The prettiest composition, part four
(translation of ਸੱਭ ਤੋਂ ਪਿਆਰੀ ਰੱਚਣਾ, ਭਾਗ ਚੌਥਾ)

51. In note 49 we had merely scratched at the geometry of the 'baby action' by linear substitutions on the space $\mathbb{R}P^n$ of all homogenous degree n *equations* in unknowns $\mathtt{x}$ and $\mathtt{y}$. This acting *baby group* $\mathcal{G}$ is of all bijections of the space $\mathbb{R}P^1$ of all *lines* of $\mathbb{R}^2$ through its origin induced by the linear isomorphisms of $\mathbb{R}^2$. So it is the quotient of the group $GL(2,\mathbb{R})$ obtained by dividing out by diagonal matrices with both entries same. So each element of our group $\mathcal{G}$ can be identified with a pair $\pm A$ of 2×2 matrices of determinant ± 1. Hence, $\mathcal{G}$ *is three-dimensiona*l, but in note 49 we had focused only on its one-dimensional subgroup given by pairs $\pm A \in SO(2,\mathbb{R})$.

Its *two fold cover* $\tilde{\mathcal{G}}$—which identifies with the group of all matrices A of determinant ± 1, or else the quotient of $GL(2,\mathbb{R})$ obtained by dividing out only by diagonal matrices with both entries same and positive—is the group of all bijections of the space S^1 of all *rays* of $\mathbb{R}^2$ from its origin, which are induced by the linear isomorphisms of $\mathbb{R}^2$. This group $\tilde{\mathcal{G}}$ acts by linear substitutions on the two fold cover of the space $\mathbb{R}P^n$, viz., the space S^n of all homogenous degree n *inequations* $a_n\mathtt{x}^n + a_{n-1}\mathtt{x}^{n-1}\mathtt{y} + \cdots + a_1\mathtt{x}\mathtt{y}^{n-1} + a_0\mathtt{y}^n \geq 0$, where at least one coefficient a_i is nonzero, in unknowns $\mathtt{x}$ and $\mathtt{y}$.

For $n \geq 2$, the n-sphere S^n is simply connected and $\mathbb{R}P^n$ is not, but S^1 is homeomorphic to $\mathbb{R}P^1$. This is why it is only for manifolds of dimension two that their symmetric powers are also manifolds, and FTA holds.

However, $(S^1,\tilde{\mathcal{G}})$ *is not geometrically equivalent to* $(\mathbb{R}P^1,\mathcal{G})$, i.e., no bijection $S^1 \to \mathbb{R}P^1$ induces an isomorphism of transformation groups $\tilde{\mathcal{G}}$ and $\mathcal{G}$:- possible fixed points of transformations in the first geometry are designated pairs, a ray and its opposite, but this is not the case for the second geometry. $\square$ We note that we'll sometimes use the word *geometry,* as we just did, in the formal sense of Klein, for a set with some specified bijections forming a group.

In fact, though $\tilde{\mathcal{G}}$ and $\mathcal{G}$ are topologically equivalent spaces, each with two components which retract to circles, algebraically they are inequivalent :- the component $\mathcal{G}_0$ of the identity of the baby group identifies with $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\pm I$, which is known − next note − to be a simple group, so it is not isomorphic to its covering group $\tilde{\mathcal{G}}_0 = SL(2,\mathbb{R})$. $\square$

On the other hand, its subgroup $SO(2,\mathbb{R})/\pm I$ is isomorphic to its cover $SO(2,\mathbb{R})$ under $A \mapsto A^2$:- for, *squaring* all matrices of $SL(2,\mathbb{R})$ maps its abelian subgroup $SO(2,\mathbb{R})$ onto itself with kernel $\pm I$. $\square$

These remarks alert us that the full baby action will be subtler, but promises to take us beyond the *foliation by circles* of note 49.

