(\eta)(y) = 0$ for $\eta \in \underline{R}^1 \subset \underline{R}^m$ and $y \notin V$, and for all η and $y \notin p^{-1}(V)$ ($p: M \to B$ being the fibration). Now $F(\eta)$ will map $M - V$ into $M - V$ and we will have no trouble in seeing that (3) holds besides (1) and (2). Thus patching a finite number of s_i constructed from such V_i will give the required 2-parametrix. (Note that if each p_i disturbs filtration by 1 unit, so does $p_1 + s_1 p_2 + s_1 s_2 p_3 + \cdots = p$; since each s_i commutes with the filtration.) We record this as

<u>Proposition 26</u>. A fibered manifold possesses a 2-parametrix for d.

E. We remark that the hypotheses used in step D of the construction hold also if the foliation has diffeomorphic leaves, is regular (i.e., each pt. has a transverse nhbd. meeting a leaf only once), and arises from the orbits of a Lie group which acts freely on the manifold. For such a foliation we can construct maps $\bar{F}$ with the same properties as in D. So such foliations arising from the free action of a Lie group also have a 2-parametrix.

b. Transverse invariant forms form the complex

$$D^0 \xrightarrow{d} D^1 \xrightarrow{d} \cdots \to D^c. \qquad (32)$$

Reinhart [26] constructed a parametrix for $D \xrightarrow{d} D$ under

the condition that one has a bundle-like metrix. And thus (since D is a HLCTVS) is follows that the homology, i.e., $E_2^{*,0}$, is finite dimensional, under this condition. Since Schwarz [28] has shown that $E_2^{*,0}$ need not be finite dimensional (32) may fail to have a parametrix. In any case if we have maps $D \xrightarrow{s} D$ such that $1 - s = dp + pd$, then they induce maps s^+, p^+ in the complex (20) of prop. 15. These induced maps still obey the identity $1 - s^+ = d\, p^+ + p^+ d$. So if the spaces of (20) can be given a Hausdorff topology, then we will have a parametrix for this complex also.

c. Now we point out the simple connection between Hausdorffness and Serre's duality. We follow the argument given by Serre [20]. The basic lemma is that if V is a Frechet space and the differential $d: V \to V$ is a topological homomorphism then the homology $H_d(V)$ is also a Frechet space (in the induced topology) and its dual is $H_{d'}(V')$ where $V' \xrightarrow{d'} V'$ is the topological dual. This is Lemma 1 in [20]. Let us now take up the forms with distributional coefficients as in [20]. This complex (K,d) also carries a natural filtration and thus gives us another spectral sequence $F_r^{p,q}$ also converging to the deRham cohomology. One now notes that $E_0^{p,q}$ has the topological dual $F_0^{c-p,1-q}$ (same proof as Prop. 4 of [20]) and that the dual map d_0' corresponds to d_0 $F_0 \to F_0$ under this isomorphism (same proof as Prop. 5 of [20]). Using these remarks one gets the following result by using Lemma 1 again and again.

<u>Proposition 27</u>. If the spectral sequence morphisms $d_0, d_1, \ldots$ are topological homorphisms (with respect to the induced topologies) then Serre's duality $E_\infty^{p,q} \cong E_\infty^{c-p,1-q}$ holds (M is of course orientable).

This proposition implies that if E_1 is finite dimensional then Serre's duality holds. And thus <u>all odd dimensional foliations with dim $E_1 < \infty$ lie only on signature zero manifolds</u>.

d. Calculations made by Kodaira and given by Reinhart [26] show that <u>for almost all irrational flows on the torus dim $E_1 < \infty$</u> (in fact $E_1^{0,0} = E_1^{0,1} = E_1^{1,0} = E_1^{1,1} = \underline{R}$). But on the other hand <u>there exist irrational flows on the torus for which dim $E_1^{1,0} = E_1^{1,1} = \infty$</u>. We will recall their argument below. Note for the moment however that an irrational toral flow is <u>ergodic</u> (i.e., any measurable set made up of complete leaves is of measure 0 or 1) and <u>minimal</u> (i.e., every leaf is dense). So neither of these hypotheses suffice to ensure dim $E_1 < \infty$.

Think of the 2 torus T^2 as the (r,s) plane satisfying $0 \le r \le 1$, $0 \le s \le 1$ with proper boundary identifications. Let us be given a 1-foliation represented by straight lines making an angle θ with the r-axis, such that $\tan\theta = \lambda$ is an irrational number. Let us denote by x the direction along the leaves. Let us now try to calculate $E_1^{1,0}$. Obviously $Z_1^{1,0}$ can be supposed to be all smooth forms $\phi = \varphi dx$. Now if ϕ lay in $B_1^{1,0}$ then $\varphi = \frac{\partial f}{\partial x}$ where f is another smooth function. Thinking of both φ, f as bi-periodic functions in r,s we have their <u>Fourier series</u> $\varphi = \Sigma\, \varphi_{mn} \exp(2\pi imr + 2\pi ins)$, $f = \Sigma\, f_{mn} \exp(2\pi imr + 2\pi ins)$. Substituting in $\varphi = \frac{\partial f}{\partial x}$ we see that we must have $\varphi_{00} = 0$; and then $f_{mn} = \dfrac{\varphi_{mn}}{2\pi i \cos\theta\,(m+n\lambda)}$ for $(m,n) \neq (0,0)$. (The denominator $m+n\lambda \neq 0$ as λ is irrational). Conversely whenever $\varphi_{00} = 0$ one can build a fourier series for f. However this series does not converge for some value of λ. But if λ is <u>irrational enough</u>--i.e., $m+n\lambda \geq f(n)$, where $f(n) = O(n^s)$, $s > 2$--it does converge to a smooth function. By a theorem in diophantine approximation theory (due to Khintchine) this condition is satisfied for all λ outside a set of measure zero.

Prof. Gromov has suggested that this example may be extendable to nilflows on <u>nilmanifolds</u> (see Auslander *et al*., Annals Studies no. 53).

If the foliation has a dense leaf note that $\dim E_1^{p,0} < \infty$. This follows since $E_1^{p,0}$ consists of transverse invariant forms; and the value of such form over a leaf is determined by its value at one point thereof.

e. The following questions are interesting:

1. Is $\dim E_1 < \infty$ for all <u>Anosov</u> flows: (See a book on dynamical systems for definitions.)

2. Is $\dim E_3 < \infty$ for all compact foliated manifolds?

As yet, the author is unable to answer them.

<u>19</u>. In this section we shall cover the general relations between connection theory and our spectral sequence.

There are two main (kinds of) definitions of connection. The first may be called 'analytical' as it is convenient in differential geometry. Here a connection on a vector bundle W (on M) is a morphism $C^\infty(W) \xrightarrow{\nabla} C^\infty(W) \otimes C^\infty(T^*M)$ obeying certain rules. [See, e.g., Kobayashi and Nomizu.]

The second is the 'algebraical' definition as it is convenient for defining characteristic classes. Here the connection is defined as an algebra morphism of a finite dimensional algebra into the exterior algebra of forms over P, the principal bundle of W. (See, e.g., H. Cartan [6].)

The two definitions are due, mainly, to Koszul [19] and to Ehresmann [12], respectively. We shall look at connection theory from both these viewpoints. First, the 'analytical' aspect.

