CHAPTER 6

The Topological Work of Henri Poincaré

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Introduction

Topology, as we know it today, started with Poincaré's "*Analysis Situs*" [59] and its five *Compléments* [61, 62, 66, 67, 69]. My objective is to describe the contents of these classics, together with some remarks relating them to further developments.

So we will not go into the details of Poincaré's life (1854–1912) and the various honours bestowed on him. However, a mathematician will enjoy Appell's anecdotes [3] about their year together in high school in Nancy, e.g., the way in which Poincaré would draw figures on a wall with his finger to explain his reasoning to his fellow students, or how he instantly gave very creative solutions to geometry problems that were posed to him (Appell gives a number of such examples). Also very informative is Darboux's eulogy [15] of 1913.

As we shall see, Poincaré covered a lot of ground in the papers mentioned above. In the light of this, it seems almost incredible that this was really *only a small part of the huge canvas on which he was working during this time*. In a series of long papers starting from 1880 he had created the *qualitative theory of ordinary differential equations*. Then, impelled with the desire to solve linear differential equations having algebraic coefficients, he had created and developed yet another theory, that of *Fuchsian and Kleinian groups*. Hard on the heels of this had come his prize-winning 1890 paper [55] on the 3-body problem, which was now being elaborated further in his three volume *treatise on celestial mechanics* [57]. Add to this dozens of courses delivered in almost every imaginable area of *theoretical physics*, and we are left gasping at the very idea that he had any time left to create and develop yet another very original mathematical theory!

Darboux tells us that Poincaré's "answers came with the rapidity of an arrow" and that "when he wrote a memoir, he drafted it at one go, limiting himself to just some crossings out, without coming back to what he had written". Despite this, Poincaré's writings are characterized by great lucidity of thought, an intuitive ability of getting at once to the heart of the matter, and clarity of exposition.

At Mittag-Leffler's request Poincaré wrote in 1901 an analysis of his own work [63]. Of these hundred odd pages only four, pp. 100–103, deal with "Analysis Situs" and its first two Complements. Here he recalls (this line occurs in the Introduction of "A.S." as well) that "*geometry is the art of reasoning well with badly made figures. Yes, without*

HISTORY OF TOPOLOGY
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Henri Poincaré (1854–1912)

doubt, but with one condition. The proportions of the figures might be grossly altered, but their elements must not be interchanged and must conserve their relative situation. In other terms, one does not worry about quantitative properties, but one must respect the qualitative properties, that is to say precisely those which are the concern of Analysis Situs." Thus his hope was that topology would render in higher dimensions much the same service which these "badly made" figures give to ordinary geometry. After mentioning the previous work of Riemann and Betti in this direction he continues as follows.

"*As for me, all the diverse paths on which I was successively engaged have led me to Analysis Situs. I had need of the ideas of this science to pursue my studies on curves defined by differential equations and for extending these to higher order differential equations and in particular to those of the three body problem. I had need of it for the study of multi-valued functions of 2 variables. I had need of it for the study of periods of multiple integrals and for the application of this study to the development of the perturbation function. Finally I glimpsed in Analysis Situs a means of attacking an important problem in the theory of groups, the search for discrete or finite groups contained in a given continuous group. It is for all these reasons that I devoted to this science a fairly long work.*"

Indeed Poincaré's other works probably contain just as much interesting "topology" – in the wide sense of the word – as "Analysis Situs" and its five Compléments! For example, his memoirs on the qualitative theory of differential equations contain the **Poincaré index formula** giving the Euler characteristic of a surface as the sum of the local degrees of a generic vector field at its isolated singularities: this was generalized later to higher dimensions by Hopf [27]. And, of course, the study of periods of multiple integrals is "de Rham–Hodge theory"; and of invariant integrals, which he introduced while doing celestial mechanics, that of "symplectic transformations"; and the work on perturbation functions of astronomy the "small divisors problem". (A seminar run by A. Chenciner has recently been analysing Poincaré's treatise on celestial mechanics.) The **last geometric theorem** [70] which Birkhoff [5] resolved shortly after Poincaré's premature death, is also equally "topology". It says that if a volume preserving diffeomorphism of the annulus moves its two bounding circles in opposite directions than it must have two fixed points. (A recent paper of Golé and Hall [24] shows that the existence of a fixed point does follow by slightly modifying Poincaré's original attempt.)

However, we shall confine ourselves in the following to "Analysis Situs" and its five "Compléments". Section 1 is a summary of "Analysis Situs". Section 2 contains notes on this summary, intended mostly to connect Poincaré's contributions with future developments. For the "Compléments" (these contain more material than "A.S." itself) we have summarized and annotated in tandem in Section 3. We shall pause for just a few remarks before we embark on this task. We have not hesitated to use modern notations, and even ideas, whenever this seemed to help in understanding Poincaré's mathematics. For example, Riemann's *connectivity* of a surface was 1 more than b_1, so Poincaré defined his Betti numbers to be 1 more than the modern ones: we have lowered them by 1. Again, we have discarded Poincaré's *congruences*, and just used $\partial w = c$ to denote a boundary. On the other hand, for homotopy between loops, we liked Poincaré's *equivalences* $A \equiv B$, and have, like him, combined these using *additive*, rather than the modern multiplicative notation. Lastly, Poincaré's grade in art class notwithstanding – see Darboux [15, p. XIX] for the surprising answer! – it is clear to anybody who reads him that he thought via *figures*: so we have added some, but we remark that, of those given below, five are his own.

1. A summary of "Analysis Situs"

Introduction. The branch of Geometry called *Analysis Situs* describes the relative situation between some points, lines, and surfaces, without bothering about their sizes. There is a similar Analysis Situs in more than three dimensions as has been demonstrated by Riemann and Betti (and which we shall develop further in this paper). We expect it will have many applications, e.g., the following three.

"The classification of algebraic curves by means of their genus is based, following Riemann, on the classification of closed real surfaces, made from the viewpoint of *Analysis Situs*. An immediate induction now tells us that the classification of algebraic surfaces and the theory of their birational transformations is intimately tied to the classification of closed real (hyper)surfaces in 5-space from the viewpoint of *Analysis Situs*. M. Picard, in a work which has been hailed by the Académie des Sciences, has already stressed this point."

"Besides, in a series of memoirs published in the *Journal de Liouville* and entitled "*Sur les courbes définis par les équations différentielles*", I have used ordinary 3-dimensional Analysis Situs to study (second order) differential equations. The same researches have also been pursued by M. Walther Dyck. One sees easily that a generalized *Analysis Situs* would permit us to similarly treat higher order equations, and in particular those of Celestial Mechanics."

"M. Jordan has analytically determined the groups of finite order which are contained in the linear group of n variables. M. Klein had previously, by a geometrical method of rare elegance, solved the same problem for the linear group of two variables. Could not one extend the method of M. Klein to a group of n variables, or even *an arbitrary continuous group*? I have not been able to do this so far, but I have thought long on this question, and it appears to me that the solution should depend on a problem of *Analysis Situs* and that the generalization of the celebrated theorem of Euler should play a role in this."

§ 1. *Première définition des variétés.* A nonempty subset V of n-space defined by p equations $F_\alpha(x_1, \ldots, x_n) = 0$ and q inequalities $\phi_\beta(x_1, \ldots, x_n) > 0$, where the functions F and ϕ are continuously differentiable, will be called a *variety of dimension* $n - p$ if the rank of the matrix $[\partial F_\alpha/\partial x_i]$ is equal to p at all points V.

When a variety is defined only by inequalities, i.e. when $p = 0$, then it is called a *domain*. Furthermore, varieties which are one-dimensional, respectively, not one-dimensional but having codimension one, are called *curves*, respectively, (hyper) *surfaces*. A variety will be called *bounded* (finie) if the distance of all its points from the origin is less than some constant.

We will only consider *connected* (continue) varieties, regarding others we only remark that they can be decomposed into a finite or infinite number of connected varieties. For example, the plane curve $x_2^2 + x_1^4 - 4x_1^2 + 1 = 0$ is the disjoint union of the two connected curves obtained by adjoining to its defining equation either the inequality $x_1 < 0$ or else $x_1 > 0$. (See Fig. 1.)

By the *complete boundary* (frontière complète) of a variety V we will mean the set of all points of n-space satisfying $\{F_\alpha = 0,\ 1 \leqslant \alpha \leqslant p,\ \phi_\beta = 0;\ \phi_\gamma > 0,\ 1 \leqslant \gamma \neq \beta \leqslant q\}$ for some $1 \leqslant \beta \leqslant q$. However, sometimes we shall think of the largest $(n - p - 1)$-dimensional variety contained in this set as the true *boundary* (we shall denote this by ∂V) of V. A *boundaryless* (illimité) variety will be one which has empty true boundary; if furthermore it is connected and bounded we shall call it *closed* (fermée).

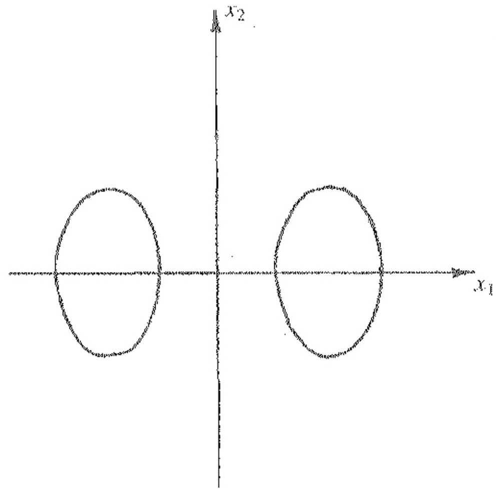

Fig. 1. $x_2^2 + x_1^4 - 4x_1^2 + 1 = 0$.

§ 2. *Homéomorphisme.* Consider the "group" formed by all maps between open subsets of n-space for which the functional determinant is always nonzero: the science whose object is the study of this and some other analogous "groups" is called *Analysis Situs*.

By a *diffeomorphism* (homéomorphisme) between two varieties of n-space we shall mean a bijection between them which extends to a differentiable bijection between open Euclidean sets obtained by replacing their defining equalities $F_\alpha = 0$ by some inequalities $-\varepsilon < F_\alpha < +\varepsilon$. A similar definition can be given for more complicated figures, made up of many varieties, of n-space.

§ 3. *Deuxième définition des variétés.* Consider first m-dimensional varieties υ of n-space satisfying a system of n equations $x_i = \theta_i(y_1, \ldots, y_m)$ with $\operatorname{rank}[\partial\theta_i/\partial y_j] \equiv m$, and some inequalities $\psi(y_1, \ldots, y_m) > 0$.

For example, the system of three equations $x_1 = (R + r\cos y_1)\cos y_2$, $x_2 = (R + r\cos y_1)\sin y_2$ and $x_3 = r\sin y_1$ defines a *torus*. (See Fig. 2).

Indeed in the following definition we may only use those υ's which, unlike that of the above example, have a one–one θ. Furthermore, we can assume these functions to be (real) analytic: this follows because we can always replace θ by an arbitrarily close real analytic θ'.

Given two such varieties υ and υ' we shall say that they are *analytic continuations* of each other iff their intersection $\upsilon \cap \upsilon'$ is also an m-dimensional variety of the above type. As per our new definition a "variety" – or sometimes, to use a different word, a *manifold* – will mean any *connected network* (réseau continu) M of varieties υ related to each other by analytic continuation (i.e. a graph whose vertices are varieties of the above type, with two vertices contiguous in the graph iff they are analytic continuations of each other).

We shall see later that such an M need not be definable by equations of the type given in § 1; however, as shown below *any variety V of § 1 is also a variety as per this second definition.*

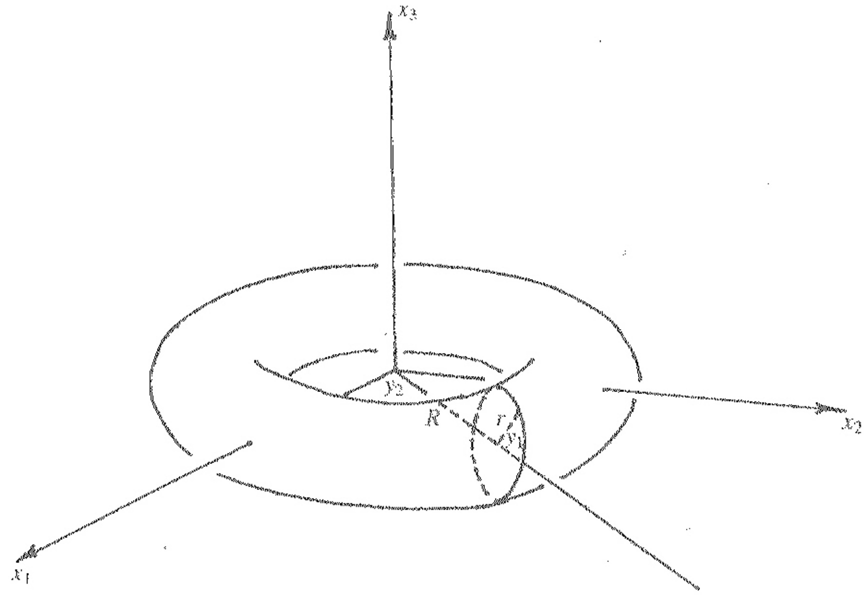

Fig. 2. $(x_1^2+x_2^2+x_3^2+R^2-r^2)^2-4R^2(x_1^2+x_2^2)=0$.

To see this we shall use the well-known result that if the n real analytic equations y_i = $F_i(x_1,\ldots,x_n)$ are such that their functional determinant is nonzero at x, then they hav real analytic solutions $x_i=\theta_i(y_1,\ldots,y_n)$ valid in some neighbourhood of $F(x)$.

Now let P be any point of V, defined as in § 1 by p equations $F_\alpha(x_1,\ldots,x_n)=0$ an some inequalities $\phi(x_1,\ldots,x_n)>0$. It clearly suffices to find an $(n-p)$-dimensiona variety ν_P of the above type such that $P\in\nu_P\subseteq V$. To see this choose any $n-p$ ad ditional analytic functions $F_{p+1},\ldots,F_n$ of n variables, which vanish at P, and are suc that the functional determinant of all the n functions F_i is nonzero at P. So we can solv the n equations $u_i=F_i(x_1,\ldots,x_n)$ to get real analytic solutions $x_i=\theta_i(u_1,\ldots,u_n$ in some neighbourhood of $F(P)=0$ specified by some inequalities $\lambda(u_1,\ldots,u_n)>$ (By making this neighbourhood smaller, if need be, we will assume also that these in equalities imply the defining inequalities $\phi(x_1,\ldots,x_n)>0$ of V. Thus the n equation $x_i=\theta_i(0,\ldots,0,\ y_1,\ldots,y_{n-p})$ and the inequalities $\lambda(0,\ldots,0,\ y_1,\ldots,y_{n-p})>0$ ar satisfied by P and imply the defining p equations $F_\alpha(x_1,\ldots,x_n)=0$ and inequalitie $\phi(x_1,\ldots,x_n)>0$ of V, and so give a ν_P such that $P\in\nu_P\subseteq V$.

§ 4. *Variétés opposées.* We will assume that if we interchange two of the defining equa tions of a V as in § 1 then we no longer get V, but the *opposite variety* $-V$. Mor generally, given any nonsingular matrix $A_{\alpha\beta}$ of functions, the ordered set of equation $\sum_\alpha A_{\beta\alpha}F_\alpha=0$ gives V, respectively, $-V$, iff $\det(A_{\alpha\beta})$ is positive, respectively, nega tive.

Likewise, for a ν as in § 3, we shall assume that interchanging any two of the m parame ters y_i no longer gives ν, but the opposite variety $-\nu$, and more generally, if the parameter undergo a transformation $y_1,\ldots,\ y_m\mapsto z_1,\ldots,z_m$, we shall assume that the resultin variety is ν or $-\nu$, depending on whether the transformation's functional determinant is positive or negative.

The two concepts will be tied to each other by stipulating that if $\nu_P \subseteq V$ as in § 3, then ν_P has the correct orientation iff the $n \times n$ functional determinant mentioned there is positive.

Also the order of the defining equations of an $(n-p-1)$-dimensional nonsingular variety occurring in the true boundary of V will be deemed to be that in which one first writes the equations of V and then puts the new equation $\phi = 0$ in the very end.

§ 5. *Homologies.* Suppose V is a subvariety of a manifold M whose oriented boundary consists of k_i copies of the variety ν_i for $1 \leqslant i \leqslant a$, and s_j copies of the variety $-\mu_j$ for $1 \leqslant j \leqslant b$. Then we shall write

$$k_1\nu_1 + \cdots + k_a\nu_a \simeq s_1\mu_1 + \cdots + s_b\mu_b,$$

and refer to this relation as a *homology* of M. These "homologies can be combined with each other just like ordinary equations" (i.e. the sum of any two homologies will also be deemed to be a homology, and we can take any term to the other side provided we change its sign, and so on).

