THE CANONICAL CHAIN HOMOTOPY OF WEIL MAPS

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1.Introduction

Let G be a connected Lie group and W(G) its "Weil algebra" ([1] and §3 below).Let ${}^1W_i \subseteq W(G)$ denote the ideal generated by polynomials of degree $\geqslant i$; also let ${}^2W_{2i} = {}^2W_{2i-1} = {}^1W_i$. If d denotes the differential of the Weil algebra , one can check that $d({}^1W_i) \subseteq {}^1W_i$ for all i . Hence $d({}^2W_i) \subseteq {}^2W_i$ for all i. Thus we have two "filtered complexes" which will be denoted by ${}^1W(G)$ and ${}^2W(G)$.

Let $\mathcal{F}$ be a smooth foliation of a smooth manifold M. An <u>invariant principal bundle</u> (P, ξ) over the foliated manifold $(M, \mathcal{F})$ is a continous G-bundle P over M, equipped with a maximal family ξ of "local trivializations "$\xi_i : U_i \times G \xrightarrow{\cong} P|U_i$ such that (a) $\{U_i\}$ is an open cover of M and (b) the associated "transition functions " $\xi_{ij} : U_i \cap U_j \to G$ are smooth functions constant on the leaves of the induced foliation of $U_i \cap U_j$. Such a ξ is called an <u>invariant structure</u> on the continous bundle P.

Given such a ξ there exists a unique smooth structure $\hat{\xi}$ on P such that (c) each ξ_i is a diffeomorphism. In general this correspondence $\xi \to \hat{\xi}$ is not one-one.

We denote by $\Lambda(P, \hat{\xi})$, or just $\Lambda(P)$, the algebra of $\hat{\xi}$-smooth forms on P. Let $\Lambda_i(\mathcal{F}_P)$ be the subspace spanned by all forms ω which vanish when degω-i+1 of the vectors project (on M) to vectors tangent to $\mathcal{F}$. If d denotes the exterior derivative, one can check that $d(\Lambda_i(\mathcal{F}_P)) \subseteq \Lambda_i(\mathcal{F}_P)$. Thus we get another filtered complex which will be denoted by $\Lambda(\mathcal{F}_P, \hat{\xi})$, or just $\Lambda(\mathcal{F}_P)$.

We recall that both W(G) and $\Lambda(P)$ are graded-commutative algebras equipped with "inner products ι_X" and "Lie derivatives L_X", with respect to each left invariant vector field X of G. A <u>connection</u> (or <u>Weil map</u>) f on $(P, \hat{\xi})$ is a graded algebra homomorphism $W(G) \xrightarrow{f} \Lambda(P)$ which commutes with the differentials and all these operations ι_X and L_X.

Such a f is entirely determined by its values on the left invariant 1-forms $A^1(G) \subseteq W(G)$. By using the trivializations ξ_i and the injection $\epsilon_i : U_i \to U_i \times G$ given by $\epsilon_i(x)=(x,1)$, we get the map $f_i = \epsilon_i^* \xi_i^* f : A^1(G) \to \Lambda^1(U_i)$. We say that f is a Bott (or basic) connection on (P, ξ) if, for all i , the one forms lying in the image of f_i vanish on vectors tangent to the leaves of $\mathcal{F}$. If furthermore these 1-forms are "invariant " (with respect to all vector fields tangent to $\mathcal{F}$), we say that f is an invariant connection on (P, ξ). One can see that a Bott connection exists if M is paracompact. However invariant connections need not exist . One can check that a Bott connection gives a " map of filtered complexes " ${}^1W(G) \xrightarrow{f} \Lambda(\mathcal{F}_P)$ i.e. $f({}^1W_i) \subseteq \Lambda_i(\mathcal{F}_P)$ for all i. Similarly an invariant connection gives a map ${}^2W(G) \xrightarrow{f} \Lambda(\mathcal{F}_P)$ of filtered complexes.

Two maps of filtered complexes are called k-chain homotopic if they are related by a chain homotopy that disturbs the filtration by at most k-1 units . Clearly this an equivalence relation and one can thus speak of k-chain homotopy classes.

