The prettiest composition, part five
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57. We have proved above a bit more than was needed. That is, $\mathcal{G}$ takes us from any ordered triple of $\widehat{L}$ to any other, while we needed only to go from any unordered triple to any other. This job is done even by the orientation preserving subgroup $\mathcal{G}_0$ of $\mathcal{G}$, which along with the other, orientation reversing component, is like $\heartsuit_3$ topologically just an *open solid torus*; further, *the baby action of* $\mathcal{G}_0$ *on it triple covers the swallowtail* $\heartsuit_3$:- we can write each transformation of $\mathcal{G}_0$ as a rotation, times a translation keeping ∞ fixed, times a homethety keeping both 0 and ∞ fixed. Besides, there are only six elements of $\mathcal{G}$ that map any ordered set (a, b, c) on its six permutations, of which three take us to the even permutations that alone are in $\mathcal{G}_0$. □

Clearly just the baby group $\mathcal{G}$ and its simple subgroup $\mathcal{G}_0$ are natural, all conjugates of the maximal compact or maximal abelian subgroup S^1 of the latter are equally worthy of our attention. So here's a bit *about the foliation of* $\heartsuit_n, n \geq 3$ *under the action of* $\mathcal{G}_0$:- this action preserves the cyclic order of the roots. The open solid 3-toral orbits are covered by the $\mathcal{G}_0$-action a divisor of n times. Which divisor depends on which subgroup of the cyclic group $\mathbb{Z}_n$ makes, upto this $\mathcal{G}_0$ action, $conv(z_1, \ldots z_n)$ most regular. □

Not only for $n = 3$ but for $n = 4$ also, the divisor 1 is not possible in the above covering result, *the action of* $\mathcal{G}_0$ *on* $\heartsuit_4$ *covers one open solid toral leaf* 4 *times, and all others twice* :- Applying to any equation a rotation and translation we get an equation with roots $\{x_1 < 0, x_2 = 0, x_3 > 0, x_4 = \infty\}$, and then after a homothety keeping 0 and ∞ fixed we can assume $x_1 x_3 = -4$. Now regard the points of S^1 tied to them $z_1, z_2 = T, z_3, z_4 = -T$. The condition $|x_1|/2 = 2/x_3$ is telling us the chord $z_1 z_3$ subtends an angle of ninety degrees on the centre $-T$ of the circular mirror. So $z_3 = -z_1$ are also antipodal. The first case occurs if and only if the diameter $z_1 z_3$ is vertical. □

So about this picture emerges, *below each solid toral* $\mathcal{G}_0$*-orbit of* $\heartsuit_4$ *there is an* S^1*-orbit of the mobius strip* $\heartsuit_2$ *going at half the speed* :- The angle $0 < \theta \leq \pi/2$ between the diameters $z_1 z_3$ and $z_2 z_4$ determines the $\mathcal{G}_0$-orbit; below on S^1-orbit that quadratic to whose roots the tied points on S^1 are $z_1^2 = z_3^2$ and $z_2^2 = z_4^2$; between which the angle is 2θ, for the middle orbit π, and towards the boundary of the strip almost zero. □

Even more partial is yet my understanding of this three-dimensional leaved *baby foliation* for $n > 4$, but this much is clear that, *each leaf of* $\heartsuit_n$ *has an equation with roots in order* $\{0, 1, x_3, \ldots, x_{n-1}, \infty\}$:- apply to an equation of the leaf whose roots in this *cyclic order* on circle $\widehat{L}$ are $\{y_1, y_2, \ldots, y_n\}$ the unique transformation of $\mathcal{G}_0$ such that $y_1 \mapsto 0, y_2 \mapsto 1, y_n \mapsto \infty$. □

This oner $y \mapsto x = \frac{y-y_1}{y_2-y_1} \cdot \frac{y_n-y_2}{y_n-y}$ is dubbed the *cross-ratio of the* 4*-tuple* (y_1, y_2, y, y_n)*,* and from this definition is clearly *invariant under* $\mathcal{G}_0$*-action.* Note for a segment or half-ray (y_1, y_n) the *cayley distance* separating any 2-tuple (y_2, y) on it was also the same ratio, but for a log etc in front to make it additive. Based on which we defined, for any euclidean open set U not containing a full line, a *relativistic distance* separating any 2-tuple. But if we don't want to be constrained by any boundedness condition *maybe giving up on* 2*-tuples we ought to consider* 4*-tuples only* : for any 4-tuple (P, Q, R, S) of a sphere $\widehat{E}$ of any dimension n we have the cross-ratio $\frac{PR}{PQ} \cdot \frac{QS}{RS}$ and the mobius transformations of $\widehat{E}$ are precisely all its bijections that preserving them.

Resuming, *the total number of equations with roots* $\{0, 1, x_3, \ldots, x_{n-1}, \infty\}$ *in any leaf of* $\heartsuit_n$ *seems to be a divisor of* n :- The invariance of cross-ratio shows our definition did not depend on which equation in the leaf we had started from, but yes, it depended on which root $y_1 \in \widehat{L}$ thereof we had deemed first on the circle. So we can start from such an equation in the leaf for whose roots the tied points of S^1 have the maximum cyclic regularity. $\square$

For example, *each leaf of* $\heartsuit_4$ *has at most two with roots* $\{0, 1, t$ *or* $\frac{t}{t-1}, \infty\}$, so the leaf space of this baby foliation is the quotient of $(1, \infty)$ under the involution $x \to \frac{x}{x-1}$:- Our definition seems to give four equations with roots $\{0, 1, t_i, \infty\}$, $i \in \mathbb{Z}/4\mathbb{Z}$ where t_i is the cross-ratio of a 4-tuple $(y_i, y_{i+1}, y_{i+2}, y_{i+3})$. Which we can take, because of the invariance of cross-ratio, the first of these equations too. With $t_1 = t$ this gives $t_2 = \frac{\infty-1}{t-1} \cdot \frac{0-t}{0-\infty} = \frac{t}{t-1}$, $t_3 = \frac{0-t}{\infty-t} \cdot \frac{1-\infty}{1-0} = t$, $t_0 = \frac{1-\infty}{0-\infty} \cdot \frac{t-0}{t-1} = \frac{t}{t-1}$. $\square$

So *the leaves of* $\heartsuit_4$ *are given by these* $t \in (1, 2]$ *or those* $\theta \in (0, \pi/2]$ *and* $\tan \frac{\theta}{2} = \sqrt{t-1}$:- Applying the translation $x \mapsto x - 1$ roots $\{0, 1, t, \infty\}$ become $\{-1, 0, t-1, \infty\}$, then homothety $x \mapsto \frac{2}{\sqrt{t-1}}$ gives $\{-\frac{2}{\sqrt{t-1}}, 0, 2\sqrt{t-1}, \infty\}$, both diagonals of whose quadrilateral in S^1 are diameters. The angle θ between them is twice the angle made with the horizontal by the line joining $(-1, 0)$ and $(1, 2\sqrt{t-1})$. $\square$

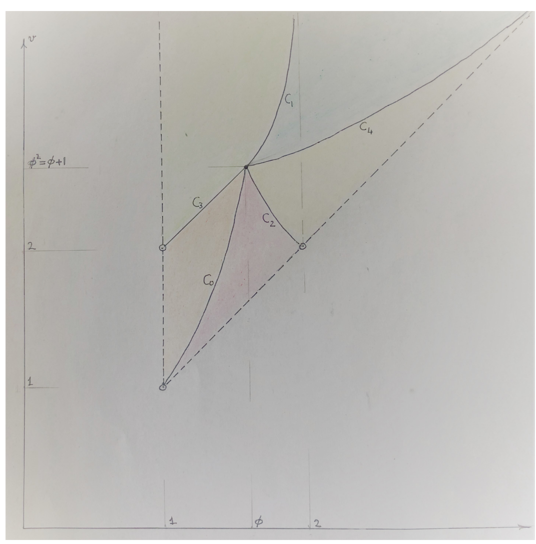
Seems with the usage of these *special equations* $\{0, 1, x_3, \ldots, x_{n-1}, \infty\}$ and some more effort we'll understand the leaf space of any $\heartsuit_n$ after all. But before going down this road, let us recall – see (50.38-9) – that *their half-turn tilings are special too,* from which it begins to dawn that for solving at least such equations *calculation of the periods of some hyperelliptic integral* should suffice. Further it is clear that instead of an equation of the affine $(n-3)$-swallowtail we can instead solve a special equation of the n-swallowtail.