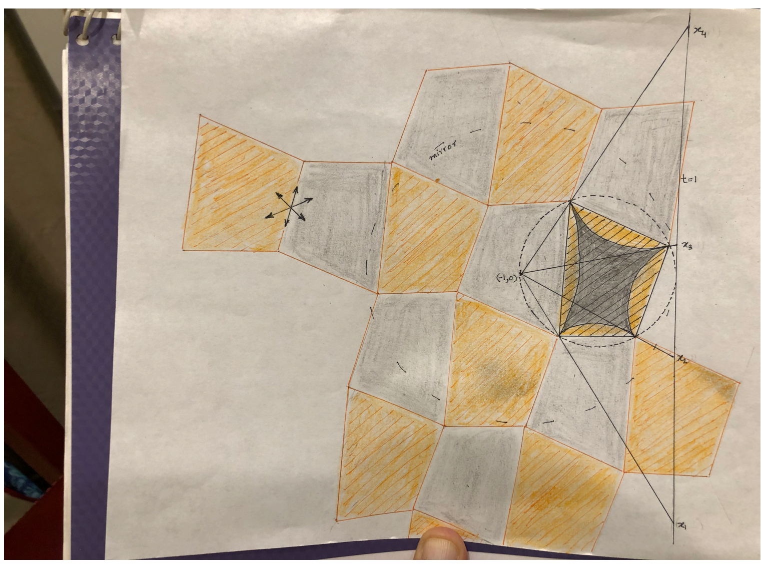
52. Regarding something we used above, it has been known for a hundred years that, *for* $n \geq 2$*, and* $\mathbb{F}$ *any field,* $PSL(n,\mathbb{F})$ *is a simple group, excepting* $PSL(2,\mathbb{F}_2) \cong S_3$ *and* $PSL(2,\mathbb{F}_3) \cong A_4$. An easy proof is given for example in a recent paper of Conrad. And strikingly, the simplicity of not only $PSL(2,\mathbb{F}_5) \cong A_5$, but of all $PSL(2,\mathbb{F}_p)$ for primes $p > 3$, goes back to Galois !

So $PSL(2,\mathbb{R})$ has no nontrivial quotient group, but it has, besides $PSL(2,\mathbb{Z})$ etc., uncountably many discrete subgroups : of symmetries of *relativistically* *regular tilings of a disk of radius* $c < \infty$! And these infinite tilings often cover, *finite regular tilings* $\{p, q\}$ *of closed surfaces.* Where now – the reader is urged to read and absorb the arguments given in Magic Carpet (2010) before going further – any $p \geq 3$ and $q \geq 3$ is possible.

Thus, *relativity destroys with a vengeance the dictum that there are only five regular solids* ! And, just like A_5 arises from the icosahedron $\{3, 5\}$, hordes of finite groups arise from these $\{p, q\}$'s, including, *all finite simple groups* ! For, it is apparently (= I did not read these papers in full) a corollary of their alleged (= no one has read these papers in full !) classification that, any finite simple group can be generated by just *two* elements, and two-dimensional relativity does give in this way, *all finite groups with two generators.*

It is moot how much of this was seen by Galois, but there is on page 380 in the 1870 treatise of Jordan, a method of solving any equation using functions, whose symmetries are those of a relativistic tiling.

Indeed, two-dimensional relativity is there in our theory of equations from its very inception, because – next note – *the relativistic geometry of a disk identifies with the baby geometry* $\mathcal{G}$ *of its bounding circle* !



53. We recall that the elements of $\mathbb{R}P^1$ are the lines through the origin of $\mathbb{R}^2$; cutting each with any line L not through the origin—say the line $t = 1$ as in the figure, the unique parallel line through the origin 'cuts' L at ∞—gives, *an isomorphism of baby geometry* $(\mathbb{R}P^1, \mathcal{G})$ *with the möbius geometry of the one-point compactification* $\widehat{L} = L \cup \infty$ *of a euclidean line* L :-

For, if a transformation of $\mathcal{G}$ is induced by the linear isomorphism $(x, t) \mapsto (ax + bt, cx + dt)$ of $\mathbb{R}^2$, then the corresponding transformation of $\widehat{L}$ maps any $(x, 1) \in L$ to ∞ or $(\frac{ax+b}{cx+d}, 1)$ depending on whether or not $cx + d = 0$, so ∞ to ∞ or $(\frac{a}{c}, 1)$ depending on whether or not $c = 0$. So it is a composition of euclidean isometries $(x,1) \mapsto (\pm x+\lambda, 1)$ and homotheties $(x,1) \mapsto (\mu x, 1), \mu > 0$ keeping ∞ fixed, and the *inversion* $(x,1) \leftrightarrow (\frac{1}{x}, 1)$ switching $(0,1)$ with ∞. $\square$

Likewise, *the möbius geometry of any compactified euclidean space* $\widehat{E} = E \cup \infty$ *is defined,* by all compositions of isometries of E and homotheties $x \mapsto \mu x$ of rays from a chosen origin keeping ∞ fixed, and the single inversion $x \leftrightarrow \frac{1}{x}$ of these rays switching this origin with ∞.

