The prettiest composition, part two
(translation of ਸਭ ਤੋਂ ਪਿਆਰੀ ਰੱਚਣਾ, ਭਾਗ ਦੂਜਾ)

12. So, note 11 ties solving cubics outside the peacockfeather to 'doubling' the disjoint half-lines $\textcircled{\alpha}$. Likewise, doubling for each $k > \frac{1}{2}$ the projection map $(x, y, z) \mapsto (x, y)$ of the shape $S \subset G$ above the segment $y = k$ ending on the red curves, is tied to solving the cubics in it. This doubling triple-wraps a circle on another circle :- both end-points of the segment also have now three pre-images, because the pre-image in the interior has two copies. □

13. The thing is that real al jabr, with trisection of angles, suffices to solve the cubics in the peacockfeather. For, the above arc $S \subset G$ is described by $x = X \cos 3t, y = k, z = Z \cos t$, where Z and $-Z/2$ are the two roots at the right end (X, k) of our segment :- $k = \frac{1}{2} + \frac{3}{8}Z^2$ and $X = \frac{1}{8}Z^3$ by note 4, so the point $(Xf(t), k, Z \cos t)$ is on the surface $2x + 2yz = z + z^3$ for all t iff $f(t) = 4\cos^3 t - 3\cos t$, i.e., $f(t) = \cos 3t$. □ To solve cubics convert each sinusoidal motion of amplitude $X = X(k)$ to three sinusoidal motions of amplitude $Z = 2X^{\frac{1}{3}}$, one third the frequency, and phase differences 120 degrees.

14. Algebra as we know it today began with Vieta's analysis of Cardano's al jabric recipe of note 11, besides he gave the above non al jabric recipe for solving the remaining cubics. Using the former, the endless whiskers of the shape S in the section $y = k$ of G have the formula $z = \Sigma\{x \pm \sqrt{x^2 - X^2}\}^{\frac{1}{3}}$:- for, any cubic $\odot = (-\frac{B}{2}, -\frac{A-1}{2})$ below the red curves has the root $\alpha = \Sigma\{\frac{-B \pm \sqrt{B^2 + 4A^3/27}}{2}\}^{\frac{1}{3}}$; now put $\alpha = z, B = -2x, A = 1 - 2k$ where $k = \frac{1}{2} + \frac{3}{2}X^{\frac{2}{3}}$. □

15. Calculations with 'imaginaries' were raging already, but it was about two hundred years later, i.e., the same time ago, when Argand let out that these 'unreal' numbers $\mathbb{C}$ were in fact twice as real : $\mathbb{R}^2$ with addition of coordinates, and a product that at once multiplies distances from the origin and adds angles. If these complex numbers $\mathbb{C}$ are used the above al jabric formula solves all cubics $\odot = (-\frac{B}{2}, -\frac{A-1}{2}) \in \mathbb{C}^2 = \mathbb{R}^4$ of any 2-plane $y = k$, with $(\pm X, k)$ now the cubics on it with two equal roots, and we sum those values of the two three-valued complex surds which extend their solutions.

16. Reverting to $\mathbb{R}$, we note that all points of G above $x = 0$, i.e. the line $\textcircled{0}$ of all $\odot$ having one root 0, satisfy $z = 0$ or $z = \pm\sqrt{2y - 1}$, two real al jabric formulas. Further, over the half-line $y \geq \frac{1}{2}, x = 0$, just the one al jabric formula $z = \pm\frac{1}{2}\sqrt{2y - 1} \pm \frac{1}{2}\sqrt{2y - 1}$ suffices, which shows that G having three sheets does not obstruct such a description. More generally, points of G above any $\textcircled{\alpha}$, i.e., the line $2x + 2\alpha y = \alpha + \alpha^3$, satisfy $z - \alpha = 0$ or $2y = 1 + z^2 + \alpha z + \alpha^2$, i.e., $z = \alpha$ or $z = -\frac{1}{2}\alpha \pm \frac{1}{2}\sqrt{8y - 4 - 3\alpha^2}$, both real al jabric formulas, at least if $\alpha \in \mathbb{Q}$. But, for $\alpha \neq 0$, we cannot describe G over the half-line $y \geq \frac{1}{2} + \frac{3}{4}\alpha^2$ of $\textcircled{\alpha}$ by *one* real al jabric formula :- because there's now, as in note 8, a topological ess obstruction to this over a segment of this half-line. □