In any $\mathcal{G}_0$ orbit of the swallowtail $\heartsuit_5$ there must be a quintic with roots in order $\{0, 1, u, v, \infty\}$ and at most four more special equations obtained by making some other three cyclically ordered roots $(\infty, 0, 1)$ respectively. Using the same cross-ratio calculation these are, second $\{0, 1, \frac{u(v-1)}{v(u-1)}, \frac{u}{u-1}, \infty\}$, third $\{0, 1, \frac{v-1}{v-u}, \frac{u(v-1)}{v-u}, \infty\}$, fourth $\{0, 1, \frac{v}{u}, \frac{v-1}{u-1}, \infty\}$ and fifth $\{0, 1, \frac{v}{v-1}, \frac{v}{v-u}, \infty\}$. *But for one leaf, with all five special equations the same, these five special equations in any leaf of* $\heartsuit_5$ *are all distinct* :- If the first and second are the same, then $v = \frac{u}{u-1}$ implies $u = \frac{u(v-1)}{v(u-1)} = v - 1 = \frac{u}{u-1} - 1 = \frac{1}{u-1}$, that is u is the golden ratio $\frac{1+\sqrt{5}}{2}$ and $v = \frac{3+\sqrt{5}}{2}$. With some more like work you can check that in fact *any two equations are the same if and only if* $(u, v) = (\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2})$. $\square$ However, in the following method of finishing off this proof the above extra work is not necessary, and at the same time time it opens the way to understanding the leaf space of any $\heartsuit_n$.

The map $(u,v) \mapsto (\frac{u(v-1)}{v(u-1)}, \frac{u}{u-1})$ *gives an action of the order five cyclic group,* done twice we get $(\frac{v-1}{v-u}, \frac{u(v-1)}{v-u})$, three times $(\frac{v}{u}, \frac{v-1}{u-1})$, four times $(\frac{v}{v-1}, \frac{v}{v-u})$ and the fifth time once again (u,v). Therefore, *since five is prime,* either the five 2-tuples are all distinct, or all else all equal, with the latter case if and only if u is the golden ratio and v one bigger. □

The map $(u_1,\ldots,u_{n-3}) \mapsto (\frac{u_1(u_2-1)}{u_2(u_1-1)}, \ldots, \frac{u_1(u_{n-3}-1)}{u_{n-3}(u_1-1)}, \frac{u_1}{u_1-1})$ *is an order* n *cyclic action on the infinite* $(n-3)$*-simplex* $\{(u_1,\ldots,u_{n-3}) : 1 < u_1 < \cdots < u_{n-3}\}$*, and its quotient the leaf space of* $\heartsuit_n, n > 4$:- If a special equation has roots $\{0,1,u_1,\ldots,u_{n-3},\infty\}$, or after a rotation $\{1,u_1,\ldots,u_{n-3},\infty,0\}$, then the unique element of $\mathcal{G}_0$ such that $1 \mapsto 0, u_1 \mapsto 1, 0 \mapsto \infty$, i.e., $t \mapsto \frac{t-1}{u_1-1} \cdot \frac{0-u_1}{0-t} = \frac{u_1(t-1)}{t(u_1-1)}$ gives the special equation $\{0,1,\frac{u_1(u_2-1)}{u_2(u_1-1)},\ldots,\frac{u_1(u_{n-3}-1)}{u_{n-3}(u_1-1)},\frac{u_1}{u_1-1},\infty\}$. The same done n times makes all the special equations of the leaf before we return to the one we started out with. □

About the $\mathbb{Z}/5\mathbb{Z}$ *action on the* 2*-simplex* $\{(u,v) : 1 < u < v\}$:- From the vertex $(1,1)$ the curve $(u,u^2), 1 < u < \frac{1+\sqrt{5}}{2}$ goes to the unique fixed point $(\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2})$. Applying to it the map $(u,v) \mapsto (\frac{u(v-1)}{v(u-1)}, \frac{u}{u-1})$ repeatedly gives in all five disjoint curves from the boundary to the fixed point. Not this map, but its cube $(u,v) \mapsto (\frac{v}{u}, \frac{v-1}{u-1})$ is what identifies each curve with the clockwise next, for instance after the above parabolic curve is the segment $(u,u+1), 1 < u < \frac{1+\sqrt{5}}{2}$. So the orbit space of the action–the leaf space of $\heartsuit_5$–is a cone over the fixed point, topologically a plane. □



Just the inversion $x \to \frac{x}{x-1}$ *has birthed all these cyclic actions,* on all tuples of numbers bigger than one! Besides, *this definition is over* $\mathbb{Q}$, so over all the rational 2-tuples of the picture we get this free $\mathbb{Z}/5\mathbb{Z}$-action. Which seems to be true also for any n prime? But first, writing some $\mathbb{Z}/6\mathbb{Z}$-orbits of some 3-tuples

(u,v,w), let us check that when n is not prime, then besides the central leaf the orbit space has also some other singularities.

About the $\mathbb{Z}/6\mathbb{Z}$-action :- Now the map is $(u,v,w) \mapsto (\frac{u(v-1)}{v(u-1)}, \frac{u(w-1)}{w(u-1)}, \frac{u}{u-1})$. If $(u,v,w) = (\frac{u(v-1)}{v(u-1)}, \frac{u(w-1)}{w(u-1)}, \frac{u}{u-1})$, then $v = \frac{u(w-1)}{w(u-1)} = w-1 = \frac{u}{u-1} - 1 = \frac{1}{u-1}$, so $u = \frac{u(v-1)}{v(u-1)} = \frac{u(2-u)}{u-1}$, so $u = \frac{3}{2}, v = 2, w = 3$, i.e., $(\frac{3}{2}, 2, 3)$ *is the unique fixed point,* note unlike the case $n = 5$ it is rational. Now the map can also be order two or three. The square of the map is $(u,v,w) \mapsto (\frac{(w-u)(v-1)}{(w-1)(v-u)}, \frac{v-1}{v-u}, \frac{u(v-1)}{v-u})$. If this is fixed then $w = uv$ and $u = \frac{uv-u}{uv-1}v$, so $uv - 1 = (v-1)v$, so *all points on the curve* $u = \frac{v^2-v+1}{v}, v > 1, w = v^2 - v + 1$ *other than* $(\frac{3}{2}, 2, 3)$ *are all the* 3*-tuples of order two.* Similar effort shows that *order three* 3*-tuples are all points other than* $(\frac{3}{2}, 2, 3)$ *on the surface* $u(v-1) = w(u-1)$. Any ray in the simplex from the beak $(1,1,1)$ cuts it in just one other point, thus it is an open 2-cell dividing the 3-simplex into two parts. The order two curve starting from the beak goes through the lower part to cut this surface in the fixed point and carries on through the upper part towards (∞, ∞, ∞).

An example of an order two orbit is $(\frac{7}{3}, 3, 7) \mapsto (\frac{7}{6}, \frac{3}{2}, \frac{7}{4}) \mapsto (\frac{7}{3}, 3, 7)$ and of order three $(2,3,4) \mapsto (\frac{4}{3}, \frac{3}{2}, 2) \mapsto (\frac{4}{3}, 2, 4) \mapsto (2,3,4)$. But the most common - making the remaining open set of this 3-simplex - are orbits of full length six, for example, $(3,4,5) \mapsto (\frac{9}{8}, \frac{6}{5}, \frac{3}{2}) \mapsto (\frac{3}{2}, 3, 9) \mapsto (2, \frac{8}{3}, 3) \mapsto (\frac{5}{4}, \frac{4}{3}, 2) \mapsto (\frac{5}{4}, \frac{5}{2}, 5) \mapsto (3,4,5)$, etc. So easy are these orbital calculations that one gets hooked on them, but making now even a rough picture is not that easy!

Anyway, on a generic orbit one has alternately $u(v-1) \lesseqgtr w(u-1)$, three tuples in the lower part and rest in the upper; while all three of an order three orbit are on the intervening cell; and of order two on the curve, one below the fixed point one above. *The orbit space is a closed solid cone* on the fixed orbit. The remaining cone boundary is order three orbits, quotient of the open 2-cell under the order three cyclic action around the curve. And the curve itself folds under the involution over the fixed point becoming a curved axis of the cone consising of all order two orbits. The remaining interior points of the cone are all the orbits of full length six. So topologically the leaf space of $\heartsuit_6$ is a three dimensional closed half space. $\square$