Or, more elegantly, *by all compositions of reflections in codimension one flat or round mirrors* :- For euclidean isometries flats suffice, translations being compositions of reflections in two parallel flats. And compositions of reflections in two concentric rounds are all homotheties. Any round mirror being the image of a chosen unit sphere under a homothety followed by a translation, conjugates of a single inversion give reflections in all rounds. $\square$

Further, *all mirrors—round of any positive radius or flat—are on the same footing.* Reflection in a round mirror makes spheres through its centre flats, with those also tangent to it becoming its tangent flats. *Plainly plane geometry suffices* to check such-like assertions : $\widehat{L}$ *is reflected to the tangent unit circle* S^1 *of* $\mathbb{R}^2$ *by the round mirror of radius* 2 *around* $-T$:-

Any spherical reflection switches points P, P' on rays from the centre C of the mirror at distances x and $\frac{1}{x}$ times its radius, that is $\frac{CP}{CT} = \frac{CT}{CP'}$, i.e., $\angle CTP = \angle CP'T$. Therefore, P is on L, i.e., $\angle CTP = 90^\circ$, if and only if $\angle CP'T = 90^\circ$, i.e., P' is on the circle with diameter CT. $\square$

So there is nothing special about ∞, and just like the euclidean geometry of E is generated by all mirrors through it, reflections in all mirrors through any point gives us, *a euclidean geometry in any point-complement* ! For example, the composition of two reflections in mirrors tangent at a point gives us a euclidean translation in its complement, etc.

These euclidean distances on point-complements are not preserved by their homotheties but *möbius transformations preserve or reverse angles* between intersecting planar circles or lines. If one cut angle is zero, if not they cut twice at equal but opposite angles, for two lines one of the cuts is at ∞.

Completing euclidean geometry in a natural way, möbius geometry remains just as *rigid.* Meaning, $\widehat{E}$ being a subset of a bigger dimensional $\widehat{F}$ fixes a möbius subgroup of the latter which restricts bijectively to all möbius transformations of the former :- any mirror of $\widehat{E}$, flat or round, is the intersection with $\widehat{E}$ of one and only one perpendicular mirror of $\widehat{F}$. $\square$

For example, the möbius group $\mathcal{G}$ of $\widehat{L}$ identifies with the möbius subgroup of $\widehat{\mathbb{R}^2}$ generated by all reflections in mirrors perpendicular to $\widehat{L}$. Hence, by using the reflection in the circle of radius 2 with centre $-T$, *we can identify the baby group* $\mathcal{G}$ *with the möbius subgroup of* $\widehat{\mathbb{R}^2}$ *generated by all reflections in circles and lines perpendicular to the unit circle* S^1.

Likewise, *the möbius geometry of any round n-sphere,* sitting in any euclidean m-space, $m > n$, is given by all its bijections that are restrictions of compositions of reflections in mirrors perpendicular to it. This definition avoids the use of ∞ and displays the homogeneity of this geometry from the outset.

There are some more transformations of the ambient $\widehat{\mathbb{R}^m}$ which preserve S^n

but for $m = n+1$ the quotient group is only $\mathbb{Z}_2$ and due to the reflection in mirror S^n itself. The möbius geometry of S^n is thus given by all transformations of $\widehat{\mathbb{R}^{n+1}}$ which preserve it as well as both components of its complement. The identified *möbius geometry of the open ball* B^{n+1} bounded by S^n is given by the restrictions to it of these very transformations.