17. We used, if a real al jabric formula is not single-valued, its graph admits a homeomorphism of order two preserving the projection map:- Multi-valuedness can stem only from the two-valued surds $\pm(\ \ )^{\frac{1}{2n}}$ in the formula. Switching the

values of any surd gives a continuous map of the graph on itself which preserves its fibers. Since our formula has more than one output for some input, at least one of these self-inverting maps is other than the identity. $\square$

18. Also we used, only the identity homeomorphism of an ess preserves its projection map:- More generally, if a projection map $S \to S'$ folds a segment an odd number of times, such a homeomorphism of S keeps both its ends fixed, so by unique continuation above $\mathrm{int}(S')$ it keeps all of S fixed. $\square$ It follows that: only the identity homeomorphism of G preserves its projection map.

19. There is an ess of G over a given line only on the bridge if any between the two red curves. So there is no ess over a parallel to the y-axis, while over a parallel to the x-axis there is an ess iff it is above the cusp, and over any other line there is an ess iff it is the parallel @ or separates it from the cusp. The two bends of the ess above @ are the vertex of the parabola with axis $z = -\alpha/2$ and its intersection with the line $z = \alpha$, the two curves whose union constitutes this plane section of G – see notes 9, 16 – and the bridge under this ess consists of all points of @ such that $\frac{1}{2} + \frac{3}{4}\alpha^2 \leq y \leq \frac{1}{2} + \frac{3}{2}\alpha^2$. We note that the second of these two bends is not smooth, both bends are smooth only for the esses that occur above the parallels to the x-axis.

20. My simple proof that G, which is the graph of an algebraic (multi-valued) function, is not the graph over some segments of any al jabric—in the sense of note 8—formula seems to be new. A quadratic-like formula for all cubics was the dream of al qaidas or cookbooks, from Khwarizmi through Cardano, that is, of al jabr, which was to slowly become algebra. The much finer work done by Khayyam et al in the seven centuries separating these two was all but lost then. Had this not been so, maybe Descartes a bit later, who had flirted with spaces of equations, would have seen an impossibility result for cubics, and algebra would have started differently ...

21. ... but all seemed possible with 'imaginaries', that is, with just the *one* extra dimension of $\mathbb{C} = \mathbb{R}^2$. Not only are all degree 3 equations solvable complex al jabrically—note 15—this extra room magically (now obviously) ensures that, complex surds $(\ \ )^{\frac{1}{n}}$ are n-valued, de Moivre. So, the hope became pervasive that all degree n equations can be solved completely in $\mathbb{C}$, *and that too al jabrically*. The first part, the fundamental theorem of algebra, was proved by d'Alembert et al in various different ways by reducing to $\mathtt{x}^n - a = 0$, but none could be honed to get the second part, because of the very good reason that, *the second part is false for $n > 4$*. One reason why Ruffini's long impossibility proof remained unread—however Cauchy thought it was correct—may have been that it came when this hope was still alive; on the other hand, Abel's clearer proof came when this hope had all but died down.

22. To understand the involutions given by switching the values of a real surd– note 17–we consider now an example $z = \pm\sqrt{x^2+y^2-1} \pm \sqrt{x^2+y^2-1}$. This formula is not defined in the unit circle, has just one value $z = 0$ on it, and three distinct values $z = \{2\sqrt{x^2+y^2-1}, 0, -2\sqrt{x^2+y^2-1}\}$ outside it. The homeomorphisms of its graph preserving the projection map permute these

three sheets in any which way, but only the transposition of the first and the third arises from its involution. So, the group of an al jabric formula, i.e., the group generated by its involutions, can be smaller than the group of covering transformations of its graph.