19A. Let W be a smooth vector bundle on M and denote by $\Lambda(W) = \Lambda^p(W)$ the vector space of smooth sections of $W \otimes \lambda T^*$. Let us try (in analogy with the exterior derivative $d: \Lambda \to \Lambda$) to build an endomorphism

$$\partial(fg\omega \wedge \sigma) = df \wedge g\omega \wedge \sigma + f\partial g \wedge \omega \wedge \sigma + fgd\omega \wedge \sigma + (-1)^p fg\omega \wedge d\sigma \qquad (33)$$

Here $f \in C^\infty(M)$, $g \in C^\infty(W) = \Lambda^0(W)$, $\omega \in \Lambda^p$, $\sigma \in \Lambda^q$ and the meaning of the various products is the natural one. Note that by (33) ∂ cannot be the zero map.

<u>Proposition 28</u>. One can find an ondomorphism $\partial: \Lambda(W) \to \Lambda(W)$ satisfying 33 [∂ is called a <u>connection</u> on W].

Proof: Locally $\Lambda(W)$ is generated by smooth functions, smooth sections of W and smooth forms by employing the various products. Hence, by (21), it suffices locally to just define $\partial : \Lambda^0(W) \to \Lambda^1(W)$, i.e., $C^\infty(W) \to C^\infty T^* \quad W)$. To do this we select, for a basis $s = s_1, s_2, \ldots$ of $C^\infty(W)$, a matrix ω of 1-forms and put

$$\partial s = \omega s \tag{34}$$

If in an overlapping locality a basis s' is chosen with $s' = gs$, we put $\quad \omega' = (dg + g\omega)g^{-1}$ (35)

Now we check that $\omega' s' = dgs + g\partial s$ which is $\partial(gs)$. Hence the definition extends globally. Actually if M is compact a finite number of such matrices ω (<u>called connection matrices</u>) are enough to define ∂. QED

Now we naturally ask if one can ask for $\partial^2 = 0$.

<u>Proposition 29</u>. One can find an endomorphism $\partial : \Lambda(W) \to \Lambda(W)$ satisfying (33) and $\partial^2 = 0$ if and only if the structure group of W can be reduced to a finite subset.

Proof: Let us first take $\partial^2 = 0$. Choosing as above a basis s for $C^\infty(W)$ in a local area this means $\partial^2 s = 0$, i.e., $\partial(\omega s) = 0$ (with above terminology), and hence by (33), $d\omega s + \partial s \wedge \omega = 0$, i.e., $d\omega s - \omega \wedge \omega s = 0$. Hence we get the equivalent condition

$$\Omega = d\omega - \omega \wedge \omega = 0 \tag{36}$$

The matrix Ω of 2-forms (depending on s) is called a <u>curvature matrix</u>. One can see that in a new basis

$s' = gs$ we shall have

$$\Omega' = g\Omega g^{-1} \tag{37}$$

and hence $\Omega = 0$ is a condition quite independent of the local basis s selected. By the Ambrose-Singer theorem (see Kobayashi and Nomizu, Chapter II, esp. p. 92) this implies that the local (or the restricted) holonomy group of W is zero. Locally trivializing W by horizontal sections we can arrange that the coordinate transformations are constant $w \times w$ matrices. Conversely let us suppose that we can cover W by a finite number of trivializations s $\left(\text{basis of } C^\infty(W)\right)$ such that the connecting matrices g are constant. Now take ω as the zero matrix in each of these. Since $dg = 0$ the required transformation law holds and we have a connection. It is clear that one has $\partial^2 = 0$. QED

Before proceeding with proving another prop. of the same kind we will indicate some additional results. Given a connection ∂ on W one attempts to compute the cohomology $H^*(M, W_\partial)$ where W_∂ denotes the sheaf of germs of horizontal sections (i.e., $\partial f = 0$) of W. In the particular case given by the above proposition this sheaf has a fine resolution by

$$0 \to W_\partial \to \mathcal{A}^0(W) \to \ldots \to \mathcal{A}^m(W) \to 0$$

and we have a generalised deRham theorem. This cohomology $H^*(M, W_\partial)$ is traditionally called 'with local coefficients W.' In the general case such a resolution is not readily available and one has to resort to methods of a different kind (see work of Nijenhuis, Spencer, etc.). It is clear that the theory of characteristic classes is intimately related to this cohomology.

Now we employ the bigrading of section 14 and denote by $\Lambda^{p,q}(W)$ the vector space formed by the smooth sections of $W \otimes \lambda^{p,q}T^*$. We denote by ∂_{01} the part of a connection which is of bidegree (0,1).

<u>Proposition 30</u>. One can find an endomorphism $\partial : \Lambda(W) \to \Lambda(W)$ satisfying (33) and $\partial_{01}^2 = 0$ if and only if W is an invariant bundle.

Proof: Suppose that $\partial_{01}^2 = 0$. Hence, with respect to a local basis s for $C^\infty(W)$, the curvature matrix Ω consists of 2 forms having filtration ≥ 1. Using the Ambrose Singer theorem we see that the local holonomy of each leaf is trivial. Now if we cover M by a finite number of trivializations s keeping each of them horizontal along leaves it follows that the connecting matrices g are constant along leaves. Conversely take some trivializations of W which are related by matrices g constant along leaves. Let ω

be the matrices of any connection. Now compare the parts of (35) of bidegree (1,0). We get $\omega_{10}' = (dg + g\omega_{10})g^{-1}$. These matrices ω_{10} give required connection. QED

An invariant bundle shall always, if not otherwise mentioned, carry such a connection. We shall call this a <u>Bott connection</u>. The above proof suggests that for each $\xi \in H^1(M, G_D)$, $G = GL(w)$, we also define the notion of a <u>ξ-Bott connection</u> on W as follows: Let us take any cocycle $(\mathcal{U}, g)$ representing . That is $\mathcal{U}$ is a covering of M by open sets U_i and g consists of sections g_{ij} over $U_i \cap U_j$ of the sheaf G_D which obey $g_{ij}g_{jk} = g_{ik}$ (see Hirzebruch's book for more details). Now on each U_i choose trivializations s_i for $C^\infty(W)$ so that $s_j = g_{ij}s_i$. Let ω_i be the connection matrices with respect to s_i. If they are of filtration ≥ 1 we say that we have a ξ-Bott connection. This is a valid definition for if some other trivializations s_i' are taken equation (35) shows ω_i' is also of filtration ≥ 1. Again if some other cocycle $(\mathcal{U}, g')$ is chosen we have $g_{ij}' = f_j^{-1}g_{ij}f_i$ where f_i is section over U_i of G_D. So we see that $s_j' = g_{ij}'s_i'$ where $s_i' = f_i^{-1}s_i$. By (35) the connection matrices with respect to s_i' will also be of filtration ≥ 1. The proof above now tells us that if the invariant bundle W is associated to $\xi \in H^1(M,G_D)$ then we have a ξ-Bott connection on W.

We shall denote by $E_1(W)$ the homology of $\Lambda(W)$ under ∂_{01}, if ∂ is a Bott connection. Also denote by $gl(w)$ the w^2-dimensional vector space of all endomorphisms of $\underline{R}^w$. The group $GL(w)$ acts on it by

$$g \cdot l = glg^{-1} \tag{38}$$

Using this action we construct a vector bundle W^2 with fibre dimension w^2 associated to W. It is clear that one can think of the <u>curvature</u> (of a connection $\partial : \Lambda(W) \to \Lambda(W)$) as an element of $\Lambda^2(W^2)$ given locally by the curvature matrices Ω encountered before. We denote this global form also by Ω. So $\Omega \in \Lambda^2(W^2)$. When W is invariant and ∂ is a ξ-Bott connection we get a part $\Omega_{1,1} \in \Lambda^{1,1}(W^2)$ which satisfies $\partial_{01}(\Omega_{1,1}) = 0$ [because $\Omega = d\omega - \omega \wedge \omega$ shows that locally, $\Omega_{1,1} = d_{01}\omega$]. Hence we get a class $[\Omega_{1,1}]$ $E_1^{1,1}(w^2)$.