In case M has a boundary, the notation $k_1\nu_1 + \cdots + k_a\nu_a \simeq \varepsilon$ will indicate that the sum of the varieties on the left is homologous to a sum of varieties contained in this boundary.

§ 6. *Nombres de Betti.* The cardinality of a maximal *linearly independent* set – i.e. one for which there is no nontrivial homology between its members – of closed r-dimensional subvarieties of M will be called the *r-th Betti number* $b_r(M)$ of M. (In the paper it is $b_r(M) + 1$ which is called the r-th Betti number and is denoted by P_r.)

Let us make these definitions clearer by an example. Let D be a domain of 3-space bounded by n disjoint surfaces S_i. Then its Betti numbers are $b_1(V) = (1/2)\sum_i b_1(S_i)$ and $b_2(V) = n - 1$, where each $b_1(S) + 1$ is necessarily odd, being the connectivity of S as defined by Riemann.

§ 7. *Emploi des intégrales.* The integral

$$\int_V \sum \omega_{\alpha_1 \ldots \alpha_r}(x_1, \ldots, x_n)\, \mathrm{d}x_{\alpha_1} \cdots \mathrm{d}x_{\alpha_r}, \quad 1 \leqslant \alpha_i \leqslant n,$$

or briefly $\int_V \omega$, over any r-dimensional variety V (which is equipped with an orientation as in § 4) of n-space, will be defined to be

$$\sum_\nu \int \sum \omega_{\alpha_1 \ldots \alpha_r}(x_1, \ldots, x_n) \det\big(\partial x_{\alpha_i}/\partial y_j\big)\, \mathrm{d}y_1 \cdots \mathrm{d}y_r,$$

where $V = \sum \nu$, and for each ν, the multiple integral is evaluated, using the equations $x_i = \theta_i(y_1, \ldots, y_r)$ of ν, between the limits of y_i prescribed by the inequalities of ν.

About the functions $\omega_{\alpha_1 \ldots \alpha_r}$ – or $\omega(\alpha_1, \ldots, \alpha_r)$ – being integrated it will be assumed that they merely change sign when any two of the indices α_i are interchanged. The result below is from paragraph 2, entitled "*Conditions d'intégrabilité*", of Poincaré [54], 1887.

The integrals $\int_V \omega$ are zero for all closed varieties V of n-space if and only if the $\binom{n}{r+1}$ cyclic sums

$$\sum (-1)^{ri} \partial/\partial x_{\alpha_i} \left[\omega(\alpha_{i+1}, \ldots, \alpha_{r+1}, \alpha_1, \ldots, \alpha_{i-1})\right],$$

are identically zero.

The proof given there shows also that if these $\binom{n}{r+1}$ conditions hold in the vicinity of a given m-dimensional submanifold M of n-space, then the above integrals are still zero over closed subvarieties V of M; in fact, for this, just $\binom{m}{r+1}$ analogous conditions suffice.

For any functions ω satisfying these conditions, one can find at most $b_r(M)$ numbers such that the integral $\int_V \omega$ of ω over any closed r-variety V of M is a linear integral combination of these numbers (we omit the proof given). In other words, *the indefinite integral $\int \omega$, of any functions ω satisfying the conditions of integrability near M, has at most $b_r(M)$ periods.* Further, it can be shown that this bound is the best possible, i.e. there exist such functions ω having exactly $b_r(M)$ periods. For $r = 1$, $m = 1$, this interpretation of the numbers $b_r(M)$ was given by Betti himself.

§ 8. *Variétés unilatères et bilatères.* A manifold M (as defined in § 3) will be called *two-sided* (bilatère) iff we can assign an orientation (as in § 4) to each of the varieties v of its connected network, in such a way that the $m \times m$ determinant $\det(\partial y_i/\partial y'_j)$ is positive whenever v is contiguous to v'.

Otherwise M will be called *one-sided* (unilatère) and deemed equal to its own opposite $-M$. This happens iff either, its network contains a contiguous pair $\{v, v'\}$ with the determinant not of the same sign in all the components of $v \cap v'$, or else, has a *one-sided circuit* $(v_1, \ldots, v_q)$, i.e. one for which making the determinant between v_i and v_{i+1} positive for $1 \leqslant i \leqslant q-1$, makes the determinant between v_q and v_1 negative.

However, to justify these definitions (i.e. to see that one- or two-sidedness is a property of the space M) one also must check (we omit the proof given) that the same alternative continues to hold if a new local parametrization v^* is added to the connected network.

Everyone knows of the one-sided surface which one obtains by folding a paper rectangle $ABCD$ and then gluing the edges AB and CD in such a way that A is glued to C and B to D. (See Fig. 3.)

Examples of two-sided manifolds are easier to give: for example, in n-space, any domain, or any curve, or any closed $(n-1)$-dimensional surface, are all two-sided. Indeed much more is true: *the varieties V of § 1 are all necessarily two-sided* (we omit the proof given).

This shows that "variétés" as defined in § 3 (i.e. manifolds) do not all satisfy equations of the type given in § 1.

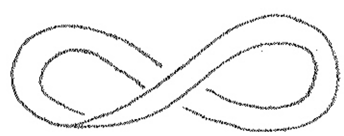

Fig. 3.

§ 9. *Intersection des deux variétés.* Given two points x and x' of n-space lying in oriented varieties v and v' of dimensions p and $n-p$ we denote by $S(x, x') \in \{-1, 0, +1\}$ the sign of the $n \times n$ determinant

$$\begin{vmatrix} \partial x_i/\partial y_j \\ \partial x'_i/\partial y'_k \end{vmatrix}, \quad 1 \leqslant i \leqslant n, \; 1 \leqslant j \leqslant p, \; 1 \leqslant k \leqslant n-p,$$

and using it deem the *algebraic number of intersections* of v and v' to be $N(v, v') = \sum\{S(x, x')\colon\ x = x'\}$. More generally one likewise (we omit the details given) counts the algebraic number of intersections of two oriented complementary dimensional submanifolds of any oriented n-manifold M (intuitively one counts an intersection x as $+1$ iff the orientation of v at x followed by that of v' agrees with that of M).

We note that $N(v, v')$ changes sign if the orientation of any one of the three manifolds $\{M, v, v'\}$ is reversed, and that

$$N(v', v) = (-1)^{\dim v \dim v'} N(v, v').$$

If closed $(n-p)$-dimensional varieties $V_i \subset M$ are such that there exists a p-dimensional cut C of M having intersection number $\sum_i k_i N(C, V_i)$ nonzero then we cannot have $\sum_i k_i V_i \simeq 0$: and conversely, if this homology does not hold, then such a cut C can be found. Here, by a *cut* (coupure) of M we mean either any closed subvariety, or else one whose boundary is contained in the boundary of M.

For case $p = 1$ and M closed (we omit the proof given for the case $\partial M \neq \emptyset$) the direct part follows because if $\partial W = V_1 + \cdots + V_t$, then the oriented closed curve C must go as many times from the complement of W into W, as it goes from W into its complement. Conversely the given conditions ensure (we omit details given) that there is no nontrivial homology amongst these V_i's. So the complement W of $V_1 \cup \cdots \cup V_t$ in M must be connected, for otherwise the boundary of any component of this complement will furnish a nontrivial homology between some of these V_i's. Now we can obtain the required closed curve C by joining the extremities y and z, of a small arc yxz cutting V_1 at x, to each other in W. (Regarding the sketched generalization see the First Complement.)

If follows that, for a closed M, the Betti numbers equidistant from the two ends are equal, i.e. that $b_p(M) = b_{n-p}(M)$ for $0 \leqslant p \leqslant n$. "This theorem has not been, I believe, ever been stated; it is, however, known to many, who have even found some applications of it."

To see this choose in M maximal sets of linearly independent p- and $(n-p)$-dimensional closed oriented varieties $\{C_1, \ldots, C_\lambda\}$ and $\{V_1, \ldots, V_\mu\}$, where $\lambda = b_p(M)$ and $\mu = b_{n-p}(M)$. If the number λ of linear equations $\sum_i x_i N(C_j, V_i) = 0$ was less than the number μ of unknowns x_i, they would have a nontrivial solution $x_i = k_i$. Then (by the direct part of the previous result) we would have $\sum_i k_i N(C, V_i) = 0$ for *all* closed r-dimensional C's. So (by the converse part of that result) we would have $\sum_i k_i V_i \simeq 0$ in M. Since this is not so we must have $\lambda \geqslant \mu$. Likewise $\mu \geqslant \lambda$.

Let us now consider the middle Betti number $b_{n/2}(M)$ for the case n even: *if $n \equiv 2 \bmod 4$ then $b_{n/2}(M)$ is even.*

To see this choose $b = b_{n/2}(M)$ linearly independent closed $(n/2)$-dimensional subvarieties $V_1, V_2, \ldots$ of M, and consider the $b \times b$ determinant $N = [N(V_i, V_j)]$, where by $N(V, V)$ we mean $N(V, V')$ for a suitable $V' \simeq V$. Since $n/2$ is odd this determinant is skewsymmetric, and so, if b were odd, it would be zero. So we would be able to find k_i's not all zero such that $\sum_j k_j N(V_i, V_j) = 0$. So as in last argument we would have $\sum_j k_j N(C, V_j) = 0$ for all $(n/2)$-varieties C, which implies $\sum_j k_j V_j \simeq 0$ in M, a contradiction.

This is no longer true if 4 divides n, nor if M is one-sided, as we shall see later by means of examples.

§ 10. *Représentation géométrique.* "There is a way of describing three-dimensional varieties situated in four space which facilitates their study remarkably", viz., as some polytope(s) P having an even number of facets, with the facets identified in pairs.

For a two-sided variety these *conjugate* facets $F \equiv F'$ are such that if we walk on P along ∂F keeping F to our left, then the corresponding walk on P along $\partial F'$ should keep F' to our right.

Let me recall something similar from ordinary space, viz. cutting a torus along a meridian and a parallel, we can describe it as a square $ABDC$ with identifications $AB \equiv CD$, $AC \equiv BD$, of its sides. Likewise we can identify pairs of facets of a cube in, for example, the following five ways, which all satisfy the above criterion for two-sidedness. (In the paper, the fifth example, i.e. $\mathbb{R}P^3$, is defined by the antipodal conjugation of the facets of an octahedron instead of the cube.) (See Fig. 4.)

Nevertheless, not all of the above facet conjugations of the cube can occur: we shall see below that Examples 1, 3, 4 and 5 are admissible but Example 2 is not.

First note that – in complete analogy with the formation of cycles in the theory of Fuchsian groups – the prescribed facet conjugations partition off the sets of edges and vertices into *cycles* consisting of edges or vertices which get identified to each other, e.g., for Example 2 these are $AB \equiv B'D \equiv C'C \equiv B'A' \equiv AC \equiv DD'$, $AA' \equiv DC \equiv C'A' \equiv B'B \equiv C'D' \equiv DB$, $A \equiv B' \equiv C' \equiv D$, and $B \equiv D' \equiv C \equiv A'$.

For each cycle α of vertices let f_α = its cardinality, e_α = half the sum of the number of facets incident to each member of α, and v_α = number of cycles of edges incident to

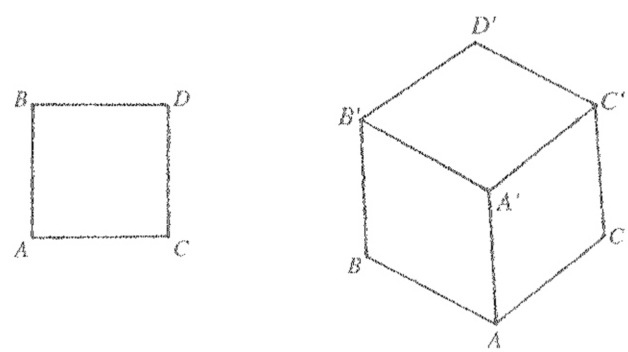

Fig. 4. Square and cube.

	Example 1	Example 2	Example 3	Example 4	Example 5
$ABDC \equiv$	$A'B'D'C'$	$B'D'C'A'$	$B'D'C'A'$	$B'D'C'A'$	$D'C'A'B'$
$ACC'A' \equiv$	$BDD'B'$	$DD'B'B$	$DD'B'B$	$BDD'B'$	$D'B'BD$
$ABB'A' \equiv$	$CDD'C'$	$DD'C'C$	$C'CDD'$	$CDD'C'$	$D'C'CD$

vertices of α, taking care to count each such edge cycle twice if both vertices of an edge are in α. We assert that *for a conjugation of facets to be admissible, it is necessary and sufficient, that one has* $v_\alpha - e_\alpha + f_\alpha = 2$ *for all* α.

To see this note that the subdivision, of the portion of the variety consisting of all points at a distance $< \varepsilon$ from any vertex α, is diffeomorphic to a *star* (aster), i.e. a figure formed by some solid angles arranged around a single vertex in such a way that each point of space belongs to one and only one of them. Since v_α, e_α, and f_α are the number of rays, faces and solid angles of this star, the required condition follows by using Euler's formula.

The above condition holds (we omit the computations given) for all our examples excepting the second, for which $v_\alpha - e_\alpha + f_\alpha = 0 \forall \alpha$.

§ 11. *Réprésentation par un groupe discontinu.* In analogy with the theory of Fuchsian groups one may sometimes describe a three-dimensional variety via a properly discontinuous group of substitutions S of ordinary space.

Indeed, consider any fundamental domain D of this group. Subdivide its boundary into surfaces F which it shares with neighbouring translates $S(D)$, with F' denoting the surface shared by P and $S^{-1}(D)$. Then the variety can be obtained, just as in the last article, from D, by gluing all its conjugate pairs of facets F, F' to each other.

EXAMPLE 6. Consider the group G_T of transformations of 3-space generated by

$$(x, y, z) \mapsto (x+1, y, z), \quad (x, y, z) \mapsto (x, y+1, z), \quad \text{and}$$
$$(x, y, z) \mapsto (\alpha x + \beta y, \gamma x + \delta y, z + 1),$$

where α, β, γ, and δ are four chosen integers with $\alpha\delta - \beta\gamma = 1$, i.e. such that $T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. One can check (we omit the proof given) that this group is discontinuous, with the unit cube P as a fundamental domain. So we obtain a variety M_T by conjugating pairs of facets of a subdivided cube P_T.

The simplest case is when $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, now $P_T = P$, and one recovers Example 1. For $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, once again $P_T = P$, but now the conjugations are that of Example 4. When $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then P has a nontrivial subdivision P_T, which with its facet conjugations is shown in Fig. 5. More generally, any P_T has the same (unsubdivided) vertical facets as P, but the number of its top and bottom cells will increase with the size of the entries of the matrix T.

§ 12. *Groupe fondamental.* Suppose given a system $\mathcal{F}$ of multiple-valued locally defined continuous functions F_α on the variety V, which return to their initial values if we trace small loops on the variety. We will denote by $g_{\mathcal{F}}$ the group of all permutations of the branches which ensue if we follow them over all closed loops starting and ending at a given base point b of the variety.

To be specific we may consider solutions F_α of an equation

$$\mathrm{d}F_\alpha = \sum_{1 \leqslant i \leqslant n} X_{\alpha,i}(x_1, \ldots, x_n; F_1, \ldots, F_\lambda)\, \mathrm{d}x_i,$$

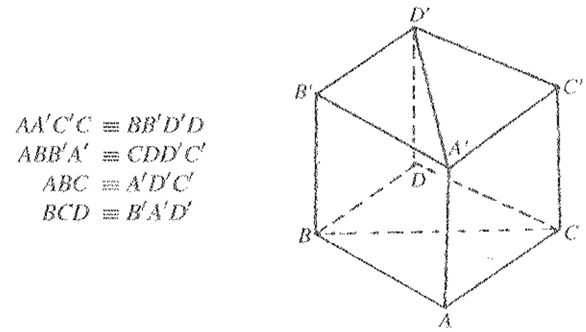

Fig. 5. P_T for $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

where the $n\lambda$ coefficients $X_{\alpha,i}$ are given functions of x_k and F_i, single valued and continuous, together with their derivatives, in a neighbourhood of V, and satisfying there the conditions of integrability

$$\frac{\partial X_{\alpha,i}}{\partial x_j} + \sum_\beta \frac{\partial X_{\alpha,i}}{\partial F_\beta} X_{\beta,j} = \frac{\partial X_{\alpha,j}}{\partial x_i} + \sum_\beta \frac{\partial X_{\alpha,j}}{\partial F_\beta} X_{\beta,i}.$$

We note that if we trace a *lacet* C, i.e. go along any path from b to c, followed by a small loop at c, and then return to b along the original path, then we only get the identity substitution $S_C \in g_{\mathcal{F}}$. Also for the loop C_1C_2, i.e. C_1 followed by C_2, one has $S_{C_1C_2} = S_{C_1}S_{C_2}$.