23. These involutions need not commute to make $(\mathbb{Z}/2)^k$ but, for sure, the group of a real al jabric formula is a twisted power of the group of two elements:- The formula's surds are nonzero on an open dense subset of its domain. On this open set its graph is that of a finite set of single-valued real continuous functions and its involutions bijections of this finite set. All such functions form a big space closed under pointwise addition, subtraction, multiplication, and division by a nowhere zero function. Also, there is a much smaller closed intermediate function space on which interchanging the values of our surds defines automorphisms, i.e., bijections preserving these four operations. To construct this we start with the functions x and y and all those made from them by using these four operations, then at each step we take, the one or two functions given by a surd not already made which is used next in our formula, and again make all functions possible by using the four operations, and so on. The group arising from the surds already included before each step preserves the function subspace made, and is either equal to the new group after this step, or else is a subgroup of index two, so a normal subgroup, of the new group. $\square$

24. Even the restriction of G to an open set of the peacockfeather is not the graph of any real al jabric formula :- If not, its three sheets form the above 'finite set' of this formula, and these three and x, y generate a closed subspace of its 'intermediate function space'. So, by the last note, the group of automorphisms of this subspace over $\mathbb{Q}(x, y)$ would only have elements of orders 2^k. On the other hand – note 31 below – on any open set of the peacockfeather any permutation of the sheets of G extends to an automorphism. $\square$

25. That abstruse object which pops up in all algebra books in this context, the general equation $\mathtt{x}^n + u_{n-1}\mathtt{x}^{n-1} + \cdots + u_1\mathtt{x} + u_0 = 0$, we'll visualize, by identifying each particular equation with $(u_0, \ldots, u_{n-1})$, as $\mathbb{R}^n$. Therefore, the general equation with sum of the roots zero, being, the space of all degree n equations with sum of the roots zero, is $\mathbb{R}^{n-1}$, and the plumage of another bird, a swallowtail, will denote, the open subset of $\mathbb{R}^{n-1}$ formed by all equations $\mathtt{x}^n + u_{n-2}\mathtt{x}^{n-2} + \cdots + u_1\mathtt{x} + u_0 = 0$ with n distinct real roots. Its closure is all such equations with roots all real but not necessarily distinct. The graph of the (multi-valued) function which associates to an equation all its real roots hovers above in one dimension more, denoting this variable by z, the graph above $\mathbb{R}^{n-1}$ is given by $z^n + u_{n-2}z^{n-2} + \cdots + u_1 z + u_0 = 0$.

26. For $n = 3$ we had used $u_0 = B, u_1 = A$, so the swallowtail is the same as the peacockfeather except for a change of coordinates, $x = -\frac{u_0}{2}, y = -\frac{u_1 - 1}{2}$, which had changed the equation of the line @ of cubics having one root α from $u_0 + \alpha u_1 + \alpha^3 = 0$ to $2x + 2\alpha y = \alpha + \alpha^3$, the perpendicular bisector of the segment joining $(0, 0)$ to (α, α^2). Likewise, degree n equations with sum of roots zero and one root α form a hyperplane @ of $\mathbb{R}^{n-1}$, viz., $u_0 + \alpha u_1 + \cdots + \alpha^{n-2}u_{n-2} + \alpha^n = 0$, which, if we put $y_0 = -\frac{u_0}{2}, y_1 = -\frac{u_1 - 1}{2}, \ldots, y_{n-2} = -\frac{u_{n-2} - 1}{2}$, becomes,

$2y_0 + 2\alpha y_1 + \cdots + 2\alpha^{n-2}y_{n-2} = \alpha + \alpha^2 + \cdots + \alpha^{n-2} + \alpha^n$. So again, $\textcircled{\alpha}$ is perpendicular to the segment joining the origin to the point $(\alpha, \alpha^2, \ldots, \alpha^{n-1})$ of a moment curve, but for $n > 3$ is not through its mid-point. On the other hand, it bisects, but not perpendicularly for $n > 3$, the segment joining the origin to the point $(\alpha, \ldots, \alpha, \alpha^2)$ of a parabola. So, maybe Khayyam's method extends to all n with a relativistic or cayley distance on $\mathbb{R}^{n-1}$? That is, we can solve any $\odot \in \mathbb{R}^{n-1}$ by drawing around it the ellipsoid of all points at the same cayley distance as an origin, and examining its cuts on a fixed moment curve? We note that in peacock coordinates the graph G over $\mathbb{R}^{n-1}$ is given by $2y_0 + 2zy_1 + \cdots + 2z^{n-2}y_{n-2} = z + z^2 + \cdots + z^{n-2} + z^n$.