We recall that in Atiyah [1] such a class was used to characterize those complex analytic bundles which can admit a complex analytic connection. Prop. 31 below gives an analagous result. For each $\xi \in H^1(M,G_D)$ a connection on W (a vector bundle associated to ξ) is called an <u>ξ-invariant connection</u> if its connection-matrices ω_i, wrt any trivializations

s_i agreeing with a cocycle $(\mathcal{U},g)$ of ξ, consist of transverse invariant forms. Note that the curvature of any such connection is of filtration ≥ 2. In other words $\Omega_{1,1} = 0$. Any connection obeying this condition will be called an <u>ξinvariant connection</u>. If the vector bundle W admits an ξ-invariant connection we shall say that it is <u>stiff</u>.

<u>Proposition 31</u>. A bundle w is stiff if and only if $[\Omega_{1,1}] \in E_1^{1,1}(w^2)$ vanishes.

Proof: The "only if" part is obvious. So only the converse needs a demonstration. W is now just an invariant bundle and we choose a Bott connection and then look at the homology class $[\Omega_{1,1}]$ lying in $E_1^{1,1}(W^2)$. We assume that it vanishes. Thus there exists a one-form $\underline{\theta} \in \Lambda^{1,0}(W^2)$ such that $\partial_{01}\underline{\theta} = \Omega_{1,1}$. Choosing local bases s which are horizontal over leaves one can think of $\underline{\theta}$ as being locally defined by matrices varying by $\theta' = g\theta g^{-1}$. So if ω denotes the connection matrices of the given Bott connection, $\omega - \theta$ is also a Bott connection. The relation $\partial_{01}\underline{\theta} = \Omega_{1,1}$ reads $d_{01}(\omega - \theta) = 0$, i.e., that these matrices consist of transverse and invariant 1-forms. This shows that W is stiff.

QED

We refer the reader to Deligne [10] for a similar viewpoint of connection theory. [He also considers connections as derivations of $\Lambda(W)$ lying above d.] Also the work of Molino [21] is closely related to the above, e.g., he has the notion of an invariant connection; though the reduction of the structure sheaf is not stressed.

19B. This section will be devoted to dualising the results of section 19A into statements about the Weil homomorphism. Also some essentially new features will be pointed out.

It is well-known--see, e.g., [17], pages 65-66--that the definition of connection in section 19A (by matrices ω obeying (35)) is equivalent to putting on P (the principal bundle of W) a smooth plane field transverse to the fibres, and of dimension m, which is preserved by the group action. With this in mind we now go over to the 'algebraical' treatment of connections.

Let G be any lie group. We'll first of all introduce the notion of a G-algebra. By this we mean a graded anticommutative algebra over $\underline{R}$ [or, more generally, over any commutative ring with unity] which is supplied with a differential d [i.e., a skew derivation of degree +1 and order 2 ($d^2 = 0$)] and for

each $X \in \underline{G}$--the lie algebra of G--is supplied with the endomorphisms i_X [which is a skew derivation of degree -1 and order two] and L_X [a derivation of degree zero] such that the following commutative rules hold

$$L_{[X,Y]} = [L_X, L_Y], \quad i_{[X,Y]} = [L_X, i_Y],$$

$$i_X d + d i_X = L_X \tag{39}$$

Note that these imply that $[L_X, d] = 0$.

Now we give an example of a real G-algebra. Let us take the algebra $W(G) = \Lambda\underline{G}^* \otimes S\underline{G}^*$ formed by tensoring the exterior and the symmetric algebras generated by $\underline{G}$. If we agree to give the grading 2p to polynomials of degree p it is anticommutative. We will use the notation $\Lambda(G)$ (resp. $S(G)$) for $\Lambda\underline{G}^*$ (resp. $S\underline{G}^*$) in the following. Now we define the 3 endomorphisms on the generating elements $\Lambda'(G)$ and $S'(G)$: this will define them everywhere as they are derivations or skew derivations. Note that any (skew) derivation will be zero on $W^0(G) = \underline{R}$. We now define i_X: for $\omega \in \Lambda^1(G)$, $i_X\omega \in \underline{R} = W^0(G)$ with $i_X\omega = \omega(X)$ while on $S^1(G)$ it vanishes. (40)

L_X: for $\omega \in \Lambda^1(G)$, $L_X\omega \in \Lambda^1(G)$ with $L_X\omega(Y) = \omega([X,Y])$. For $\varphi \in S^1(G)$, $L_X\varphi \in S^1(G)$ with $(L_X\varphi)(Y) = \varphi([X,Y])$ (41)

d: Let h denote the canonical isomorphism $\Lambda^1(G) \to S^1(G)$. Then for $\omega \in \Lambda'(G)$, $(d - h)\omega \in \Lambda^2(G)$ is defined by $\big((d - h)\omega\big)(X,Y) = \frac{1}{2}\omega([X,Y])$: for $\varphi \in S'(G)$

$$d\varphi \in \Lambda^1(G) \otimes S^1(G) \text{ with } i_X d\varphi = L_X\varphi \qquad (42)$$

Note that the last part of (42) means that if $X_1, X_2, \ldots$ is a basis for $\underline{G}$ then $d\varphi_i = \sum_j \varphi_j \otimes L_{X_j}\varphi_i$ for $\varphi_j \in S^1(G)$. We refer the reader to Cartan [6] for more details regarding these definitions. There it is also shown that the commutation rules (39) hold. This $W(G)$ is called the <u>Weil Algebra</u> of G. [Note that it is really $\underline{G}$ that is important: we can start off with any lie algebra and do the above construction.]

Now we will define a <u>connection</u> to be a G-algebra morphism from $W(G)$ to some other real G-algebra. Hereby a G-algebra morphism we mean that the entire G-algebra structure is preserved under the map.

To point out its relationship to the definition above we first of all see that the space of all smooth forms on a principal bundle P with group G is in fact a G-algebra: Denote this space by $\Lambda(P)$. Then $d:\Lambda(P) \to \Lambda(P)$ is the exterior derivative. On the other hand for each $X \in \underline{G}$ we get a canonical vector field along the fibres of P. By taking the interior

product with respect to this vector field (which shall also be denoted by X) we define $i_X:\Lambda(P) \to \Lambda(P)$. Finally Lie differentiation with respect to this vector field yields us the third endomorphism. All the equations of [39] are valid by standard results--see, e.g., [17].