THEOREM. Let (P, ξ) be a principal bundle over the foliated manifold $(M, \mathcal{F})$. Then

(A) all its Bott connections ${}^1W(G) \xrightarrow{f} \Lambda(\mathcal{F}_P, \hat{\xi})$ lie in the same 1-chain homotopy class, and

(B) all its invariant connections ${}^2W(G) \xrightarrow{f} \Lambda(\mathcal{F}_P, \hat{\xi})$ lie in the same 2-chain homotopy class.

These homotopy classes will be denoted by $[A](\xi)$ and $[B](\xi)$ respectively.

For any two connections f , g on the smooth bundle P, we will define the canonical chain homotopy p(f,g). It is a linear map p: $W(G) \to \Lambda(P)$ which, besides satisfying dp+pd=g-f, has many other pleasing properties. For instance, if f and g are Bott (resp. invariant) connections then $\rho({}^1W_i) \subseteq \Lambda_i(\mathcal{F}_P)$ (resp. $\rho({}^2W_i) \subseteq \Lambda_{i-1}(\mathcal{F}_P)$) for all i. This verification will prove the theorem.

Each filtered complex has a "spectral sequence " [2]; we will denote the spectral sequences of ${}^1W(G)$, ${}^2W(G)$ and $\Lambda(\mathcal{F}_P)$ by ${}^1E_i(G)$, ${}^2E_i(G)$ and $E_i(\mathcal{F}_P)$ respectively. We recall (p. 321 of [2]) that k-chain homotopic maps induce the same spectral sequence homomorphisms from the kth term onward. Thus whenever (P, ξ) admits a Bott --- resp. invariant--- connection then we have homomorphisms $A_i(\xi): {}^1E_i(G) \to E_i(\mathcal{F}_P)$, $i \geq 1$ ---resp. $B_j(\xi): {}^2E_j(G) \to E_j(\mathcal{F}_P)$, $j \geq 2$--- which depend only on the invariant structure ξ.

Further remarks and bibliographic comments will be made in no.4 below.

2. The canonical chain homotopy

Throughout we will use the words "graded algebra ", "derivation" , "bracket " etc. in their usual sense. (See, e.g. [8]).

Let $\Lambda(M)$ be the graded algebra of smooth skewsymmetric covariant tensors-- or "forms"-- on M, with the multiplication being defined as in [7]. For each vector field X on M one defines [7] the Lie derivative $L_X : \Lambda(M) \to \Lambda(M)$ by

$$L_X\omega = \lim_{s\to 0} \frac{\omega - (\psi_{-s})^*(\omega)}{s}. \tag{1}$$

Here ψ_s is the local 1-parameter group of diffeomorphisms--- or the "flow " --- which corresponds to X. The interior product $\imath_X : \Lambda(M) \to \Lambda(M)$ is the derivation which on 1-forms is given by $\imath_X\omega = \omega(X)$.

When M is replaced by a Lie group G and X is a left invariant vector field thereon then each ψ_s is a right translation of G and thus L_X maps the subalgebra of left invariant forms $A(G) \subseteq \Lambda(G)$ into A(G). More generally given a smooth principal G-bundle P, right action by ψ_s defines a flow on P; the corresponding "canonical vertical vector field " on P is also denoted by X. In this way we have the maps $\imath_X, L_X : \Lambda(P) \to \Lambda(P)$.

We recall [1] that a connection $W(G) \xrightarrow{f} \Lambda(P)$ restricts to a map $f: A^1(G) \to \Lambda^1(P)$ such that

$$\imath_X f(\sigma) = \sigma(X)$$
$$\text{and} \quad L_X f(\sigma) = f(L_X \sigma) \,, \tag{2}$$

for all $\sigma \in A^1(G)$; and conversely, any map $f: A^1(G) \to \Lambda^1(P)$ obeying (2) extends in a unique way to a connection.

Let $\mathbb{R}$ denote the real numbers. Now $P \times \mathbb{R}$ is a principal G-bundle over $M \times \mathbb{R}$. We denote by $\frac{\partial}{\partial x}$ the vector field on $P \times \mathbb{R}$ which corresponds to the flow $\zeta_s(a,r) = (a, r+s)$. Also we define, for each $t \in \mathbb{R}$, the map $\varphi_t : P \to P \times \mathbb{R}$ by $\varphi_t(a) = (a,t)$.