Each swallowtail has one and only one leaf with only one special equation, meaning, *any cyclic action above has just one fixed point.* In fact, for any continuous *involution* $x \mapsto \overline{x}$ (think $\overline{x} := \frac{x}{x-1}$) of numbers bigger than one, the maps $(x, u_2, \ldots, u_m) \mapsto (\overline{x} \div \overline{u_2}, \ldots, \overline{x} \div \overline{u_m}, \overline{x})$ on increasing m-tuples of such numbers have unique fixed points :- $x \mapsto \overline{x}$ is a *decreasing homeomorphism* of $(1, \infty)$, so the maps are well-defined. If $(x, u_2, \ldots, u_m) = (\overline{x} \div \overline{u_2}, \ldots, \overline{x} \div \overline{u_m}, \overline{x})$ then $u_m = \overline{x}$, $u_{m-1} = \overline{x} \div \overline{u_m} = \overline{x} \div x$, $u_{m-2} = \overline{x} \div \overline{u_{m-1}} = \overline{x} \div \overline{(\overline{x} \div x)}$, $u_{m-3} = \overline{x} \div \overline{(\overline{x} \div \overline{(\overline{x} \div x)})}$, and finally $x = f_m(x)$. Here $f_1(x) = \overline{x}$ whose graph above $(1, \infty)$ decreases from ∞ to 1, so cuts the line $y = x$ in a unique point (s_1, s_1). And inductively $f_{i+1}(x) = \overline{x} \div \overline{f_i(x)}$ whose graph above $(1, s_i)$ decreases from ∞ to 1, so this also cuts the line $y = x$ in just one point (s_{i+1}, s_{i+1}). So for any m just one m-tuple is fixed. $\square$

But, *there are involutions whose map on* 2*-tuples has very long orbits*:- Any decreasing order two bijection of a finite subset of $(1,\infty)$ can be extended to an involution. And for example we can keep on lengthening the orbit of say $(2,3)$, using such a bijection so defined that every new number has a *generic valid value.* To locate valid values subdivide $(1,\infty)$ by all already used finitely many numbers and note the sub-interval containing the new number is, and generic means out of the infinitely many numbers between the values of its end points omit all fractions of the already used numbers. $\Box$

This shows the inversion $\overline{x} := \frac{x}{x-1}$, on which we'll stay focussed, is pretty nice! We know not only that its induced maps on m-tuples have a unique fixed point, but also that the length of any other orbit is a positive divisor of $m+3$. And it seems for $m \geq 2$ there is an orbit of each such length ?

The leaf space of the swallowtail $\heartsuit_n$ for primes $n > 5$:- This is the orbit space of the infinite $(n-3)$-simplex $1 < u_1 < \cdots < u_{n-3} < \infty$ under the $\mathbb{Z}/n\mathbb{Z}$-action. The complement of the unique fixed point has the homotopy of an $(n-4)$-sphere, simply connected since $n > 5$. Further n is prime, so the orbit of each point in it has length n, i.e., on it the $\mathbb{Z}/n\mathbb{Z}$-action is free and the quotient map is an n-fold unbranched covering. So in the orbit space the *complement of the fixed orbit has fundamental group $\mathbb{Z}/n\mathbb{Z}$,* and only near this one point is this space not a manifold. In contrast to the case $n = 5$, when this unique singularity was only geometric, now it is a topological singularity.

We can put on the infinite simplex a *$\mathbb{Z}/n\mathbb{Z}$-invariant riemannian metric* by averaging any riemannian metric under this finite group action. The wavy level surfaces of this distance function from the fixed points are topological spheres S^{n-4}, and the gradient curves of this function normally cutting these surfaces rays emanating from the fixed point. The quotient of any level surface under this action is a closed manifold M^{n-4} with fundamental group $\mathbb{Z}/n\mathbb{Z}$ and *the leaf space is a cone of over M^{n-4}.* $\Box$

Is this M^{n-4} the quotient obtained if we limit the circle action of Hopf on the odd sphere S^{n-4} to an order n cyclic subgroup?

It remains, *is the fixed point irrational for primes $n > 5$ also ? For $n = 7$ yes* :- $f_1(x) = \frac{x}{x-1}$, $f_2(x) = \frac{1}{x-1}$, $f_3(x) = \frac{x(2-x)}{x-1}$, $f_4(x) = \frac{x(-x^2+x+1)}{(x-1)(2x-x^2)}$ and $s_1 = 2, s_2 = \phi, s_3 = \frac{3}{2}$, so s_4 is between 1 and $\frac{3}{2}$ a root of $x = f_4(x)$ that is a root of $x^3 - 4x^2 + 3x + 1 = 0$. Secondly, this polynomial has no root in $\mathbb{Q}$, for then – the lemma of Gauss – it would be in $\mathbb{Z}$, which is easily checked to be not correct. So s_4, the first coordinate of the 4-tuple which is fixed under the $\mathbb{Z}/7\mathbb{Z}$-action, is irrational. $\Box$

For some time using reals we have been confirming results that we had really found by making $\widehat{L}$ in the extended plane into the unit circle S^1. By means of a reflection in a round mirror, so de facto using complex algebra. *From now on we'll openly and unhesitatingly use complex numbers,* for it is not wise to keep ourselves away from the two-dimensional intuition that had shown us these one-dimensional results. For instance, the leaf of $\heartsuit_n$ having just one special equation is that which has the equation whose roots seen in the unit circle are the nth roots of 1. Using this, is quickly resolved the above hard problem!

Indeed, *for any prime $n \geq 7$ too the fixed point is irrational* :- If $\omega = e^{2\pi i/n}$

the nth roots of 1 are $\{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$ and from these we can calculate the remaining $n-3$ roots $1 < u_i < \infty$ of the unique central special equation of $\heartsuit_n$ by applying the complex mobius transformation such that $\omega \mapsto \infty, \omega^2 \mapsto 0, \omega^3 \mapsto 1$, that is, $z \mapsto \frac{z-\omega^2}{\omega^3-\omega^2} \cdot \frac{\omega-\omega^3}{\omega-z}$. For example, the image of 1 is $u_{n-3} = \frac{1-\omega^2}{\omega^3-\omega^2} \cdot \frac{\omega-\omega^3}{\omega-1} = \frac{(1-\omega^2)^2}{\omega(\omega-1)^2} = \frac{(1+\omega)^2}{\omega} = \frac{1}{\omega} + 2 + \omega = 2(1 + \cos 2\pi/n)$. Because $1/\omega = \omega^{n-1}$ and ω both satisfy $z^n - 1$, a monic integral polynomial, u_{n-3} also satisfies some such polynomial. So – Gauss lemma – if it is in $\mathbb{Q}$ then it is in $\mathbb{Z}$, but $2(1+\cos 2\pi/n)$ lies in the interval $(3, 4)$ for $n \geq 7$. □ *So, for primes $n \geq 5$ the $\mathbb{Z}/n\mathbb{Z}$-action is free on the $\mathbb{Q}$-points of the $(n-3)$-simplex.*

58. So keeping our full focus still on line $\mathbb{R}$ we'll now be also taking shorts cuts through plane $\mathbb{C}$. Not forgetting that basic are the mobius manifolds of extended line, plane, space, so on: all numbers and their addition, subtraction, multiplication, division, calculus too, all are but means used by local observers confined in their frames of reference seeking global truths! Invaluable from this relativistic viewpoint are ideas and results allowing us to paste things local into something global. As I've explained before, indefinite integration, thanks to the change of variables formula, is exactly such an invaluable idea.

Our goal is to solve any given equation $f(x) = 0$ of $\heartsuit_n$. Which manifolds are tied to its n or $n-1$ unknown roots? At once $\mathbb{R}$ minus roots comes to mind but it is not connected, so lets start with *integral* $\int \frac{dz}{f(z)}$ *on* $\mathbb{C}$ *minus roots.* It is uncanny how musing on this simple integral very quickly and naturally came back to me *memories, one after another, of many famous things* :-

Thanks to change of variables, on a manifold an i-fold indefinite integral is the same as a differential *i-form.* If it is *closed,* and only then, the value of the integral on any trivial i-cycle is zero. So, the rank of the free abelian *period group* formed by its values on all i-cycles is at most the *betti number* β_i. And "generically" this rank is equal to β_i. For example,

On plane minus roots the 1*-form* $\frac{dz}{f(z)}$ *is closed* :- If $\frac{1}{f(z)} = P(x, y) + iQ(x, y)$, $\omega = (P + iQ)(dx + idy) = (Pdx - Qdy) + i(Qdx + Pdy)$ has *exterior derivative* $d\omega = (-\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})dx \wedge dy + i(-\frac{\partial Q}{\partial y} + \frac{\partial P}{\partial x})dx \wedge dy = 0$ because $\frac{1}{f(z)}$ is *holomorphic,* i.e., *Cauchy-Riemann equations* $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$ hold. □

And, the fact that on any trivial 1-cycle the value of any holomorphic, so closed, 1-form is zero was *Cauchy's theorem.* So the rank of the period group of $\int \frac{dz}{f(z)}$ is at most $\beta_1 = deg(f)$. Further, *Cauchy's formula* $\oint \frac{dz}{z-a} = 2\pi i n, n \in \mathbb{Z}$, gives the period group for $\deg(f) = 1$.