Now take any G-algebra morphism $W(G) \xrightarrow{f} \Lambda(P)$. Being an algebra morphism it is determined uniquely by its values on $\Lambda^1(G)$ and $S^1(G)$. But by (42) for any $\varphi = h\omega \in S^1(G)$ we have $f\varphi = f(h\omega) = df\omega - f(d - h)\omega$. Hence it is determined simply by its restriction $\Lambda^1(G) \xrightarrow{f} \Lambda^1(P)$. This restriction commutes with L_X and with the maps $i_X:\Lambda^1(G) \to \underline{R}$. Thus if $\omega_1, \omega_2, \ldots$ is a basis of $\underline{G}^* = \Lambda^1(G)$ then $f(\omega_1), f(\omega_2), \ldots$ will give us smooth 1-forms on P which are equivariant under the right action of G [$\because$ f commutes with L_X] and also transverse [$\because$ f commutes with i_X]. Then $\ker f(\omega_1) \cap \ker f(\omega_2) \cap \ldots$ will be the required m-dimensional smooth plane field which is transverse to the fibres and which is preserved by the group action. Thus we have rejoined the standard definition of connection recalled above. One can of course retrace the arguments back and interpret any such plane field as a G-algebra morphism.

The restriction map $S^1(G) \xrightarrow{f} \Lambda^2(P)$ can be interpreted also as a $\underline{G}$ valued 2 form on P which is equivalent under right action of G. It is the <u>curvature</u> of the connection f. When we are thinking of P as foliated by the fibration we will write P_f to distinguish it from the normal case when P is foliated in codimension c by using the given foliation on M.

<u>Proposition 32</u>. The image of $S^1(G) \xrightarrow{f} \Lambda^2(P)$ lies in $\Lambda^2_2(P_f)$.

Proof: The proposition simply states that the curvature form is horizontal. QED

We shall say that a connection $W(G) \xrightarrow{f} \Lambda(P)$ is a <u>Bott Connection</u> if the image of its curvature map $S^1(G) \xrightarrow{f} \Lambda^2(P)$ lies in $\Lambda^2_1(P)$, i.e., if the curvature is of filtration ≥ 1 with respect to the codimension c foliation. By the remarks made above this agrees with the definition used in section 19A. On the other hand the principal bundle P shall be called <u>invariant</u> if we can cover M by trivializations of P in which the coordinate transformations $U_i \cap U_j \to G$ are constant on leaves. We shall also set up the <u>1-filtration</u> of $W(G)$ by saying that an element is of 1-filtration $\geq i$ if it lies in the subspace $W(G) \otimes S^{2i}(G)$, i.e., all those things which contain polynomials of degree $\geq 2i$.

When we are considering $W(G)$ with this filtration we shall write it as $(1)W(G)$ and $(1)W_i(G)$ shall denote things which have 1-filtration $\geq i$.

<u>Proposition 33</u>. The differential $d: W(G) \to W(G)$ preserves the (1)-filtration.

Proof: This follows by examining the definition of d. QED

The dual of prop. 30 is now the following.

<u>Proposition 34</u>. A principal G-bundle P is invariant if and only if we have a connection $f: (1)W(G) \to \Lambda(P)$ commuting with the filtrations (i.e., a Bott connection).

We now examine in more detail the relationship between the structure of the invariant bundle P and Bott connections.

The isomorphism classes of principal G-bundles over M form the set $H^1(X, G_S)$--see, e.g., Hirzebruch [15]. Choose now an element ξ of the cohomology set $H^1(M, G_D)$. Then if f is an ξ-Bott connection we emphasize its relationship to ξ by writing $f(\xi)$.

<u>Proposition 35</u>. All ξ-Bott connections $f(\xi): (1)W(G) \to \Lambda(P)$ lie in the same 1-chain homotopy class.

Proof: Let f_1, f_2 be two such Bott connections.

Choose a cocycle $(\underline{U}, U_i \cap U_j \xrightarrow{g_{ij}} G)$ of ξ in which both f_1 and f_2 can be represented by connection matrices of filtrations $\geq$. We now use the 1-foliation of $M \times I$ (see section 9), i.e., we foliate $M \times I$ in codimension c in the obvious way. We will have the principal bundle $P \times I$ sitting above $M \times I$ in the natural way. Now we can define a third connection. $F:(1)W(G) \to (1)\Lambda(P \times I)$ as follows: $\big(F(\varphi)(\theta_t^* X)\big) = \big(f_1(\varphi) + t\big(f_2(\varphi) - f_1(\varphi)\big)\big)(x)$ and $\big(F(\varphi)\big)(P_x^* \frac{\partial}{\partial t}) = 0$. Here X is a tangent vector to P, $\theta_t = P \to P \times I$ is $x \mapsto x,t$ and $\frac{\partial}{\partial t}$ is the standard vector on I. Further $\varphi \in \Lambda^1(G)$ and $P_x : I \to P \times I$ is $t \mapsto x,t$. Since it obviously commutes with ig, Lg, for all $g \in \underline{\mathfrak{g}}$, this definition $F:\Lambda^1(G) \to \Lambda^1(P \times I)$ extends to a G-algebra morphism. One sees that with respect to the cocycle above the connection matrices of F will also be of filtration ≥ 1. So it is in fact a Bott connection. We now define the chain homotopy $s:W(G) \to \Lambda(P)$ of degree -1 by the formula $s(\varphi) = \int_0^1 \theta_t^* i_{\frac{\partial}{\partial t}} F(\varphi)dt$ for all $\varphi \in W(G)$. Just as in section 10 we compute $(ds + sd)(\varphi) = \int_0^1 \theta_t^* d i_{\frac{\partial}{\partial t}} F(\varphi)dt + \int_0^1 \theta_t^* i_{\frac{\partial}{\partial t}} dS(\varphi)dt$ since $d\theta_t^* = \theta_t^* d$ ($\because$ d commutes with

induced maps) and $F(d\varphi) = d(F\varphi)$ ($\because$ F commutes with d). Hence the right hand side equals $\int_0^1 \theta_t^* L_{\frac{\partial}{\partial t}} F(\varphi)dt$.

For $\varphi \in \Lambda^1(G)$ it is clear from the definition of $F(\varphi)$ that it equals $f_1(\varphi) - f_2(\varphi)$. Again for $\varphi \in S^1(G)$ the curvature map $F:S^1(G) \to \Lambda^2(P \times I)$ is given by $F(\varphi) = f_1(\varphi) + t\big(f_2(\varphi) - f_1(\varphi)\big) + \overline{f_2(\varphi) - f_1(\varphi)} \wedge dt$ where the bar denotes horizontalization ([17], p. 76-77). (We use the projection $P \times I \to P$ and think of $f_1(\varphi)$, $f_2(\varphi)$ as forms on $P \times I$.) So $L_{\frac{\partial}{\partial t}}$ of this is $f_2(\varphi) - f_1(\varphi)$ + forms containing dt. Hence $\theta_t^* L_{\frac{\partial}{\partial t}}$ is just $f_2(\varphi) - f_1(\varphi)$. Thus the relation $ds + sd = f_2 - f_1$ holds. Finally from the definition of s it is clear that it preserves filbration. So s is the required 1-chain homotopy, in the terminology of section 9. QED

An immediate consequence of this result is that for $r \geq 1$ the induced spectral sequence maps $f(\xi):(1)E_r(G) \to E_r(P)$ do not depend on the choice of the Bott connection f. We can record this as

<u>Corollary 36</u>. Each invariant structure $\xi \in H^1(M, G_D)$ gives us a well-defined map

$$\bar{\xi} : E_r(G) \to E_r(P) \text{ for } r \geq 1 \tag{43}$$

Now the dual of Proposition 31 of section 19A is the following

Proposition 37. A bundle $\xi \in H^1(M,G_D)$ is stiff if and only if the map $\bar{\xi}: E_1^{1,1}(G) \to E_1^{1,1}(P)$ vanishes.