Given two connections $f,g : W(G) \to \Lambda(P)$ we define a map $F: A^1(G) \to \Lambda^1(P \times \mathbb{R})$ so that,

$$F(\sigma)\left(\frac{\partial}{\partial x}\right) = 0$$
$$\text{and} \quad \varphi_t^* F(\sigma) = t . f(\sigma) + (1-t) . g(\sigma) \,, \tag{3}$$

for all $\sigma \in A^1(G)$, $t \in \mathbb{R}$. One can check that this map F obeys the conditions (2) and so extends to a connection $F: W(G) \to \Lambda(P \times \mathbb{R})$.

Next we define $p(f,g): W(G) \to \Lambda(P)$ by

$$p(\sigma) = \int_0^1 \varphi_t^* \imath_{\frac{\partial}{\partial x}} F(\sigma) \, |\mathbb{R}| \,. \tag{4}$$

Here $|\mathbb{R}|$ denotes the Lebesgue measure and $\sigma \in W(G)$.

LEMMA. <u>p is a chain homotopy from f to g i.e. $[d, p] = g - f$.</u>

Proof. Since $d\varphi_t^* = \varphi_t^* d$, $dF = Fd$ and $d\imath_{\frac{\partial}{\partial x}} + \imath_{\frac{\partial}{\partial x}} d = L_{\frac{\partial}{\partial x}}$ we see from (4) that

$$(dp + pd)(\sigma) = \int_0^1 \varphi_t^* L_{\frac{\partial}{\partial x}} F(\sigma) \, |\mathbb{R}| .$$

Using (1) it follows that

$$\varphi_t^* L_{\frac{\partial}{\partial x}} F(\sigma) = \lim_{s \to 0} \frac{\varphi_t^* F(\sigma) - \varphi_t^* \zeta_{-s}^* F(\sigma)}{s} .$$

But $\zeta_{-s} \varphi_t = \varphi_{t-s}$. So the above expression equals $\frac{d}{dt}\left[\varphi_t^* F(\sigma)\right]$. Substituting this into the integral we get

$$(dp + pd)(\sigma) = \varphi_1^* F(\sigma) - \varphi_0^* F(\sigma).$$

But $(\varphi_t)_*$ maps the canonical vertical vector field X of P into the corresponding vector field X of $P\times\mathbb{R}$. So $(\varphi_t)^*$ commutes with i_X and L_X and the maps $\varphi_t^*F: W(G)\to\Lambda(P)$ are connections. Since, by (3), φ_1^*F coincides with g on $A^1(G)$ we have $\varphi_1^*F = g$. Similarly $\varphi_0^*F = f$.

3. The filtrations

Let S(G) denote the algebra of left invariant symmetric covariant tensors -- or "polynomials" -- on the Lie group G. A tensor of rank i is given the grading 2i ; with this convention S(G) is graded-commutative. For each left invariant vector field X we define $L_X: S(G)\to S(G)$ by using (1); on the other hand $i_X: S(G)\to S(G)$ is the derivation which on $S^1(G)$ is given by $i_X(\sigma)=\sigma(X)$.

A(G) (resp. S(G)) will now be identified with the subalgebra $A(G)\otimes 1$ (resp. $1\otimes S(G)$) of the graded tensor product $W(G)=A(G)\otimes S(G)$. We extend the four derivations $i_X, L_X: A(G)\to A(G)$ and $i_X, L_X: S(G)\to S(G)$ to all of W(G) by defining them to be zero on the remaining generators. This gives us, respectively, the derivations i_X^A, L_X^A, i_X^S and L_X^S in W(G). We have the linear isomorphism $h: A^1(G)\to S^1(G)$ given by $h(\omega\otimes 1)=1\otimes\omega$. We now extend h to a derivation of W(G), and also define three more derivations of W(G), by