But, when $deg(f) \geq 2$, the rank of the period group of $\int \frac{dz}{f(z)}$ *is less than* β_1 :- Consider $\oint \frac{dz}{f(z)}$ on a circle around the origin of a radius R so big that all roots of $f(z)$ are within it. Cauchy's theorem tells us that its value does not change as $R \to \infty$. But the circumference of the circle is $2\pi R$ while the order of the absolute value of the integrand is $1/R^2$. So the value of this definite integral is zero. □ A like argument gives *the fundamental theorem of algebra* :- if f(z) has no root, then $u(z) = \int_0^z \frac{dz}{f(z)}$ is a non-constant bounded holomorphic function on $\mathbb{C}$, which is not possible by a *theorem of* Liouville. □ Further,

The full period group is $\oint \frac{dz}{f(z)} = 2\pi i \sum_j n_j A_j$ where $n_j \in \mathbb{Z}$ and A_j are the *residues* on the unknown roots a_j of $f(z) = 0$ i.e., $\frac{1}{f(z)} = \sum_j \frac{A_j}{z-a_j}$, so $A_j = \prod_{k\neq j} \frac{1}{a_j - a_k}$. Obviously it has rank ≥ 1, and because of $\sum_j A_j = 0$ i.e., *the identity* $\sum_j (\prod_{k\neq j} \frac{1}{a_j-a_k}) = 0$, which is what we had proved presently using integration (!) it is always $\leq \beta_1 - 1$. In fact,

Only for the case $deg(f) = 1$ *is* $\frac{dz}{f(z)}$ *singular at* $z = \infty$:- Excepting this case the 1-form $\phi(w)dw$ obtained if we put $z = \frac{1}{w}$ and $dz = -\frac{dw}{w^2}$ has $w = 0$ as an ordinary point. □ *So for* $deg(f) \geq 2$ *we should consider this* 1*-form on* $\widehat{\mathbb{C}}$, and the first betti number of this complex manifold is one less than β_1.

Returning to $\mathbb{C}$ minus roots a_j, if we perturb the numerators A_j to make them *independent over* $\mathbb{Q}$, we get very close to $\frac{dz}{f(z)}$ a holomorphic 1-form whose rank is exactly β_1. So we can calculate the betti numbers $\beta_0 = 1$ and β_1 of this complex manifold by using instead of its *de Rham complex* $\Lambda^* \xrightarrow{d} \Lambda^{*+1}$ the *basic subcomplex* $E_1^{*,0} \xrightarrow{d} E_1^{*+1,0}$ of holomorphic forms. On the other hand,

Extreme nondegeneracy is possible in this *spectral sequence* $E_k^{p,q}$ *of a complex manifold* (or of a foliated manifold). But, all n-dimensional manifolds, and quite a few other simplicial complexes K^n too, are found embedded in $2n$-space $\mathbb{C}^n$. *Question*: can we then calculate the betti numbers $\beta_i(K^n)$ from the *neighbouring holomorphic forms* of $\mathbb{C}^n$? Further, can for any such the Heawood inequality $\alpha_n(K^n) < (n+2) \cdot \alpha_{n-1}(K^n)$ be proved by just complexifying the arguments of this linked paper? But yes, for any $K^m \subset \mathbb{C}^n$ we can calculate the betti numbers from the *neighbouring smooth forms* :- because it has arbitraily small neighbourhoods U having the same homotopy type. □

In this context let us recall that $\mathbb{C}^n$, and for $\mathbb{C}$ all, and for $\mathbb{C}^n, n > 1$ some open sets are *Stein manifolds,* and for these open compex manifolds it is even true that *Dolbeault cohomology* $E_1^{p,q}$ is zero for $q > 0$, so only the holomorphic forms $E_1^{*,0}$ survive. But, *there are open sets of* $\mathbb{C}^n, n > 1$, *that are not Stein* :- for example a tubular neighbouhood U of $S^3 \subset \mathbb{C}^2$ has betti number $\beta_3(U) = 1$ but U has no holomorphic 3-form. □

This *Dolbeault vanishing,* for $\mathbb{C}^n$ called *Grothendieck lemma,* even for our simple manifold $\mathbb{C}$ minus points a_j is not so simple. We do know that E_2 term is final with $E_2^{0,1} = E_2^{1,1} = 0$, so $d_1 : E_1^{0,1} \cong E_1^{1,1}$, but why are they zero:-

Because *the Cauchy-Riemann p.d.e.* $\frac{\partial g}{\partial \overline{z}} = f$ (for the notation used see below) we can solve on any open set U of the plane, i.e., $\frac{\partial}{\partial \overline{z}} : C^\infty(U) \to C^\infty(U)$ is surjective with kernel all holomorphic functions.

If $f(z)$ is compactly supported then *convolution* $g(\zeta) = \frac{1}{2\pi i} \int\int \frac{f(z)}{z-\zeta} dz \wedge d\overline{z}$ is smooth on U and $\frac{\partial g}{\partial \overline{\zeta}} = f(\zeta)$. Its proof uses *Stokes' formula,* whose proof in turn uses, *the fundamental theorem of calculus,* i.e., $\frac{d}{dx} : C^\infty \to C^\infty$ is surjective on any open set of the line $\mathbb{R}$ with kernel all locally constant functions.

For each $f \in C^\infty(U)$ and compact set $K_i \subset U$, there exist compactly supported functions f_i which on K_i are equal to f, but the above g_i such that $\frac{\partial g}{\partial \overline{z}} = f$, usually on the limit U of these sets K_i only give us a *distribution* $g \in \mathcal{D}(U)$ solving $\frac{\partial g}{\partial \overline{z}} = f$.

But there exist also smooth solutions - *the Mittag-Leffler theorem* - which are the limit points of the subset {all functions g_i plus holomorphic functions} in the t.v.s. $C^\infty(U)$. For more details see the first chapter of the book of Hörmander on complex analysis. $\square$

More on the notation : It is but the points (x, y) of the plane $\mathbb{R}^2 = \mathbb{C}$ that are complex numbers $z = x + iy$, e.g., $(0, 1) = i$ (when not an index!) and *the talk using these complex numbers is still about the mobius geometry of the 2-sphere* $\widehat{\mathbb{R}^2} = \widehat{\mathbb{C}}$, reflection in the flat mirror x-axis is conjugation $\overline{z} = x - iy$. Our functions and forms are all smooth and complex valued, so for any function g on the plane we can write $dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy$ also in the new basis $dz = dx + idy, d\overline{z} = dx - idy$ of 1-forms, which gives $dg = \frac{\partial g}{\partial z}dz + \frac{\partial g}{\partial \overline{z}}d\overline{z}$, where $\frac{\partial g}{\partial z} := \frac{1}{2}(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y})$ and $\frac{\partial g}{\partial \overline{z}} := \frac{1}{2}(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y})$. So if in real and imaginary parts $g(z) = P(x, y) + iQ(x, y)$ then $\frac{\partial g}{\partial \overline{z}} = \frac{1}{2}((\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}) + i(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}))$ and indeed $\frac{\partial g}{\partial \overline{z}} = 0$ means g holomorphic. $\square$

A complex manifold is covered by charts related holomorphically, so must be even dimensional and orientable, and the key to many other conditions is its *Hodge bigrading* : $E_0^{p,q}$ means all $(p+q)$-forms which in charts $(z_1, \ldots, z_m)$ use wedge of exactly p dz_i and q $d\overline{z_j}$, so its rank is $\binom{m}{p}\binom{m}{q}$ over all functions $E_0^{0,0}$. On which $dg = \partial g + \overline{\partial} g$ where $\partial g = \frac{\partial g}{\partial z}dz$ and $\overline{\partial} g = \frac{\partial g}{\partial \overline{z}}d\overline{z}$. So on all forms is available the $\mathbb{C}$-linear splitting $d = \partial + \overline{\partial}$ into two maps of bidegree $(1, 0)$ and $(0, 1)$, and $d^2 = 0$ is equivalent to $\partial^2 = \partial\overline{\partial} + \overline{\partial}\partial = \overline{\partial}^2 = 0$.

So removing the first column $p = 0$ of E_0, then the first two columns $p = 0, 1$, etc., gives a decreasing sequence of de Rham subcomplexes, and (E_k, d_k) above is the spectral sequence of this filtration (the spectral sequence defined by removing rows is isomorphic under complex conjugation) so $d_0 = \overline{\partial}$.