Finally, *the möbius geometry of* S^n *identifies with the relativistic geometry of* B^{n+1} :- For, a radial *self-homeomorphism* of this closed $(n+1)$-ball identity on $\partial B^{n+1} = S^n$—that we met in Hyperbolic manifolds (2012)—magically and abruptly straightens all its curved mirrors cutting the boundary S^n normally, such that, reflections in them become restrictions of linear isomorphisms of $(n+2)$-space preserving the cone over the ball. □

So indeed the baby geometry $\mathcal{G}$ of S^1 identifies with the relativistic geometry of B^2. Also plainly it suffices to construct the above self-homeomorphism for this plane case. What all came out from a non-magical and non-abrupt way of doing this we'll consider in some notes below.

54. *Trying to understand entails giving the same thing different names and different things the same name.* These constantly varying nuances require the expressive power and informality of natural language to convey. But yes, now and then, we need to define this or that more formally, and make a proof or two to allay doubts. How far I have come so far in my attempt to (re)understand things from scratch in a cartesian manner is of course for you to judge, but from here on I plan to add remarks to clarify especially some seemingly odd usage that I have slipped into during this journey.

The adjective *relativistic* signals a natural change in the euclidean geometry of an open set that ought to be made if we are confined to it. The prototypical example is the geometry above of an open n-ball B of radius $c < \infty$, which for $n = 3$ along with a similar geometry of the cone over it is the one used in special relativity, and which for any n is rather turgidly often called 'the klein model of n-dimensional hyperbolic geometry'.

But there are far more general examples too in PG&R (2013) and its sequels. Let U be any connected open euclidean set such that any segment when extended exits it on at least one side, so it has a definite *cayley length,* then the relativistic distance between two points of U is the infimum of the cayley distance over all paths joining them made from finitely many segments.

Returning to the homogenous ball geometry, *in my opinion even the möbius geometry of a microphysical* B *should be deemed relativistic.* Maybe not for the denizens of this microworld, but for a macro observer probing from outside. For her that self-homeomorphism amounts to nothing, because it does not change the möbius geometry of the spherical interface, so thanks to its rigidity, this quantum physicist can now and then hear or see something of what is going on in this microworld as a möbius tiling, etc.

It is uncanny how often Arnol'd had already said aloud what I was wondering why no one had said before, but his 'mathematics is a part of physics' is fine with me only because for me *physics means cartesian physics.*

It is worth mentioning again in this context how because of and within a cartesian motion can be seen all closed manifolds living various diverse lives, and if this motion is relativistic they admit lipschitz charts, but it remains to tell you how this birthing motion equips these closed manifolds as well with smoothing operators, thanks to which the index formulas of Atiyah and others also come within the purview of cartesian physics.

Amid the pervasive tosh about modern high standards of rigour, it was refreshing to read Arnol'd saying drily, '*As far as I know, the criteria of rigor have not changed from the time of* Euclid.' Whose thirteen books are priceless because what is in them is about all we know for sure about the mathematics discovered till then by his predecessors. For example, his fifth and deepest book recorded the ideas of Eudoxus, who was even more rigorous.

But alas! the focus of attention shifted soon from the invaluable substance of these books to Euclid's style, his axiomatic method. Even here, with a little less focus, it would have dawned a lot sooner that, *Euclid III.36 gives a homogenous geometry not satisfying Euclid's fifth postulate* :-

Consider a circular arc in the open disk cutting its boundary normally. Its centre O, the intersection of the tangents to the boundary at these two cuts, lies outside the disk, and its radius r is the equal length of these two tangents from O. The cited proposition of Euclid tells us that a line from O between these tangents cuts the boundary of the disk in P and P' such that $OP \cdot OP' = r^2$, i.e., $OP/r = r/OP'$, i.e., the boundary and the disk are preserved by the reflection in the round mirror with centre O and radius r.

These round reflections together with those in the diameters of the open disk generate the 'baby group' $\mathcal{G}$ which acts transitively on all *points* of the open disk as well as all these *lines,* viz., circular arcs normal to boundary and diameters. In this geometry the parallel postulate is obviously false, but all arguments of Euclid not depending on it still hold. $\square$

In fact for *any* O at a distance R from the centre of our disk of radius c it is true that $OP \cdot OP' = R^2 - c^2$:- because the remaining case $R < c$ follows from the immediately preceding proposition *Euclid III.35.* $\square$

A less stern account written at the same level would have given us a far better idea of the mathematics of those times, for it is likely that many attractive facts known to its author were ruthlessly excluded because of his axiomatic method from Euclid's treatise.