$$\begin{aligned} h(\sigma) &= \sum_i h(X_i').i^A_{X_i}\sigma, \\ k(\sigma) &= \sum_i X_i'.i^S_{X_i}\sigma, \\ d^A(\sigma) &= \sum X_i'.L^A_{X_i}\sigma, \\ \text{and}\quad d^S(\sigma) &= \sum_i X_i'.L^S_{X_i}\sigma, \end{aligned} \tag{5}$$

for all $\sigma\in W(G)$. Here X_i' is a basis of $A^1(G)$ and X_i the dual basis of left invariant vector fields. It is clear that this definition does not depend on the choice of the basis X_i'. We recall [1] that the derivations i_X, L_X, d: $W(G)\to W(G)$ are defined by $i_X=i_X^A$, $L_X=L_X^A+L_X^S$ and $d=h+d^A+d^S$. It follows immediately from (5) that the ideal ${}^1W_i$, generated by polynomials of degree $\geq i$, is mapped into itself by d. The following relations are also proved in [1]:

$$[L_X, L_Y] = L_{[X,Y]} \qquad [L_X, \imath_Y] = \imath_{[X,Y]} \qquad [L_X, d] = 0 \tag{6}$$
$$[\imath_X, \imath_Y] = 0 \qquad [\imath_X, d] = L_X \qquad [d, d] = 0$$

In particular, d is a differential. So ${}^1W(G)$ and ${}^2W(G)$ are filtered complexes.

If $\mathcal{F}$ is a smooth foliation of M we denote by $\Lambda_1(\mathcal{F})$ the ideal of $\Lambda(M)$ generated by those 1-forms which vanish on vectors tangent to leaves. The fact that vector fields tangent to leaves form a sub Lie algebra can be re-stated as $d(\Lambda_1(\mathcal{F})) \subseteq \Lambda_1(\mathcal{F})$. So if $\Lambda_i(\mathcal{F})$ denotes the ith power of $\Lambda_1(\mathcal{F})$, we have $d(\Lambda_i(\mathcal{F})) \subseteq \Lambda_i(\mathcal{F})$ for all i; this filtered complex is denoted by $\Lambda(\mathcal{F})$. The given principal invariant bundle (P, ξ) over $(M, \mathcal{F})$ will be equipped with the $\hat{\xi}$-smooth foliation $\mathcal{F}_P$ whose leaves are the inverse images of the leaves of $\mathcal{F}$; this will give us the filtered complex $\Lambda(\mathcal{F}_P)$ of the introduction. Again, for each open set $U \subseteq M$ we have the induced foliation $\mathcal{F}_U$ and so the filtered complex $\Lambda(\mathcal{F}_U)$.

Let $W(G) \xrightarrow{f} \Lambda(P)$ be any connection. For each $g \in G$ we have the right translations R_g of G and P. They induce automorphisms R_g^* in $W(G)$ and $\Lambda(P)$. Since G is connected one can prove by using (1) that, for all $g \in G$,

$$f R_g^* = R_g^* f \, . \tag{7}$$

Next we examine how the chain maps $f_i = \epsilon_i^* \xi_{i}^{*} f : W(G) \to \Lambda(U_i)$ are related to each other. If v is a tangent vector at the point x of $U_i \cap U_j$, and $\sigma \in A^1(G)$, then

$$f_j(\sigma)(v) = f_i\left(R^*_{\xi_{ij}(x)}\sigma\right)(v) + \xi_{ij}^*(\sigma)(v) \, . \tag{8}$$

To prove this we note that $\xi_i^{-1}\xi_j\epsilon_j(x) = (x, \xi_{ij}(x))$. Hence

$$\left(\xi_i^{-1}\xi_j\epsilon_j\right)_*(v) = \left(R_{\xi_{ij}(x)}\epsilon_i\right)_*(v) + \overline{(\xi_{ij})_*(v)}$$

where $\overline{(\xi_{ij})_*(v)}$ denotes the canonical vertical vector corresponding to the left invariant vector field of G which, at $\xi_{ij}(x) \in G$, equals the vector $(\xi_{ij})_*(v)$. Using this, (7), and the fact that for a canonical vertical vector X, $f(\sigma)(X) = \sigma(X)$, we get (8).