For example, *when* $m = 1$, only $E_0^{0,0}, E_0^{0,1}, E_0^{1,0}, E_0^{1,1}$ are nonzero in E_0, all four have rank one over functions, and $g \mapsto \frac{\partial g}{\partial \overline{z}}(g)d\overline{z}$, $gdz \mapsto \frac{\partial g}{\partial \overline{z}}d\overline{z} \wedge dz$ gives d_0. So indeed the result above was the same as saying that for any open set of $\mathbb{C}$, it is only $E_1^{0,0}, E_1^{1,0}$ that are nonzero in E_1, and contain respectively all holomorphic functions and 1-forms. Further, because the homology E_2 of this holomorphic de Rham complex $d_1 = d : E_1^{0,0} \to E_1^{1,0}$ is final, $E_2^{0,0} \cong \mathbb{C}^{b_o}, E_2^{1,0} \cong \mathbb{C}^{b_1}$ where b_0 and b_1 are the Betti numbers of the open set, and $d_2 = 0$. In fact, all this is true also for any *open Riemann surface,* but if it is closed then just from $b_2 = 1$ it is clearthat we don't have Dolbeault vanishing.

Returning to $\int \frac{dz}{f(z)}$, where f has degree $\delta > 1$, we had seen its period group has rank not bigger than the betti number $\beta_1 = \delta - 1$ of $\{\widehat{C}$ minus roots$\}$, but it remains to work out how it depends on the equation $f = 0$ of the swallowtail $\heartsuit_\delta$. Before this we recall that for *degree even* one way of compactifying this manifold is to use its *'double' after* $\frac{\delta}{2}$ *disjoint cuts.* Which gives a *closed riemann surface* of genus $\frac{\delta}{2} - 1$ and in the last section we'll see that *Jordan's method of solving equations is tied to this compactification* and uses the periods of $\int \frac{dz}{\sqrt{f(z)}}$, so this now is a 'warm-up' by familiarizing ourselves with an easier integral! Note $\heartsuit_\delta$ contains too affine degree $\delta - 1$ equations—those with one root infinity—so there is *no loss of generality in assuming degree is even.*

Similarly it seems for $\mathbb{C}^n$ *if open set U is Stein* – maybe of points nearby any $K^n \subset \mathbb{C}^n$? – *then a 'double' will be a closed complex Kähler manifold?* So its homology – that should shine a light on the combinatorics of K^n, perhaps may even give its Heawood inequalities? – would be very special : now Dolbeault cohomology E_1 is final and Poincaré-Serre duality $E_1^{p,q} \cong E_1^{n-p,n-q} \cong E_1^{n-q,n-p}$ is given by multiplications of a basic $(1,1)$ class, so again this *hard Lefschetz theorem* will be used for combinatorics.

The roots of $f = 0$ give the periods of $\int \frac{dz}{f(z)}$, but conversely they can give at most *the differences of roots* $a_j - a_k$, because periods were $2\pi i \sum_j n_j A_j$ where $A_j = \prod_{k \neq j} \frac{1}{a_j - a_k}$; anyway all these differences suffice, for $\sum_j a_j$ we can at once read from the equation $f = 0$. For degree two the job is easy, $a_1 - a_2 = \frac{1}{A_1}$, but then for degree three some method for writing *squares of differences* only seems at hand, for example :- square $(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)A_1 = (a_2 - a_3)$ etc., writing the *discriminant* $(a_1 - a_2)^2(a_1 - a_3)^2(a_2 - a_3)^2$, a symmetric function of the roots, in terms of the coefficients of $f = 0$. $\Box$

The field $\mathbb{F}$ generated by the coefficients of $f = 0$, has as smallest extension containing all the differences $a_j - a_k$, the same as the one containing all the roots a_j, but it usually bigger than that with all A_j. *How much bigger can be the field* $\mathbb{F}(a_j)$ *compared to the sub-field* $\mathbb{F}(A_j)$*?* For degree three we saw above it is at most a quadratic extension, and for example for $x^3 - 5x = 0$ the dimension of $\mathbb{F}(a_j) = \mathbb{Q}(\sqrt{5})$ over $\mathbb{F}(A_j) = \mathbb{Q}$ is two. However, the sub-extension $\mathbb{F}(A_j)$ *is also preserved by permutations of roots* :- $a_j \leftrightarrow a_k$ induces the transposition $A_j \leftrightarrow A_k$. $\Box$ So the symmetric functions of A_j can also be written in terms of the coefficients of $f = 0$, for example $\sum_j A_j = 0$ we proved already, and it is easily checked that $(-1)^{\binom{n}{2}} \prod_j \frac{1}{A_j}$ equals the discriminant d.

In möbius geometry however *the ratios of differences* are more natural, and, for degree three $\frac{A_1}{A_2} = \frac{a_2 - a_3}{a_1 - a_3}$, for degree four $\frac{A_1 A_3}{A_2 A_4} = -(\frac{a_2 - a_4}{a_1 - a_3})^2$, for degree six $\frac{A_1 A_3 A_5}{A_2 A_4 A_6} =$ etcetera, so one begins to hope that, *we can write these ratios using the surds of the periods?* It seems always $\mathbb{F}(a_j)$ is *solvable* over $\mathbb{F}(A_j)$, *and it may be that only compass and ruler suffice,* that is, a chain of quadratic extensions takes us from the smaller to the bigger field?

Generically the fixed field of $\tilde{\mathbb{F}}(a_i)$–where $\tilde{\mathbb{F}}$ is the extension of $\mathbb{F}$ by the square roots $\pm\sqrt{d}$ of the discriminant–under all the even permutations of $a_1, \ldots, a_n$ seems to be $\tilde{\mathbb{F}}$, but *for* $n > 4$ this permutation group is simple, so $\tilde{\mathbb{F}}(A_i) = \tilde{\mathbb{F}}(a_i)$? That is to say, using addition subtraction multiplication division and one square root we can write any root a_i of the equation $f = 0$ in terms of its coefficients and the integrals $\frac{1}{2\pi i} \oint \frac{dz}{f(z)}$. Explicitly, what are these formulas solving any equation? Further, tied to $f = 0$ is a *faux tiling* – meaning with all vertices on the boundary – of the open disk folding which gets made $\hat{\mathbb{C}} \setminus \{a_i\}$ and which is associated to the aforementioned Cauchy periods ...

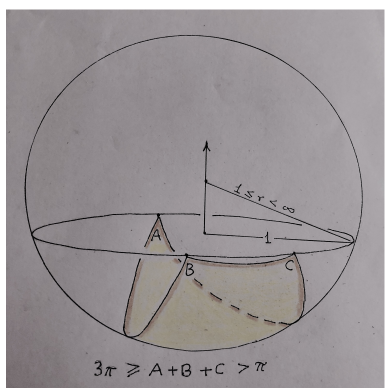
A natural *true tiling* – meaning with compact tiles – we have also tied to any degree $n > 4$ equation $f = 0$ long ago–see part four, also notes 50–and the next note in fact is an addendum written at about the same time.

59. In part four we made using half turns, starting from any $n \geq 4$ points on the unit circle, an $\{n, n\}$ tiling of concentric radius $c > 1$ – with $c = \infty$ only for $n = 4$ – crystallographic with respect to its geometry. Meaning, all tiles are congruent to the seed n-gon with the given n vertices inscribed in the unit circle, and the union of any two tiles sharing an edge is symmetric around its mid-point, and that there are n tiles on each vertex of the tiling.

Somewhat similarly we can make *a spherically crystallographic tetrahedron* $\{3, 3\}$ *starting from any* $n = 3$ *points on the unit circle* :- Consider a sphere of radius $r \geq 1$ through the unit circle with centre right above its centre. Join the three points with its great circle 'segments'. As r increases from 1 towards ∞, the sum of the angles of this spherical triangle decrease continuously from 3π towards π. Take that r for which it is 2π. Now apply around the mid-points of the edges spherical half-turns, to get this tetrahedron. $\square$

So triangles of our tetrahedon have area πr^2, agreeing with $(A+B+C-\pi)r^2$, the area of a spherical triangle. Second note this r with angle-sum is 2π, depends on the three points on the circle we started with. Being the smallest if the distances between them are equal, when we get *regular tetrahedron.*

In this case all three angles are equal for all r, and when $\pi/2$ and $2\pi/5$, we get a *regular octahedron and regular icosahedron,* and then at limit $r = \infty$ a regular tiling $\{3, 6\}$ of the entire plane. In all other cases we'll not run into any other semi-deformed platonic solid, but at limit $r = \infty$ is a *planar* $\{3, 6\}$ *tiling by half-turns starting from any triangle.* From this tiling is obtained another elliptic, meaning on $\mathbb{C}$ a doubly periodic meromorphic, function.



Notes : (a) All four triangles are spherical *convex hulls* of their vertices–see figure–there being no antipodal pairs in them. The triangle DCB obtained by rotating ABC by a half-turn about the midpoint α of its side BC is also in the complement of the antipode $-\alpha$ and intersection of the two this side. From this fact and area is firm the existence of $ABCD$. (b) This half-turn gives the permutation $DCBA$, the side AD has the same length as BC, and the other end $-\alpha$ of the axis its mid-point. (c) The three sides of its triangles being usually of different lengths, *its symmetry group has just four elements,* identity and half turns $DCBA, CDAB$ and $BADC$. (d) On the other hand half turn motions of the plane around the limiting flat ABC generate an infinite group and tiling

$\{3,6\}$, six triangular tiles at each vertex. (e) So now the compositions of the three half turns are not identity, only their square is identity. The triangles of the tiling generated by these three fold compositions are of double the size, in each four smaller triangles, and folding the plane under this index four subgroup gives again the same crystallographic tetrahedron.