For example, *repeated half turns around the midpoints of their sides will tile the entire plane by congruent copies of any given quadrilateral* :- experimentation suggests this is so, because the sum of the angles of a quadrilateral is 2π, but we need also the *simple connnectivity* of the plane. $\square$

So, *to each degree four homogenous equation with distinct extended real roots is tied such a quadrilateral tiling of the plane,* see figure :- the seed being the quadrilateral inscribed in S^1 whose vertices become, under reflection in the round mirror with radius 2 and centre $(-1, 0)$, these roots on $\widehat{L}$. $\square$

If instead we reflect relativistically the seed quadrilateral—or any inscribed polygon with three or more sides—in its sides and continue in this vein we can 'tile' the open unit disk B^2 with relativistically congruent copies of the seed minus its vertices, but these faux tiles are *not compact.*

So, *for any* $n \geq 5$ *we'll partially straighten the* n *sides of the inscribed curved* n*-gon,* with sides circular arcs normal to the boundary of the unit disk, so that they now become circular arcs normal to the boundary of a bigger concentric disk, of a radius c such that the sum of the angles of our new curved n-gon is exactly 2π, and then use this as seed, *to tile a bigger open disk of radius* c *by* n *möbius half turns around the midpoints of its sides* :-

55. Given $n > 4$ points on the unit circle, there exists a unique $1 < c < \infty$ which makes the n angles sum to 2π, for, as we straighten its sides more and more, the angle sum of the initial curved n-gon having these n points as vertices increases continuously from 0 towards $(n-2)\pi > 2\pi$.

We'll stop at this c, but note as $c \to \infty$ the concentric disks become the entire plane, and the limit of an increasing sequence of radial self-homeomorphisms of the closed unit disk straightens all its arcs.

This seed curved n-gon propagates to a crystallographic tiling in the möbius geometry of this open disk of radius c by using möbius half turns—conjugates of ordinary half turns around the centre—reversing one side of a laid tile. For, there is no local fitting problem since the angle sum is 2π.[1] And, if thus laying the tiles we reach a point along two different paths, the tile on it will be laid in the same way, since the disk is simply connected. □

If we like we can now use the radial self-homeomorphism $r \mapsto \tilde{r}$ of this disk of radius $c < \infty$, which maps circular arcs perpendicular to its boundary on line segments having the same end points, to change this tiling by curved n-gons to one by n-gons. The straightened seed n-gon has all its vertices on a concentric circle of a bigger radius $\tilde{1}$, and this relativistic tiling sprouts from it by *repeated central symmetries in midpoints of sides,* the only difference from the infinite radius or euclidean or classical $n = 4$ case being that we are now talking of the cayley distance of this open disk of radius c. □

But, *no single* $1 < c < \infty$ *works for all cardinality* n *subsets of the unit circle if* $n > 4$:- For, the closed disk of radius one, which covers only a finite *hyperbolic area* of this bigger open disk, has infinitely many such disjoint n-gons, which would then all have the same positive hyperbolic area prescribed – see §16.4 of my jail-book – by their *angle defect* from the euclidean case. □

Knowing someone's jail-book is helpful! The reason why I was struggling to understand some cutting-edge physics had dawned on me only after I learnt that the marooned-on-an-island-book of one of its leading exponents was a handbook on good old *special functions.*

The 2008 paper to which I just linked has, besides footnote 17 on this nickname for Coxeter's *Introduction to geometry* (1969), many other lovely lulus, for example this on the unity of mathematics on its page 10 : *the tiniest living bit of mathematics is enough to clone back the entire beast!*

Not only that, from a specific and barely 'living bit'—a wrong (!) formula for

[1] Neither is there any problem if the angle sum is $2\pi/j$, so this usage of half turns associates to each finite subset σ of the unit circle of cardinality $n \geq 3$ *an infinite discrete spectrum of tilings!* Tilings again of disks of radii $c_{\sigma,j} > 1$, but for $n = 3$ note $j \geq 2$. Except for cases $n = 3, j = 2$ and $n = 4, j = 1$ this $c_{\sigma,j}$ is finite, and as $j \to \infty$ it has limit one.

the area of a quadrilateral room—had sprouted naturally many threads of ideas, so fast and so diverse that, in my humble opinion, this criss-crossing *cartesian ideation,* if continued, would create the 'entire beast'.