Conversely given maps $f_i : A^1(G) \to \Lambda^1(U_i)$ satisfying (8) we can in an obvious way, define f: $A^1(G) \to \Lambda^1(P)$ so that (2) holds. (See also [7], vol. I, p.66). Using this remark we prove that if M is paracompact, then there exists a Bott connection on (P,ξ). Due to the paracompactness we know (see, e.g. [7]) that a connection f exists. Let t be any smooth 1-1 tensor on M such that $t^2 = t$ and, at each point, ker(t) is precisely the tangent space of the foliation. We define $f_i' : A^1(G) \to \Lambda^1(U_i)$ by $f_i'(\sigma)(v) = f_i(\sigma)(tv)$. Since $\xi_{ij} : U_i \cap U_j \to G$ is constant on leaves, $\xi_{ij}^*(\sigma) \in \Lambda_1(\mathcal{F}_{U_i \cap U_j})$, and is infact invariant along leaves. So, because v-tv $\in$ ker(t) is tangent to a leaf, we see that $\xi_{ij}^*(\sigma)(v) = \xi_{ij}^*(\sigma)(tv)$. Thus the equation (8) is true even if f_i, f_j are replaced by f_i', f_j'. The resulting connection f′ is Bott.

Given a Bott connection f we now want to verify that it gives a map ${}^1W(G) \xrightarrow{f} \Lambda(\mathcal{F}_P)$ of filtered complexes. Clearly it suffices to check that f maps $S^1(G)$ into $\Lambda_1(\mathcal{F}_P)$. Each element of $S^1(G)$ is of the form hσ, $\sigma \in A^1(G)$. But $i_X f(h\sigma) = f(i_X^A h\sigma) = 0$. So it suffices to verify that each $f_i : W(G) \to \Lambda(U_i)$ maps $S^1(G)$ into $\Lambda_1(\mathcal{F}_{U_i})$. Now, by (5),

$$\begin{aligned} f_i(h\sigma) &= f_i(d\sigma - d^A\sigma) \\ &= df_i(\sigma) - \sum_j f_i(X_j') f_i(i^A_{X_j}\sigma), \end{aligned} \tag{9}$$

for each $\sigma \in A^1(G)$. Since f_i maps $A^1(G)$ into $\Lambda_1(\mathcal{F}_{U_i})$, $df_i(\sigma) \in \Lambda_1(\mathcal{F}_{U_i})$ while the second term lies in $\Lambda_2(\mathcal{F}_{U_i})$.

Similarly an invariant connection gives us a map ${}^2W(G) \xrightarrow{f} \Lambda(\mathcal{F}_P)$ of filtered complexes. To prove this we have to verify that each f_i maps $S^1(G)$ into $\Lambda_2(\mathcal{F}_{U_i})$. But, for any vector field V along the leaves, $i_V df_i(\sigma) = L_V f_i(\sigma) = 0$; so $df_i(\sigma) \in \Lambda_2(\mathcal{F}_{U_i})$ and using (9) the result follows.

Let $\{\xi_\alpha\} \subseteq \xi$ be such that $\{U_\alpha\}$ is an open cover of M. We remark that a connection f will be Bott (resp. invariant) if, for all α and all $\sigma \in A^1(G)$, $f_\alpha(\sigma)$ is a form (resp. invariant form) of $\Lambda_1(\mathcal{F}_{U_\alpha})$. This follows from (8).

<u>Proof of theorem.</u> Let ${}^1\mathcal{F}$ -- resp. ${}^2\mathcal{F}$ -- denote the foliation of $M\times\mathbb{R}$ whose leaves are $L\times\mathbb{R}$ --resp. $L\times\{t\}$ -- where L is any leaf of $\mathcal{F}$. We equip the bundle $P\times\mathbb{R}$ with the set $\eta=\{\eta_i\}$ of local trivializations $\eta_i:(U_i\times\mathbb{R})\times G\to(P\times\mathbb{R})|U_i\times\mathbb{R}$, where $\eta_i(x,t,g) =(\xi_i(x,g),t)$; clearly the associated transition functions are constant on the leaves of ${}^1\mathcal{F}$ (and so of ${}^2\mathcal{F}$). We can enlarge η to the maximal set ${}^1\eta$ -- resp. ${}^2\eta$ -- to get a principal invariant bundle $(P\times\mathbb{R},{}^1\eta)$ over $(M\times\mathbb{R},{}^1\mathcal{F})$ -- resp. $(P\times\mathbb{R},{}^2\eta)$ over $(M\times\mathbb{R},{}^2\mathcal{F})$.