Motivated by this we shall set $C \equiv 0$ for all lacets, and $C_1 + C_2 \equiv C_1C_2$. A general *equivalence*

$$k_1C_1 + k_2C_2 + \cdots \equiv k_\alpha C_\alpha + k_\beta C_\beta + \cdots$$

will be between integral combinations of loops based at b. One adds them just like homologies but the order of the terms cannot be interchanged. So, e.g., $A \equiv B$ and $C \equiv D$ implies $A + C \equiv B + D$ but not $C + A \equiv B + D$. Also note that from $2A \equiv 0$ one does not have the right to conclude $A \equiv 0$. Another difference from homologies is that a base point b is involved in their definition.

(The above careful distinction between equivalences and homologies notwithstanding at one point it is erroneously written that the boundary of a two dimensional variety of V is equivalent to zero; also see [66, p. 390]; analogously on p. 293 of [61] $C_1 \equiv 0$ should be $C_1 \simeq 0$. This surprising error is rectified in [69], see pp. 451–452.)

For any $g_{\mathcal{F}}$, we obviously have (1) $C \equiv C_1 + C_2 \Rightarrow S_C = S_{C_1}S_{C_2}$ and (2) $C \equiv 0 \Rightarrow S_C = \mathrm{Id}$. We shall denote by G the *fundamental group* G of substitutions S_C satisfying (1) and the stronger (2′) $C \equiv 0 \Leftrightarrow S_C = \mathrm{Id}$. There is thus an epimorphism from G onto any $g_{\mathcal{F}}$. This can be one–one, but is in general not so, because some loop C, which is not decomposable into lacets, may still give the identity substitution in $g_{\mathcal{F}}$.

§ 13. *Équivalences fondamentales.* One can always find some *fundamental loops* $C_1, \ldots, C_p$ such that any loop is equivalent to a combination of these. The relations subsisting between them, which determine the form of the group G, will be called *fundamental equivalences*.

For a variety described as in § 10, using just one polytope P, each pair of conjugate facets $\{F, F'\}$ gives a fundamental loop C: proceed along a straight line from the base point $b \in \text{int}P$ to a point $x \in F$, and then along another straight line from the conjugate point $x' \in F'$ back to b. Denoting each edge of P as the product of its two incident facets any cycle of edges is of the type $F_1 F'_\mu \equiv F_2 F'_1 \equiv \cdots \equiv F_\mu F'_{\mu-1}$. Each of these gives us a fundamental equivalence $C_1 + C_2 + \cdots + C_\mu \equiv 0$.

Ignoring the order of the terms in these fundamental equivalences, one obtains the *fundamental homologies* between these loops. These give $b_1(M)$, which for these three-dimensional varieties M, also equals $b_2(M)$.

For Example 3 of § 10 (computations are also given for Examples 1, 4 and 5) this method gives fundamental equivalences $2C_1 \equiv -2C_2 \equiv 2C_3$, $4C_1 \equiv 0$, which show that G is isomorphic to the *hypercubic* order eight group $\langle i, j, k \rangle$, and that $b_1(M) = b_2(M) = 0$.

For Example 6 of § 11 the fundamental group is evidently isomorphic to G_T. Denoting (the loops inducing) the three defining substitutions of this group, respectively, by C_1, C_2, and C_3, we see that

$$C_1 + C_2 \equiv C_2 + C_1, \quad C_1 + C_3 \equiv C_3 + \alpha C_1 + \gamma C_2, \quad \text{and}$$
$$C_2 + C_3 \equiv C_3 + \beta C_1 + \delta C_2.$$

These are fundamental equivalences, because using them, any member of G_T can be written $m_3 C_3 + m_1 C_1 + m_2 C_2$, which can be checked to be the identity substitution iff $m_1 = m_2 = m_3 = 0$. The fundamental homologies are thus $(\alpha - 1)C_1 + \gamma C_2 \simeq 0$ and $\beta C_1 + (\delta - 1)C_2 \simeq 0$. These homologies are trivial iff $T = I$. In this, and only this, case does one have $b_1 = b_2 = 3$. For $T \neq I$, the above homologies are proportional iff the determinant $\begin{vmatrix} \alpha - 1 & \gamma \\ \beta & \delta - 1 \end{vmatrix}$ vanishes, i.e. iff $\text{tr}(T) = \alpha + \delta = 2$. So in this case, and only in this case, the Betti numbers are $b_1 = b_2 = 2$. In all other cases the homologies are nontrivial and nonproportional, and so we have $b_1 = b_2 = 1$.

§ 14. *Conditions de l'homeomorphisme.* One knows that *closed 2-manifolds are diffeomorphic iff their Betti numbers are same*. This follows, for example, from the study of the periods of Abelian functions. In any Riemann surface R with z as variable, one can introduce a new complex variable t, such that z is a Fuchsian function of t and that t, considered as function of z, has no singular point on the surface R. The Fuchsian group is obviously nothing else but the fundamental group G of R. This rules out the possibility that some cycle of vertices of the Fuchsian polygon, R_0 or $R_0 + R'_0$, has angle sum $2\pi/n$ with $n > 1$, for then we would get a nonidentity substitution as we describe a lacet around this point of the variety. The possibility of a non simply connected Fuchsian polygon $R_0 + R'_0$ is ruled out because then a nontrivial loop C of this polygon would yield the identity substitution of the Fuchsian group. Thus only Fuchsian groups of the first kind with angle sums 2π at all cycles of vertices can occur. All of these groups which are of the same genus are isomorphic, and it is for this reason that all closed two-dimensional manifolds having the same Betti number are diffeomorphic.

However, in dimensions > 2 the questions of *Analysis Situs* become much more complicated and, as we shall see, it is no longer the case that closed varieties having the same Betti numbers must be diffeomorphic.

Let us return to our sixth example (§§ 11 and 13). We shall say that our $T \in SL(2, \mathbb{Z})$ is *hyperbolic, parabolic*, or *elliptic*, according as its two eigenvalues are real distinct, equal, or imaginary. If $G_{T'} \cong G_T$ then a pair of elements of G_T can correspond to the elements C_1' and C_2' of $G_{T'}$ only if they are commuting but linearly independent. So let $a_3C_3 + a_1C_1 + a_2C_2$ and $b_3C_3 + b_1C_1 + b_2C_2$ be any two elements of G_T such that no nonzero multiple of either equals a multiple of the other. *For T hyperbolic these elements commute iff $a_3 = 0$ and $b_3 = 0$, and for T elliptic or $T = -I$ this happens iff $a_3 \equiv 0 \bmod v$ and $b_3 \equiv 0 \bmod v$, where $v > 1$ is such that $T^v = I$* (we omit the proof given).

One can say more: *the groups G_T and $G_{T'}$ are isomorphic iff T is in the same conjugacy class as T'*. To check this (we omit the very long details) we choose generators C_1, C_2, C_3 of G', such that C_1 and C_2 are commuting but linearly independent and such that $C + C_3 \equiv C_3 + T(C)$ for all $C \in \langle C_1, C_2 \rangle$. The idea of the proof is that, in these relations, T gets replaced by the *similar* (transformée) matrix UTU^{-1} if we replace C by $U(C)$. We give, for various cases, sequences of such elementary moves by means of which we finally replace T by T' in these relations.

The number of these conjugacy classes is infinite because similar matrices must have the same *trace*; however conversely, just like noncongruent quadratic forms can have the same determinant, two linear substitutions can be nonsimilar even if they have the same trace.

Thus there are infinitely many nondiffeomorphic M_T's. Since, on the other hand, their Betti number $b_1 = b_2$ can be only 1, 2, or 3, it follows that, *for two closed varieties to be diffeomorphic, it does not suffice that the Betti numbers be the same*. This follows equally because, for our third example G was of order 8, for the fifth ($\cong \mathbb{R}P^3$) of order 2, and for the unit sphere of 4-space it is of order 1, yet for all of these $b_1 = b_2 = 0$. So it seems natural that only those varieties should be called *simply connected* for which G is null.

It would be interesting to know which fundamental equivalences can actually arise, and how one can construct these varieties, and whether two varieties having the same G must be diffeomorphic? Such questions need a difficult and long study, so I will not pursue these here.

However, I do want to draw attention to one point. Riemann had studied algebraic curves as two-dimensional varieties, likewise algebraic surfaces are four-dimensional varieties. M. Picard has shown that for all but some very special algebraic surfaces one always has $b_1 = 0$. This paradoxical looking result appears less so now: the group G can be quite complex and yet the Betti numbers can be very small.

§ 15. *Autres modes de génération.* One may give other definitions of varieties which are, so to speak, intermediate between the two given before, e.g., if the equations of § 1 depended on q parameters, then the dimension of our variety would increase by q, or, if the parameters of § 3 were subject to λ equations, the dimension would decrease by λ.

Also, given a variety W, and a group G which preserves it, one may construct a variety V, to each point of which corresponds one and only one *orbit* (système de points) of W. This variety will be two-sided iff the functional determinants of the substitutions of G are positive with respect to compatible parametrizations of the two-sided variety W.

EXAMPLE 7. Let V be the sphere $y_1^2 + y_2^2 + y_3^2 = 1$ of ordinary space and let G be $\langle (y_1, y_2, y_3); (-y_1, -y_2, -y_3) \rangle$. If then, e.g.,

$$x_1 = y_1^2, \quad x_2 = y_2^2, \quad x_3 = y_3^2, \quad x_4 = y_2 y_3, \quad x_5 = y_3 y_1, \quad x_6 = y_1 y_2,$$

the x's will not change if the y's change signs. This two-dimensional variety V of six-dimensional space is one-sided because the spherical substitution $(\phi, \theta); (\phi + \pi, \pi - \theta) \in G$ has functional determinant -1.

EXAMPLE 8. Let W be the $(2q-2)$-dimensional variety W of $2q$-dimensional space given by the equations $y_1^2 + \cdots + y_q^2 = 1$ and $z_1^2 + \cdots + z_q^2 = 1$; its points correspond to ordered pairs (Q, Q') of points of the hypersphere $y_1^2 + \cdots + y_q^2 = 1$. The $q(q+3)/2$ combinations

$$y_i + z_i, \qquad y_i z_i, \qquad y_i z_k + z_k y_i,$$

give us $n = q(q+3)/2$ new variables $x_1, x_2, \ldots, x_n$ which do not change if we interchange the y's and the z's, i.e. a variety V whose points correspond to unordered pairs $\{Q, Q'\}$ of points of the hypersphere S^{q-1}.

For $q = 2$ this V is not closed, however, for $q \geqslant 3$ it does have an empty boundary ∂V; furthermore V is one-sided for q even and two-sided for q odd (we omit the proofs given).

The *nonzero Betti numbers of* W *are* $b_0(W) = b_{2q-2}(W) = 1$ *and* $b_{q-1} = 2$. By duality it suffices to compute b_i, $i \leqslant q-1$. Any subvariety of dimension less than $q-1$ can be deformed into the ball $W \setminus (U_1 \cup U_2)$, where U_1, respectively, U_2 denotes all $(Q, Q') \in W$ such that $Q = Q_0$, respectively, $Q' = Q_0$. In dimension $q-1$, U_1 and U_2 are linearly independent: for, if J is the usual *volume element* of S^{q-1} in spherical coördinates, then $\int (J_1 + \lambda J_2)$ satisfies the conditions of integrability of § 7, and, for λ irrational, its periods over U_1 and U_2 are integrally independent. Lastly (we omit the proof given) any closed $(q-1)$-dimensional variety v of W is homologous to some $mU_1 + nU_2$.

On the other hand, *for* $q \geqslant 3$, *the nonzero Betti numbers of* V *are* $b_0(V) = b_{q-1} = b_{2q-2}(V) = 1$ (we omit the argument). This shows, as announced in § 9, that there exist one-sided, respectively, two-sided, varieties of dimensions $4k+2$, respectively, $4k$, having middle Betti number odd.

§ 16. *Théorème d'Euler.* This tells us that if S, A and F are, respectively, the number of vertices (sommets), edges (arêtes) and faces of a convex polyhedron, then one must have $S - A + F = 2$. This theorem has been generalized by M. l'amiral de Jonquières to nonconvex polyhedra. One now has $S - A + F = 2 - b_1$, where b_1 denotes the Betti number of the bounding surface. The fact that the faces are planar is of no importance, and the same result is true for any subdivision of a closed surface into *cellular* (simplement connexe) regions.

We shall generalize this result to an arbitrary closed variety V of dimension p. This will be subdivided into some varieties v_p of dimension p which are not closed, and the boundaries of these v_p's will be made up of some varieties v_{p-1} which are not closed,

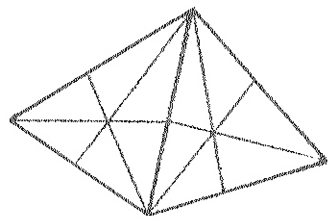

Fig. 6. Derived tetrahedron.

and so on till some points v_0. If the regions v_i's are all cellular, then we shall call such a subdivided V a *polyhedron*. We propose to calculate the number

$$N = \alpha_p - \alpha_{p-1} + \cdots \pm \alpha_0,$$

where α_q denotes the number of the v_q's in the polyhedron.

Two polyhedra arising from the same V will be called *congruent*. Further, if the regions of the first are contained in those of the second, then the first will be said to be a *derived* of the second, e.g., we can derive a tetrahedron into 24 triangles as shown in Fig. 6.

We shall show that *the number N is the same for any two congruent polyhedra*. Since two congruent polyhedra have a common derived it suffices to show $N(P') = N(P)$ for any derived P' of P. If a v_i, $i \leqslant q-2$, is incident to exactly 2 v_{i+1}'s we shall say that v_q is a *singular region* of the polyhedron. We allow these because then we can go (we omit the argument given) from P' to P, in a number of steps, done in order of increasing dimension, each involving erasing a v_i having exactly two incident v_{i+1}'s, followed by an annexation of these three regions. Clearly N is unchanged after each of these steps.

However, this argument is open to objections, e.g., during the above operations the regions may not remain cellular? Before modifying our proof so as to overcome these objections, let us compute some N's.

For the boundary V of any $(p+1)$-cell one has $N = 2$ if p is even and $N = 0$ if p is odd. By the above we can use, e.g., the boundary of a generalized tetrahedron (we omit this calculation), or a generalized cube $-1 \leqslant x_i \leqslant +1$, $+1 \leqslant i \leqslant p+1$: for the latter $\alpha_q = 2^{p+1-q}\binom{p+1}{q}$, so $(1-2)^{p+1} = 1 - \alpha_p + \cdots \pm \alpha_0 = 1 - N$, i.e. $N = 1 - (-1)^{p+1}$.

Our rigorous proof of the invariance of N will be by induction on p. The regions v_i, $i > q$, incident to a given $v_q \in P$ will be said to constitute the *star* (aster) of v_q. The induction hypothesis, and the above computation, imply that, *for the star of any $v_q \in P$ one has*

$$\gamma_p - \gamma_{p-1} + \cdots \pm \gamma_{q+1} = 1 + (-1)^{p-q-1} \tag{A}$$

where γ_i denotes the number of v_i's in the star.

Next, let us take a *quadrillage*, i.e. a cubical subdivision of n-space by n pencils of nonaccumulating hyperplanes parallel to the coordinate planes, $x_i = a_{i,k}$, $1 \leqslant i \leqslant n$. Then, *if the mesh of this quadrillage is small, the intersection of each of its $(n-t)$-cubes D_{n-t} with V is a $(p-t)$-cell v_t, and these cells give us a polyhedron Q covering V*. Let P' be a polyhedron which is a derived of P and of Q.

To see $N(P') = N(P)$ we go from P' to P by erasing the hyperplanes $x_i = a$, one by one. Let δ_q denote the number of q-cells of P' on this plane, δ'_q the number contiguous to it on the $(x_i < a)$-side of the plane, and δ''_q the number contiguous to it on the other side. Since $\delta'_q = \delta_{q-1} = \delta''_q$ (with also $\delta'_0 = 0 = \delta''_0$ and $\delta_p = 0$), the suppression of this hyperplane decreases each α_q by $\delta_q + \delta_{q+1}$, and since the alternating sum over q of these numbers is zero, N remains same.

To see $N(P') = N(Q)$ we can assume (by making the mesh small) that the interior c of each cell of Q intersects only *one* least dimensional cell v_q of P: thus the cells of P intersecting c are precisely those that have v_q on their boundary. We now go from P' to Q by erasing all cells of P' which are in p-cells c of Q but which have lesser dimension than p. So in each c we are erasing the least dimensional v_q and, for each $p > t > q$, γ_t incident cells of dimension t. Moreover, the number of p-cells within c was γ_p before and 1 after. Thus the total decrease in N is $-1 + \gamma_p - \gamma_{p-1} + \cdots \pm \gamma_{q+1} \mp 1$, which by the result above is zero. Next we erase all cells of P' which are in $(p-1)$-cells of Q but which have lesser dimension than $p - 1$, and so on. The same verification shows that N remains same at each step.