Anyways, this was the cartesian genesis of at least Four Half Turns (2010), a roof-top tiling then, re-born now from degree four equations, a new context which has us seeing clearly in the same tiling, *a doubly periodic holomorphic function from the plane $\mathbb{C}$ to the riemann sphere S^2* :-

We still won't equip $\mathbb{R}^2$ with complex multiplication, for it suffices to make only a *cartoon*—cf., nice (2002)—of this meromorphic function.

The group generated by half turns around the midpoints of sides acts simply transitively on the tiling. Dividing the plane by its action gives S^2 because folding the four sides of a tile over midpoints gives 5 vertices, 4 edges and 1 cell, so euler number is $5-4+1=2$. This *quotient map* $\mathbb{R}^2 \to S^2$ is one-to-one near all points except these midpoints, near them it is two-to-one.

Further, we had observed that the roof-top tiling can also be laid by sliding along the quadrilateral's diagonals, so the quotient map $\mathbb{R}^2 \to S^2$ has these two vectors as its periods. Also, a fundamental region—other than the hackneyed parallelogram—exhibiting this double periodicity is drawn in the figure.[2] The quotient factorizes $\mathbb{R}^2 \to T^2 \to S^2$ into the usual unbranched infinite covering of the torus followed by *a two-fold covering of the sphere by the torus branched four times,* once over four of its five vertices. □

We're strolling towards roughly this *method for solving degree four* (similarly degree $n > 4$) *equations* :- from the equation we can write following Legendre (or Jacobi for $n > 4$) a suitable *line integral* of a closed one-form, amongst whose Cauchy *periods* around its singularities are the above two vectors, but the unit circle has at most two chords equal to a vector, so up to this ambiguity integration gives the seed quadrilateral and so the roots of the biquardatic, and a little extra work should overcome this ambiguity too.

56. *Swallowtail* ♡ means space of all homogenous equations of degree n in $\mathtt{x}$ and $\mathtt{y}$ with n distinct extended real roots. But to its affine subspace of degree n equations in $\mathtt{x}$ was also given the same name. And before that for $n=3$ the still smaller yellow subspace of Khayyam where sum of roots is zero was *morpankh,* partly because of shape, partly because it goes with his poetry.

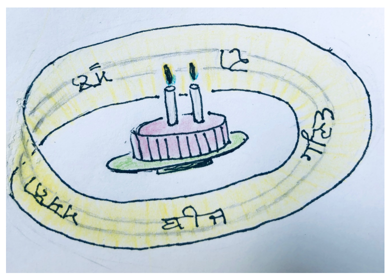
But I did not like the word peacock-feather, so during translation I changed the bird itself and started using swallowtail, even though I knew that Thom had used it before, but for a related singularity. Of this change of usage I have spoken before, it is somewhat like previously the common meaning of the word ball was what is now its boundary.

Amending Thom's ideas Arnol'd proved general classification theorems about singularities and wrote a wonderful book *Catastrophe theory* (1984). In it too swallowtails are various singular sets but it is clear—see pages 34, 37, 85, 89,

[2] Alas! the most natural fundamental region for this index two subgroup is not drawn, viz., *union of two adjacent tiles!* For $n=4$ a hexagon whose opposite sides identified give a torus, while for $n \geq 5$ it gives similarly a factorization $B^2 \to M^2 \to S^2$ of the quotient map into the universal covering map of a surface of higher genus, followed by a two-fold covering of the sphere branched once over n (if it is even) or $n+1$ points.

90, 101, 109, and exercises 5, 6, 15, 41, 52, 69, 70 and especially 77 given at the end of the 2004 English edition—that its author was aware of aspects of spaces of equations on which we have said not a word so far.