60. Folding compact $\{n,n\}$ tilings fully gives the 2-sphere $\widehat{\mathbb{C}}$ but if we divide only by the *index two subgroup* of even compositions of the n half turns then is obtained universal covering of a *closed Riemann surface* M^2. The remaining division gives a holomorphic *double branched covering* $M^2 \to \widehat{\mathbb{C}}$. Full quotient map is a, *periodic under the group* of all n half turns, meromorphic that is holomorphic with values in $\widehat{\mathbb{C}}$ function. When $n = 3, 4$ the index two subgroup is generated by two independent translations of the plane, so this is a doubly periodic meromorphic that is an *elliptic function* $z(u)$.

Does this $z(u)$ invert the *multi-valued function* $u(z) = \int^z \frac{dz}{\sqrt{f(z)}}$ given by all *path integrals* of $\mathbb{C} \setminus \{a_i\}$ starting from some *base point* till z? Since the integrand is holomorphic the ambiguity of $u(z)$ is limited to the *periods* $\oint \frac{dz}{\sqrt{f(z)}}$, which make for each equation $f = 0$ of the 4-swallowtail a *lattice* of the plane $\mathbb{C}$, see Goursat volume 2 page 120. And before that from its page 114 and sequel that for any equation $f = 0$ of any *even* n-swallowtail that, *all periods* $\oint \frac{dz}{\sqrt{f(z)}}$ *make a subgroup of* $\mathbb{C}$ *of rank* $\leq \frac{n}{2}$.

For $n > 4$ we cannot hope for a plane lattice but, as for our warm-up case, that theorem on page 380 of the Traité of 1870 of Jordan is telling us perhaps that these periods of $f = 0$ give us for the same $\tilde{\mathbb{F}}$ another *Galois extension* only a bit smaller than $\tilde{\mathbb{F}}(a_i)$ and generically the same? So again the same question: in a clear manner *what are these formulas* giving for any equation its *roots in terms of the coefficients and hyperelliptic periods?*

To return to the tiling we need to treat the *two-fold ambiguity of the square root* correctly. For the warm-up one-form $\frac{\mathrm{d}z}{f(z)}$ the domain $\widehat{\mathbb{C}} \setminus \{a_i\}$ is correct, but not for $\frac{\mathrm{d}z}{\sqrt{f(z)}}$. To correctly define it, like Riemann we make $\frac{n}{2}$ *disjoint cuts* joining pairs of roots, and then pasting this cut space with a copy *double* it. We have got the same Riemann surface M^2, and when we regard these cuts on its universal space the same tiling. But only got till topology : geometry we'll get when and only when we would have understood conceptually the formulas about which we have raised the question above.

Per Mumford's Theta II, page xi, Umemura has appended to this book "simple expressions" solving equations, that he has obtained by developing a method of Jordan. Like music mathematics too is done in a variety of very different moods! True, if fully changing gears we abruptly switch to a formal mood, these fearsome formulas do slowly become simpler. But to fit in a natural way this mathematics done in a different mood into our own evolving theory of equations should obviously be our goal.

So without fully changing gears let me give *a concise synopsis* fairly different from the above books. The story began with the trigonometric method of Vieta for solving cubics – see Notes 13, 14 – which is what is recalled by Jordan in 1870 in Note 505 of Traité :- *Translate* variable to make the second coefficient–sum of roots–zero. If the third coefficient–sum of products of roots taken two at a time–is not zero, *scale* variable to get cubic $4z^3 - 3z + a = 0$. *The trisection of a circular arc into three equal parts* is tied to which, more precisely, if we put $a = \sin u$, the roots are $z = \sin \frac{u}{3}, \sin \frac{u+2\pi}{3}, \sin \frac{u+4\pi}{3}$. □

Remarks : (ੳ) Translation making some other coefficient zero requires effort, to make constant zero a root itself of the equation. (ਅ) Scaling to adjust the third coefficient involves extraction of a *square root,* if we were to adjust the fourth then a cube root, etc. (ੲ) *Scaling by positive square root is retraction of that peacock feather of* Khayyam *on segment* $(4z^3 - 3z + 1 = 0, 4z^3 - 3z - 1 = 0)$, the roots of the boundary points of which, $\{+\frac{1}{2}, -1, +\frac{1}{2}\}$ and $\{-\frac{1}{2}, +1, -\frac{1}{2}\}$ are also given by these formulas; but all three don't give the unique root 0 of its cusp $z^3 = 0$. (ਸ) *The method in full* for all cubics is this : if post translation the third coefficient is zero – this doesn't occur on peacock feather, but consider for instance $z^3 + a = 0$ – then we take *cube roots* which can all three be complex; and if not we use *complex sine.* (ਹ) Namely, *the periodic function* $z(u)$ *inverting the multivalued 'function'* $u(z) = \int_0^z \frac{dz}{\sqrt{1-z^2}}$:- note $\frac{dx}{\sqrt{1-x^2}} = \sqrt{dx^2 + dy^2}$ if $y = \sqrt{1-x^2}$, so its real integration from 0 to $x \in (-1, +1)$ gives the *circular arc* $u \in (-\frac{\pi}{2}, +\frac{\pi}{2})$ such that $\sin u = x$. The full *domain of* $\frac{dz}{\sqrt{1-z^2}}$ *is* M^2 made for example by making the cut $(-1, +1)$ on the plane with a scissors and then with some glue pasting it to this cut of a copy. Starting from the base point, 0 of first sheet, and avoiding (for the moment) ± 1, *all paths of* M^2 *are used to find these values* $u(z)$ *of the* 1*-form.* But, *on all such trivial loops of this cylinder we have* $\oint \frac{dz}{\sqrt{1-z^2}} = 0$. Because, we can replace such a loop of M^2 going only around -1 once, by a small loop formed by going over the same small circle around it on the first sheet and then again on the second sheet. So the restriction that the path avoid ± 1 was not needed, in particular, *on the non-trivial loop of* M^2 *formed by going on the cut from* -1 *to* $+1$ *on the first sheet and then back from* $+1$ *to* -1 *on the second we have* $\oint \frac{dz}{\sqrt{1-z^2}} = 2\pi$, the perimeter of the unit circle. So on any loop the integral is 2π times an integer, *this inverse function* $\sin : \mathbb{C} \to M^2$ *wraps the plane around the cyclider with periodicity* 2π. □ (ਕ) Wrapping this method of Vieta in full in a single "simple formula" we can using *analytic extension* check that for input any cubic equation $\odot \in \mathbb{C}P^3$ its output shall be the (one, two or three complex) roots of $\odot$. (ਖ) We have no intention intend of dressing up such formulas, interesting is that the job of inverting $u = \int^z \frac{dz}{\sqrt{f(z)}}$ has now commenced, but instead of the given cubic we are still talking in it only of a quadratic's square root.

For a quadratic equation completing the square (discriminant's square root) *suffices, nor do we learn anything about its roots from the periods* $\oint \frac{dz}{\sqrt{z^2+az+b}} = 2\pi i m, m \in \mathbb{Z}$:- thanks to analytic extension it is enough to check this when roots are two real numbers, so it reduces to checking $\int_{-r}^{+r} \frac{dx}{\sqrt{r^2-x^2}} = \pi$, which is true, because the semi-circle $y = \sqrt{r^2 - x^2}$ of radius r has arc element $\sqrt{dx^2 + dy^2} =$