§ 17. *Cas où p est impair.* For any polyhedron P (subdividing V as in § 16) we shall denote by $\beta_{\lambda\mu}$ the sum, over all v_λ, of the number of v_μ's which are incident to v_λ. Note that $\beta_{\lambda\lambda} = \alpha_\lambda$ and $\beta_{\lambda\mu} = \beta_{\mu\lambda}$.

If the dimension p of V is even, the number N depends on the Betti numbers of V (see § 18) but *if p is odd, then a closed variety V always has N = 0*. To see this consider the following tableau

$$\begin{array}{lllll}\beta_{p,p-1} - \beta_{p,p-2} & + \beta_{p,p-3} & - \beta_{p,p-4} & + \cdots \\ \quad\; + \beta_{p-1,p-2} & - \beta_{p-1,p-3} & + \beta_{p-1,p-4} & - \cdots \\ & + \beta_{p-2,p-3} & - \beta_{p-2,p-4} & + \cdots \\ & & \cdots\cdots\cdots & \cdots\cdots \\ & & & \cdots\cdots \end{array}$$

The sum of the first row is the sum of the N's of the bounding $(p-1)$-spheres of the α_p p-cells of P, so it equals $2\alpha_p$. Likewise that of second row is zero and that of third is $2\alpha_{p-2}$, etc. Thus the sum of the tableau is twice $\alpha_p + \alpha_{p+2} + \cdots$. On the other hand the sums of the columns are $2\alpha_{p-1}$, 0, $2\alpha_{p-3}$, 0, $\ldots$ by Eq. (A) of § 16. Thus the sum of the tableau is also twice $\alpha_{p-1} + \alpha_{p-3} + \cdots$. Equating the two values one gets $N = 0$.

§ 18. *Deuxième démonstration.* This proof will tell us how N depends on Betti numbers. I will first give an exposition of it for the *Case* $p = 2$, i.e. an ordinary polyhedron P with α_0 vertices, α_1 edges and α_2 faces, and show that $N = 2 - b_1$.

Assign to each of the α_0 vertices any number, and to each of its oriented α_1 edges the difference δ of the numbers of its two vertices. These α_1 numbers δ depend on the α_0 numbers, and conversely determine them up to an additive constant, so *there are in all* $\alpha_1 - \alpha_0 + 1$ *linear relations between the* δ*'s*. These linear relations are given by setting equal to zero, the algebraic sum of the δ's, of some cycle K of edges. Firstly, each of the oriented α_2 faces furnishes a cycle, viz. its perimeter Π. Secondly, from any chosen b_1 homologously independent cycles C of V, we construct cycles C'' of edges as follows – see Fig. 7.

We assert that any relation between the δ's is a linear combination of the $\alpha_2 + b_1$ relations given by the Π's and the C''s. To see this, let K be any cycle of edges. Adding a suitable linear combination of the C''s to it we get L, which is homologous to zero. Being a cycle

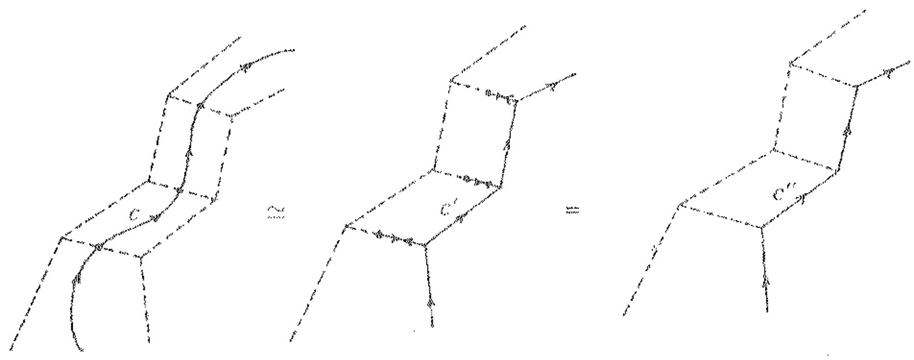

Fig. 7.

of edges of P, this L must be the boundary of a sum of faces of P, and so a sum of their perimeters Π. Also, the sum of *all* the oriented perimeters is zero, but no *partial sum* of the Π' is zero. Thus our new count shows $\alpha_2 + b_1 - 1$ linearly independent relations between the δ's. Equating this with $\alpha_1 - \alpha_0 + 1$ gives the required formula.

For any subdivision P of a closed p-dimensional variety V one has

$$\alpha_p - \alpha_{p-1} + \alpha_{p-2} - \cdots = b_p - b_{p-1} + b_{p-2} - \cdots .$$

This follows by a generalization (we omit the proof which is written out for the *Case* $p = 3$ only) of the above argument. Since the Betti numbers equidistant from the extremes are equal the above formula again shows that $N = 0$ when the dimension p is odd.

2. Notes

With reference to the Introduction of A.S.

NOTE 1. Starting with his dissertation [73], 1851, Riemann had visualized the graph of a multi-valued analytic function – e.g., the function y of x defined by a *polynomial equation* $f(x, y) = 0$ *in two variables* – as a surface obtained by gluing some complex "sheets" to each other along some "cuts", and in [74] he showed that the connectivity of this surface characterizes a nonsingular algebraic curve up to birational equivalence. This connectivity is defined on p. 11 of [73], and in [75] he left some ideas about a similar notion in higher dimensions; these higher connectivities were defined by Betti [4]. In 1880, Picard launched an analogous programme for *polynomial equations* $f(x, y, z) = 0$ *in three variables*, leading eventually to the famous treatise [49] on algebraic surfaces which he wrote with Simart.

NOTE 2. Poincaré developed his *qualitative theory of differential equations* in the three part memoir [51]. The index formula for generic vector fields is proved for all surfaces on pp. 121–125 of the second part – the 2-sphere case is there even in his C.R. note [50] of 1880 – while pp. 192–197 of the third part deal with the case of all n-spheres, $n \geqslant 3$. For Dyck's work on Analysis Situs see [19]. We note also that Poincaré's "A.S." was preceded by two Comptes Rendus notes [56, 58].

NOTE 3. The assertion that the (next to impossible) problem of *classifying the finite subgroups of* GL$(n, \mathbb{C})$ had been solved is of course wrong, however, Jordan [33] had given (modulo two groups of order 168 and 169 which he missed) the classification of all finite subgroups of GL$(3, \mathbb{C})$. The case $n = 2$ had been done previously by Klein [35]. The classification is now known for all $n \leqslant 7$: for references, and other information on this subject, see Dixon [17].

With reference to § 1 of A.S.

NOTE 4. *Poincaré's uses of the word "variety" have at least four modern connotations.* For instance a "closed variety" V, or rather its closure, can be thought of as a closed **pseudomanifold**, e.g., for $q \geqslant 4$, Poincaré's eighth example (§ 15) only gives pseudomanifolds. The "variety" M defined in § 3 as a "reseaux connexe" is more or less today's abstract closed **manifold**, while the special kind of "varieties" v used in its definition are the **local parametrizations** of M. In § 4 "varieties" are oriented, and then in §§ 5 and 6 Poincaré considers integral or rational linear combinations of the oriented varieties of an M to define its homologies and Betti numbers: in this context it is best to think of his "varieties" as **smooth chains** of M.

With reference to § 2 of A.S.

NOTE 5. This definition of Analysis Situs is in harmony with Klein's famous *Erlangen Program* [34] of 1872, even though now the "group" in question is really only a pseudogroup or a groupoid.

With reference to § 3 of A.S.

NOTE 6. The reader will note that Poincaré's "analytic continuation" works equally well with C^1 or C^∞ charts, and is just the way one would nowadays define an **abstract manifold** M, together with an immersion in n-space. Poincaré's focus will always be on the abstract M, he never enters into questions related to the immersion, and only exploits the convenience of n-space to present without fuss some important ideas whose simplicity is obscured if one insists – the book of Milnor [45] being a beautiful exception – on a totally intrinsic treatment.

The idea of an abstract manifold goes back to Riemann, but became popular only much later after Weyl [88], 1918.

With reference to § 5 of A.S.

NOTE 7. Interpreting ∂ as the oriented boundary of smooth oriented chains Poincaré's homologies are the same as those of Eilenberg [22]. It was probably because of this that Eilenberg remarks, on p. 408 of his 1942 paper [21] on **singular homology**, that the singular method of defining homology *"is as old as topology itself"*.

NOTE 8. At this stage Poincaré's "just like equations" is confusing, for it is not clear that he allows division by nonzero integers, i.e. whether he wants to use integral or rational coefficients? However, this point gets clarified in the first "Complément".

With reference to § 6 of A.S.

NOTE 9. As Poincaré pointed out in [60], the numbers defined by Betti [4] himself were not the same as these! In modern terms Betti had considered *the number of elements required to generate* $H_r(M;\mathbb{Z})$, rather than the rank of the free part of this group: see the First Complement.

With reference to § 7 of A.S.

NOTE 10. Poincaré's indefinite integrals are uniquely determined by their skewsymmetric integrands or *differential forms* ω. This section of "Analysis Situs" inspired É. Cartan [12, 13]. Following him, the **exterior derivative** $d\omega$, of an r-form ω of a tubular neighbourhood U of a manifold $M \subset \mathbb{R}^n$, is the $(r+1)$-form defined by

$$(d\omega)(\alpha_1,\ldots,\alpha_{r+1}) = \sum_i (-1)^i \partial/\partial x_{\alpha_i}\big[\omega(\alpha_1,\ldots,\widehat{\alpha_i},\ldots,\alpha_4)\big].$$

One has $d \circ d = 0$, i.e. these differential forms constitute a *cochain complex* $(\Omega(U), d)$. Now Poincaré's "conditions d'intégrabilité" read $d\omega = 0$, and the result, nowadays called **Poincaré's Lemma**, which he quotes from his earlier paper, says that $H^*(\Omega(\mathbb{R}^n), d)$ vanishes in all positive dimensions. Given any r-form satisfying $d\omega = 0$, integration over cycles gives an additive group homomorphism $H_r(U;\mathbb{R}) \to (\mathbb{R},+)$ (likewise $H_r(M;\mathbb{C}) \to (\mathbb{C},+)$ if one uses complex valued forms) whose image is called the *period group* of ω. Poincaré checks that this free Abelian group has rank $\leqslant b_r(M)$, and asserts without proof that this bound is the best possible. For simple cases – like, e.g., $U = \mathbb{C} \setminus \{\text{some points}\}$ when one can use *Cauchy's integral formula*, also see Poincaré's use of the volume form J in § 15 – one can check this by giving explicit closed r-forms having $b_r(M)$ periods. The assertion is true in general, and equivalent to a generalization of the Poincaré lemma proposed by Cartan [13], which soon became **de Rham's theorem** [72], viz., the cohomology $H^*(\Omega(U), d)$ defined via differential forms is isomorphic to $H^*(M;\mathbb{R})$.

NOTE 11. As observed in Sarkaria [76], dropping the requirement that the components of ω be skewsymmetric with respect to the indices gives a bigger cochain complex $(\Omega_{\text{assoc}}(U), d)$ with d defined exactly as above, and furthermore, intermediate between $\Omega(U)$ and $\Omega_{\text{assoc}}(U)$ one has yet another, $(\Omega_{\text{cycl}}(U), d)$, consisting of all ω's skewsymmetric with respect to rotations of their indices. The cohomology of $(\Omega_{\text{assoc}}(U), d)$ is also $H^*(M;\mathbb{R})$, but the **cyclic cohomology** is somewhat different, being

$$H^*\big(\Omega_{\text{cyc}}(U), d\big) \cong \bigoplus_{j \geqslant 0} H^{*-2j}(M;\mathbb{R}).$$

Cyclic subcomplexes were first observed by Connes [14], however, the cyclic manner in which Poincaré displayed his "conditions d'intégrabilité" *could have* suggested $(\Omega_{\text{cyc}}(U), d)$ even to Cartan?

With reference to § 8 of A.S.

NOTE 12. *Orientability is not sufficient to ensure that a manifold can be defined as in § 1.* Such a $V^{n-p} \subset \mathbb{R}^n$ has a trivial normal bundle, so all its *characteristic classes* must vanish.

For example, since $p_1(\mathbb{C}P^2) \neq 0$, the complex projective plane, i.e. for $q = 3$, Poincaré's eighth example, cannot be so defined.

With reference to § 9 of A.S.

NOTE 13. Here *rational coefficients are necessary*, e.g., the double of a nonbounding 1-cycle V of Poincaré's fifth example $\mathbb{R}P^3$ (see § 10) bounds, so $N(C, V) = 0$ for all two cycles C. With rational coefficients it is true that a $(n-p)$-cycle is nonbounding iff its intersection number with some p-dimensional cut is nonzero. The proof which Poincaré gives of this assertion for the case $p = 1$ is okay, however, for $p \geqslant 2$ the sketched generalization is flawed: see the First Complement.

NOTE 14. Picard had used $b_1 = b_3$ for complex surfaces. Besides **Poincaré duality**, i.e. $b_p = b_{n-p}$ for closed oriented manifolds, the above assertion about cuts also implies *Lefschetz duality*, i.e. $b_{n-p}(\operatorname{int} M) = b_p(M, \operatorname{bd} M)$ for oriented manifolds with boundary. More generally, one has $b_{n-p}(M \setminus A) = b_p(M, A)$ if $M \setminus A$ is an orientable n-manifold, e.g., one has *Alexander duality* between the Betti numbers of a closed $A \subset S^n$ and its complement. This generalized *Jordan curve theorem* shows, e.g., that the Betti numbers of the example of § 6, i.e. of $S^3 \setminus$ {some bouquets of circles}, are indeed those given by Poincaré.

NOTE 15. Recall that two integral $n \times n$ matrices A and B are called *congruent* iff $A = PBP'$ for some $P \in \mathrm{GL}(n, \mathbb{Z})$. The unimodular **intersection matrix** $N(V_i, V_j)$ of size $b_{n/2}(M)$ which Poincaré considers is well defined up to congruence, and is especially important for $n = 4k$ when it is symmetric. For example, Whitehead [91] showed that the homotopy type of a closed simply connected 4-manifold is determined by the congruence class of this matrix, and a theorem of Donaldson [18] says that, if definite, such a matrix must be congruent to $\pm I$.

Combining this with Freedman [23] it follows, e.g., that there are about a 100 million distinct simply connected closed four-dimensional **topological manifolds** with $b_2 = 32$, and having intersection matrix – now of course defined via *cup products* – positive or negative definite, yet only two of these manifolds can carry a smooth structure! *Topological manifolds, or for that matter all of point set topology, came long after "Analysis Situs"*: unless explicitly stated otherwise, these notes also are about smooth manifolds and polyhedra.

With reference to § 10 of A.S.

NOTE 16. The assumption that manifolds are obtainable from polytope(s) by facet conjugations is equivalent to their **triangulability**. In § 16 Poincaré suggests a cell subdivision via "quadrillages", and in the first "Complément" (§ XI) he gives yet another with more details. For proofs of triangulability of smooth manifolds see Cairns [10, 11], and Whitehead [89]. For topological manifolds, triangulability is a *much* more delicate question, e.g., Casson has shown that some such closed 4-manifolds (related to Poincaré's homology 3-sphere) are not homeomorphic to any simplicial complex: see Akbulut and McCarthy [1, p. xvi].

NOTE 17. *There are in all seven orientable closed 3-manifolds obtainable by conjugating opposite facets of a cube*, these are listed in Sarkaria [77]. We note also that instead of Poincaré's star criterion one may simply check that the 3-complex resulting from the facet conjugations has Euler characteristic zero, then it will automatically – see Seifert and Threlfall [78, p. 216] – be a manifold. For *topological triangulations* stars can be funny, e.g., Edwards [20] gives a simplicial subdivision of S^5, in which one of the edges has as link the 3-manifold of Poincaré's "Cinquième Complément".

With reference to § 11 of A.S.

NOTE 18. Poincaré's sixth example amounts to identifying the 2 ends of $\mathbb{T}^2 \times [0, 1]$ using the diffeomorphism of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ defined by $T \in SL(2, \mathbb{Z})$. Starting instead with two copies of a solid torus, and identifying their bounding tori using T, one obtains the **lens spaces** L_T of Tietze [87]. For $T = \begin{bmatrix} 1 & 0 \\ h & 1 \end{bmatrix}$ this toral diffeomorphism commutes with the projections of the two solid tori onto the 2-disk, so yielding all the *circle bundles* $L_T \to S^2$: cf. Steenrod [84]. As against this, Poincaré's M_T's are $\mathbb{T}^2$-bundles over S^1, and in the "Troisième Complément" he will also consider other surface bundles over S^1.

NOTE 19. One can check that *the top and the bottom squares of* P_T *are each made up of exactly* $|\alpha| + |\beta| + |\gamma| + |\delta| - 1$ *facets.*

With reference to § 12 of A.S.