But returning to our story, *n-swallowtail $\heartsuit_n$ surjects on the circle $\widehat{L}$ with fiber $(n-1)$-ball and is not orientable for n even* :- As projection we can take the sum of the n distinct roots x_i where if one of them is ∞ sum will be ∞. All preimages are topological $(n-1)$-balls, that of ∞ being all equations with one root ∞. Deleting this last fibre gives us the affine n-swallowtail, an n-ball, which we can orient by the natural order of its finite real roots $x_1, \ldots, x_{n-1}, x_n$, but when the biggest root crosses infinity it becomes the smallest, so if the swallowtail were orientable its orientation should also be given by $x_n, x_1, \ldots, x_{n-1}$, but this has a different parity for n even. □ Indeed our story started two years ago with this very möbius strip for $n = 2$:-



The projection above, *the sum of the roots x_i*, is preserved by the translations of L keeping ∞ fixed; it does not show *how* the biggest root crosses infinity to become the smallest. In möbius geometry ∞ is just like any other point of the circle $\widehat{L}$, so it is better to think of $\widehat{L}$ after reflection in that round mirror of $\mathbb{R}^2$ as the unit circle S^1, and further, identify as usual the rotation subgroup of the baby group $\mathcal{G}$ acting on it with S^1 itself. So we have on S^1, thanks to this baby subgroup, a *baby product*—serious mathematicians call it complex multiplication!—and we'll now use as projection from the n-swallowtail, *the baby product of the points z_i of S^1*, corresponding to the n roots. This surjection is tied to and enables us to see the foliation by circles :-

The projection $\prod z_i$ covers the circle after angle $2\pi/n$ but only one S^1-orbit of the swallowtail closes so quickly, others may go as many times round as any divisor of n; *how many circuits an orbit makes before closing depends on how regular is the n-gon with vertices z_i;* one round sufficing iff this n-gon is fully regular, these equations being on one orbit which we'll call central; for example, *if n is prime all other orbits go n times around the central orbit.* □

Even the case $n = 3$ of cubics considered since Khayyam is interesting from the viewpoint of this topology, *the S^1-orbits on the boundary of the 3-swallowtail are type $\{3, 2\}$ torus knots* :- If z_1 and z_2 are tied respectively to a simple and a double root the periodicity of the projection $(z_1 z_2^2, z_1 z_2)$ on $S^1 \times S^1$ is $(2\pi/3, \pi)$. From what we saw before in the limit in the first direction the orbit will go around three times, in the other direction two rounds are needed, even when z_1

and z_2 are antipodal, because after rotation by π the multiplicities switch. On cubics with a triple root z, that is on the cuspidal orbit, the limit projection (z^3, z^2) is again type $\{3, 2\}$. $\square$

Zeeman's *The umbilic bracelet and the double-cusp catastrophe* (1976) treats a related geometry on the double cover S^3 – the action being now by matrices $A \in SO(2, \mathbb{R})$ rather than by pairs $\pm A$ – of the space $\mathbb{R}P^3$ of all cubic equations; the torus knots lift to bracelet unknots $\{3, 1\}$; and this catastrophe is close to the graph G on $\mathbb{R}P^3$ defined by the real roots of the cubics.

The *theory of equations* – yes, polynomial equations – that I am developing from the beginning in a novel way is of equations in one unknown $\mathtt{x}$ only. True to the adage, a spade should be called a spade, I have seeing this context been talking unhesitatingly from the outset of this or that *space of equations,* but I don't know why, all people whose work seemed close, I have found them all shirking from this natural usage!

Language becomes simpler without the explicit mention of the dual unknown $\mathtt{y}$ (or time $\mathtt{t}$). The thing is that instead of equations of degree n it is more natural to consider equations of all degrees $\leq n$. Then, provided we don't forget the unique *equation of degree zero,* these spaces are *compact.* For starters, the space of equations of degree one is a line – because we have these real numbers only so far – L, but those of degree ≤ 1 make a circle $\widehat{L}$, and proceeding onwards, we saw all of degree $\leq n$ make $\mathbb{R}P^n$.

But for $n \geq 2$ our focus is limited to an n-dimensional submanifold-with-boundary of $\mathbb{R}P^n$, *the closed n-swallowtail* $\overline{\heartsuit}_n$*,* the space of all equations in $\mathtt{x}$ of degrees $0 \leq j \leq n$ with all j roots real. We can deem its equations homogenously of degree n, saying that the remaining $n-j$ roots have gone to ∞. For example, the degree 0 equation $1 = 0$ has 0 real roots, but as an element of the closed n-swallowtail, it has ∞ as a root of multiplicity n.