27. A kissing circle of $y = x^2$ is a khayyam circle only at $(0, 0)$:- for it has a contact of order 3 but the sum of the 3 roots is zero. $\square$ The cusp – note 2 – is on the evolute $y = \frac{1}{2} + \frac{3}{4}(2x)^{\frac{2}{3}}$—all centres of kissing circles—of the parabola, but the two red curves—centres of all circles through $(0, 0)$ which touch the parabola at some other point—are above it. All lines tangent to the red curves are $\textcircled{\alpha}$, $\alpha \neq 0$ – note 4 – they are not all lines normal to the parabola.

28. There is a cusp and 'red curves' on the boundary of any swallowtail, viz., the equation with all n roots 0, and all those with a nonzero root α repeated $n-1$ times (for $n > 3$ there is also much else). These two curves are traced for $\alpha < 0$ and $\alpha > 0$ by the functions $u_0(\alpha), \ldots, u_{n-2}(\alpha)$ defined by $\mathtt{x}^n + u_{n-2}\mathtt{x}^{n-2} + \cdots + u_1\mathtt{x} + u_0 \equiv (\mathtt{x} - \alpha)^{n-1}(\mathtt{x} + (n-1)\alpha)$. The hyperplanes $\textcircled{\alpha}$, $\alpha \neq 0$, are the osculating planes of these curves :- for $\textcircled{\alpha}$ has a contact of order $n-1$ with the curve at the point $(\ldots, u_i(\alpha), \ldots)$. $\square$ The real roots of any equation $\odot \in \mathbb{R}^{n-1}$ are given by the osculating planes through it, in particular, if $\odot$ is in the swallowtail there are n of these. This weak generalization of Khayyam's method is not quite solving equations, but shows again like note 5 that, graph G is the disjoint union of the parallel $(n-2)$-dimensional flats α^* above $\textcircled{\alpha}$ at height α. This implies that G is homeomorphic to $\mathbb{R}^{n-1}$, but for n even, its projection—the complement of the open set of these equations with no real root—is homeomorphic to a closed $(n-1)$-dimensional half space.

29. The point $(0, 0)$ of $y = x^2$ is really not all that special, in fact Khayyam's method works for all degree four equation $\mathtt{x}^4 + A\mathtt{x}^2 + B\mathtt{x} + C = 0$, but we have been looking only at the 2-plane $\textcircled{0}$ of these equations with 0 as a root, i.e., the subspace $C = 0$ of 'cubic equations'. Indeed, any four points (α, α^2), (β, β^2), (γ, γ^2), (δ, δ^2) with $\alpha + \beta + \gamma + \delta = 0$ lie on a circle :- Rewrite $\alpha^4 + A\alpha^2 + B\alpha + C = 0$, etc., as $-C - B\alpha - (A-1)\alpha^2 = \alpha^2 + \alpha^4$, etc., so $-B(\beta - \alpha) - (A-1)(\beta^2 - \alpha^2) = (\beta^2 - \alpha^2) + (\beta^4 - \alpha^4)$, etc., which shows that $(-\frac{B}{2}, -\frac{A-1}{2})$ is on the right bisectors $2(\beta - \alpha)x + 2(\beta^2 - \alpha^2)y = \beta^2 - \alpha^2 + \beta^4 - \alpha^4$ of the segments joining (α, α^2) to (β, β^2), etc. $\square$ So, if we think of $\odot = (-\frac{B}{2}, -\frac{A-1}{2}, -\frac{C}{2})$ as the above equation, it is on the line common to the right bisecting planes $2(\beta - \alpha)x + 2(\beta^2 - \alpha^2)y = \beta^2 - \alpha^2 + \beta^4 - \alpha^4$, etc., of the segments joining the pairs of points $(\alpha, \alpha^2, 0)$ given by its real roots. To solve the equation, draw with $\odot$ as centre the 2-sphere with diameter $\sqrt{B^2 + (A-1)^2 + (C-1)^2 - 1}$ and look for all points $(t, t^2, 0)$ on it, these values of t will give all the real roots.