$dx\sqrt{1+(\frac{dy}{dx})^2} = \frac{r\,dx}{\sqrt{r^2-x^2}}$. □ (ग) *But the domain M^2 of $\frac{dz}{\sqrt{z^2+az+b}}$ does depend on the roots $\alpha, \beta \in \mathbb{C}$ $z^2+az+b=0$,* it is made again making a cut along segment $\alpha\beta$ and then pasting it to the same cut of a copy. Not only can we check that its topology is that of a cylinder but also that, *the loop integrals of this* 1*-form over M^2 have values $2\pi i$ times an integer.* (घ) In this degree$(f)=2$ case the 1-form blows up at infinity, that's why this domain is not compact, but its points are equally smooth and we can choose *any $0 \in M^2$ as base point,* then inverting path integrals $u(z) = \int_0^z \frac{dz}{\sqrt{z^2+az+b}}$ gives *a holomorphic function $z(u)$ of $\mathbb{C}$ onto M^2 such that $z(u+2\pi i) = z(u)$ and $z(0)=0$.* (ङ) Consider these integrals over loops in one copy, that is loops of $\mathbb{C}$ not going through cut $\alpha\beta$, as $\beta \to \alpha$, this becomes the famous formula $\oint \frac{dz}{z-\alpha} = 2\pi i m, m \in \mathbb{Z}$ of Cauchy. (च) The *pull-back in $\mathbb{C}$ of the loop $\alpha\beta\alpha$ of M^2 made from the cuts gives a line on which the pre-images of α and β occur alternately at distance π,* meaning, in this case of two roots the seed tile is a 1-simplex $\alpha_0\beta_0$ and we make that line by repeated *relections* in its two facets. (छ) However–note (50.02)–

dividing by all even compositions of the half-turns which restrict to these reflections gives this universal covering $\mathbb{C} \to M^2$, and if we divide by the full group a further double covering $M^2 \to \mathbb{C}$ branched over α and β :- Betwee any two parallel lines going through the ends α and β of the tile $\alpha\beta$ is a fundamental 2-cell of the group, which folds these lines over α and β, so the quotient has Euler number $2-2+1=1$, etc. □ (ज) *The full map $\mathbb{C} \to M^2 \to \mathbb{C}$ is the square of the 'generalized sine function' $z(u)$* :- The images $z(u)$ of the first map are the points $(z, \pm\sqrt{z^2+az+b})$ of the double sheeted Riemann graph M^2, and $z^2(u)$ the points (z, z^2+az+b) of a single valued graph over $\mathbb{C}$. □ (झ) Only on which root in which sheet we start depends the integral over the segment

$\int_{\alpha\beta} \frac{dz}{\sqrt{z^2+az+b}} = \pm\pi i$:- The same $\int_{-r}^{+r} \frac{dx}{\sqrt{r^2-x^2}} \equiv \pi$ and analytic extension tells us that *the integral is constant over the affine* 2*-swallowtail $\mathbb{C}\Omega^2$,* for example, if we come back after making such a tour to the same equation ⊙ that its roots $\{\alpha, \beta\}$ get interchanged, then at the same time that square root in the integrand–which rotates at half speed–changes its sign. □ (ञ) If imprisoned in the line $\mathbb{R}$ *always distinct n particles* remain in the same order, so real n-swallowtail $\mathbb{R}\Omega^n$ is an open n-cell, and over it *the graph G of roots* has n disjoint sheets. But in the freedom of the plane $\mathbb{C}$ all $n!$ permutations are possible, so *not only is G connected the group of its covering transformations has full order $n!$,* and on it surjects *the fundamental group the complex affine n-swallowtail $\mathbb{C}\Omega^n$.* □ (ट) Any path of $\mathbb{C}\Omega^n$ viewed in $\mathbb{C} \times \mathbb{R}$ shows the *paricles weaving a braid with n strands.* so this fundamental group is the nth *braid group* of Artin, so its action or *monodromy* on the covering space G births the generic Galois group of degree n equations. □ (ठ) The current $n=2$ case, i.e.,

quadratic equations have very interesting topology! The graph G is now *the deleted square* $\mathbb{C}^2$ minus diagonal, of all disjoint ordered pairs (α, β) and all unordered $\{\alpha, \beta\}$ is $\mathbb{C}\Omega^2$, both spaces are topologically $(\mathbb{C} \setminus 0) \times \mathbb{C} \simeq S^1$ and the covering map $(\alpha, \beta) \mapsto \{\alpha, \beta\}$ is up to $\mathbb{Z}_2$-homotopy type the antipodal so squaring map of S^1. □ (ड) *The swallowtail $\mathbb{R}P\Omega^2$ of homogenous real quadratic* *equations with distinct extended real roots is an open* Möbius *strip* :- After an inversion of the same Möbius, along S^1 instead of an $\widehat{\mathbb{R}}$, we meet any antipodal pair of roots again after a half, and the other pairs after a full trip (likewise $\mathbb{R}P\Omega^n$'s topology, etc., but remains to do cyclic or S^1-homology). $\Box$ (ਢ) The closure of this strip is far smaller than the $\mathbb{R}P^2$ formed by all homogenous real quadratics but its magical that the school formula for the equations of the 2-cell $\mathbb{R}\Omega^2$ solves also any $\odot \in \mathbb{C}P^2$ (and thanks to the same Möbius rigidity or analytic extension it suffices to solve equations of $\mathbb{R}\Omega^n \subset \mathbb{C}P^n$, that from the viewpoint of geometry we have almost done for all $n > 2$ using the periods of meromorphic quotients tied to those n half turn tilings, but remain some things and relation with formulas of Mumford et al). $\Box$ (ਣ) *The swallowtail* $\mathbb{C}P\Omega^2$ *of homogenous quadratic complex equations* $\cong$ *the tangent bundle of* $\mathbb{R}P^2$:- After a Möbius inversion of $\widehat{\mathbb{R}^3}$ the roots become pairs of the round S^2 instead of $\widehat{\mathbb{C}}$, all antipodal making $\mathbb{R}P^2$ and remain fixed under the fiber map, which takes any other pair $\{\alpha, \beta\}$ to the $\pm\gamma$ on the unique great circle passing through them normal to mid-point δ. $\Box$ So $\mathbb{C}P\Omega^2 \simeq \mathbb{R}P^2$ the *stable subspace of all repelling pairs* on the round S^2 (but remains for n repelling particles this compact deformation retract of $\mathbb{C}P\Omega^n$). (ਤ) *Our swallowtails are complementary to those of* Thom, these are all $\odot$ with less roots that is discriminant $= 0$: $\mathbb{C}P\Omega^2$ *is* $\mathbb{C}P^2$ *minus quadratics* $az^2 + bzw + cw^2 = 0$ *such that* $b^2 - 4ac = 0$ *whose neighbourhood* $\mathbb{C}P^2$ *minus an* $\mathbb{R}P^2$ *is a bundle over an* S^2 :- The fiber of the quadratic with both roots $\alpha \in \widehat{\mathbb{C}} \cong S^2$ contains all great circles through it minus $-\alpha$. $\Box$ (ਥ) *Whose pull-back under* $S^2 \times S^2 \to S^2 * S^2 \overset{\cong}{\to} \mathbb{C}P^2$ *is* $\cong$ *tangent bundle of* S^2 :- All normal pairs (α, β) make a $T_1(S^2) \cong \mathbb{R}P^3$ which in $S^2 \times S^2$ separates its diagonal and stable 2-sphere, and below in $S^2 * S^2$ a $T_1(\mathbb{R}P^2)$ separating the diagonal S^2 from the stable $\mathbb{R}P^2$. $\Box$ (ਦ) Assessing old mathematics correctly is beyond its ordinary historians, à la Arnol'd

$S^2 * \cdots * S^2 \overset{\cong}{\to} \mathbb{C}P^n$ *should be called Vieta's theorem* :- Sure in that olden time leave alone manifolds, complex numbers were far in the future, but that coefficients are the *elementary symmetric functions* of the roots was known to this ancient! Taking S^2 to be $\widehat{\mathbb{C}}$ this is the definiion of our map, and clearly it is one-one and continuous from a closed pseudomanifold to a connected closed manifold of the same dimension, so this map is also onto and the pseudomanifold in fact a manifold. $\Box$ (ਧ) This F T A written in symmetric powers $\mathbb{C} * \cdots * \mathbb{C} \overset{\cong}{\to} \mathbb{C}^n$ tells us that *continuous functions of* n *distinct roots and continuous functions of the coefficients are the same* :- The composition of $\mathbb{C} \times \cdots \times C \to \mathbb{C} * \cdots * \mathbb{C}$ and this Vieta homeomorphism extends the *principal* $n!$*-fold covering map* associated to the covering $G \to \mathbb{C}\Omega^n$ of equations with n distinct roots. $\Box$ (ਨ) *And this result remains valid if instead of 'continuous' there is 'rational' or 'polynomial'* :- The complex analyticity etcetera of the covering map implies this over $\mathbb{C}$ that is for all polynomials. But then, for any subfield $\mathbb{F}$ of $\mathbb{C}$, we can take only all those whose values on $\mathbb{F}$-points are also in $\mathbb{F}$. $\Box$ (ਪ) It is by a recollection of this over $\mathbb{Q}$ purely al-jabric result—which by the time of Newton was firm—for "quelconque" $\odot \in \mathbb{C}\Omega^n$ *starts the third Livre of the "Traité" of Jordan* : 'we know that any (rational) symmetric function of the roots is a rational function of the coefficients ... ’