(A). If f and g are Bott connections then the connection F, defined in (3), is easily seen to be a Bott connection on $(P\times\mathbb{R},{}^1\eta)$. To see this apply above remark to the subset $\eta\subseteq{}^1\eta$. So $F:{}^1W(G)\to\Lambda({}^1\mathcal{F}_{P\times\mathbb{R}})$ is a map of filtered complexes. In the definition (4) of p, the vector field $\frac{\partial}{\partial x}$ is tangent to ${}^1\mathcal{F}_{P\times\mathbb{R}}$ while $\varphi_t:P\to P\times\mathbb{R}$ maps leaves of $\mathcal{F}_P$ into those of ${}^2\mathcal{F}_{P\times\mathbb{R}}$, and so of ${}^1\mathcal{F}_{P\times\mathbb{R}}$. Hence $p({}^1W_i(G))\subseteq\Lambda_i(\mathcal{F}_P)$ for all i.

(B). If f and g are invariant connections then the connection F is an invariant connection on $(P\times\mathbb{R},{}^2\eta)$. So now we have the map $F:{}^2W(G)\to\Lambda({}^2\mathcal{F}_{P\times\mathbb{R}})$ of filtered complexes. This time the interior product $\iota_{\frac{\partial}{\partial x}}$ can disturb filtration by one unit, because $\frac{\partial}{\partial x}$ is not tangent to ${}^2\mathcal{F}_{P\times\mathbb{R}}$. Hence $p({}^2W_i(G))\subseteq\Lambda_{i-1}(\mathcal{F}_P)$ for all i.

4. Concluding Remarks

(a). Let $c(\mathcal{F})$, or just c, denote the codimension of the foliation $\mathcal{F}$. We construct <u>a contravariant functor $(M,\mathcal{F})\to(P(\mathcal{F}),\xi(\mathcal{F}))$ from the category of foliated manifolds to the category of principal invariant bundles</u> as follows. (The morphisms in the two categories are defined in an obvious way). $P(\mathcal{F})$ is the principal $GL(c(\mathcal{F}))$-bundle associated to the bundle of 1-forms vanishing on the foliation. Each pair (U,ω)-- where $U\subseteq M$ is an open set and $\omega=(\omega_1,\omega_2,\ldots,\omega_c)$ a set of linearly independent invariant 1-forms on U-- enables us to construct a trivialization $U\times GL(c)\cong P(\mathcal{F})|U$. $\xi(\mathcal{F})$ is the set of all such trivializations. $(P(\mathcal{F}),\xi(\mathcal{F}))$ is called the <u>principal normal</u>

bundle of $(M,\mathcal{F})$. Note that if $0<c(\mathcal{F})<\dim M$ then $\xi(\mathcal{F})$ is by no means the only invariant structure compatible with the smooth structure of $P(\mathcal{F})$.

(b). On the foliated manifold $(M,\mathcal{F})$ consider a smooth bundle P with differentiable structure η. If it admits a connection f which is a map of filtered complexes ${}^1W(G)\xrightarrow{f}\Lambda(\mathcal{F}_P,\eta)$ then there exists an invariant structure ξ on P such that (i) $\hat{\xi}=\eta$ and (ii) f is a Bott connection on (P,ξ). One can prove this statement by defining ξ_i to be the local trivializations given by sections of P which are "parallel" over each leaf of $\mathcal{F}_{U_i}$. Note that all such connections lie in the same ∞-homotopy class. Let us denote this, possibly empty, class by $[C](\eta)$.