Proof: This map becomes simply $[\Omega_{1,1}]$ if we interpret it in the definitions of section 19A. QED

In the present terminology a connection $f: W(G) \to \Lambda(P)$ is called an invariant connection if the image of its curvature map $f: S^1(G) \to \Lambda^2(P)$ lies in $\Lambda_2^2(P)$. (Note--by Prop. 32--that any connection is invariant with respect to the foliation of P arising from the fibration.) We now define the 2-filtration of W(G) by setting $(2)W_{2i-1}(G) = (2)W_{2i}(G) = (1)W_i(G)$. It follows from Proposition 33 that the differential d of W(G) also preserves this filtration. The following proposition is obvious from the preceding developments.

Proposition 38. A principal G-bundle P is stiff if and only if there is a connection $f: (2)W(G) \to \Lambda(P)$ commuting with the filtration (i.e., an Invariant connection).

Further a detailed examination analogous to the above is possible. Take an element $\theta \in H^1(M,G_D)$, and let f be an θ-invariant connection. To emphasize

this relationship we use the notation $f(\theta)$.

Proposition 39. All θ-invariant connections $f(\theta): (2)W(G) \to \Lambda(P)$ lie in the same 2-chain homotopy class.

Proof: We proceed exactly as in the proof of Prop. 35 except that we now employ the 2-foliations of $M \times I$ (see section 9). With this change F will also be an invariant connection. Now the chain homotopy S will disturb filtration by one unit. QED

This result generalizes a well known theorem of Weil. First, we see that the induced spectral sequence maps $f(\theta): (2)E_r(G) \to E_r(P)$, for $r \geq 2$, do not depend on the choice of the invariant connection $f(\theta)$. We thus have the following.

Corollary 40. Each stiffness structure $\theta \in H^1(M,G_D)$ gives us a well-defined map

$$\bar{\theta}: (2)E_r(G) \to E_r(P), \text{ for } r \geq 2. \tag{44}$$

To get the classical case we assume that we have the point foliation on M. One can see easily from the definition of the Weil algebra that $(2)E_2^{*,0}(G)$ are simply the symmetric invariant polynomials of G--see [6]. Also it is clear that $E_r^{*,0}(P) = E_r^{*,0}(M)$; and so, for a point foliation $E_r^{*,0}(P_f) = H^r(M)$. Thus in this case $\bar{\theta}: (2)E_2^{*,0} \to H^*(M)$

is simply the Chern Weil homomorphism. Note that for a point foliation $G_D = G_S$ and so $\vartheta \in H^1(M, G_S)$ is any differentiable structure for P over M.

Kamber-Tondeur [16] also interpret Weil morphisms as spectral sequence morphisms, but they do not give the homotopy invariance results (Propositions 35 and 39).

We shall end this section by pointing out that one can employ the Chern-Simons modification in the above discussion. If $f = W(G) \to A(P)$ is a Bott connection we note that $f\left(W_{c+1}(G)\right) = 0$; thus we can "throw away" terms involving polynomials of degree $> 2c$. More precisely we replace $W(G)$--whenever we are dealing with Bott connections--by the quotient $W(G)/W_{c+1}(G)$. The rest of the treatment is precisely as above. The advantage of this is that though $E_\infty(G) = 0$ without truncation (see [6] where it is shown that $H\left(W(G)\right) = 0$) $E_\infty(G) \neq 0$ with this mofication. In fact one can compute $E_\infty(G)$--with modification--to be equal to the Gelfand-Fuks cohomology of the formal vector fields on $\underline{R}^c$ [See, e.g., Theorem 2.1 in Guillemin's paper in Advances in Mathematics, 1973]. In fact Godbillon, Vey, Bott and Haefliger have approached the problem of understanding the "exotic"

characteristic classes from the viewpoint of this Gelfand Fuks cohomology.

20. Now the most important example of an invariant bundle is the bundle $D^\perp \subset T^*$ of covectors which kill the involutive distribution $D \subset T$. Any connection on a subbundle of T or T* can be extended to the whole bundle; it is such linear connections--i.e., connections defined on the principal bundle $L(M)$ of tangent frames--that shall concern us now. We shall not assume that D is necessarily involutive. Any connection on T* reducible to $D^\perp$, such that the cxc part of the curvature is of filtration ≥ 1 w.r.t. D shall be called a Bott Connection. Here if Ω^i_j is the curvature its $c \times c$ part is given by $i, j > c$.

Proposition 41. Any connection on T* which is reducible to $D^\perp$ and which has zero torsion must be a Bott connection; and then D is involutive.

Proof: Without recalling the definition of torsion we simply recall one of the consequences of 'zero-torsion,' viz., the equation

$$d\theta = A(\nabla\theta) \tag{45}$$

[See Cor. 8.6 in Kobayashi and Nomizu, Ch. III.] . Here A denotes alternation and ∇ denotes covariant differential. Now use this equation with ω a section of $D^\perp$ and $X \in C^\infty(D)$, $Y \in C^\infty(T)$:

$$d\omega(X,Y) = \tfrac{1}{2}[(\nabla\omega)(X,Y) - (\nabla\omega)(Y,X)]$$

$$= \tfrac{1}{2}[(\nabla_Y\omega)(X) - (\nabla_X\omega)(Y)]$$

Since our connection is reducible to $D^\perp$ both $\nabla_Y\omega$ and $\nabla_X\omega$ are sections of $D^\perp$. Thus $(\nabla_Y\omega)(X) = 0$. Thus

$$(\nabla_X\omega)(Y) = -2d\omega(X,Y) \tag{46}$$

whenever $\omega \in C^\infty(D^\perp)$, $Y \in C^\infty(T)$. But eqn. (46) is precisely the condition by which Bott [3] first defined a connection whose curvature has filtration ≥ 1; provided D was involutive. But if we take $Y \in C^\infty(D)$ in (46) we get $d\omega(X,Y) = 0$, i.e., $\omega([X,Y]) = 0$ (as $\omega(x) = \omega(Y) = 0$) for all $\omega \in C^\infty(D^\perp)$. So $[X,Y] \in D$ proving the required involutivity. QED

The proof exhibits clearly the following

Corollary 42. T^* admits a connection reducible to $D^\perp$ and obeying (46) if and only if D is involutive.

We shall say that a connection on T^* (resp. T) of zero torsion which is reducible to $D^\perp$ (resp. D) is a Walker connection; for the reason for this name, see [37]. Then we can extend the above corollary to the following

Proposition 43. A Walker connection exists if and only if D is involutive

Proof: Clearly it suffices to prove that a Walker

connection exists on T when D is involutive (T, T^* are associated to the sampe principal bundle: the induced connection on T^* will supply a Walker connection there). Denote by Γ the fiber-bundle with fiber $\mathbb{R}^{n^3}$, structure group Diff ($\mathbb{R}^{n^3}$), and coordinate transformations

$$\bar{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{i}_{jk} \frac{\partial x^j}{\partial \bar{x}^\beta} \frac{\partial x^k}{\partial \bar{x}^\gamma} \frac{\partial \bar{x}^\alpha}{\partial x^i} + \frac{\partial^2 x^i}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} \frac{\partial \bar{x}^\alpha}{\partial x^i} \tag{47}$$

as we go from local coordination x_i to $\bar{x}_\alpha$. A linear connection is a section of Γ. One has the relation

$$\nabla_{\frac{\partial}{\partial x_j}} \left(\frac{\partial}{\partial x_i}\right) = \Gamma^k_{ji} \frac{\partial}{\partial x_k} \tag{48}$$

with the first definition. [See Kobayashi and Nomizu, [17], Ch. III, section 7 for more details.] Since we want our connection to be reducible to D for $1 \leq i \leq l$ the RHS should not contain terms with $k > l$. So

$$\Gamma^k_{ji} = 0 \text{ for } k > l,\ i \leq l \tag{49}$$

while the condition for zero torsion is [op. cit.],

$$\Gamma^k_{ji} = \Gamma^k_{ij} \tag{50}$$

Both these conditions are compatible with (47). Thus we get a subbundle W of Γ whose fiber is $\mathbb{R}^{cl(c+1)/2}$. Choose any section of the same. QED