NOTE 20. Poincaré gives four approaches to his groups g and G. Firstly, as all *deck transformations* of a **covering space** over M, viz. that whose projection map is the inverse of the multiple valued function F_α (one should allow the number of values to be infinite also). Secondly, his differential equations definition – which plays a major rôle in Sullivan [85] – gives g as the **holonomy group** of a "curvature zero" or *integrable connection* on a vector bundle over M (for nonintegrable connections holonomy groups need not be quotients of $\pi_1(M)$). Thirdly, his definition using "loops", "equivalences" and "lacets" amounts to that which one usually finds in most text books. Lastly, in § 13, for any M obtained from a polytope by facet conjugations, Poincaré defines $\pi_1(M)$ via some simple and elegant (yet intriguing) cyclic relations.

Much later Hurewicz [31], 1935, defined his *higher homotopy groups* as fundamental groups of iterated *loop spaces*: $\pi_i(M) = \pi_1(\Omega^{i-1}M)$. That $\pi_3(S^2)$ is nontrivial was seen by using the *Hopf map* [29], i.e. the projection $L_T \to S^2$ (see Note 18) for case $T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, when $L_T = S^3$.

With reference to § 13 of A.S.

NOTE 21. *It looks curious to a modern reader that Poincaré speaks of the "fundamental group" but never of the (first) "homology group"*, and this even though he speaks of "fundamental equivalences" in tandem with "fundamental homologies"! This is because – cf. [69, p. 450] – at that time, the word "group" was used in a more restricted sense: one spoke of groups of *transformations* (= substitutions = permutations etc.) but not of groups of *points*. For example, $\mathbb{R}$ equipped with addition was seldom called a group, but one spoke of the group of translations of $\mathbb{R}$. For equivalences, Poincaré had given such an interpretation via substitutions induced by monodromy, for homologies he had not. This undefinable distinction between transformations and points was discarded later, likewise function and path spaces entered topology.

With reference to § 14 of A.S.

NOTE 22. The theory of Fuchsian groups was created and developed by Poincaré: see, e.g., [52]. Examples (of all three kinds) of these discontinuous groups of motions of the *non-Euclidean plane* will occur later in course of the arguments of the third, fourth and fifth Complements. (This last also contains a more topological argument – via *Morse theory*! – for the classification of surfaces.) Poincaré also started work on the harder theory of discontinuous groups of motion of the *non-Euclidean space* – see, e.g., [53], also see [68, pp. 64–68] for his popular account of a non-Euclidean world – and probably got interested first in 3-manifolds while examining fundamental domains of these groups. The recent work of Thurston [86] shows that going back to these geometric "roots" may lead to a classification of 3-manifolds.

NOTE 23. As observed in Sarkaria [77] the main result of this section needs to be corrected slightly: *the groups G_T and $G_{T'}$ are isomorphic iff T or T^{-1} is in the same conjugacy class, in* $\mathrm{GL}(2,\mathbb{Z})$, *as T'*. This is also then easily seen to be necessary and sufficient for the manifolds M_T and $M_{T'}$ to be diffeomorphic to each other.

Since $\mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$ is isomorphic to a free product of $\mathbb{Z}/2$ and $\mathbb{Z}/3$, it follows that the *finite orders* ν which can occur are 1, 2, 3, 4 or 6. There is just one conjugacy class of $\mathrm{SL}(2,\mathbb{Z})$ corresponding to each of these ν's, viz. those of

$$I,\ -I,\ \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ \text{and}\ \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},$$

respectively. The conjugacy classes of *parabolic elements* are also easy and are given by Poincaré: representatives are

$$\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & h \\ 0 & -1 \end{bmatrix}, \quad h \in \mathbb{Z}.$$

However, it is not easy to make Poincaré's classification of the M_T's more explicit, because a *complete enumeration of the conjugacy classes of* $\mathrm{GL}(2,\mathbb{Z})$ *is unknown*, but one does know that the number of hyperbolic conjugacy classes having a fixed trace t equals – see [77] – the **class number** of the real quadratic field $\mathbb{Q}[(t^2-4)^{1/2}]$. For a different connection between the topology of Poincaré's M_T's and number theory, read Hirzebruch–Zagier [26, pp. ix–xii].

NOTE 24. *One of the questions asked by Poincaré in § 14 can be answered by using the analogous manifolds L_T* (see Note 18) for which an explicit classification was found by Reidemeister [71], 1935: L_T is homeomorphic to $L_{T'}$ iff $\gamma = \gamma'$ and either $\delta \equiv \pm\delta' \bmod \gamma$ or $\delta\delta' \equiv \pm 1 \bmod \gamma$. Here $\pi_1(L_T) \cong \mathbb{Z}/\gamma$. So one obtains nonhomeomorphic closed 3-manifolds having the same fundamental group. Indeed, by Whitehead [90], one also has nonhomeomorphic L_T's having the same homotopy type.

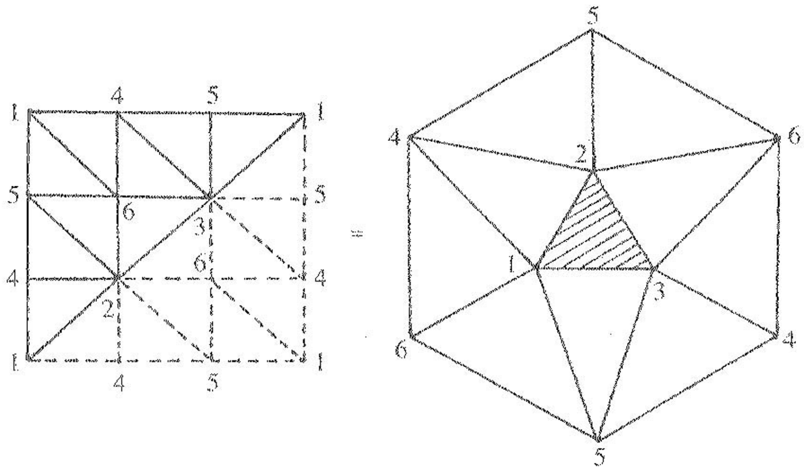

Fig. 8. Symmetric square of S^1.

With reference to § 15 *of A.S.*

NOTE 25. The image of the map of Example 7 is actually *contained in a 4-sphere* – this follows because $x_1 + x_2 + x_3 = 1$ and $x_1^2 + x_2^2 + x_3^2 + 4x_4^2 + 4x_5^2 + 4x_6^2 = 1$ – thus showing that $\mathbb{R}P^2$ embeds in the 4-sphere; more generally by Whitney [92] *any closed M^n embeds in S^{2n}*. We note also that Kuiper [38] has checked that the image of the analogous map $(y_1, y_2, y_3) \mapsto (y_1\bar{y}_1, y_2\bar{y}_2, y_3\bar{y}_3, y_2\bar{y}_3 + \bar{y}_2 y_3, y_3\bar{y}_1 + \bar{y}_3 y_1, \bar{y}_1 y_2 + \bar{y}_1 y_2)$, from the unit sphere of $\mathbb{C}^3$ to $\mathbb{R}^6$, *is equal to a 4-sphere*, thus showing that $\mathbb{C}P^2$ mod complex conjugation is S^4. See also Massey [43].

NOTE 26. For $q \geqslant 3$ the link of the diagonal points of the V of Example 8 is $S^{q-2} * \mathbb{R}P^{q-2}$, so for $q \geqslant 4$, V is only a pseudomanifold. For $q = 3$ one gets a manifold, viz. $\mathbb{C}P^2$. More generally $\mathbb{C}P^n$ *is diffeomorphic to the space of all unordered n-tuples of points of the 2-sphere* – see, e.g., Shafarevich [79, p. 402] – and likewise the symmetric n-th power of any 2-manifold is a $2n$-dimensional manifold. For $q = 2$, V is the Möbius strip, as is shown by the simplicial identifications made below. (See Fig. 8.)

Adding the shaded triangle to the Möbius strip gives the minimal triangulation $\mathbb{R}P_6^2$ of $\mathbb{R}P^2$. Analogously, the minimal triangulation $\mathbb{C}P_9^2$ of $\mathbb{C}P^2$ – see Kühnel and Banchoff [37] – is close to the result of Kuiper and Massey mentioned in the last note.

With reference to § 16 *of A.S.*

NOTE 27. The reference for the cited work of Admiral de Jonquières is [32]. Incidentally I do not know of any higher-ranking topologist!

NOTE 28. Though in § 14 Poincaré gave the now current definition of **simply connected**, mostly he used it – see p. 275 of this section, or p. 297 of the first "Complément" – to mean a *cell* or, sometimes, its bounding *sphere*: e.g., while asking, on p. 498 of [69], the famous

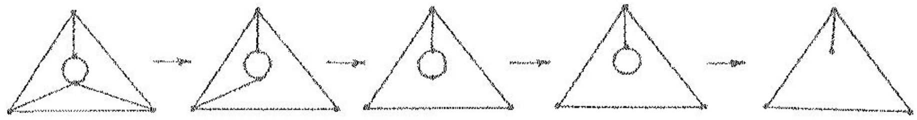

Fig. 9.

question which is now mistakenly called the *Poincaré conjecture*. The Euler–Poincaré formula is of course false if one only demands that the regions be simply connected in the modern sense of the phrase.

NOTE 29. The figure above (Fig. 9: here P is an unsubdivided triangle) shows that Poincaré's "erasing/annexing algorithm" may stop before reaching P.

Indeed *any* algorithm of this kind would give a positive answer to a problem which, to the best of my knowledge, is still open, viz. *is there a combinatorial characterization of the set of all simplicial complexes realizable as geometric subdivisions of a given simplicial complex* (cf. Hudson [30, p. 14])? However, a fundamental **theorem of M.H.A. Newman** [48] does identify the *equivalence relation* generated by "have a common geometric subdivision" with that generated by "have a common stellar subdivision". The invariance of N follows because clearly elementary stellar moves preserve it. This argument is close in spirit to the one being tried by Poincaré in his second attempt in this section.

With reference to § 17 of A.S.

NOTE 30. For **simplicial complexes** (these made their appearance in the first Complement) $\beta_{\mu\lambda} = \alpha_\lambda\binom{\lambda+1}{\mu+1}$ for all $\mu \leqslant \lambda$, so then column summation of Poincaré's "tableau" gives the **Dehn–Sommerville equations** [16, 82],

$$\alpha_p\binom{p+1}{\mu+1} - \alpha_{p-1}\binom{p}{\mu+1} + \cdots \pm \alpha_{\mu+1}\binom{\mu+2}{\mu+1} = \left(1+(-1)^{p-\mu+1}\right)\alpha_\mu,$$

which, for a simplicial sphere, are equivalent to saying that the polynomial $\zeta(z) = \alpha_p z^{p+1} - \alpha_{p-1}z^p + \cdots \pm \alpha_0 z \mp 1$ must obey the *functional equation* $\zeta(z) = \zeta(1-z)$. A complete characterization of these polynomials is now known: see Stanley [83].

With reference to § 18 of A.S.

NOTE 31. This attempt – the two first "Compléments" will push it further – at the invariance of N is the one which affected future developments the most. It gives (implicitly) *a new definition of Betti numbers which uses a cell subdivision* P of M and Poincaré is trying to show – with ideas which clearly foreshadow **simplicial approximation** – that these coincide with those of § 6. This programme, in which Brouwer – see, e.g., [9] – played a big rôle, culminated in Alexander [2], 1915, which contains an elegant proof of $H_*(P) \cong H_*(M)$. After this it remains only to check, as Poincaré does, that the alternating sum of the face numbers α_i equals the alternating sum of the Betti numbers b_i of P. This lemma came to fruition with Hopf [28] and Lefschetz [41].

3. Complements

3.1. *The First Complement*

The First Complement opens with a reference to the "très remarquable" work of Heegaard [25], 1898, who had deemed the duality $b_p = b_{n-p}$ of "Analysis Situs" inexact and the proof of it given there without any value. Before examining Heegaard's specific objections Poincaré points out that **Betti's numbers** [4] were quite different from those of "A.S.". For Betti homologies were between *distinct* varieties together forming the boundary of some V, while Poincaré had considered arbitrary ∂V's. At this point Poincaré states that it is convenient to even allow division by nonzero integers (see Note 8). What an example given by Heegaard, or for that matter Example 3 of "A.S." itself, showed (see Note 13), is only that the *duality is false for Betti's numbers*, on the other hand the *duality is very much true for* **Betti numbers** (as defined in "A.S."), and the main object of this paper is to give a new proof of this using the polyhedra P of § 16 of "Analysis Situs".

As for the previous proof (§ 9 of "Analysis Situs"), after showing for $c = 1$ that a homologously nontrivial codimension c cycle V of an oriented closed manifold M admits a c-dimensional transversal C which intersects it nontrivially, Poincaré had hurriedly sketched that the general case could be done by finding an M' of one dimension more which contains V, then using case $c = 1$ to get a one-dimensional transversal cut C' in M', and finally enlarging C' to a complementary dimensional cycle C which intersects M' in C'. **Heegaard's objections** to this were two: how can one find M', or even if one can, how can one enlarge C' to a cycle C of the required kind? Poincaré admits the validity of at least the second of these objections.

Given a polyhedron P its **schema**, i.e. how it is built up from the v_i's, is determined by its *incidence numbers*: one has $\varepsilon^q_{ij} = 0$ if the j-th (oriented) $(q-1)$-cell is not incident to the i-th q-cell, and $= \pm 1$ otherwise, sign depending on whether or not the orientation of the $(q-1)$-cell agrees with the boundary orientation (§ 4 of "A.S.") of the q-cell. Poincaré observes the all-important necessary condition

$$\varepsilon^q \varepsilon^{q-1} = 0 \quad (\text{i.e. } \partial \circ \partial = 0),$$

but points out that this is not all, one has, e.g., the star condition of "A.S.", § 10. Poincaré poses the problem of characterizing schemas of manifolds (Newman's Theorem, Note 29, answers this partially).

Next, given a cell subdivision P of our manifold, we can consider the **reduced Betti numbers** $b_q(P) \leqslant b_q$ defined as the maximum number of linearly independent cellular cycles. Note that here "linearly independent" is still in the sense of § 6 of "A.S.", i.e. the homologies are not required to be cellular. However, Poincaré asserts that *all homologies are generated by the cellular ones*, and an intricate proof of this – only for the case of 3-manifolds – is given in Section VI. (As remarked in Note 31 a full proof, using Brouwer's simplicial approximation, was given much later by Alexander.)

However, before this he shows in Section III how the above assertion implies (this is along lines already sketched in § 18 of "A.S.") the **Euler–Poincaré formula** for the reduced Betti numbers,

$$\alpha_m(P) - \alpha_{m-1}(P) + \cdots = b_m(P) - b_{m-1}(P) + \cdots$$

(i.e. he checks that $\sum_i (-1)^i \dim C_i = \sum_i (-1)^i \dim H_i$ for any chain complex $\cdots \to C_i \to C_{i-1} \to \cdots$ over $\mathbb{Q}$).

In modern parlance, Section IV checks that, by imaging each q-cell of P to the sum of all the smaller compatibly oriented q-cells of a subdivision P', one gets the **chain subdivision map** $C(P) \to C(P')$.

In Section V he proves (again using the assertion that cellular homologies suffice) that the **reduced Betti numbers are subdivision invariants**. For this he deforms each q-cycle of P' over h-cells of P, in order of decreasing h, till finally it is contained in the q-skeleton of P. Then the coefficients of all smaller cells belonging to the same q-cell of P being the same, chain subdivision identifies it with a q-cycle of P. So $b_q(P)$ does not depend on P.

At the end of Section V, Poincaré asserts that given any closed (smooth) cycle one can always subdivide P so that the cycle becomes cellular in this subdivision. Using this **triangulability** assertion it follows that the **reduced Betti numbers coincide with those of "Analysis Situs"**, § 6. He enters into these intricacies in Section X (no use is now made of the "quadrillages" of § 16, "A.S.", instead there is an interesting idea involving *joins of simplicial complexes*) and declares at the end that "on est ainsi débarassé des dernier doutes" about triangulability. (We note that the "simple triangulation method" of Cairns [11] is almost the same as Poincaré's previous method of § 16, "A.S.", viz. intersecting M with a sufficiently fine "quadrillage".)

In Section VII Poincaré puts a vertex $\widehat{\sigma}$ in each $\sigma \in P$, and subdivides inductively by coning the already subdivided boundary of σ over $\widehat{\sigma}$. This gives a **simplicial complex**, viz. the *barycentric derived* P' of P. If one transfers the incidence relation amongst the cells of P to their barycentres, one sees that the simplices of P' have as vertices all totally orderable sets of barycentres, and that a cell σ of P consists of all simplices of P' having *highest* vertex $\widehat{\sigma}$.

Poincaré now defines his **dual cells** σ^* by inductively coning over $\widehat{\sigma}$ the already defined dual cells of the higher dimensional cells incident to σ (that σ^* is indeed a cell follows by the star criterion of "A.S.", § 10). So σ^* consists of all simplices of P' whose *lowest* vertex is $\widehat{\sigma}$.