(c). The canonical chain homotopy $p(f,g)$ commutes with all the operations i_X and L_X. This follows easily from formula (4). We note that a Bott (resp. invariant) connection has the vanishing property: ${}^1W_{c+1}\subseteq \ker f$ (resp. ${}^2W_{c+1}\subseteq \ker f$). One can check that if f and g are Bott connections then $p(f,g)$ also has the vanishing property. These remarks enable one to construct a host of chain homotopy classes ; e.g. we will have a ∞-homotopy class $[C](\eta):{}^1W(G)/{}^1W_{c+1}\to\Lambda(\mathcal{F}_P,\eta)$.

(d). To give another application we identify $\Lambda(M)$ with the image of the algebra monomorphism $\pi^*:\Lambda(M)\to\Lambda(P)$. Here $\pi:P\to M$ denotes the projection map of the principal invariant bundle (P,ξ). In this way $\Lambda(M)$ becomes the basic subcomplex [1] of $\Lambda(P)$; i.e. the set of elements annihilated by the operations i_X and L_X. On the other hand the basic subcomplex of $W(G)$ is $I_S(G)$, the algebra of symmetric invariant polynomials equipped with the zero differential. If M is paracompact we have the non-empty 1-chain homotopy class $[A](\xi):{}^1I_S(G)\to\Lambda(\mathcal{F})$; one thus has the maps $A_i(\xi):{}^1I_S(G)\to E_i(\mathcal{F})$, $i\geqslant 1$. If P admits an invariant connection we also have a non-empty 2-chain homotopy class $[B](\xi):{}^2I_S(G)\to\Lambda(\mathcal{F})$; one can see that this happens only if the maps $A_i(\xi)$ are trivial. In this case we have, for $j\geqslant 2$ the maps $B_j(\xi):{}^2I_S(G)\to E_j(\mathcal{F})$

We note that for $i = j = \infty$ these maps depend only on the differentiable structure $\hat{\xi}$ of P. Further, if $\mathcal{F}$ is the foliation of M by points then $E_2(\mathcal{F}) = E_\infty(\mathcal{F}) = H(M)$ and the map B_2 is the well-known Chern-Weil homomorphism. Note that, in this case, every connection is an invariant connection.

(e). We note that in the formula (5) for the derivation d^A of W(G) we do not have the factor $\frac{1}{2}$ of [1], because we are using the 'other' definition of exterior multiplication. One can easily verify that the derivation $[d,k]$ of W(G) multiplies each tensor of W(G) with its rank. This implies that W(G) is acyclic. Infact (5) also shows that $k({}^1W_i) \subseteq {}^1W_{i-1}$. So the second term of the spectral sequence of ${}^1W(G)$ is trivial. Similarly ${}^2E_3(G)$ is also trivial.

(f). In the category of principal invariant bundles one has of course a notion of equivalence. Equivalence classes of principal G-bundles on $(M,\mathcal{F})$ form the cohomology set $H^1(M, G_{s(\mathcal{F})})$. Here $G_{s(\mathcal{F})}$ denotes the sheaf of germs of smooth G-valued functions on M which are constant on leaves. It is a subsheaf of G_s, the sheaf of germs of all smooth G-valued functions. In this language the existence of an invariant structure determines a reduction of the structure sheaf of the bundle [4] from G_s to $G_{s(\mathcal{F})}$. One should also note that the chain homotopy classes [A] and [B] depend, upto a natural equivalence, only on the equivalence class of the invariant bundle (P,ξ).

(g). The results of this paper are taken from my thesis ([10], §19), done under the guidance of A.V.Phillips. On the other hand Molino [9] and Kamber and Tondeur [5][6] have investigated the concept of a 'foliated bundle' i.e. a principal bundle equipped with a 'partial flat connection over the leaves'. Clearly this is about the same thing as an invariant bundle. It seems to me that the reduction of the structure sheaf is a more basic concept; so I have put it in the forefront and also used my original terminology "invariant bundle". We point out that the 'CTP' connections of Molino are the invariant connections of this paper. We refer the reader to his paper for examples of foliations for which $P(\mathcal{F})$ does not admit invariant connections. Kamber and Tondeur also view connections as maps of filtered complexes and, in [6], give some homotopy invariance results by using very different and algebraical constructions. Our approach is nearer to that in [7], vol.II, and the paper of Chern and Simons [3]. Various other chain homotopies are also considered in my thesis and in [11].

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