This proof is due to Willmore [38]. Using the equations (49) and (50) above we can given another proof of the fact that a Walker connection must be a Bott connection (Prop. 41). In fact if one puts [17],

$$R^i_{jks} = \frac{\partial \Gamma^i_{jk}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^s} + \Gamma^t_{sj}\Gamma^i_{kt} - \Gamma^t_{kj}\Gamma^i_{st}, \tag{51}$$

then the curvature matrix is

$$\Omega^i_j = R^i_{jks} dx^k \wedge dx^s \tag{52}$$

in the given coordinates $x_1, \ldots, x_m$ compatible with the foliation. Using (49), $R^i_{jks} = 0$ for $i > 1$, $j < 1$; i.e., $\Omega^i_j = 0$ for these values. Using (49) and (50) one sees that $R^i_{jks} = 0$ for $i > 1$, $j > 1$, and $k, s \leq 1$; i.e., for $i, j > 1$ the 2-forms Ω^i_j are of filtration ≥ 1. QED

We call a connection which satisfies the hypothesis of Cor. 42 a basic connection. For a linear connection

Walker $\Rightarrow$ basic $\Rightarrow$ Bott.

So far we have looked for cohomological obsturction by using Bott connection only. By proposition 41 it seems reasonable to see if a Walker connection leads to anything new.

For this purpose we reformulate prop. 43. Given a subbundle $D \subset T$ let $P \subset L(M)$ be the principal bundle of frames whose first 1 entries span D. the group of this bundle is denoted G (all automorphisms

$\underline{R}^m \to \underline{R}^m$ keeping $\underline{R}^1$ invariant). The bundle of affine framers A(M) is a m-dimensional vector budnle over L(M). Let P denote the part sitting over P, and $\tilde{G}$ the group of $\tilde{P}$. It is clear that $\mathfrak{a}'(\tilde{G}) = \mathfrak{a}'(G) \oplus \underline{R}^m$ with $\mathfrak{a}'(G)$ a Lie subalgebra of $\mathfrak{a}'(\tilde{G})$. Let ω be the connection form and denote by $\theta : \underline{R}^m \to \Lambda^1(P)$ the canonical form, i.e., for a tangent vector X at $(x; e_1, \ldots, e_m)$ of P where $x \in M$ and $e_1, \ldots, e_m$ is a basis of T_x, the vector $\theta_1(X)e_1 + \cdots + \theta_m(X)e_m$ lies under X. So ω and θ gives us a map $A^1(G) \xrightarrow{(\omega,\theta)} \Lambda^1(P)$ and $\Omega (= \bar{\omega})$ and 0 give us a map $S^2(G) \xrightarrow{\Omega} \Lambda^2(P)$. Thus we get an algebra morphism. $W(G) \xrightarrow{(\omega,\theta,\Omega)} \Lambda(P)$. Again we have the connection $A(G) \xrightarrow{(\omega,\theta)} \Lambda(P)$ and the related Weil map.'

<u>Proposition 44</u>. D is involutive if and only if the algebra morphism $W(\tilde{G}) \xrightarrow{(\omega,\theta,\Omega)} \Lambda(P)$ can be lifted to the Weil map $W(\tilde{G}) \xrightarrow{\overline{(\omega,\theta)}} \Lambda(\tilde{P})$ for some ω.

Proof: By prop. 3.4, Kobayashi and Nomizu, section 3, Ch. III [17] the curvature Ω of (ω, θ) sits over $\Omega + \Theta$ where Θ is the torsion. Now we use prop. 41 QED

Suppose more generally that on L(M) we have a torsionless connection which is reducible to a subbundle with fiber G, a subgroup of GL(m). Then we say that we have a <u>torsionless G structure on M</u>. The study of such structures covers most of geometry. (By prop. 43 a foliation is such a structure.) The problem of finding necessary conditions for a torsionless G-structure to exist is of great interest. (The Bott vanishing theorem should be seen in this context.)

According to a result of Cartan, Kobayashi etc. [18] if G is one of the groups

$$GL(m),\ O(m),\ CO(m);\ GL(1,m),\ GL(1,m,k) \tag{53}$$

we always have such torsionless connections (affine, riemannian, conformal structures; flows, parametrized flows respectively). Such structures shall be called trivial since there is no obstruction. Conversely it is known also that if all G structures on M can be made torsionless, then, for $m > 2$, G must lie in the above list (53). In fact for the group $O(m)$ of automorphisms of $\underline{R}^m$ preserving a non-degenerate scalar product we know more: every $O(m)$-structure can be made torsionless by a unique connection and conversely if every G structure on M admits such a unique zero torsion connection, then $G = O(n)$.

Another famous torsionless structures occurs when G is composed of matrices of the type $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. This is called a complex structure on M. We suppose m is even; this subgroup of $GL(m)$ is called $GL(\frac{m}{2},\underline{C})$. Again, if out of these matrices we take those which are orthogonal we get a still smaller group $U(\frac{m}{2})$. A torsionless structure with this group is called a Kaehler structure. It is known that then we have a closed 2-form which represents an integral cohomology class if and only if the manifold is algebraic (Kodaira's theorem). These examples thus show the great importance of torsionless G-structures.

By an integrable G-structure we mean that we can cover M by charts so that the jacobians lie in G. The Frobenius theorem thus says that a torsionless $GL(1,m)$ structure is an integrable $GL(1,m)$ structure; while the Newlander-Nirenberg theorem makes the same statement with the subgroup $GL(\frac{m}{2},\underline{C})$. But a torsionless $O(m)$-structure is of course not integrable: we need the vanishing of another tensor, the curvature tensor. The general problem of finding necessary and sufficient conditions for the integrability of a G-structure (in terms of vanishing of certain tensors) has been pursued by Spencer, Guillemin [14] and others.

21. In this section we will show how the existence of certain torsionless G-structures enables us to construct certain 2-parametrices.

a. As before, let M be foliated. So we can assume that we have a torsionless $GL(l,m)$ structure on M. Now the Lie algebra $\mathfrak{g}L(l,m)$ of this group consists of homomorphisms $\underline{R}^m \to \underline{R}^m$ which preserve $\underline{R}^l$. (We think of $\underline{R}^m = \underline{R}^l \oplus \underline{R}^c$ as usual.) We now define a smaller Lie algebra $\mathfrak{g}(l,m)$ consisting of homorphisms $\underline{R}^m \to \underline{R}^m$ whose image lies in $\underline{R}^l$. We assume that we have a torsionless G-structure on M where G is a lie subgroup of $GL(m)$ whose lie algebra lies in $\mathfrak{g}(l,m)$. (Example: If there exist c globally defined vector fields $\bar{x}_1, \ldots \bar{x}_c$ transverse to the foliation such that we can cover M by neighborhoods $x_1, \ldots, x_l, x_{l+1}, \ldots, x_m$ such that $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_l}$ span D, $\frac{\partial}{\partial x_{l+1}} = \bar{x}_1, \ldots, \frac{\partial}{\partial x_m} = \bar{x}_c$.) Let Q be the principal bundle in question (it is a sub-bundle of the bundle P of frames compatible with the foliation); then we shall foliate Q by pulling back the foliation on M.