So the n-swallowtail $\heartsuit_n$ is the space of all equations in $\mathtt{x}$ with n distinct extended real roots, meaning, of degree n or $n-1$ with roots distinct reals. The closure of the first *open n-cell* Σ_n is $\overline{\heartsuit}_n$ and of the second $(n-1)$-cell Σ_{n-1} its subset $\overline{\heartsuit}_{n-1} \subset \overline{\heartsuit}_n$. This inclusion is not preserved by the baby action, but the infinite increasing union gives *the space* $\overline{\heartsuit}_\infty$ *of all equations in* $\mathtt{x}$ *with all roots real,* which is *contractible.* In our opinion—see notes 50—the cyclic baby action on n-swallowtails should suffice for many results that have been proved by first complexifying it to the *Picard-Lefschetz action.*

Were we to use the dual unknown the n-swallowtail $\heartsuit_n$ would stay put but the cells Σ_n and Σ_{n-1} of which it is a disjoint union would change. For, one gets the dual of any equation by changing $\mathtt{x}$ to $1/\mathtt{x}$ and clearing denominators. That is, it is the equation with reciprocal roots from school. Similarly we changed the sign of all roots by changng $\mathtt{x}$ to $-\mathtt{x}$, subtracted the same number a from all roots by $\mathtt{x}$ to $\mathtt{x} + a$, and divided all roots by the same nonzero number b by $\mathtt{x}$ to $b\mathtt{x}$. So, *from school algebra it is clear that in the context of equations we should augment numbers by* ∞ *and use möbius or baby geometry* $\mathcal{G}$*.*

The homogeneity of baby geometry on the extended number line became clear when, using a round reflection of the extended plane, we started considering it as a geometry on the unit circle. And this gave us as well a subgroup of rotations $S^1 \subset \mathcal{G}$. Whose baby—i.e., by above substitutions from school—action is what foliates the swallowtail into circles. *Each S^1-orbit of $\heartsuit_n$ cuts the open $(n-1)$-cell Σ_{n-1}, and can cut it as many times as any divisor of n* :- for example, Σ_1 is cut by the central orbit $z_1 = -z_2$ of the möbius strip $\heartsuit_2$ just once, and twice by all other orbits; likewise for any n how often an orbit cuts depends on how regular is the n-gon $\mathrm{conv}(z_1, \ldots, z_n)$. □

The circle $\overline{\heartsuit}_1 \subset \overline{\heartsuit}_2$ meets the boundary of the latter in only the equation of degree zero, their union is thus a *figure eight.* The union of such a figure eight on the boundary of the solid torus $\overline{\heartsuit}_3$ and the 2-cell Σ_2 in its interior makes the closed möbius strip $\overline{\heartsuit}_2 \subset \overline{\heartsuit}_3$. *Any given point of the swallowtail $\heartsuit_3$ is the intersection of three such möbius strips* :- the cell Σ_2 comprised all cubics with one root ∞; take the three like cells of all cubics with one root a, b or c where these are the three distinct roots of the given cubic. □

The same game is played in the n-swallowtail $\heartsuit_n$. For each extended real number a there is the $(n-1)$-cell comprising equations having a as a root. As against the pairwise disjoint $(n-1)$-cells arising as the fibers of the projection map $z_1 \cdots z_n$, this family of cells is only $(n+1)$-wise disjoint, but each point of $\heartsuit_n$ is the intersection of n such cells. Further, *the baby group $\mathcal{G}$ acts transitively on the family of all intersections of triples of these $(n-1)$-cells, so if $n \leq 3$ then $\heartsuit_n$ is a single $\mathcal{G}$-orbit* :- any one of three numbers can be mapped to ∞ by an inversion, keeping ∞ fixed we can then use an isometry of the line to map any one of the other two on 0 such that the third is positive, finally a homothety keeping ∞ and 0 fixed maps this third to 1. □

I am very grateful to the Creator for his blessings, and if it so pleases him will try to push this story a bit further.

K S Sarkaria 11-30 April, 2019