(ਫ) From this Newton theorem is quickly defined for ‘chaque’ $\odot \in \mathbb{C}\Omega^n$ that permutation group of roots to which we now give the name of Galois (abstract groups and the word field are far in the future, they are not in Traité) := *the smallest such that rational functions of coefficients and of roots invariant under these permutations are the same.* □ (ਬ) Wild it surely is but *the Galois group of* $\odot$ *is of maximum order* $n!$ *on a dense subset of* $\mathbb{C}\Omega^n$:- Addition, subtraction, multiplication, division make only countably many rational functions of roots, and any such non-symmetric can be a function of the coefficients—so symmetric—only if there some al-jabric relation between the roots, so only on a closed nowhere dense set. □ The disjoint nonempty sets on which the order of the Galois group is the same divisor of $n!$, are these all not only dense in $\mathbb{C}\Omega^n$ but also uncountable and non-measurable? (ਭ) ‘Lemme III’ of Traité (in today’s language that the splitting field can be generated by just one element) : *all roots of* $\odot$ *can be written rationally in terms of the coefficients and just one integral combination of the roots* :-

“Lemme II : Almost all such combinations $V_1 = M_1x_1 + M_2x_2 + \cdots$ have $n!$ distinct values V_α under the permutations α.’ So the degree $(n-1)!$ equation in the unknown V with roots $\{V_\alpha : \alpha(x_1) = x_1\}$ is unchanged under these $(n-1)!$ permutations, and under any other becomes one with totally different roots. Further its coefficients can be written (rationally) in terms of x_1 and the symmetric functions of $x_2, x_3, \ldots$, so in terms of x_1 and the coefficients of $\frac{F(x)}{x-x_1} = 0$, so in terms of x_1 and the coefficients of $F(x) = 0$. Now read this rational identity $f(V, x_1) = 0$ as $f(V_1, x) = 0$ with x unknown. Only one root, x_1, of it is shared with $F(x) = 0$. Meaning $x - x_1$ is their h.c.f. That we can find using Euclid’s method, so we can write x_1 rationally in terms of V_1 and the coefficients of $F(x) = 0$.’ □ (ਮ) For any rational function V_1 of the roots, and any $\odot \in \mathbb{C}\Omega^n$ on which it has $n!$ distinct values V_α, is valid this ‘Théorème fondamental’ : *the factorization on* $\odot$ *of the degree* $n!$ *equation of Galois* $\prod_\alpha (V - V_\alpha) = 0$ *gives us all the orbits of its Galois group* :-

‘The coefficients of this equation are symmetric, so in the field $\mathbb{F}$ of rational functions of the coefficients of $\odot$, and we are speaking of full factorization over $\mathbb{F}$. Using ‘Lemme III’ we can write rational function of roots as a rational function $\psi(V_\alpha)$ over $\mathbb{R}$ of any one root. The permutations interchanging the V_β in the factors of V_α make a group. If $\psi(V_\alpha)$ is invariant under it, that is it is a symmetric rational over $\mathbb{F}$ in these V_β, then *using Newton’s therem over* $\mathbb{F}$, it is a rational function over $\mathbb{F}$ of the coefficients of the factor. But the coefficients of the factor are in $\mathbb{F}$, so $\psi(V_\alpha)$ equals a function of $\mathbb{F}$: because V_α is a shared root of this equality and its irreducible factors over $\mathbb{F}$, so–‘Lemme I’–this equality holds as well for all V_β of this factor.’ □ (ਯ) *Galois’s method for understanding the subgroups of Galois*$(\odot)$ *or “Galois theory”* :- Over the fields of coefficients and some *adjoined numbers* the irreducible factors of the above equation of degree $n!$ can be made smaller and smaller, the permutations of their V_β give all subgroups, etcetera. □ (ਰ) *All Galois*$(\odot)$ *for the equations of any set* $K \subset \mathbb{C}\Omega^n$ *generate the group Galois*(K), so if one Galois$(\odot)$ has order

$n!$ then $Galois(K)$ contains all covering transformations. This 'al-jabric group' does not depend on how K is described by some *parameters* k*,* but,

transformations of $Galois(K)$ *which over the space of parameters* k *map any sheet into the same component form a normal subgroup,* and these 'monodromy subgroups' are often smaller :- *Example* : quadratics with root-sum zero $z^2 - k = 0, k \neq 0$ make a punctured plane $K \subset \mathbb{C}\Omega^2$ on which the graph of roots is connected, so with this $1-1$ parameter k monodromy is full $Galois(K) \cong \mathbb{Z}_2$; but if we think of K as equations $z^2 - k^2 = 0, k \neq 0$ then over the space of this 2-1 parameter k there are two disjoint sheets of pulled back roots, so monodromy is now trivial. The monodromy is always normal because any transformation g of the $n!$-fold covering of the k-space maps components to components, so if h preserves all components, then ghg^{-1} is also of the same kind. $\square$ (ऋ) This simple example of *Traité* generalizes all the way: *the square root of the discriminant* is a connected 2-fold cover of $\mathbb{C}\Omega^n$ over which the pulled back $n!$-fold covering space has two components with sheets related by even permutations of the roots, so under these parameters the monodromy subgroup has order $n!/2$:-

These parameters are the coefficients of the varying $\odot \in \mathbb{C}\Omega^n$ and a square root of their function equal to the symmetric function of roots $(\prod(x_i - x_j))^2$, so now it is impossible that making a round will just interchange two roots because with this sign of $\prod(x_i - x_j)$ also changes. For example for $\odot \in \mathbb{C}\Omega^2$ the parameters are the coefficients b, c of these quadratics $x^2 + bx + c = 0$ and $\pm\sqrt{b^2 - 4c}$, but now, since the pulled back 2!-covering space is trivial, both roots x_1 and x_2 stay put on the pull-back, so there is a rational formula for them in the parameters. Likewise for any n *to solve all equations* $\odot \in \mathbb{C}\Omega^n$ *it is necessary and sufficient : a trivial pull-back of its principal* $n!$*-fold covering space defined using coefficients.* Further *for any* n *there is a beautiful determinant giving the discriminant* in terms of the coefficients, see Burnside Panton, and its square root unfolds each top most stratum of the *fundamental partition of* $\mathbb{R}\Omega^n$ to make it from a half to a full n-space $\mathbb{R}^n$. $\square$ (ह) For example it was by further unfolding by cube roots this double unfolding of all real cubics with root sum zero and discriminant bigger than it, that the renaissance disciples of Khowarazmi made that al-jabric *Cardan formula of Note 11*, on the second page only of this work, but now from the prettiest composition of Khayyam, from which it was clear also the shape of the full graph of real roots!

In this graph over those $\mathcal{S}$ making the peacock feather on the closure of the double cover are circles and *trisection of just one period,* Note 13, gave a formula applicable by analytic extension to all of $C\Omega^3 \simeq S^1$, that is this 3-1 parameter for the double cover makes the monodromy group A_3 of $Galois(\mathbb{C}\Omega^3) \cong S_3$ fully trivial. For degree $n = 4$ equations too using the same means there is the *formula of Ferrari thanks to this special fact that only* $A_n, n = 4$ *has another normal subgroup namely the Viergruppe of Klein* :- The monodromy A_4 after using the square root of the discriminant becomes on a 3-1 cover $\mathbb{Z}/2 \times \mathbb{Z}/2$ and at the same time is solved *the reducing cubic*–see Burnside Panton–from the square roots of whose roots gets made easily a 4-1 cover that makes the monodromy fully trivial. $\square$ It remains the relationship of the method of Hachtroudi to this, but clearly this method exploiting a special feature of A_4 won't generalize. (ञ) It was perhaps this realization that led Abel to search for *doubly periodic* functions, which he found by inverting Legendre's integrals.

In 506 of Traité is a method for degree four which directly makes monodromy trivial from A_4 using periods of elliptic functions of Jacobi :- It had served as 'warm-up' for Hermite before his using these periods to do degree five equations (but is more natural than his and Kronecker's $n = 5$ methods which turn around the icosahedron nothing like which is there in $n > 4$ euclidean space). *This is somewhat like our four half turns method* : on the 2-1 graph of the square root of the discriminant the periods of the doubled tiling made from the roots of each $\odot$ are given by elliptic integrals from the coefficients, so their bisection gives the roots of the equation. □ For cubics with root-sum zero too we have a doubly periodic parametrization by the $\wp(z)$ of Weierstatass, but all becomes clear moving ahead, *for any degree $n > 4$ this method of n half turns works, and of itself relativity arises* :- This because to make a tiling from the roots of $\odot$ demands angle sum 2π, so sides become concave circular and this group ranges on an open disk of a finite radius: *ends the euclidean pretense of infinitely extended plane!* On the other hand the periods of the doubled tiling are given by similar hyperelliptic integrals made from the coefficients, and so to speak their bisection solves by analytic extension any $\odot \in \mathbb{C}\Omega^n$. □

Another orbit completed today taking God's name I'm posting this part five, but remain to elaborate the last remarks and to translate.

K S Sarkaria 11th April, 2020