Points of Q are of the type $(x; e_1, \ldots, e_m)$ where $x \in M$ and $e_1, \ldots, e_m$ is a tangent frame for T_x. For each $\eta \in \underline{R}^m$ we now define a vector field η^* on Q in the following way: η^* at $(x; e_1, \ldots, e_m)$ is the horizontal vector which lies above $\eta_1 e_1 + \cdots + \eta_m e_m$. (In the terminology of Kobayashi and Nomizu, η^* is a canonical horizontal vector field.) Let us suppose that our connection is <u>complete</u>, i.e., that the vector field η^* generates a one parameter group $F(\eta t)$ of diffeomorphisms of Q. $\big(t \in \underline{R},\ F(\eta t_1) \cdot F(\eta t_2) = F(\eta t_1 + \eta t_2)\big)$. For $t = 1$ this gives the diffeomorphism $F(\eta)$ of Q.

<u>Proposition 45</u>. For each $\eta \in \underline{R}^m$ the diffeomorphism $F(\eta): Q \to Q$ maps leaves into leaves.

Proof: For each $A \in \mathfrak{g}(l,m)$ we get a canonical vertical vector field A^* in Q (see e.g., [17]). And if $A_1, \ldots, A_h$ is a basis $A_1^* \ldots A_h^*$ shall be a basis for the tgt. space of fibres. By Prop. 2·3, Ch. III of [17] we have the relation $[A^*, \eta^*] = (A\eta)^*$; and since torsion is zero we see from prop. 5·4, Ch. III of [17] that $[\eta_1^*, \eta_2^*]$ is always vertical. Since $[A^*, \eta^*] = \lim_{t \to 0} \frac{F(\eta t) A^* - A^*}{t}$ and $A\eta \in \underline{R}^l$ the first equation tells us that $F(\eta t)$ maps A^* into a vector tgt. to a leaf of Q. The second says that $F(\eta_2 t)$ maps η_1^* (for $\eta_1 \in \underline{R}^l$) into a vector tgt to a leaf of Q. Thus the proposition follows. QED

Now on Q we have a complete global parallelism (in the sense of p. 66) given by

$$\bar{F}: \underline{R}^m \oplus \mathfrak{g}(l,m) \to C^\infty(TQ) \tag{54}$$

where $\bar{F}(\eta) = \eta^*$ for $\eta \in \underline{R}^m$ and $\bar{F}(A) = A^*$ for

$A \in \mathfrak{g}(l,m)$. By the above proposition we have seen that $\overline{F}(\eta) \in C_l^\infty(TQ)$ if $\eta \in \underline{R}^m$. On the other hand if $\eta \in \underline{R}^l$, $[A^*, \eta^*] = (A\eta)^*$ with $A\eta \in \underline{R}^l$; and if $A_1 \in \mathfrak{g}(l,m)$, $A_2 \in \mathfrak{g}(l,m)$ we have $[A_1^*, A_2^*] = [A_1, A_2]^*$ with $[A_1, A_2] \in \mathfrak{g}(l,m)$. Thus the vertical vector fields A^* also preserve the foliation of Q, so the image of (54) lies in $C_l^\infty(TQ)$. Arguing as on pp. 63-pp. 68 we obtain

<u>Proposition 46</u>. Suppose that a compact manifold M has a torsionless G-structure such that the lie algebra $\mathfrak{g} \subset \mathfrak{g}(l,m)$. Let Q denote the principal bundle, and let the connection be complete. Then--with respect to the induced foliation on Q--we can find a 2-parametrix $A_Q \xrightarrow{p} A_Q$ by putting

$$p\omega = \int_{\underline{R}^m \oplus \mathfrak{g}(l,m)} \int_1^0 \left(i_{\frac{\partial}{\partial t}} F(\theta t)^* \omega \right) f(\theta)\, d\theta\, dt \qquad (55)$$

Here $F(\theta) = F_1(\theta)$ if $F_t(\theta)$ is the one-parameter group of $\overline{F}(\theta)$ in (64). Thus we have $1 - s = dp + pd$ where the smoothing operator $A_Q \xrightarrow{s} A_Q$ is given by

$$s\omega = \int_{\underline{R}^m \oplus \mathfrak{g}(l,m)} F(\theta)^* \omega \cdot f(\theta)\, d\theta \qquad (56)$$

Here $f(\theta)$ is a smooth function with sufficiently small support near the zero of $\underline{R}^m \oplus \mathfrak{g}(l,m)$.

We have already pointed out in propositions 21 and 27 the relation between the existence of a 2-parametrix and finiteness and Serre duality. Note that although Q is not compact, its cohomology can be calculated using forms having a compact support; and the map s will be compact on the space of such forms. Another remark to be made is that $E_2^{*,0}(Q) \cong E_2^{*,0}(M)$. Hence if the foliation arises from a torsionless G-structure with $\mathfrak{g} \subset \mathfrak{g}(l,m)$ then $E_2^{*,0}(M)$ is finite dimensional.

b. We now consider the more general case of a torsionless GL(l,m) structure (i.e., a foliation), with the bundle P of frames. Note that $\mathfrak{g}L(l,m)$ decomposes as $\mathfrak{g}(l,m) \oplus \mathfrak{g}L(c)$ where both parts are Lie algebras and the first part is preserved by brackets with respect to the second. Choose any basis $A_1, \dots, A_{lm}$; $B_1, \dots, B_{c^2}$ of L(l,m) agreeing with this decomposition. Also choose a basis $\eta_1, \dots, \eta_l$; $\xi_1, \dots, \xi_c$ of $\underline{R}^m$ agreeing with the decomposition $\underline{R}^m = \underline{R}^l \oplus \underline{R}^c$. We define $\overline{D}$ to be the $l + lm$ dimensional plane field spanned by $A_1^*, \dots, A_{lm}^*$; $\eta_1^*, \dots, \eta_l^*$. Then the following proposition gives us a $l + lm$ dimensional foliation of P sitting over the foliation of M:

<u>Proposition 47</u>. The plane field $\overline{D}$ is

involutive.

Proof: We know that $[A_i^*, A_j^*] = [A_i, A_j]^*$ and that $[A^*, \eta^*] = (A\eta)^*$. So it only remains to show that if $\eta_1, \eta_2 \in \underline{R}^1$ then $[\eta_1^*, \eta_2^*]$ is a linear combination of the η_i^* and A_i^*. Since torsion is zero, by 5.4, III, [17], this is a vertical vector. Hence it is enough to show that $\omega([\eta_1^*, \eta_2^*])$ lies in $\mathfrak{g}(1,m)$. Here ω is the connection form, and obeys $\omega(A^*) = A$. But η_1^*, η_2^* being horizontal vectors we see from 5.3, II, [17], that

$$\omega([\eta_1^*, \eta_2^*]) = -2\Omega(\eta_1^*, \eta_2^*) \tag{57}$$

where Ω is the curvature form. But the $c \times c$ part of Ω is of filtration ≥ 1 by Prop. 41. Hence the right side lies in $\mathfrak{g}(1,m)$. QED

Now the question arises whether $F(\eta): P \to P$ preserves this foliation of P. In general it cannot. But if the normal bundle $D^\perp$ is stiff (see section 19), then this is so. Then one can assume that our torsionless connection is invariant restricted to $D^\perp$:

<u>Proposition 48</u>. $F(\eta): P \to P$ maps leaves into leaves for each $\eta \in \underline{R}^m$, if the connection is invariant. Also the same statement is true for the diffeomorphisms $F(A): P \to P$ for any $A \in \mathfrak{gL}(1,m)$. ($F_t(A)$ being the 1-parameter group of A^*.)