30. Thom's swallowtail, the subset of $\mathbb{R}^3$ given by all $\mathtt{x}^4 + A\mathtt{x}^2 + B\mathtt{x} + C = 0$

with a multiple real root—ours is a component of its complement, the other two have all equations with no or two real roots—is the union of the tangent lines of the 'red curves' :- for, the intersection of osculating planes ⓤ∩@ is all equations with u and α as roots, and approaches a tangent line when $u \to \alpha$. $\square$ Similarly, the subset of $\mathbb{R}^{n-1}$ of all $\mathtt{x}^n + u_{n-2}\mathtt{x}^{n-2} + \cdots + u_1\mathtt{x} + u_0 = 0$ with a multiple real root is the union of codimension two flats kissing the red curves, and such flats of higher codimensions stratify this singularity. From Arnol'd's famous (save this I know little about it so far!) classification theorem it seems that if our interest is only in the topology of singularities we can take many u_i to be zero. Khayyam's method also works for degree n equations with many $u_i = 0$, but if $n > 4$ then for a complete solution we should perhaps use this natural linear structure on the swallowtail?

31. Long before Thom, Cayley had seen his swallowtail arise in an ellipsoidal wavefront, and some decades later Kronecker had fully described it. The latter also opened an easier path to Ruffini-Abel by showing that any polynomial splits in a 'unique' field—see Artin, *Galois Theory,* pages 29-32, and its pages 74-76 on applications by Milgram—which implies that any permutation of its n roots extends to an automorphism. From note 25 it is evident that 'a general equation' is a topological notion, so Ruffini and Abel were grappling with a problem of topology from the outset (this point was made by Arnol'd in some lectures but—alas!—he did not write their notes himself) and the argument given on the cited pages is at heart topology. Making some obvious changes it shows that – note 24 – on any open set of a swallowtail, any permutation of the n sheets of G extends to an automorphism.

32. As in note 6, if we delete from the pull-back of the closed swallowtail in G the points above its boundary there remain, n copies of our swallowtail, all tied to it by $(u_0, \ldots, u_{n-2}, z) \mapsto (u_0, \ldots, u_{n-2})$. Above the points of the swallowtail with sums of the positive and negative roots ± 1 they give the n open top cells of a subdivided open $(n-2)$-ball. Of their $n!$ permutations very few preserve the lower strata, but starting from a doubling as in note 12, maybe even Vieta's method of note 13 can be extended to all n-swallowtails ? [Today 23/04/18 I still can't say, but it is for sure that

33. Khayyam's method extends to all n, using Euclid's distance only, but a broader notion of a moment curve :- This curve P of $\mathbb{R}^{n-1}$ is made by the mirror images in the hyperplanes @ of a fixed $p \in \mathbb{R}^{n-1}$. In swallow coordinates @ has equation $\alpha^n + u_{n-2}\alpha^{n-2} + \cdots + u_1\alpha + u_0 = 0$ and the line through $p = (p_0, p_1, ..., p_{n-2})$ perpendicular to it is $u_0 = p_0 + t, u_1 = p_1 + \alpha t, \ldots, u_{n-2} = p_{n-2} + \alpha^{n-2} t$. The value $t(\alpha)$ of the parameter t at their intersection is therefore $t(\alpha) = -\frac{\alpha^n + p_{n-2}\alpha^{n-2} + \cdots + p_1\alpha + p_0}{\alpha^{2(n-2)} + \cdots + \alpha^2 + 1}$ and the mirror image of p in @ is $P(\alpha) = (p_0 + 2t(\alpha), p_1 + \alpha 2t(\alpha), \ldots, p_{n-2} + \alpha^{n-2} 2t(\alpha)), \alpha \in \mathbb{R}$. The ratio of the successive coordinates of $P(\alpha) - p$ being all α, we can read the real roots α of any equation $\odot \in \mathbb{R}^{n-1}$ from the cuts made on the curve P by the $(n-2)$-sphere with centre $\odot$ passing through p. $\square$ If $n = 3$ and $p = (0, 1)$ then P is Khayyam's own parabola, and if $n = 4$ and $p = (1, 0, 1)$ essentially that of note 29, etc. Can we

get rid of this distortion $t(\alpha)$ by using a spherical metric to now solve all degree n homogenous real equations $u_n \mathtt{x}^n + u_{n-1}\mathtt{x}^{n-1}\mathtt{y} + \cdots + u_1 \mathtt{x}\mathtt{y}^{n-1} + u_0 \mathtt{y}^n = 0$ in the same manner ? But for this shortcoming, and yes but for complex roots, it is very interesting that one solution of the fundamental problem of algebra is obtained by simply extending Omar's prettiest composition !]

A lot remains that I'd planned to say, but now this story shall continue—with God's grace—during my next orbit around our star. April 11, 2018.