The dual cells σ^* constitute the **polyèdre réciproque** P^* (Poincaré dual cell complex): note that P and P^* have common subdivision P', so just the subdivision invariance of Betti numbers gives $b_q(P) = b_q(P^*)$. We shall orient the dual cells so that, under the incidence reversing correspondence $\sigma \leftrightarrow \sigma^*$ between the schema of P and P^*, one has

$$\varepsilon^q_{ij}(P) = \varepsilon^{n-q}_{ji}(P^*)$$

(i.e. the boundary ∂ of P becomes the *coboundary* δ of P^*, thus giving at once the modern $H_q(P) \cong H^{n-q}(P^*)$).

Using this duality of incidences Poincaré obtains his **duality** $b_q(P) = b_{n-q}(P^*)$ by showing, in the course of Section VIII, that the reduced Betti numbers can be computed from the schema by using

$$b_q(P) = \alpha_q(P) - r\big(\varepsilon^q(P)\big) - r\big(\varepsilon^{q+1}(P)\big),$$

where $r(A)$ denotes the *rank* of the matrix A. (For the sake of simplicity Poincaré prefers to write all details, starting with the definition of P^*, only for 3-manifolds; however, the general versions can be found in §§ 1 and 3. of the Second Complement.)

Section VIII actually gives an **algorithm** which computes the Betti numbers of a schema. For this he sets up his **tableaux**

$$\begin{bmatrix} I_{\alpha_q} & \varepsilon^q \\ \varepsilon^{q+1} & 0 \end{bmatrix},$$

and then using *elementary row and column operations*, triangularizes the top right and bottom left corners. Working over $\mathbb{Z}$, instead of $\mathbb{Q}$ which suffices for *his* Betti numbers, enables him to show in Section IX that b_q *coincides with Betti's q-th number iff the greatest common divisor of the largest sized nonzero minors of the* $(q+1)$*-th incidence matrix is* 1.

Section IX also contains a cellular version of a result of § 9 of "Analysis Situs", viz. that *it is possible to find a p-cycle* V_1 *in* P *such that* $N(V_1, V_2)$ *is nonzero if and only if the* $(n-p)$*-cycle* V_2 *of* P^* *is not homologous to zero over* $\mathbb{Q}$. This is easy algebra because σ and σ^* have intersection number $N(\sigma, \sigma^*) = \pm 1$. Then, using triangulability, Poincaré again claims the previous results of § 9, "A.S.", in full.

3.2. *The Second Complement*

The Second Complement is only, says Poincaré, to simplify and clarify results already in hand. He begins by precising that, with dual cells oriented as above, one has

$$N(\sigma, \sigma^*) = (-1)^{q(q+2)/2}, \quad \text{where } q = \dim(\sigma).$$

In the previous paper Poincaré had only triangularized the corners of his "tableaux" because he was unaware of Smith [81], 1861, where it had been shown that *a rank r integer matrix can be reduced, by elementary operations over* $\mathbb{Z}$*, to a unique matrix of the type*, $\operatorname{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0)$, $d_1 | d_2 | \ldots | d_r$. Still unaware of Smith's work, he now re-discovers, and gives a nice proof of this result in § 2.

In terms of these important new **torsion invariants** d_i of the schema P he then works out in § 3 that **Betti's** q**-th number exceeds** b_q **by the number of invariants of** ε^{q+1} **bigger than** 1, and that the product of these invariants gives the number of "distinct" cycles whose multiples bound (i.e. the order of the torsion part of $H_q(P)$).

As mentioned in Note 21 Poincaré did not *speak* of homology groups, but of course knowledge of Betti numbers and torsion invariants is equivalent to knowing homology or cohomology groups:

$$H_q(P) \cong \bigoplus \{\mathbb{Z}/d_i(\varepsilon^{q+1})\mathbb{Z} \colon d_i(\varepsilon^{q+1}) > 1\} \oplus \mathbb{Z}^{b_q},$$
$$H^q(P) \cong \bigoplus \{\mathbb{Z}/d_i(\varepsilon^{q})\mathbb{Z} \colon d_i(\varepsilon^{q}) > 1\} \oplus \mathbb{Z}^{b_q}.$$

Computations (§ 4). Poincaré gives the Betti numbers and torsion invariants of Examples 1, 3, 4, and 5 of § 10 of "Analysis Situs" by diagonalizing over $\mathbb{Z}$ the incidence matrices of the *cell complexes* (so ε^q_{ij} can be integers other than 0, ± 1) given by the facet conjugations (a more sophisticated cell complex H was used later for the homological computations of the Fourth Complement).

He also computes the same for Heegaard's [25] example, viz. **the singular link of the complex surface** $z^2 = xy$. Curiously, though it is all but apparent from the cell subdivision which he uses, he fails to notice that *Heegaard's example is actually the same as his own Example 5 of "Analysis Situs"*, i.e. diffeomorphic to $\mathbb{R}P^3$. (See Milnor [46] for more on singular links of complex hypersurfaces.)

Poincaré also computes his new invariants for the manifolds M_T of Example 6 (§ 11 of "A.S."). In modern terms he shows that $H_1 \cong \mathbb{Z}\oplus\mathbb{Z}/d_1(T-I)\mathbb{Z}\oplus\mathbb{Z}/d_2(T-I)\mathbb{Z}$ and $H_2 \cong \mathbb{Z}^{b_1}$. This time his method is to note (using § 13 of "A.S.") that H_1 is $\langle C_1, C_2, C_3\rangle$ mod the relations

$$(\alpha - 1)C_2 + \gamma C_3 \simeq 0,$$
$$\beta C_2 + (\delta - 1)C_3 \simeq 0,$$

and the result follows by reducing the coefficient matrix.

In § 5 (which perfects Section X, and end of Section VII, of the First Complement) there is a direct combinatorial proof of the particular case $b_q(P) = b_q(P^*)$ of the invariance theorem.

In § 6 it is shown that *if the q-skeleton of P has no "one-sided circuits" in the sense of § 8 of "Analysis Situs", then its $(q-1)$-th homology is torsion free*. This condition amounts to saying that if we consider any circuit, with some entries of the q-th incidence matrix as its vertices, and with edges alternatively horizontal and vertical, then the product of its vertices, if nonzero, is $+1$ or -1 depending on whether the length of the circuit is, respectively, 0 or 2 mod 4. (See Fig. 10.)

From this Poincaré deduces that all minors of ε^q must be 0 or ± 1 which of course implies that H_{q-1} is free. (We note that the vanishing of a *Stiefel–Whitney class* w_{n-q} can likewise be interpreted as a milder "orientability condition" on the q-skeleton.)

Poincaré ends by conjecturing that if Betti numbers and torsion invariants are all trivial then the manifold is a sphere. As is well known, he later disproved this via the famous example of the Fifth Complement, but it is to be noted that he already has at least examples of orientable 3-manifolds having the same homology groups but different fundamental groups, e.g., take M_T with $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$.

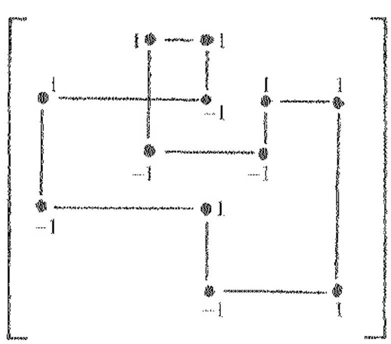

Fig. 10. A one-sided circuit.

3.3. *The Third Complement*

The Third Complement computes $\pi_1(V)$ of some V's satisfying the polynomial equation $z^2 = F(x, y)$. (In [64] Poincaré mentions that he got interested in these surfaces because of a problem regarding perturbation functions of celestial mechanics.) It is assumed that, but for finitely many *singular points* $\{A_1, \ldots, A_q\}$ of the complex y sphere, $F(x, y) = 0$ has $2p + 2$ distinct roots $x_0(y), x_1(y), \ldots, x_{2p+1}(y)$, $p \geqslant 1$.

First consider y as a constant $\neq A_i$, $1 \leqslant i \leqslant q$. Then $z^2 = F(x, y)$ gives a **complex curve** V of genus p, and the coordinates x and z of its points are *Fuchsian functions* of an auxiliary variable $u = \xi + i\eta$, having a Fuchsian polygon R of the first type with angle sum 2π, such that opposite pairs of its $4p$ edges get identified under transformations $S_k(u) = \phi_k(\xi, \eta) + i\psi_k(\xi, \eta)$, $1 \leqslant k \leqslant 2p$, generating the Fuchsian group G' admitted by these functions. The curve being *hyperelliptic* (it has the involution $(x, y, z) \leftrightarrow (x, y, -z)$) one can choose an R which admits a central symmetry (non-Euclidean, if $p \geqslant 2$) and is made up of two symmetric halves R' and R'', with R' being such that its $2p+1$ vertices lie above x_0 and the mid-points of its edges correspond to the remaining roots $x_1, \ldots, x_{2p+1}$ of $F(x, y) = 0$. Each of these tiles R' covers the complex x sphere with the two halves of each of its $2p+1$ edges imaging onto the two "lips" of a cut going from x_0 to $x_1, \ldots, x_{2p+1}$ (the genus p surface is obtained by identifying two copies of this cut sphere). (See Fig. 11.) Each member of $\pi_1(V) = G'$ is a product of an even number of central symmetries s_k through points above x_k (e.g., $S_1 = s_1 s_{2k+1}$). These symmetries obey, besides $s_k^2 = 1$, the *Fuchsian relation* $s_0 s_1 \cdots s_{2p+1} = 1$.

What happens if we now let y vary and describe a simple closed curve in $S^2 \setminus \{A_1, \ldots, A_q\}$? Our Fuchsian group will vary in a continuous way, likewise the roots $x_0, x_1, \ldots, x_{2p+1}$, and the Fuchsian polygon R. After y has described the closed curve, the group will return to the original G', but the points x_i will in general get permuted amongst themselves, so that R might become a different, but still *equivalent* polygon R_1, i.e. still generating G'.

The three-dimensional variety V defined by $z^2 = F(x, y)$, with y-constrained on such a closed curve is then analyzed via **monodromy**, i.e. as y varies we shall make the Fuchsian variable u vary continuously in such a way that vertices of the original tiling go to vertices of the new (but homeomorphic) tiling, edges to edges, and congruent points go to congruent points. We introduce three real variables ξ, η, and ζ: the first two being the real and imaginary parts of the initial u, and the last a function of y alone which augments by 1 as we describe the closed curve. We can then represent V (see "A.S.", § 11) by the discontinuous group G generated by the $2p + 1$ substitutions

$$(\xi, \eta, \zeta) \mapsto \big(\phi_k(\xi, \eta), \psi_k(\xi, \eta), \zeta\big),$$
$$(\xi, \eta, \zeta) \mapsto \big(\theta(\xi, \eta), \theta_1(\xi, \eta), \zeta + 1\big),$$

where $u_1 = \theta(\xi, \eta) + i\theta_1(\xi, \eta)$ denotes final position of $u = \xi + i\eta$ (so the 3-manifold V is the **mapping torus** of the diffeomorphism of the surface of genus p induced by the monodromy $u \mapsto u_1$). Note that $\pi_1(V) = G$ contains a normal subgroup isomorphic to G' which, together with the last substitution Σ, generates it. For the Euclidean case $p = 1$ we have

$$\theta(\xi, \eta) = \alpha\xi + \beta\eta, \quad \theta_1(\xi, \eta) = \gamma\xi + \delta\eta, \quad \text{where } T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

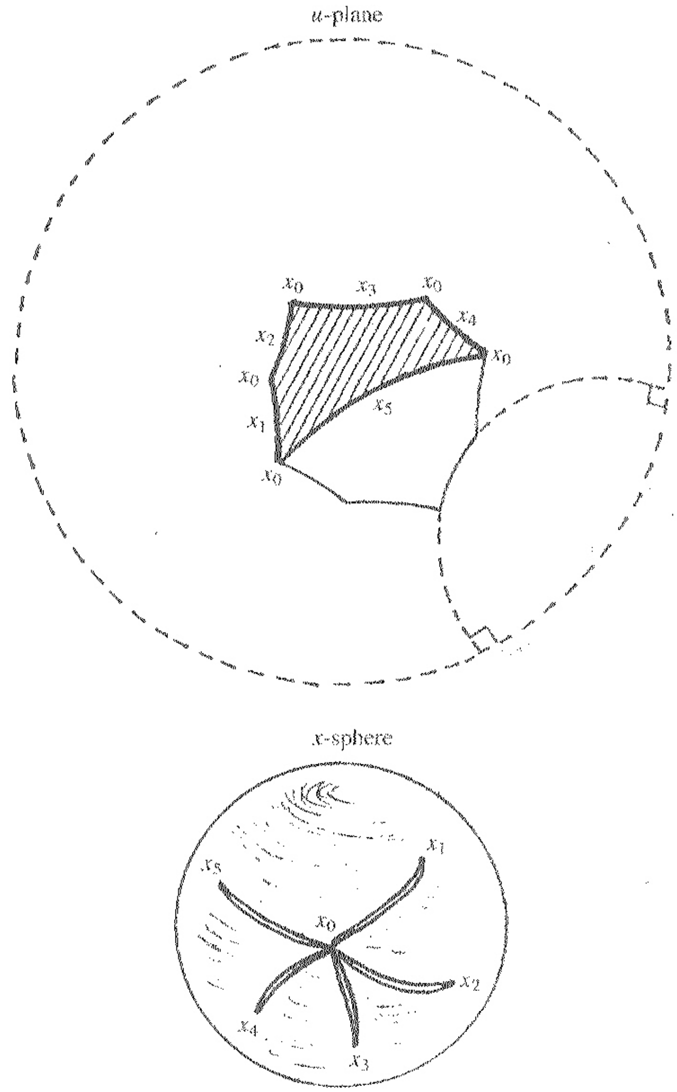

Fig. 11. Hyperelliptic curve (half) of genus two.

images the parallelogram R onto the equivalent R_1, so for $p = 1$ we have again the variety M_T and the group G_T of **Example 6** of "A.S.", § 11.

Let us consider next the four-dimensional variety V defined by $z^2 = F(x, y)$, *with y only constrained to be outside q small circles guarding the singular points* $\{A_1, \ldots, A_q\}$. To analyze V we shall join a chosen ordinary point O of the complex y sphere to these points by means of q disjoint cuts $OA_1, \ldots, OA_q$. Indeed we shall think of y as a Fuchsian function of a new auxiliary variable $\zeta + i\zeta' \in \Delta$, invariant with respect to a Fuchsian group Γ generated by a Fuchsian polygon Q of the second type whose q cusps α_i correspond to the

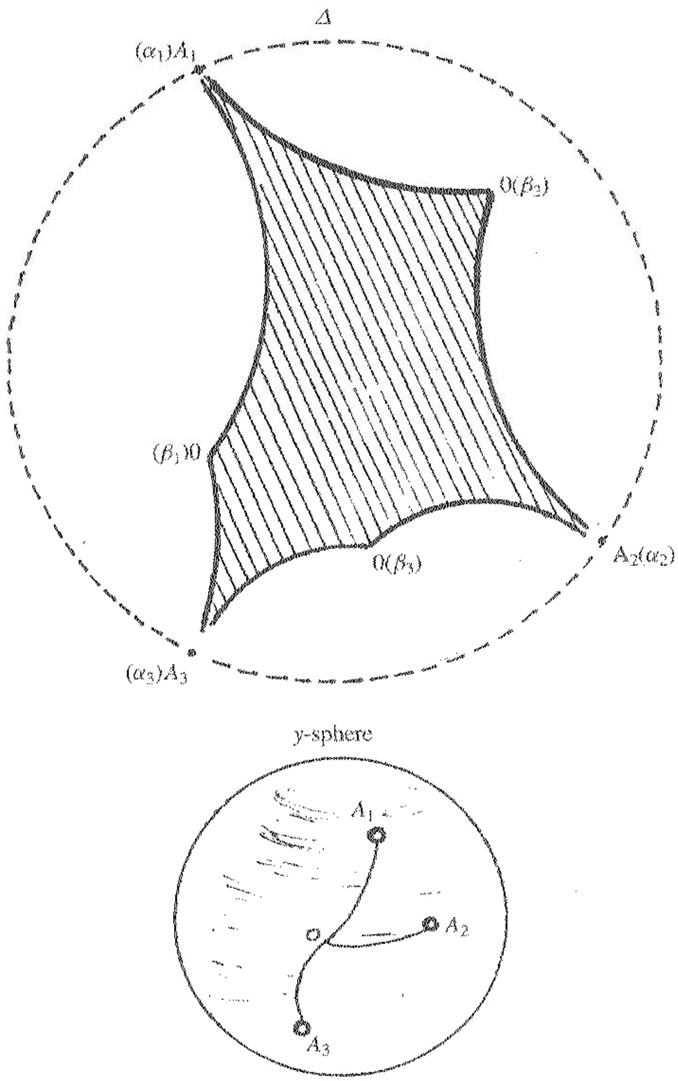

Fig. 12. A Fuchsian tile of the second kind.

singular points A_i, while its remaining vertices β_i all correspond to the ordinary point O. (See Fig. 12.) The sum of the tile's angles is again 2π (the angle at each cusp being zero) and the group Γ, i.e. the free rank $q-1$ fundamental group of $S^2 \setminus \{q \text{ points}\}$, is generated by q motions Σ_i which identity pairs of edges incident to the same cusp, and one has the relation $\Sigma_1 \Sigma_2 \ldots \Sigma_q = 1$.