Proof: In this case the $c \times c$ part of the curvature is

of filtration ≥ 2. So the RHS of (57) lies in $\mathfrak{g}(1,m)$ if $\eta_1 \in \underline{R}^1$ and $\eta_2 \in \underline{R}^m$. Hence $[\eta_1^*, \eta_2^*]$ is equal to A^* for some $A \in \mathfrak{g}(1,m)$. Again if $A \in \mathfrak{g}(1,m)$ and $\eta \in \underline{R}^m$, $[A^*, \eta^*] = [A\eta]^*$ with $A\eta \in \underline{R}^1$. These 2 remarks give the first part. On the other hand if $A \in \mathfrak{gL}(1,m)$ and $\eta \in \underline{R}^1$ then $[\eta^*, A^*]$ lies in $\overline{D}$ being equal to $-(A\eta)^*$. And if $A_1 \in \mathfrak{gL}(1,m)$, $A_2 \in \mathfrak{g}(1,m)$ then $[A_1, A_2] \in \mathfrak{g}(1,m)$--or $\mathfrak{g}(1,m)$ is an ideal in $\mathfrak{gL}(1,m)$. This shows that $[A_1^*, A_2^*] \in \overline{D}$. And thus we have the second part. QED

Note that foliations which can be supplied with a bundle-like metric are a fortiori invariant, but the converse is not true.

<u>Proposition 49</u>. In case the foliation is invariant we can find a smoothing map $\Lambda_P \xrightarrow{s} \Lambda_P$ which preserves the foliation (tgt. to $\overline{D}$) together with a paraemtrix $\Lambda_P \xrightarrow{p} \Lambda_P$ which disturbs the filtration by one unit. These are related as usual by $1 - s = dp + pd$.

Proof: Use the formulae (55) and (56) with $\underline{R}^m \oplus \mathfrak{g}(1,m)$ replaced by $\underline{R}^m \oplus \mathfrak{gL}(1,m)$ together with the discussion of pp. 63-68. QED

Again this 2-parametrix will have the familiar consequences regarding the finiteness of the E_2 terms of the foliated manifold P.

Let us notice that prop. 48 implies that all the leaves in P are diffeomorphic to each other. Instead of P we could work in the bundle P' of frames mod D (i.e., the principal bundle of T/D), where this foliation would collapse to the horizontal 1-dimensional foliation; for each $\eta \in \underline{R}^c$ the canonical vector field η^* in P' will preserve this foliation. This yields a theorem of Reinhart [27], Molino [21]: <u>any foliation which can be provided with a complete invariant connection can be covered by another foliation of the same dimension with diffeomorphic leaves</u>.

c. Let P be a principal bundle with graph G sitting over M. Now G acts freely on P from the right, and so for each $a \in G$ we have a diffeomorphism. $R_a : P \to P$. We denote by Λ_P the vector space of smooth forms on P, and by Λ_G the subspace of <u>right invariant</u> forms, i.e., forms ω such that $R_a^*\omega = \omega$ for all $a \in G$. Clearly if $R_a^*\omega = \omega$ then $R_a^* d\omega = \omega$, so Λ_G is a subcomplex of Λ_P.

Now let us equip G with a left invariant normalized Haar measure $\mu(g)$. Then we define a linear map $\Lambda_P \xrightarrow{Av} \Lambda_G$ by

$$Av\omega = \int_G R_g^* \omega\mu(g) \tag{58}$$

(We will assume for the time being that G is compact--which is a severe restriction. Later on these definitions will be amended for more general cases.)

Note that $R_a^*(Av\omega) = \int_G R_a^* R_g^* \omega\mu(g)$ $= \int_G R_{ag}^* \omega\mu(g) = \int_G R_{ag}^*(\omega)\mu(ag)$, as the measure is left-invariant, and so equals $Av\omega$: thus $Av\omega$ is right-invariant as stated above. Also from (58) it is clear that

$$d(Av\omega) = Av(d\omega) \tag{59}$$

It is clear also that Av is a continuous map.

Suppose given a parametrix for Λ_P, i.e., 2 maps $\Lambda_P \underset{p}{\overset{s}{\rightrightarrows}} \Lambda_P$ such that s is smoothing and $1 - s = dp + pd$. Then we can define 2 maps $\Lambda_G \underset{p'}{\overset{s'}{\rightrightarrows}} \Lambda_G$ by $s' = Av\, p$. By virtue of (58) it is clear that we will still have $1 - s' = dp' + p'd$, and also s' will be a smoothing map. In other words by composing with Av we can turn a parametrix for Λ_P into a parametrix for Λ_G.

(When G is not compact we take a Haar measure on G and replace (58) by

$$Av\omega = \lim_{i \to \infty} \frac{1}{|G_i|} \int_{G_i} R_g^* \omega\,\mu(g) \tag{58'}$$

where G_i is a finite measured subset of G which $\to$ G as $i \to \infty$. Clearly the limit will be finite if we work only on bounded forms in P, i.e., we have a map $Av: \Lambda_P^{bdd} \to \Lambda_G$. The equation (59) will also hold. By composing with Av we'll be able to change any parametrix on Λ_P^{bdd} to one on Λ_G.)

We now return to the case which was being treated above: M is foliated, P is the principal GL(1,m) bundle of compatible tangent frames and is also foliated by the foliation of prop. 47. Let us assume that we can find a parametrix (s,p) of P such that s preserves the filtration while p destroys it by one unit (we accomplished this when M carried a complete invariant connection). Then the following proposition will allow us the same kind of parametrix on Λ_G.

Proposition 50. The map $\Lambda_P \xrightarrow{Av} \Lambda_G$ preserves the filtration given by the foliation of prop. 47.

Proof: It will suffice to check that $R_g: P \to P$ preserves the plane field $\overline{D}$, for each $g \in G$. For $\eta \in \underline{R}^1$, $R_g(\eta^*) = (g^{-1}\eta)^*$ (2.2, III, [17]). Since $g^{-1} \in GL(1,m)$, $g^{-1}\eta$ will lie in $\underline{R}^1$ also. On the other hand if $A \in \mathfrak{g}(1,m)$, then $A^* \in \overline{D}$. Let ω be the connection form on P. We have $\omega(R_g A^*) = ad(g^{-1})\omega(A^*) = ad(g^{-1})A$. (By 1.1, II, [17]). The last term lies

in $\mathfrak{g}(1,m)$ as $\mathfrak{g}(1,m)$ is an ideal in $\mathfrak{gL}(1,m)$. Hence the vertical vector $R_g A^*$ lies in $\overline{D}$. QED

Let us denote the spectral sequence resulting from Λ_G by $E_r(P_G)$. The existence of a 2-parametrix for Λ_G gives us finiteness information about $E_2(P_G)$, e.g., that $E_2^{*,0}(P_G)$ is finite dimensional.

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