To each point of V assign four variables ξ, η, ζ and ζ', of which ξ and η are the real and imaginary parts of the u_0 above a base point y_0 obtained from u by monodromy over any path from y to y_0 which does not cross the cuts OA_i. This is independent of the path chosen, however, if y makes a loop around A_i resulting in the substitution $\Sigma_i(\zeta, \zeta') = (\kappa_i(\zeta', \zeta''), \kappa_i'(\zeta', \zeta''))$ of Γ, then the variables ξ and η can change, to say $\theta_i(\xi, \eta)$ and $\theta_i'(\xi, \eta)$. Our variety is thus represented by the discontinuous group G of 4-space determined by the $2p + q$ substitutions,

$$(\xi, \eta, \zeta', \zeta'') \mapsto (\phi_k(\xi, \eta), \psi_k(\xi, \eta), \zeta', \zeta''),$$
$$(\xi, \eta, \zeta', \zeta'') \mapsto (\theta_i(\xi, \eta), \theta_i'(\xi, \eta), \kappa_i(\zeta', \zeta''), \kappa_i(\zeta', \zeta''))$$

and this G is its fundamental group (Poincaré checks via the usual argument which identifies π_1 with all covering transformations of a simply connected cover). Note that it has G' as a normal subgroup which generates it together with the last q substitutions $T_i = (\theta_i, \theta_i', \kappa_i, \kappa_i')$. (That $\pi_1(V)$ is an extension of G' by the free group Γ can be seen also by using the *homotopy sequence* of V as a **fibration** over $S^2 \setminus \{q \text{ points}\}$ having the surface of genus p as its fiber.)

Getting rid of the circles guarding the points A_i and supposing that x and y can take arbitrary complex values we now consider the ***algebraic surface*** *V defined by $z^2 = F(x, y)$.* It will be assumed that as y approaches an A_i some two of the roots, say $x_a(y)$ and $x_d(y)$, approach a common value x_{ad}, but the other $2p$ roots all remain distinct, so the (possible) **singularities** of our V are $(x_{ad}, A_i, 0)$ only. Poincaré shows that *V is simply connected* (as against Picard who had shown $b_1(V) = 0$ for a generic complex projective surface V). For this note that $\pi_1(V)$ is a quotient of the above G. Also that, as y makes a small loop around A_i, while x remains constant, we get a small loop on V, so $T_i \simeq 1 \forall i$. With y moving as before, now let x also make a small loop around both $x_a(y)$ and $x_d(y)$. This augments the angle of $z^2 = F(x, y)$ by 4π, so giving us another small loop on V, this time around the singularity $(x_{ad}, A_i, 0)$. Thus $T_i s_a s_d \simeq 1$, so giving $s_a s_d \simeq 1 \forall a, d$.

Lastly, let V be the ***nonsingular part*** *of the above complex surface.* Since we can no longer deform past $(x_{ad}, A_i, 0)$ we cannot conclude $s_a \simeq s_d$ in the above manner. We, however, still have $T_i \simeq 1 \forall i$, so $\pi_1(V)$ is at most a quotient of G'. For $p = 1$, G' is Abelian, so then Picard's result implies that *$\pi_1(V)$ is finite*. Poincaré shows this in general by writing down some more relations using the fact that, in $\pi_1(V)$, the monodromy action of T_i must become the identity.

We illustrate this for $p = 2$, first *if $x_a(y)$ and $x_d(y)$ interchange* as y makes a small loop around A_i. Shown in Fig. 13 are the initial (full) and final (dotted) positions of cuts, from an ordinary point 0, to these two roots; the other four cuts do not change as y makes this loop. Now s_b (or just b for short) corresponds to a loop which intersects only one initial cut, viz. Ob. Observing in order the final cuts which this loop intersects we get $b \simeq dabad$. Likewise $a \simeq d$, $c \simeq dacad$, $d \simeq dad$, $e \simeq e$, and $f \simeq f$. So we again have $s_a \simeq s_d$. Indeed $(x_{ad}, A_i, 0)$ is a **removable singularity** of V, i.e. its link is S^3: this follows because, near it, our surface is like $z^2 = y - x^2$ near the origin. *If $x_a(y)$ and $x_d(y)$ remain distinct* then the picture can be as in Fig. 14. The same method now gives $b \simeq dadabadad$, $a \simeq dad$, $b \simeq dadabadad$, $c \simeq dadacadad$, $d \simeq dadad$, $e \simeq e$ and $f \simeq f$. So we only obtain $(s_a s_d)^2 = 1$. Now $(x_{ad}, A_i, 0)$ is a **conical singularity**: near it the surface is like $z^2 = y^2 - x^2$ (or *Heegard's example* $z^2 = xy$) and the link is $\mathbb{R}P^3$. From these considerations it is easy to see that

$$\pi_1(V) \cong (\mathbb{Z}/2)^{n-1} \quad or \quad (\mathbb{Z}/2)^{n-2},$$

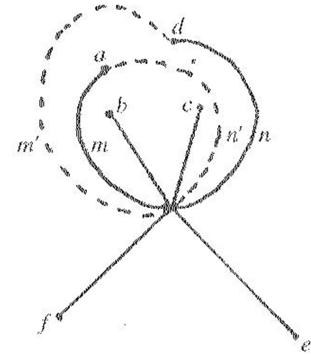

Fig. 13. Removable singularity.

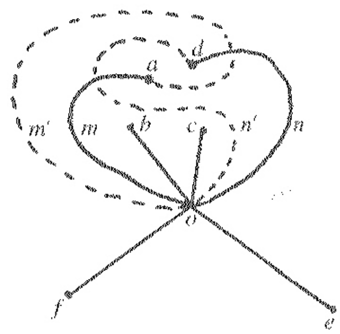

Fig. 14. Conical point.

where n denotes the number of irreducible factors of $F(x, y)$. Indeed if the roots a and b belong to the same factor, then we can identify the corresponding symmetries s_a and s_d, and otherwise identify them up to an ambiguity of order two, with the Fuchsian relation giving one more relation unless all factors are of even degree.

3.4. *The Fourth Complement*

The Fourth Complement opens with a mention of the pioneering "beaux travaux" of Picard, and goes on to show how monodromy can be used to find all the Betti numbers of a smooth complex two-dimensional variety V (Poincaré works over $\mathbb{Q}$ but it is asserted in [65] that this method will also give the torsion invariants). We shall suppose V represented as an algebraic surface $f(x, y, z) = 0$ having only "ordinary singularities", such that for each fixed $y \neq A_1, \ldots, A_q$ our equation determines a smooth complex curve $S(y) \subset V$ of constant genus p, but the genus of the q exceptional curves $f(x, A_i, y) = 0$ can be lower.

A cell subdivision H *of* V(§ 1): this projects, on the y-sphere, to a $2q$-gon Q with pairs of sides $\beta_i\alpha_i \equiv \beta_{i+1}\alpha_i$ covering the two lips of the cuts OA_i (see a picture above), and induces, for each fixed $y \notin S^2 \setminus \{\text{cuts } OA_i\}$, a subdivision P of $S(y)$ which is preserved by monodromy over paths not crossing the cuts. Further, as y approaches a point M on a cut OA_i from its two sides, P can approach two quite different subdivisions, MP and (MP), of the same Riemann surface $S(M)$: we shall assume that now H induces the common refinement P' of MP and (MP). Likewise, as y approaches O from within any of the q sectors, P can tend to q different subdivisions of $S(O)$: now H induces the common refinement P'' of all q of these. Finally, as M approaches A_i along OA_i, MP and (MP) approach coincidence, and at the same time some cells get identified, to give the cells of H covering the lower genus curve $f(x, A_i, z) = 0$. The faces, vertices, and edges of P (respectively P', respectively P'') are denoted F_i, B_j, C_k (respectively F'_i, B'_j, C'_k, respectively F''_i, B''_j, C''_k), and those of H by pre-multiplying these with the appropriate faces of Q.

Computation of $H_3(V)$ (§ 2). We sketch below the argument which is given to show (in modern terms) that $H_3(V)$ is *isomorphic to the subgroup of* $H_1(S)$ *which remains fixed under the action of the* **Picard group** (so named in [65] by Poincaré), i.e. the image of the monodromy induced group homomorphism $\pi(S^2 \setminus q\,\text{pts}) \to \text{Aut}(H_1(S)) \cong \text{GL}(2p, \mathbb{Z})$. (Also, the parity of $b_3(V) = b_1(V)$ is always *even*, being double the **irregularity** of V, as was shown by Picard using transcendental methods.)

Let $\omega = \sum_j c_j QB_j + \sum_{k,i} c_{ki}\alpha_i\beta_i F'_k$ be any 3-cycle of H, then – look at terms of left side of $\partial\omega = 0$ involving cells with first factor Q – $\Omega = \sum_j c_j B_j$ must be a 1-cycle of P, and it is easily seen that $\omega \simeq \omega'$ implies $\Omega \simeq \Omega'$. Also – look at the remaining terms of $\partial\omega = 0 - \sum_j c_j\alpha_i\beta_{i+1}B_j - \sum_j c_j\alpha_i\beta_i B_j$ is a boundary $\forall i$, which implies, on intersecting with $S(M)$, that the copies of Ω in the subdivisions MP and (MP) of $S(M)$, $M \in OA_i$, must be homologous (in P' after subdivision). In other words the 1-cycle Ω of P is **invariant** (up to homology) under monodromy. Further if Ω bounds then so must ω: to see this note that now we can add a boundary to ω to get a 3-cycle of the type $\sum_{k,i} c_{ki}\alpha_i\beta_i F'_k$, but then it has to be zero, for otherwise, for some i, we are saying that the fundamental cycle of $S(M)$, $M \in OA_i$, goes to 0 as M approaches A_i. Poincaré also checks that every invariant 1-cycle Ω arises from a 3-cycle ω in the above way. For this purpose he chooses on $S(M)$ *a region* R bounded by Ω and $T_i(\Omega)$ – here T_i denotes monodromy about A_i –which approaches 0 as M approaches A_i. Using this it follows that the boundary of $\omega = Q\Omega + \sum_i \alpha_i\beta_i R_i$ contains at most terms involving cells with first factor β_i. *This 2-cycle* $\partial\omega$ *cannot cover all of* $S(0)$ *and so must be zero*. To see this note, because of our choice of R, that it covers the area of $S(O)$ "swept out" by $\Omega \subset S(y)$, monodromed back to 0, as y describes the flower shaped contour of Fig. 15. Our monodromy (cf. Third Complement) results from the movement of the *branch points* $x_k(y)$ – i.e. the common roots of $f(x, y, z) = 0 = \partial f/\partial z$ – with y, so our sweeper curve is at all times "fleeing" away from these moving branch points, and thus $\partial\omega$ cannot cover all of $S(0)$.

As this sketch indicates the argument depends heavily on the nature of H above the points A_i. In § 5 Poincaré elaborates on this by giving two examples: in both cases the subdivisions MP and (MP) are exhibited, the invariant 1-cycle Ω and the aforementioned "vanishing region" R explicitly given, and (in §§ 3 and 4 this is assumed in general) it is shown that a 1-cycle **vanishing** at A_i is necessarily of type $\Omega - T_i(\Omega)$.

Computation of $H_2(V)$ (§ 3). Using arguments similar to those sketched above a complete list of homologically distinct 2-cycles of H is displayed. Their number $b_2(V)$ is given (there are some misprints here) in terms of the numbers of homologically distinct invariant and vanishing cycles of S and the rank of a certain matrix defined using monodromy (the

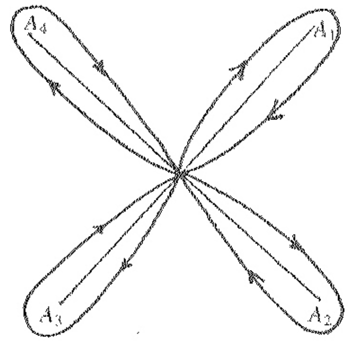

Fig. 15.

correct computation – see [40, p. 40] – identified the **Zeuthen–Segre invariant** of V with its Euler characteristic).

Computation of $H_1(V)$ (§ 4). As Poincaré points out $b_1(V)$ had already been computed by Picard who had shown (in modern terms) that $H_1(V)$ *is isomorphic to* $H_1(S)$ *mod the subgroup generated by the vanishing cycles* $\Omega - T_i(\Omega)$ (cf. a similar result, about π_1 of the smooth part of a surface, in the Third Complement). Poincaré gives another proof of this statement using arguments like those sketched above. Then, using the skewsymmetric intersection form of the genus $2p$ Riemann surface S, he shows that *the sum of the numbers of homologically distinct invariant and vanishing cycles of* S *is* $2p$, thus verifying Poincaré duality $b_1(V) = b_3(V)$ for the orientable smooth 4-manifold V. (In fact a "stronger Poincaré duality" holds for V, viz. the so-called **hard Lefschetz theorem** [40, p. 29]: *there exists a basis of* $H_1(S)$ *represented by cycles which are either invariant or vanishing*. The arguments of Lefschetz's book, which also contains generalizations for smooth complex projective varieties V of arbitrary dimension, are like those sketched above. However – see Lamotke [39] – a complete topological proof of this stronger duality still remains elusive, the best proof being via *Hodge theory*. Incidentally – see [40] – these transcendental methods were also pioneered by Picard and Poincaré.)

3.5. *The Fifth Complement*

The Fifth Complement is mostly about 2- and 3-manifolds but the method used (now called **Morse theory**) is, as Poincaré puts it, "sans doute d'un usage plus général". (For example, Morse [47] and Lusternik and Schnirelmann [42] generalized this method to *path spaces*, furnishing the tool used by Bott [6] to compute $\pi_i(U(n)) \forall i < 2n$.)

In § 2 Poincaré sections any smooth $(m+1)$-dimensional manifold $V \subset \mathbb{R}^k$ into m-dimensional subvarieties $W(t)$ by means of a one-parameter family of real hypersurfaces $\phi(x_1, \ldots, x_k) = t$. In general $W(t)$ has no singularities, but for finitely many values t_0 of t it is allowed to have one **singular point**. Poincaré notes that *the diffeomorphism type of* $W(t)$ *changes only when* t *crosses an exceptional value* t_0. If, near its singular point, the section $W(t_0)$ looks like say $\phi_1(y_1, \ldots, y_{m+1}) = 0$ (we shall take $\phi_1 = \phi - \phi(t_0)$) near the origin, then *we can always assume, after perturbing* ϕ *slightly if need be, that the* *second degree terms of ϕ_1 give a* **nondegenerate** *quadratic form*. Choosing coordinates which diagonalize this quadratic form, we see thus that near its singularity $W(t_0)$ is, for some $0 \leqslant q \leqslant m+1$, like the hypersurface

$$y_1^2 + \cdots + y_q^2 - y_{q+1}^2 - \cdots - y_{m+1}^2 = 0,$$

near the origin (so $\lambda = m+1-q$ is the **index** of the singularity). *When $q = 0$ or $q = m+1$ the* **singular link** *C of $W(t_0)$ is empty, otherwise it is diffeomorphic to $S^{q-1} \times S^{m-q}$*: this follows because C is given by the above equation and $|y_1|^2 + \cdots + |y_{m+1}|^2 = 1$. (Note that Poincaré had used a similar method even in the last two Complements, viz. sectioning a complex variety by a pencil of hypersurfaces depending on a complex parameter y. For this *holomorphic Morse theory* a singular link is given by the complex equations $z_1^2 + \cdots + z_n^2 = 0$ and $|z_1|^2 + \cdots + |z_n|^2 = 1$, and thus is the *tangent sphere bundle of a sphere*: see Lamotke [39, p. 37], for the rôle which this fact plays in this theory.)

Each $W(t)$ can have many components $w_i(t)$. Poincaré defines the **squelette** (a graph in 3-space) of V by collapsing each $w_i(t)$ to a single point. If $q = 0$ or $m+1$ then one is on a **cul-de-sac**, and if w_i splits into two (or vice versa) as we move past this t, on a **bifurcation** of the squelette. In general there are also other singular values of t which too are marked appropriately on the squelette.

SURFACES V. *Now any singularity must be a cul-de-sac or a bifurcation.* To see this let $W(0)$ have a singularity with $q = 1$ – so C consists of 4 points – near which it is the union of the intersecting arcs 13 and 24. If 1 were **associated** to 3, i.e. joinable to it in $W(0)$ without passing the singularity, then 2 must be associated to 4. Now there is no bifurcation (see Fig. 16 which shows a part of V, which we think of as a polygon with pairwise conjugation of its boundary edges) but V would be *one-sided* (for AB gets conjugated to

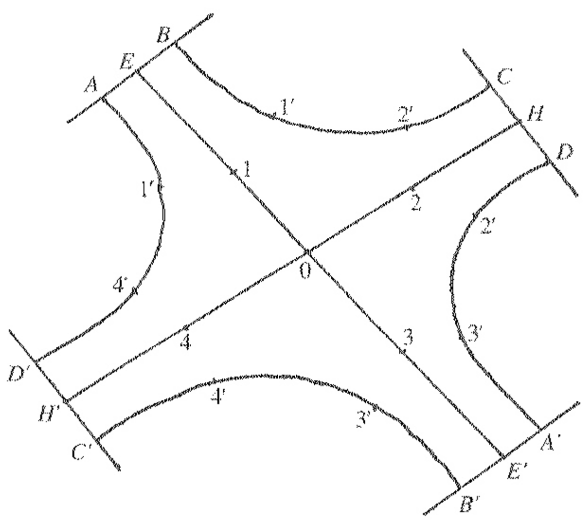

Fig. 16. A one-sided singularity.

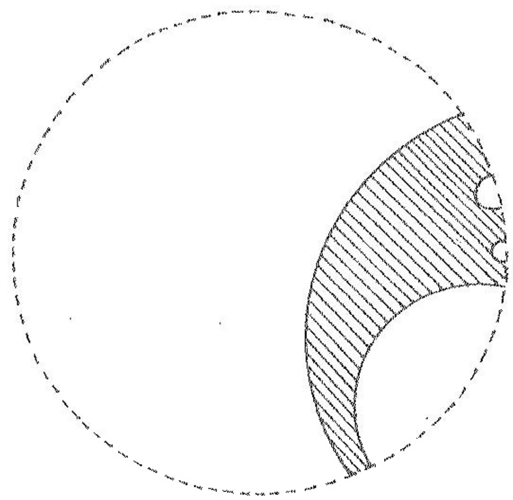

Fig. 17. Half of a Fuchsian polygon of the third kind.

$A'B'$). So 1 can only be associated to 2 or 4, and so 3 to 4 or 2, respectively, and in either of these cases we get a bifurcation of the squelette.

In § 3 this is used to sketch *a Morse theory proof of the classification of surfaces*. Choose p points on the squelette whose removal would get rid of all its circuits but keep it connected. From the above discussion one can deduce that if we cut V along the $w_i(t)$'s corresponding to these p points then we would be left with *a planar region R bounded by* $2p$ *circles*. The uniqueness of this model follows because clearly $p = b_1(V)$. One may think of R as one of infinitely many congruent **Fuchsian polygons of the third kind** tiling the plane, with conjugations realized via elements of the Fuchsian group. (See Fig. 17.) Another model of V is a **normal polygon** R' (geometrically a Fuchsian tile of the first kind) of $4p$ sides: e.g., for $p = 2$ it is an octagon 12345678 with boundary identifications giving the sole equivalence $C_1 + C_2 - C_1 - C_2 + C_3 + C_4 - C_3 - C_4 \equiv 0$ between the **fundamental cycles** $C_1 = 12,\ C_2 = 23,\ C_3 = 56,\ C_4 = 67$. For $p = 2$ (see Fig. 18) one can go (§ 4) from R to R' by **cutting** the region between DMD and $-B$ and **pasting** it to $+B$. (An algorithm for **normalizing** any polygonal representation of V was given by Brahana [17]; in many text books the classification of triangulated surfaces is proved via some such algorithm.)

ORIENTABLE 3-MANIFOLDS V. If $w(0)$ has a singular point other than a cul-de-sac, the singular link C is a union of two disjoint circles. We note (see Fig. 19) that the **throat** ("ellipse de gorge") K of $w(+\varepsilon)$ shrinks (under the **gradient flow** of the Morse function) to the singularity 0 as t decreases to 0 and then disappears. In case the two circles of C are not in the same component of $w(0) \setminus 0$, then K disconnects $W(+\varepsilon)$ and so is a boundary, *now there is bifurcation but* $w(+\varepsilon)$ *and* $w(-\varepsilon)$ *have the same* b_1. In case the two circles of C are in the same component of $w(0) \setminus 0$, then *there is no bifurcation but the* b_1 *of* $W(+\varepsilon)$ *is* 2 *more than that of* $W(-\varepsilon)$. A reduction by 1 occurs because the throat K, which is now

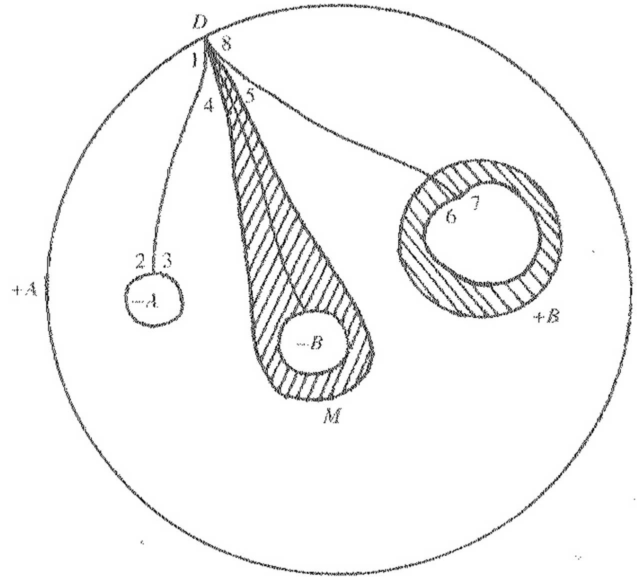

Fig. 18. Cutting and pasting.

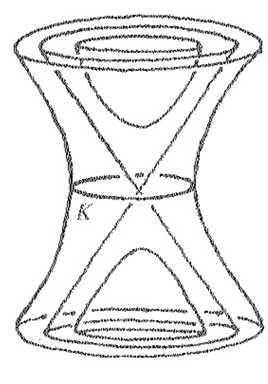

Fig. 19. Throat.

homologically nontrivial, disappears, and by another 1 because a cycle C of $W(+\varepsilon)$ with $N(C, K) \neq 0$ also disappears. There is no further reduction because if $N(C_1, K) = k_1$ and $N(C_2, K) = k_2$, then $k_2C_1 - k_1C_2$ is homologous to a cycle not cutting K, and so cannot disappear. (One obtains $W(+\varepsilon)$ from $W(-\varepsilon)$ by doing a **surgery** of type λ, or equivalently $W(\leqslant +\varepsilon)$ from $W(\leqslant -\varepsilon)$ by attaching a **handle** of index λ.)

To motivate the questions which Poincaré tackles next in §§ 3 and 4 we note that in § 5 he is going to fix (via the gradient flow), for each singular value t_q a copy of its throat on $W(t)\forall t > t_q$. Thus one needs to look at systems of *non self intersecting* ("non bouclé") cycles K_i of $W(t)$ which *do not intersect each other*. (In higher dimensions too, to simplify a Morse function, one needs to analyse systems of *spherical cycles*, of at most half dimension, on the generic level surfaces.)

For example, he checks in § 3 that *a 1-cycle of a surface W is homologous to a non self intersecting cycle iff it is a combination, with relatively prime coefficients, of the fundamental cycles*, i.e. iff it represents a **primitive element** of $H_1(W) \cong \mathbb{Z}^{2p}$. This is deduced as a corollary of a theorem which, in modern terms, says that **the map** $\mathrm{Diff}(W) \to \mathrm{Aut}_F(H_1(W))$, $f \mapsto f_*$, **is surjective.** Here $\mathrm{Aut}_F(H_1(W))$ consists of all automorphisms of $H_1(W)$ which preserve the **intersection form**. Recall that *any integral skewsymmetric matrix having determinant 1 is congruent over $\mathbb{Z}$ to*

$$F = \mathrm{diag}\left(\ldots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots\right).$$

Choosing such a basis $\mathrm{Aut}_F(H_1(V))$ is same as $\mathrm{Symp}(2p, \mathbb{Z})$, i.e. all $A \in \mathrm{GL}(2p, \mathbb{Z})$ such that $AFA' = F$. Poincaré lists some matrices over $\{-1, 0, +1\}$ which he asserts – this was verified by Brahana [8] – generate $\mathrm{Symp}(2p, \mathbb{Z})$. The theorem is proved by a long cutting and pasting argument which shows that these generators of $\mathrm{Aut}_F(H_1(W))$ arise from diffeomorphisms of W.

In § 4 Poincaré deals with some analogous questions for equivalences, e.g., *when is a given cycle of the surface equivalent to one which is non self intersecting*? Considering W as Δ/G, where $G \cong \pi_1(W)$ is a Fuchsian group of the first kind, he lifts the given cycle C to an arc of Δ going from say M to SM, $S \in G$, and denotes by α, $\beta \in \partial\Delta$ the two fixed points of this hyperbolic transformation S. He shows that *C is improperly equivalent to a non self intersecting cycle iff the non-Euclidean line $\alpha\beta$ does not intersect the corresponding line $\alpha'\beta'$ of any conjugate S' of S*. Here improper equivalence $A \equiv B$ (impr.) means that base point can move (i.e. the loops are **freely homotopic**). Poincaré points out that $A + B + C \equiv B + C + A$ (impr.), so now *cyclic reordering* is allowed, as against equivalences when no reordering may be valid, or as against homologies when all reorderings are valid.

He also gives *a rule to check if a combination of the fundamental cycles of W is equivalent to a non self intersecting cycle*. The complicated details are written out only for $p = 2$ for which case it shows, e.g., that *of all the combinations involving C_1 and C_3, only C_1, C_3, $C_1 + C_3$ and $C_3 + C_1$ are equivalent to non self intersecting cycles*. (As against this any $aC_1 + bC_3$ with $(a, b) = 1$ was homologous to a non self intersecting one; this anomaly between homologies and equivalences disappears when one uses Morse theory in dimensions $\geqslant 5$.)

The next § 5 examines an orientable 3-manifold V (with boundary $W = W(1)$) generated by connected $W(t)$'s, $0 \leqslant t \leqslant 1$, with p exceptional t_i's, at each of which b_1 increases by 2. The q-th throat fixes a non self intersecting cycle K_q on each $W(t)$ with $t > t_q$ and these cycles K_q of $W(t)$ do not intersect each other. As t increases from t_q each K_q sweeps out a ball B_q around the q-th singularity, whose final position at $t = 1$ is called A_q. Two parallel disjoint 2-balls B'_q and B''_q (which approach coincidence as t approaches 1) are then taken on either side of B_q and we denote by K'_q and K''_q their intersections with $W(t)$. We cut from $W(t)$ the small area S_q between K'_q and K''_q and paste to these two circles the 2-disks B'_q and B''_q. This new surface $W_1(t)$ *is a 2-sphere for all t bigger than* 0. To see this note that cutting out the S_q's from $W(t)$ gives a planar region R bounded by some circles and by pasting the disks we have filled in all the holes including that of the outer circle. The variety U generated by $W_1(t)$'s is thus a 3-ball U with $2p$ **scars** ("cicatrices") on its boundary, the 2 lips of the **cut** A_q which have to be identified in pairs to make V. It follows that *V is diffeomorphic to the genus p* **handlebody**, i.e. the region bounded by a genus p surface embedded in 3-space, and that this is independent of the embedding of the surface (i.e. that surfaces do not **knot** in 3-space).

Poincaré checks that any cycle of V is equivalent to one on W and that any equivalence of V is a consequence of $K_1 \equiv 0, \ldots, K_p \equiv 0$ (i.e. that $\pi_1(V)$ is the free group on p generators). Also he checks that p non self intersecting cycles $K'_1, \ldots, K'_p$ of W, which do not intersect each other, can arise in the above way only if they are equivalent to a combination of conjugates of the cycles $K_1, \ldots, K_p$ (alternatively cutting along them should give a planar region bounded by $2p$ circles).

The final § 6 considers an orientable 3-manifold V generated by connected $W(t)$'s, $0 \leqslant t \leqslant 1$, with $2p$ exceptional values of t; at the first p of these, which lie in $(0, 1/2)$, b_1 increases by 2, and at the remaining p, which lie in $(1/2, 1)$, it decreases by 2. Our V thus decomposes into two handlebodies V' and V'', the first over $[0, 1/2]$, the other over $[1/2, 1]$. The manifold is determined by the genus p surface $W = W(1/2)$ together with the two systems of **principal cycles** $K'_1, \ldots, K'_p$ and $K''_1, \ldots, K''_p$ of these handlebodies. (*Every 3-manifold admits such a Morse function*, i.e. a **Heegaard decomposition** into two handlebodies of some genus p. The least such p is called its *Heegaard genus*, and a two-dimensional description of the kind mentioned a *Heegaard diagram* of V: see [25]. A manifold has Heegaard genus 1 iff it is one of the L_T's of Notes 18 and 24, but *classification is unknown for any Heegaard genus* $\geqslant 2$.)

Poincaré shows that any cycle of this closed 3-manifold V is equivalent to one lying on W and that any equivalence is a consequence of the obvious equivalences $K'_1 \equiv 0, \ldots, K'_p \equiv 0$, and $K''_1 \equiv 0, \ldots, K''_p \equiv 0$ (this determines $\pi_1(V)$). Writing the principal cycles as combinations of the fundamental cycles and reordering one gets the homologies

$$m'_{i,1}C_1 + \cdots + m'_{i,p}C_{2p} \simeq 0,$$
$$m''_{i,1}C_1 + \cdots + m''_{i,p}C_{2p} \simeq 0,$$

which determine the Betti number and torsion coefficients of V. So these are the same as a 3-sphere, i.e. V is a **homology sphere**, *iff the $2p \times 2p$ determinant formed by the above integer coefficients is* ± 1. However, as the example below shows this need not be a *homotopy sphere*.

Poincaré defines his homology 3-sphere via a Heegaard diagram: $p = 2$ and W is represented as a planar region R bounded by four circles, he takes $K'_1 = C_1$, $K'_2 = C_3$ while K''_1 and K''_2 are given, respectively, by the unions of the full and dotted segments. (See Fig. 20.)

He computes using the above method to see that $\pi_1(V)$ is generated by C_2 and C_4 subject to the equivalences $4C_2 + C_4 - C_2 + C_4 \equiv 0$ and $-2C_4 - C_2 + C_4 - C_2 \equiv 0$. The corresponding homologies $3C_2 + 2C_4 \simeq 0$ and $2C_2 - C_4 \simeq 0$ have determinant 1. On the other hand $\pi_1(V)$ is nonzero because on adjoining the first of the following equivalences one has

$$-C_2 + C_4 - C_2 + C_4 \equiv 0, \qquad 5C_2 \equiv 0, \qquad 3C_4 \equiv 0.$$

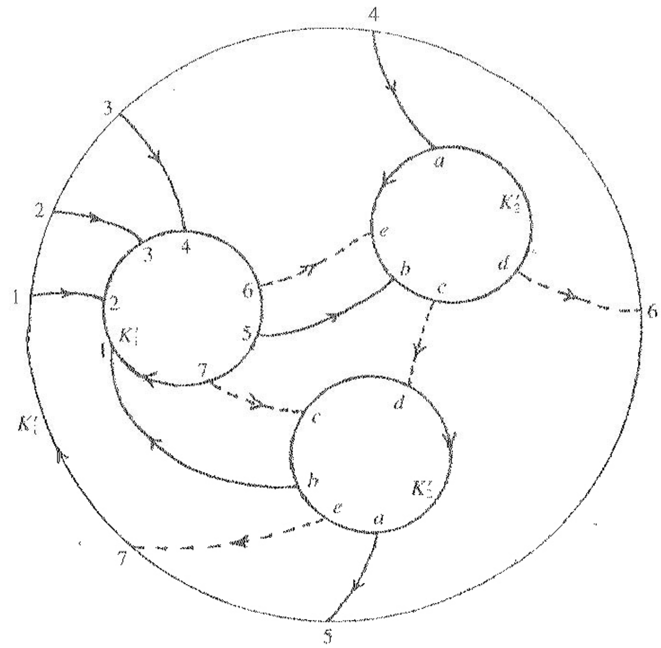

Fig. 20. Poincaré's homology 3-sphere.

which are the defining relations of the **icosahedral group**. (This, and Example 3 of "Analysis Situs", already suggest what Kneser [36] later checked: V can be obtained by conjugating facets of a dodecahedron.)

Then comes *the famous query*: "is it possible that the fundamental group of V reduces to the identity substitution, and yet V is not diffeomorphic to a sphere?" ... "*But this question will drag us too far*". (**Poincaré's conjecture** still seems to be open, but we note that Poincaré's method, i.e. Morse theory, did enable Smale [80] to show, *in dimensions* $n \geqslant 5$, that any homotopy n-sphere is necessarily *homeomorphic* to the n-sphere; by Milnor [44] it need not be diffeomorphic.)

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